

Deformations of Minimal Lagrangian Submanifolds with Boundary

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Abstract

Let L be a special Lagrangian submanifold of a compact Calabi-Yau manifold M with boundary lying on the symplectic, codimension 2 submanifold W . It is shown how deformations of L which keep the boundary of L confined to W can be described by an elliptic boundary value problem, and two results about minimal Lagrangian submanifolds with boundary are derived using this fact. The first is that the space of minimal Lagrangian submanifolds near L with boundary on W is found to be finite dimensional and is parametrized over the space of harmonic 1-forms of L satisfying Neumann boundary conditions. The second is that if W' is a symplectic, codimension 2 submanifold sufficiently near W , then under suitable conditions, there exists a minimal Lagrangian submanifold L' near L with boundary on W' .

1 Introduction and Statement of Results

A minimal Lagrangian submanifold of a symplectic manifold M is at once minimal with respect to the metric of M and Lagrangian with respect to the symplectic structure of M . Furthermore, when M is a Calabi-Yau manifold, Harvey and Lawson showed in their seminal paper [3, Section III] that minimal Lagrangian submanifolds are also *calibrated*. A consequence of this property is that minimal Lagrangian submanifolds satisfy a relatively simple geometric PDE (simple relative to the equations of vanishing mean curvature and symplectic form, which they would satisfy by virtue of minimality and being Lagrangian separately). Together, the minimal *and* Lagrangian conditions lead to the following system of equations: $L \subset M$ is minimal Lagrangian if and only if

$$\begin{aligned} \operatorname{Im}(e^{i\theta}\alpha)|_L &= 0 \\ \omega|_L &= 0, \end{aligned} \tag{1}$$

for some real number θ . Here, ω is the symplectic form of M and α is the canonical, non-vanishing, holomorphic $(n, 0)$ -form guaranteed by the Calabi-Yau structure of M .

The *calibration form* defined on M is in this case $\operatorname{Re}(e^{i\theta}\alpha)$ and thus $\operatorname{Re}(e^{i\theta}\alpha)|_L = \operatorname{Vol}_L$. The submanifold L is also referred to as *special Lagrangian with phase angle θ* in the literature, while if L is minimal Lagrangian with phase angle $\theta = 0$ then L is simply called *special Lagrangian*.

Harvey and Lawson and others, for example, have exploited the geometric structure implicit in the calibration condition in order to tackle questions related to the existence of minimal Lagrangian submanifolds. Harvey and Lawson themselves produce several examples of minimal Lagrangian submanifolds and give certain general constructions of such objects. More recently, Schoen and Wolfson [17] have presented another construction based on variational methods and are investigating the singularities that can arise there, while Haskins [4] has constructed new examples of special Lagrangian submanifolds and cones. The topic of singularities of special Lagrangian has recently

seen many further advances, for examples in papers by Joyce [6, 7, 8, 9, 10, 11, 12], stimulated by recent developments in string theory and mirror symmetry [15].

Another approach for producing minimal Lagrangian submanifolds involves studying the *deformations* of a given minimal Lagrangian candidate L and selecting those deformations of L which preserve the minimal Lagrangian condition. A deformation of a submanifold $L \subset M$ is a C^1 family of embeddings $f_t : L \rightarrow M$ of L , where f_0 is the canonical embedding. The goal of this analysis is to characterize of the space of submanifolds *near* L which are still minimal and Lagrangian, and is carried out by analysing the equations satisfied by minimal Lagrangian submanifolds using perturbative techniques in the form of the Implicit Function Theorem.

The first results in this area were obtained by McLean [14, Section 3] and extended by Hitchin [5]. Using equations (1), McLean identified the first order deformations of a special Lagrangian submanifold in a Calabi-Yau manifold and developed a method for integrating them. He used this to characterize the space of special Lagrangian submanifolds near L according to the following theorem.

Theorem (McLean, 1996). *Let M be a compact, Calabi-Yau manifold. The space of special Lagrangian submanifolds sufficiently near a given candidate $L \subset M$ is finite dimensional and is parametrized over the set $\mathcal{H}^1(L)$ of harmonic one-forms of L .*

McLean's work has been extended to the case where M is symplectic by Salur [16]. The work presented in this paper extends McLean's result in another direction — this time to the case of a minimal Lagrangian submanifold of a Calabi-Yau manifold M with non-empty boundary. This will be done by first creating a framework for incorporating boundary conditions into the minimal Lagrangian differential equations. A theorem characterizing the space of minimal Lagrangian submanifolds with boundary near a given candidate can then be formulated that is analogous to McLean's result for submanifolds with empty boundary.

More precisely, the boundary conditions will arise through geometric restrictions on the deformations of the special Lagrangian submanifolds, and an object that will be called a *scaffold* will be used for this purpose.

Definition 1. Let L be a submanifold of M with boundary ∂L and inward unit normal vector field $N \in \Gamma(T_{\partial L}L)$. A *scaffold* for L is a smooth submanifold W of M with the following properties:

1. $\partial L \subset W$;
2. $N \in \Gamma(T_{\partial L}W)^\omega$ (here, S^ω denotes the symplectic orthogonal complement of a subspace S of a symplectic vector space V , defined as $S^\omega \equiv \{v \in V : \omega(v, s) = 0 \forall s \in S\}$);
3. The bundle $(TW)^\omega$ is trivial.

REMARKS: Condition (2) is a transversality condition that ensures that JN is perpendicular to W , where J is the complex structure of M . Since W is symplectic, N can not be parallel to W . It seems reasonable to expect the Main Theorem to hold with condition (2) replaced by unconstrained transversality of N, JN to the tangent space of W along ∂L , but this weaker assumption leads to technical problems later on. In particular, the boundary value problem appearing in the analysis of the linearized operator in Section 3.2 is the Hodge system with *oblique* boundary conditions rather than the Hodge system with Neumann boundary conditions. Since this BVP is more difficult to deal with and leads to geometrically less natural results, the Author has avoided it here. Furthermore, it is possible that the most geometrically natural type of scaffold is when W is a *complex* submanifold of M (i. e. the tangent spaces of W are invariant under J) and this automatically satisfies the transversality condition (2) [19].

FURTHER REMARKS: Condition (3) will be used in the sequel in order to make certain constructions on W possible; also, the above definition of a scaffold has already been used in [2].

The main theorem to be proved in this paper uses a scaffold to introduce a boundary condition according to the following statement.

Main Theorem (Boundary Deformation Theorem). *Let L be a special Lagrangian submanifold of a compact Calabi-Yau manifold M with non-empty boundary ∂L and let W be a symplectic, codimension two scaffold for L . Then the space of minimal Lagrangian submanifolds sufficiently near L (in a suitable $C^{1,\beta}$ sense to be defined later on) but with boundary on W is finite dimensional and is parametrized over the harmonic 1-forms of L satisfying Neumann boundary conditions.*

This theorem, the analogue of McLean's theorem for special Lagrangian submanifolds with boundary, confines ∂L to move only along the scaffold W , and imposes the boundary condition in the following way. If $f_t : L \rightarrow M$ is a boundary-confining deformation, then $f_t(\partial L) \subset W$ for all t . Consequently, the deformations field $V = \frac{d}{dt} f_t|_{t=0}$ can not be arbitrary: it must be tangent to W at ∂L .

REMARK: Another important difference between the Boundary Deformation Theorem and its predecessor is that deformations amongst all *minimal* Lagrangian submanifolds are allowed here and not just amongst special Lagrangian submanifolds. This will turn out to be a necessary ingredient of the proof.

At the end of this paper, the Boundary Deformation Theorem will be used to prove an existence result for minimal Lagrangian submanifolds with boundary in M . A corollary will be proved which demonstrates the existence of minimal Lagrangian submanifolds near L with boundaries on neighbouring scaffolds.

Corollary (Scaffold Deformation Theorem). *Let L be a special Lagrangian submanifold of a Calabi-Yau manifold M and let W be a symplectic, codimension two scaffold for L . Furthermore, suppose that the topology of L is such that its first Betti number $b^1(L)$ vanishes (and thus L has no non-trivial harmonic one forms with Neumann boundary conditions). Then if W' is any symplectic, codimension two submanifold of M that is sufficiently near W in the same sense as in the Main Theorem, then there is a minimal Lagrangian submanifold L' , near L and with boundary on W' .*

The remainder of this paper will be organized in the following manner. In Section 2, the boundary value problem describing minimal Lagrangian submanifolds with boundary on a scaffold is formulated, and in Section 3, the proof of the main theorem is undertaken by solving this boundary value problem. Since the Implicit Function Theorem is to be used for this purpose, the linearized operator corresponding to the BVP must be calculated there and shown to be surjective with finite dimensional kernel isomorphic in a suitable sense to the harmonic 1-forms of L . The corollary of the Main Theorem is then proved in Section 4 using the machinery constructed in the preceding sections.

2 Formulating the Boundary Value Problem

2.1 Introduction

For the remainder of this paper, assume that L is a given, fixed *special* Lagrangian submanifold with boundary that is contained in an ambient Calabi-Yau manifold M , and that M possesses a metric g , a symplectic form ω , and compatible complex structure J . Denote by α the canonical, holomorphic, non-vanishing $(n,0)$ -form of M . Furthermore, assume that L is connected; the results for non-connected L follow simply by considering each component of L separately. The equations (1) satisfied by minimal Lagrangian submanifolds suggest the definition of a map whose zero set corresponds to the minimal Lagrangian submanifolds near L . Suppose that the dimension of L is n . Let $Emb(L, M)$ denote the set of embeddings of L into M (worry about regularity later) and denote by $\Omega^k(L)$ the k -forms of L . Now define $\Phi : Emb(L, M) \times \mathbf{R} \rightarrow \Omega^1(L) \times \Omega^n(L)$ by

$$\Phi(f, \theta) = (f^* \omega, f^* \mathbf{Im}(e^{i\theta} \alpha)). \quad (2)$$

Since L itself is special Lagrangian, $\Phi(i_L, 0) = (0, 0)$, where i_L is the canonical embedding of L . Another minimal Lagrangian embedding of L , with calibration angle θ , is an embedding f satisfying $\Phi(f, \theta) = (0, 0)$.

The main theorem of this paper consists of finding those embeddings of L near i_L which satisfy $\Phi(f, \theta) = (0, 0)$ for some θ by means of the Implicit Function Theorem. The precise version of the theorem that will be employed is the following.

Theorem 2 (Implicit Function Theorem). *Let $F : B \rightarrow Z$ be a C^1 map of Banach spaces with $F(0) = 0$. Suppose that there exist closed Banach subspaces X and Y of B so that $B = X \oplus Y$. If $D_X F(0)$ is bijective, then there is a neighbourhood \mathcal{U} of 0 in Y and a C^1 map $\phi : \mathcal{U} \rightarrow X$ so that $\phi(0) = 0$ and $F(y + \phi(y)) = 0$ for all $y \in \mathcal{U}$.*

See [1, Section 2.5] for an excellent discussion of this theorem as well as its proof. The Implicit Function Theorem thus provides families of solutions of the equation $F(b) = 0$, parametrized over the Banach subspace which complements the subspace on which the linearization of F at 0 is bijective. Note that in the case where $DF(0)$ is surjective with finite dimensional kernel K , then the Implicit Function Theorem holds with $Y = K$ and X equal to any Banach subspace, necessarily closed, that complements K . The main theorem will be proved using this special case, while the corollary will be proved using the more general statement of the Implicit Function Theorem.

The map Φ , as defined above, does not yet involve Banach spaces. Thus in order to apply the Implicit Function Theorem to Φ , a sufficiently large class of embeddings of L near i_L must be parametrized over a Banach space, and the equation $\Phi(f, \theta) = (0, 0)$ must be solved in this Banach space. An added difficulty is that the elements of the Banach space must satisfy a boundary condition which ensures that Φ acting on the Banach space is elliptic.

2.2 Imposing Boundary Conditions with a Scaffold

In order to understand why boundary conditions must be imposed on the deformations of L , one must consider the linearization of the operator Φ at the point $(i_L, 0)$.

Proposition 3. *Let $\Phi : Emb(L, M) \times \mathbf{R} \rightarrow \Omega^1(L) \times \Omega^n(L)$ be the operator defined in (2). The linearization of Φ at the point $(i_L, 0)$ is given by*

$$D\Phi(i_L, 0)(V, a) = (d\eta, d \star \eta + a \text{Vol}_L), \quad (3)$$

where V is a vector field defined on L , a is a real number and $\eta = i_L^*(V \lrcorner \omega)$.

Proof. Let $f_t : L \rightarrow M$ be a family of embeddings with $f_0 = i_L$ and $\frac{d}{dt} f_t|_{t=0} = V$; and let a_t be a family of real numbers with $a_0 = 0$ and $\frac{d}{dt} a_t|_{t=0} = a$. Now,

$$D\Phi(i_L, 0)(V, a) = \left. \frac{d}{dt} \Phi(f_t, a_t) \right|_{t=0}.$$

The calculation of the derivative of Φ in the f_t direction has already been carried out by McLean in [14]. It remains only to perform the calculation in the a_t direction. This can be done by differentiating

$$\begin{aligned} \left. \frac{d}{dt} \Phi(0, ta) \right|_{t=0} &= \left(0, - \left. \frac{d}{dt} \mathbf{Im}(e^{-ita} \alpha) \right) \right|_{t=0} \\ &= (0, i^*(\mathbf{Im}(ia\alpha))) \\ &= (0, a i^*(\mathbf{Re}(\alpha))) \\ &= (0, a \text{Vol}_L), \end{aligned}$$

by definition of a calibration form. This calculation, in combination with McLean's result, completes the proof of the proposition. \square

The reason boundary conditions are necessary is that the Hodge operator $\eta \mapsto (d\eta, d \star \eta)$ is *not* elliptic unless it acts upon a space of differential 1-forms satisfying certain boundary conditions. From the Hodge theory on manifolds with boundary [20], it is known that one such boundary

condition is the *Neumann boundary condition*: the Hodge operator is elliptic when acting on forms η which satisfy $\eta(N) = 0$ along ∂L , where N is the unit normal vector field of ∂L in L . In the case under consideration here, η arises as the 1-form associated to a deformation of a special Lagrangian submanifold, and is thus of the form $\eta = V \lrcorner \omega$, where $V = \frac{d}{dt} f_t|_{t=0}$ is the corresponding deformation vector field. The Neumann boundary condition thus translates into the condition

$$i_L^*(V \lrcorner \omega)(Y) = 0 \quad \Leftrightarrow \quad \omega(V, Y) = 0 \quad (4)$$

on V . The following proposition shows that this boundary condition arises naturally if the deformations of L that are considered force the boundary of L to remain on a scaffold as described in the introduction.

Proposition 4. *Let L be a special Lagrangian submanifold of M and let W be a scaffold for L . In addition, suppose W is scaffold for L which is also symplectic and has codimension 2 in M . Let $f_t : L \rightarrow M$ be any deformation of L satisfying $f_t(\partial L) \in W$ for all t . Then the deformation vector field $V = \frac{d}{dt} f_t|_{t=0}$ corresponding to f_t satisfies the elliptic boundary condition $\omega(V, N) = 0$, where N is the unit normal vector field of ∂L in L .*

Proof. The vector field V must be parallel to W along ∂L as indicated in the introduction. But according to the definition of a scaffold, $N \in (T_x W)^\omega$ for every $x \in \partial L$. Therefore $\omega(N, V) = 0$. \square

2.3 Constructing Scaffold Preserving Deformations

In the proof of McLean's Theorem, deformations of L are parametrized over the Banach space of $C^{1,\beta}$ sections of the normal bundle of L using the exponential map. That is, for every section V of the normal bundle of L , the exponential map defines an embedding of L via $\exp(V) : L \rightarrow M$. Exponential deformations are, however, not suitable for the present purpose, because in general $\exp(V)(\partial L)$ will not lie on W because W is in general not totally geodesic. Indeed, if $p \in \partial L$ and the geodesic starting at p and heading in the direction of V does not lie in W , then $\exp(V)(p) \notin W$. Another means of deforming L is thus necessary if ∂L is to remain confined to the scaffold under deformation. One way to avoid the difficulty described above is to consider the exponential map of a *different* metric \hat{g} — one in which W is totally geodesic. The normal bundle of L with respect to this new metric, denoted by $\hat{N}L$, will then be used to parametrize submanifolds near L with boundary on W .

Before the metric \hat{g} can be constructed, a lemma concerning the local structure of W near ∂L is needed. This is essentially a version of the Lagrangian Neighbourhood Theorem [13, page 99] that is valid for Lagrangian submanifolds with boundary.

Lemma 5. *Let W be a symplectic submanifold of codimension 2 in M and suppose that L is a Lagrangian submanifold with boundary $\partial L \subset W$. Then there exists a tubular neighbourhood \mathcal{U} of the boundary and a symplectomorphism $\psi : \mathcal{U} \rightarrow T^*(\partial L \times \mathbf{R})$ with the following properties:*

1. $\psi(W \cap \mathcal{U}) \subset T^*(\partial L) \times \{0, 0\}$;
2. $\psi(\partial L) = \partial L \times \{0, 0\}$;
3. $\psi(L \cap \mathcal{U}) \subset \partial L \times \mathbf{R}_+ \times \{0\}$; and
4. Let E be any non-zero section of $(TW)^\omega$ and denote by (s^1, s^2) the coordinates of the \mathbf{R}^2 factor. Then ψ can be constructed so that $\psi_*(E) = \frac{\partial}{\partial s^1}$.

Here, $T^*(\partial L \times \mathbf{R})$ has been identified with $T^*(\partial L) \times \mathbf{R}^2$.

Proof. Because W is symplectic, the symplectic form $\omega|_W$ makes W a symplectic manifold in its own right. Since ∂L is an isotropic submanifold of M with respect to ω , it is a compact Lagrangian submanifold of W with respect to $\omega|_W$. Consequently, the usual Lagrangian Neighbourhood Theorem can be applied to ∂L as a submanifold of W to produce a neighbourhood \mathcal{U}_0 and a symplectomorphism $\psi_0 : \mathcal{U}_0 \rightarrow T^*(\partial L)$. The desired symplectomorphism will be found by extending ψ_0 off W in a suitable way.

The Symplectic Neighbourhood Theorem [13, page 98] will be used to complete the extension. The theorem applies to two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) containing symplectic submanifolds W_1 and W_2 respectively. It states that if there exists a symplectic vector bundle isomorphism $\Psi : (TW_1)^\omega \rightarrow (TW_2)^\omega$ that covers a symplectomorphism $\psi : W_1 \rightarrow W_2$, then there exist neighbourhoods \mathcal{U}_1 and \mathcal{U}_2 of W_1 and W_2 respectively, along with a symplectomorphism $\psi_e : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ that extends ψ (that is, $\psi_e|_{W_1} = \psi$).

Let $M_1 = M$, $W_1 = W$, $M_2 = T^*(\partial L) \times \mathbf{R}^2$ and $W_2 = T^*(\partial L) \times \{0, 0\}$. Let s^1 and s^2 be the coordinate functions in the \mathbf{R}^2 factor. One of the defining conditions for a scaffold is that its symplectic normal bundle $(TW)^\omega$ is trivial. Hence it is possible to choose two vector fields E and F which span $(TW)^\omega$ and satisfy $\omega(E, F) = 1$. Extend this basis to the neighbourhood \mathcal{U}_0 and continue to denote the extended vector fields by E and F . Define an isomorphism $\Psi : (TW)^\omega \rightarrow \mathbf{R}^2$ of symplectic vector bundles by:

$$\Psi(E_x) = \frac{\partial}{\partial s^1}(\psi_0(x), 0, 0) \quad \text{and} \quad \Psi(F_x) = \frac{\partial}{\partial s^2}(\psi_0(x), 0, 0) \quad (5)$$

at any $x \in \mathcal{U}_0$. This clearly covers the symplectomorphism ψ_0 and is a symplectic map.

The Symplectic Neighbourhood Theorem can now be invoked to yield a symplectomorphism ψ_1 extending ψ_0 between some tubular neighbourhood of ∂L and a neighbourhood of $\partial L \times \{0, 0\}$ in $T^*(\partial L) \times \mathbf{R}^2$. Consequently, $\psi_1(W \cap \mathcal{U}_0) \subset T^*(\partial L)$ and $\psi_1(\partial L) = \partial L \times \{0, 0\}$. The third requirement on the symplectomorphism has not yet been met, however, but the desired property can be obtained by composing with a suitable symplectomorphism that acts as a translation in the transverse Lagrangian directions to L . \square

The Darboux coordinates adapted to L guaranteed by Lemma 5 can be used to construct the metric \hat{g} . The vector field E should be chosen to be the unit normal vector field N of ∂L in this case. The construction of \hat{g} is accomplished in three separate steps.

STEP 1: Let \mathcal{U} be the tubular neighbourhood of ∂L provided by Lemma 5 and $\psi : \mathcal{U} \rightarrow T^*(\partial L) \times \mathbf{R}^2$ the symplectomorphism. Suppose s^1 and s^2 are the Darboux coordinates for the \mathbf{R}^2 factor of the direct product, and furthermore, one can suppose that $\psi_*\left(\frac{\partial}{\partial s^1}\right) = N$. Now define the metric g_1 at the point $(x, y, s^1, s^2) \in T^*(\partial L) \times \mathbf{R}^2$ as follows:

$$g_1(x, y, s^1, s^2) = (\psi^{-1})^*(g|_W(\psi(x, y, 0, 0))) + ds^1 \otimes ds^1 + ds^2 \otimes ds^2. \quad (6a)$$

STEP 2: Without loss of generality, the form (6a) can be taken for an entire tubular neighbourhood \mathcal{U}_1 of W . This is because the topological assumption made on W — that W has trivial normal bundle — is enough to guarantee the extension of the coordinates s^1 and s^2 to the entire tubular neighbourhood. The crucial difference between the present coordinates and the ones used in Step 1 is that the new coordinates are not necessarily symplectic *everywhere* (but they remain symplectic near ∂L).

STEP 3: Let $\eta : M \rightarrow \mathbf{R}$ be a positive, C^∞ cut-off function which equals 1 inside a tubular neighbourhood \mathcal{U}'_1 of ∂L contained in \mathcal{U}_1 , and equals 0 outside \mathcal{U}_1 . Now define the metric \hat{g} by:

$$\hat{g} = \eta g_1 + (1 - \eta)g. \quad (6b)$$

It remains to verify that the metric \hat{g} brings about the following properties. First, it must be true that if V is any section of the \hat{g} -normal bundle $\hat{N}L$ satisfying the boundary condition $\omega(V, N) = 0$, then V must be tangent to W ; this encodes the boundary condition. Second, W must be \hat{g} -totally geodesic (at least in a neighbourhood of ∂L) so that $\exp(V)(\partial L) \subset W$ whenever V is sufficiently small. The following two propositions deal with these issues.

Proposition 6. *The submanifold W is totally geodesic with respect to the metric \hat{g} constructed in equations (6).*

Proof. Let $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{2n-2}}$ be a set of local coordinate vector fields for the relatively open neighbourhood $W \cap \mathcal{U}'_1$. Then $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{2n-2}}, \frac{\partial}{\partial s^1}, \frac{\partial}{\partial s^2}$ determines a set of local coordinate vector fields for the neighbourhood \mathcal{U}'_1 . In these coordinates,

$$\hat{g} = \begin{pmatrix} (g|_W(z))_{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix}.$$

Now,

$$\begin{aligned} \left\langle \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial s^k} \right\rangle &= \frac{1}{2} (\hat{g}_{z^i s^k, z^j} + \hat{g}_{z^j s^k, z^i} - \hat{g}_{z^i z^j, s^k}) \\ &= -\frac{1}{2} \frac{\partial}{\partial s^k} (g|_W)_{ij} \\ &= 0. \end{aligned}$$

This implies that the second fundamental form of W with respect to \hat{g} vanishes; and this, in turn, is equivalent to the fact that W is totally geodesic. \square

Proposition 7. *Let L be a special Lagrangian submanifold with boundary on the symplectic scaffold W and let N be the unit normal vector field of ∂L in L . Construct the metric \hat{g} according to equations (6). Suppose V is a section of $\hat{N}L$ that satisfies the boundary condition $\omega(V, N) = 0$. Then V is tangent to W over ∂L .*

Proof. Choose a point x in ∂L and Darboux coordinates at x as in the constructions above. Furthermore, assume that $T_x \partial L$ is spanned by $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{n-1}}$ and that $\frac{\partial}{\partial z^n}, \dots, \frac{\partial}{\partial z^{2n-2}}$ are orthogonal to these vectors. Since N equals $\frac{\partial}{\partial s^1}$ in these coordinates, it is now easy to see that the \hat{g} -normal bundle of L at x is spanned by the vectors

$$\frac{\partial}{\partial z^n}, \dots, \frac{\partial}{\partial z^{2n-2}} \quad \text{and} \quad \frac{\partial}{\partial s^2} + \lambda \frac{\partial}{\partial s^1},$$

for some $\lambda \in \mathbf{R}$. So, if $V \in \hat{N}_x L$ and $(\sum dz^i \wedge dz^{n-1+i} + ds^1 \wedge ds^2)(V, N) = 0$, then clearly the $\frac{\partial}{\partial s^2} + \lambda \frac{\partial}{\partial s^1}$ component of V must vanish and as a result, $V \in T_x W$. \square

From elementary metric geometry, it is known that $\widehat{\exp}$ is a local diffeomorphism on $\hat{N}L$. Thus the conclusion to be drawn from Proposition 6 and Proposition 7 is that sufficiently small \hat{g} -exponential deformations of sections of $\hat{N}L$ satisfying the boundary condition imposed by the scaffold W are in one-to-one correspondence with submanifolds near L with boundary on W that project onto L via \hat{g} -nearest point projection. This parametrization will now be used to apply the Implicit Function Theorem to the problem of selecting those sections $V \in \hat{N}L$ which give rise to minimal Lagrangian submanifolds near L with boundary on W .

3 Proof of the Main Theorem

3.1 Defining the Differential Operator

The apparatus created in the previous section for deforming the special Lagrangian submanifold $L \subset M$ such that its boundary remains confined to the scaffold W can now be used to set up a differential equation whose solutions correspond to minimal Lagrangian submanifolds near L with boundary on W . Construct the metric \hat{g} and the \hat{g} -normal bundle $\hat{N}L$ of L as in the previous section, let N denote the unit normal vector field of ∂L , and define the Banach space

$$\mathcal{X} = \left\{ V \in C^{1,\beta}(\Gamma(\hat{N}L)) : \omega(V, N) = 0 \right\}$$

of $C^{1,\beta}$, \hat{g} -normal vector fields satisfying the Neumann boundary condition imposed by W . The notation used here is the following. If B denotes any bundle over L , then $\Gamma(B)$ denotes the sections

of B and $C^{k,\beta}(\Gamma(B))$ denotes the set of sections whose k covariant derivatives exist and are bounded in the $C^{k,\beta}$ norm, which is given by

$$\|u\|_{C^{k,\beta}} = \sum_{i=0}^k \|\nabla^i u\|_0 + [\nabla^k u]_\beta,$$

for any section $u \in \Gamma(B)$, where $\|u\|_0$ is the supremum norm of a section u over L and $[u]_\beta$ is its Hölder coefficient.

Next, denote by $d\Omega^k(L)$ the set of *exact* $(k+1)$ -forms of L and define the operator

$$\Phi : \mathcal{X} \times \mathbf{R} \rightarrow C^{0,\beta}(d\Omega^1(L)) \times C^{0,\beta}(d\Omega^{n-1}(L))$$

by

$$\Phi(V, \theta) = (\widehat{\exp}(V))^*(\omega, -\mathbf{Im}(e^{-i\theta}\alpha)), \quad (7)$$

where $\widehat{\exp}$ is the \hat{g} -exponential map as defined in the previous section. Note that elements of $C^{0,\beta}(d\Omega^k(L))$ are necessarily of the form $d\eta$ for some $\eta \in C^{1,\beta}(\Omega^{k-1}(L))$ by the Poincaré Lemma and the basic properties of Hölder spaces.

Since L is special Lagrangian and $\widehat{\exp}(0) = i_L$ (the standard embedding of L) then $\Phi(0, 0) = (0, 0)$. The proof of the Main Theorem of this paper consists of showing that the Implicit Function Theorem, stated in Theorem 2, can be applied to the operator Φ in order to find solutions of the equation $\Phi(V, \theta) = (0, 0)$.

NOTE: The range of Φ is indeed the set of *exact* 1- and n -forms. This is any element of the range is homotopic to the zero 1- and n -forms, and exactness is preserved under homotopy.

3.2 Analysis of the Linearized Operator

In order to apply the Implicit Function Theorem to the map Φ in the vicinity of the point $(0, 0)$, it is necessary to show that Φ is a C^1 map of Banach spaces and the linearization $D\Phi(0, 0)$ is bounded and surjective, and that its kernel is isomorphic to the finite dimensional set of harmonic 1-forms of L that satisfy Neumann boundary conditions.

The continuous differentiability of Φ as a Banach space map is straightforward. Recall now the expression of the linearization of the minimal Lagrangian equations from Proposition 3; since $\frac{d}{dt}\widehat{\exp}(tV)|_{t=0} = V$,

$$D\Phi(0, 0)(V, a) = (d\eta, d\star\eta + a\text{Vol}_L),$$

where $\eta = i_L^*(V \lrcorner \omega)$ as before. This is clearly a bounded operator, and due to its relative simplicity, surjectivity is easy to verify.

Proposition 8. *The operator $D\Phi(0, 0) : \mathcal{X} \times \mathbf{R} \rightarrow C^{0,\beta}(d\Omega^1(L)) \times C^{0,\beta}(d\Omega^{n-1}(L))$ is surjective.*

Proof. Let N be the unit normal vector field of ∂L and let $\alpha \in C^{1,\beta}(\Omega^1(L))$ and $\beta \in C^{1,\beta}(\Omega^{n-1}(L))$. Consider the system of equations $D\Phi(0, 0)(V, a) = (d\alpha, d\beta)$ in the space $\mathcal{X} \times \mathbf{R}$; or in other words, consider

$$\begin{aligned} d\eta &= d\alpha \\ d\star\eta &= d\beta + a\text{Vol}_L \\ \eta(N) &= 0. \end{aligned} \quad (8)$$

Hodge theory for a manifold L with boundary (see [20] for a thorough explanation of all the details of this theory) shows that a k -form satisfying the equations

$$\begin{aligned} d\eta &= \sigma \\ d\star\eta &= \tau \\ \eta(N) &= 0 \end{aligned}$$

and possessing a given degree of Hölder regularity can be found if and only if the following conditions are met:

1. $d\sigma = 0$;
2. $d\tau = 0$;
3. $\tau(E_1, \dots, E_{k+1})|_{\partial L} = 0$ for any collection of vectors E_i tangent to ∂L ;
4. $\int_L \sigma \wedge \star \lambda = 0$ for every harmonic $(k+1)$ -form λ of L satisfying Neumann boundary conditions;
5. $\int_L \star \tau \wedge \star \kappa = 0$ for every harmonic $(k-1)$ -form κ of L satisfying Neumann boundary conditions.

This list of conditions is given in [20, page 123]. Note that these results are actually only stated for k -forms with Sobolev regularity. But they extend fairly easily to Hölder regularity by standard techniques of elliptic theory (as explained in [18], for example).

Because of form of the equations (8), only condition (5) above imposes any restriction on the solvability of these equations. Thus a 1-form η that solves this system of equations (and that possesses the correct degree of regularity) can be found if and only if the integrability condition

$$\int_L d\beta + a \int_L \text{Vol}_L = 0$$

can be satisfied. But since $\int_L \text{Vol}_L = \text{Vol}(L) \neq 0$, it is possible to choose a equal to

$$a = -\frac{\int_L d\beta}{\text{Vol}(L)}.$$

Hence the integrability condition can be met, proving that $(\eta, a) \mapsto (d\eta, d\star\eta + a\text{Vol})$ is surjective. This, in turn, implies that $\text{D}\Phi(0, 0)$ is surjective. \square

In order to complete the proof of the Main Theorem, it remains to find the kernel of the linearized operator. Suppose that the equations

$$\begin{aligned} d\eta &= 0 \\ d\star\eta + a\text{Vol}_L &= 0 \\ \eta(N) &= 0 \end{aligned} \tag{9}$$

are satisfied by a 1-form η on L and a real number a . Integrating the second equation over L yields:

$$\begin{aligned} a\text{Vol}(L) &= -\int_L d\star\eta \\ &= -\int_{\partial L} i_{\partial L}^*(\star\eta) \\ &= -\int_{\partial L} \star(\eta(N)) \\ &= 0 \end{aligned}$$

where $i_{\partial L}$ is the standard embedding of ∂L in M . The calculations above hold by Stokes' Theorem as well as by the properties of the Hodge star operator at the boundary of L (these properties are derived in [20, Sections 1.2 and 2.1]). Hence $a = 0$ and η satisfies the Hodge system $d\eta = \delta\eta = 0$ with the boundary condition $\eta(N) = 0$. The solutions of these equations are the harmonic 1-forms with Neumann boundary conditions. This is a finite dimensional space of dimension equal to $b^1(L)$. Again, this result can be found in [20, Section 2.6].

All of the hypotheses required by the Implicit Function Theorem are thus satisfied by the map $\Phi : \mathcal{X} \times \mathbf{R} \rightarrow C^{0,\beta}(\text{d}\Omega^1(L)) \times C^{0,\beta}(\text{d}\Omega^{n-1}(L))$. Thus if

$$K = \{V \in \mathcal{X} : \text{D}\Phi(0, 0)(V, 0) = (0, 0)\}$$

is the finite dimensional kernel of $\text{D}\Phi(0, 0)$, there is a C^1 map $f : \mathcal{U} \rightarrow \mathcal{X} \times \mathbf{R}$, where $\mathcal{U} \subset K$ is a neighbourhood of 0, that satisfies $\Phi(f(k)) = 0$ for every $k \in \mathcal{U}$. This completes the proof of the Main Theorem. \square

4 Deformations of the Scaffold

The main theorem answers the question of the existence of minimal Lagrangian submanifolds with boundary on the scaffold W which are near the given candidate L . A relatively simple extension of the theory that has been developed so far can be used to answer the question of the existence of minimal Lagrangian submanifolds on *neighbouring* scaffolds. If W' is a symplectic scaffold near W and there is a special Lagrangian submanifold L with boundary $\partial L \subset W$, one asks whether there is a special (or minimal) Lagrangian submanifold L' near L with boundary $\partial L' \subset W'$. There is an affirmative answer to this question and it is provided by once again by the Implicit Function Theorem.

To prove results about the minimal Lagrangian submanifolds on neighbouring scaffolds, it is necessary first to parametrize nearby scaffolds over a Banach space in some way. The symplectic structure preserving *Hamiltonian deformations* of W will be used for this purpose: a procedure will be developed which associates a time-one Hamiltonian flow to each element of the set of $C^{2,\beta}$ sections of the two-dimensional bundle $(TW)^\omega$.

In order to understand the details of this construction, let X be a $C^{2,\beta}$ section in $\Gamma((TW)^\omega)$ and suppose \mathcal{U} is a tubular neighbourhood of W which is symplectomorphic to $W \times \mathbf{R}^2$. Furthermore, suppose that the Darboux coordinate vector fields $\frac{\partial}{\partial s^1}$ and $\frac{\partial}{\partial s^2}$ of the \mathbf{R}^2 factor coincide with the unit normal vector field N and the vector field JN , respectively, over the boundary $\partial L \subset W$. (The existence of such coordinates follows from Lemma 5.) Write X in these coordinates as:

$$X(q) = a^1(q) \frac{\partial}{\partial s^1} + a^2(q) \frac{\partial}{\partial s^2}$$

where $q \in W$ and the a^i are functions of W . Now let $\eta : M \rightarrow \mathbf{R}$ be a positive, C^∞ cut-off function equal to zero outside \mathcal{U} and equal to one inside a smaller tubular neighbourhood of W , and define the function $H_X : M \rightarrow \mathbf{R}$ by:

$$H_X(q, s) = \eta(q, s) (-a^2(q)s^1 + a^1(q)s^2)$$

for $(q, s) \in \mathcal{U}$ and make H_X equal to zero elsewhere. Because the symplectic form of $W \times \mathbf{R}^2$ is equal to $\omega|_W + ds^1 \wedge ds^2$, it is easy to see that the Hamiltonian vector field associated to H_X is equal to X when $s^1 = s^2 = 0$; that is, on the submanifold W itself. Finally, let $\phi_X : M \rightarrow M$ denote the time-one Hamiltonian flow associated to the function H_X . By elementary properties of the flow, it is clear that

$$\left. \frac{d}{dt} \phi_{tX} \right|_{t=0} = J\nabla H_X, \quad (10)$$

and if this quantity is restricted to W , then it equals X .

The map $X_q \mapsto \phi_X(q)$ for $X_q \in (T_q W)^\omega$ is a local diffeomorphism because equation (10) implies that its linearization at the zero section is the identity. Without loss of generality, one can assume that it is the tubular neighbourhood \mathcal{U} that is diffeomorphic to a neighbourhood of the zero section in $(T_q W)^\omega$. Hence, any scaffold W' sufficiently near W and sufficiently C^1 -regular (to ensure that W' projects onto W) is a Hamiltonian deformation of the form $W' = \phi_X(W)$ for some vector field $X \in \Gamma((TW)^\omega)$ that is sufficiently close to the zero section. The $C^{1,\beta}$ sections of the bundle $(TW)^\omega$ can thus be used to parametrize scaffolds sufficiently close to W .

This parametrization of nearby scaffolds leads to the following deformation operator. Define the map $\Phi_1 : C^{1,\beta}(\Gamma((TW)^\omega)) \times \mathcal{X} \times \mathbf{R} \rightarrow C^{0,\beta}(d\Omega^1(L)) \times C^{0,\beta}(d\Omega^{n-1}(L))$ by

$$\Phi_1(X, V, \theta) = (\phi_X \circ \widehat{\exp}(V))^* (\omega, -\mathbf{Im}(e^{-i\theta} \alpha)). \quad (11)$$

If $\Phi_1(X, V, \theta) = (0, 0)$ then the submanifold $L' = \phi_X \circ \widehat{\exp}(V)(L)$ is minimal Lagrangian with calibration form $\mathbf{Re}(e^{-i\theta} \alpha)$. Furthermore, $\partial L'$ is contained in $W' = \phi_X(W)$ because the deformation $\widehat{\exp}(V)$ preserves W . The parametrization and deformation operator constructed here now lead to the final result of this paper.

Theorem 9. *Let L be a special Lagrangian submanifold of a Calabi-Yau manifold M whose boundary lies on a symplectic, codimension two scaffold W . Furthermore, suppose that the topology of L forces its first Betti number $b^1(L)$ to vanish. Then if W' is any symplectic, codimension two submanifold of M that is sufficiently near W in the sense that W' can be written as $\phi_X(W)$ for some $X \in C^{1,\beta}(\Gamma((TW)^\omega))$ which is sufficiently small, then there is a minimal Lagrangian submanifold L' near L and with boundary on W' .*

Proof. The linearization of Φ_1 in the $\mathcal{X} \times \mathbf{R}$ directions remains the operator from equation (3), and is thus an isomorphism because the triviality condition $b^1(L) = 0$ has been assumed. Therefore, the Implicit Function Theorem implies that there is an open set \mathcal{U} of 0 in $C^{1,\beta}(\Gamma((TW)^\omega))$ and a map $G : \mathcal{U} \rightarrow \mathcal{X} \times \mathbf{R}$ satisfying

$$\Phi_1(X, G(X)) = (0, 0). \quad (12)$$

Suppose $G(X) = (V(X), \theta(X))$. Then equation (12) is equivalent to the statement that the submanifold

$$\phi_X \circ \widehat{\exp}(V(X))(L) \quad (13)$$

is minimal Lagrangian, calibrated by the differential form $\mathbf{Re}(e^{-i\theta(X)}\alpha)$ and has boundary on the scaffold $\phi_X(W)$ (this is symplectic because ϕ_X is a symplectomorphism).

Consequently, if W' is any codimension 2, symplectic submanifold of the form $\phi_X(W)$ with $X \in \mathcal{U}$, then the minimal submanifold with boundary on W' required to prove the theorem is simply (13). \square

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References

- [1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, second ed., Springer-Verlag, New York, 1988.
- [2] Adrian Butscher, *Regularizing a singular special Lagrangian variety*, Submitted February 2001.
- [3] Reese Harvey and H. Blaine Lawson, Jr., *Calibrated geometries*, *Acta Math.* **148** (1982), 47–157.
- [4] Mark Haskins, *Constructing special Lagrangian cones*, math.DG/0005164.
- [5] Nigel J. Hitchin, *The moduli space of special Lagrangian submanifolds*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), no. 3-4, 503–515, Dedicated to Ennio De Giorgi.
- [6] Dominic Joyce, *Constructing special Lagrangian m -folds in \mathbf{C}^m by evolving quadrics*, math.DG/0008154. To appear in *Mathematische Annalen*.
- [7] ———, *Evolution equations for special Lagrangian 3-folds in \mathbf{C}^3* , math.DG/0010036. To appear in the *Annals of Global Analysis and Geometry*.
- [8] ———, *Lectures on Calabi-Yau and special Lagrangian geometry*, math.DG/0108088.
- [9] ———, *Ruled special Lagrangian 3-folds in \mathbf{C}^3* , math.DG/0012060. To appear in the *Proceedings of the London Mathematical Society*.
- [10] ———, *Singularities of special Lagrangian fibrations and the SYZ-conjecture*, math.DG/0011179.
- [11] ———, *Special Lagrangian 3-folds and integrable systems*, math.DG/0101249.
- [12] ———, *Special Lagrangian m -folds in \mathbf{C}^m with symmetries*, math.DG/0008021.
- [13] Dusa McDuff and Dietmar Salamon, *Introduction to Symplectic Topology*, Second ed., The Clarendon Press Oxford University Press, New York, 1998.
- [14] Robert C. McLean, *Deformations of calibrated submanifolds*, *Comm. Anal. Geom.* **6** (1998), no. 4, 705–747.
- [15] David R. Morrison, *Mathematical aspects of mirror symmetry*, *Complex Algebraic Geometry* (Park City, UT, 1993), Amer. Math. Soc., Providence, RI, 1997, pp. 265–327.
- [16] Sema Salur, *Deformations of special Lagrangian submanifolds*, *Commun. Contemp. Math.* **2** (2000), no. 3, 365–372.
- [17] R. Schoen and J. Wolfson, *Minimizing volume among Lagrangian submanifolds*, *Differential Equations: La Pietra 1996* (Shatah Giaquinta and Varadhan, eds.), *Proc. of Symp. in Pure Math.*, vol. 65, 1999, pp. 181–199.
- [18] Richard Schoen, *Lecture Notes in Geometric PDEs on Manifolds*, Course given in the Spring of 1998 at Stanford University.
- [19] ———, *Private communication*, Will be part of the MSRI Lecture Notes for the Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces of the Summer of 2001.
- [20] Günter Schwarz, *Hodge Decomposition—A Method for Solving Boundary Value Problems*, Springer-Verlag, Berlin, 1995.