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Plane-wave Matrix Theory from $\mathcal{N}=4$ Super Yang-Mills on $\mathbb{R}\times S^3$

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Abstract

Recently a mass deformation of the maximally supersymmetric Yang-Mills quantum mechanics has been constructed from the supermembrane action in eleven dimensional plane-wave backgrounds. However, the origin of this plane-wave matrix theory in terms of a compactification of a higher dimensional Super Yang-Mills model has remained obscure. In this paper we study the Kaluza-Klein reduction of $D=4$, $\mathcal{N}=4$ Super Yang-Mills theory on a round three-sphere, and demonstrate that the plane-wave matrix theory arises through a consistent truncation to the lowest lying modes. We further explore the relation between the dilatation operator of the conformal field theory and the hamiltonian of the quantum mechanics through perturbative calculations up to two-loop order. In particular we find that the one-loop anomalous dimensions of pure scalar operators are completely captured by the plane-wave matrix theory. At two-loop level this property ceases to exist.

1 Introduction

At present the most promising candidates for a microscopic description of M-theory are given in terms of supersymmetric gauge quantum mechanical models, which are believed to provide a light-cone quantization of M-theory in suitable backgrounds [1]. Concretely in the case of a flat Minkowski background the associated matrix model is the maximally supersymmetric Yang-Mills quantum mechanics [2], obtained through the discretization of the Minkowski background supermembrane action in light-cone gauge [3]. In the same spirit a similar matrix model has been proposed in [4] for M-theory in the maximally supersymmetric plane-wave background first discovered by Kowalski-Glikman [5] in 1984,

$$ds^2 = -2\,dx^+\,dx^- + \sum_{I=1}^{9}(dx^I)^2 - \left[\sum_{a=1}^{3}\tfrac{\mu^2}{9}\,(x^a)^2 + \sum_{i=4}^{9}\tfrac{\mu^2}{36}\,(x^i)^2\right](dx^+)^2 \tag{1}$$

with non-vanishing four-form field strength $F_{123+} = \mu$. The corresponding matrix model is given by a mass deformation of the flat space matrix theory and takes the relatively simple form in the conventions of [6]

$$S = S_{\text{flat}} + S_M \tag{2}$$

with

$$\begin{aligned} S_{\text{flat}} &= \int dt\ \operatorname{tr}\Big[\tfrac{1}{2}(D_tX_I)^2 - i\theta D_t\theta + \tfrac{1}{4}[X_I,X_J]^2 + \theta\Gamma^I\,[X_I,\theta]\Big]\,, \\ S_M &= \int dt\ \operatorname{tr}\Big[-\tfrac{1}{2}(\tfrac{m}{3})^2\,(X_a)^2 - \tfrac{1}{2}(\tfrac{m}{6})^2\,(X_i)^2 + \tfrac{m}{4}i\,\theta\Gamma_{123}\theta + \tfrac{m}{3}i\varepsilon_{abc}\,X_a\,X_b\,X_c\Big]\,. \end{aligned}$$

Here X and θ denote bosonic and fermionic Hermitian $N \times N$ matrices, respectively. The transverse SO(9) index $I = 1, \ldots, 9$ is decomposed into $a = 1, 2, 3$ and $i = 4, \ldots, 9$ reflecting the SO(3) $\otimes$ SO(6) split of the transverse sector induced by the background geometry. In the above m denotes the dimensionless parameter $m = \mu\alpha'/(2R)$, R is the radius of the compactified eleventh direction and the covariant derivative is given by $D_t\mathcal{O} = \partial_t\mathcal{O} - i[\omega,\mathcal{O}]$ with the gauge field ω. For $m \to 0$ ones recovers the usual matrix model in a flat background.

This model possesses sixteen dynamical and sixteen kinematical supersymmetries inherited from the maximal supersymmetry of the plane-wave background

$$\begin{aligned} \delta X^I &= 2\theta\Gamma^I\epsilon(t) \\ \delta\theta &= \Big[iD_tX_I\Gamma^I + \tfrac{1}{2}[X_I,X_J]\Gamma^{IJ} + \tfrac{m}{3}iX_a\Gamma^a\Gamma_{123} - \tfrac{m}{6}iX_i\Gamma^i\Gamma_{123}\Big]\epsilon(t) + \eta(t) \\ \delta\omega &= 2\theta\epsilon(t) \end{aligned}$$

with time-dependent parameters $\epsilon(t) = e^{-\frac{m}{12}\Gamma_{123}t}\epsilon_0$ and $\eta(t) = e^{\frac{m}{4}\Gamma_{123}t}\eta_0$.

Plane-wave matrix theory exhibits a number of interesting properties compared to the model without mass deformation. The usual matrix theory has a continuous spectrum due to flat potential valleys [7], whereas by virtue of the mass terms the energy spectrum of the plane-wave matrix theory is discrete [4, 8]. Furthermore, the introduction of the dimensionless parameter m allows for a perturbative study of the spectrum of the plane-wave matrix theory for $m \gg 1$ [8]. As noted independently in [6] and [9] the spectrum contains an infinite series of protected multiplets, whose energies are exactly given by their free field value, a property that may be shown using the representation theory of the classical Lie superalgebra $SU(2|4)$, the symmetry algebra of the M-theory plane-wave, as done in [9] and [10]. For further works on the plane-wave matrix model see [11, 12].

Now, whereas the higher dimensional origin of the usual flat space matrix theory as the trivial reduction of the maximally supersymmetric $D = 4$ Yang-Mills theory to $1 + 0$ dimensions (or equivalently as the reduction of $\mathcal{N} = 1$ Super Yang-Mills in $1 + 9$ to $1 + 0$ dimensions) is obvious by taking all fields to be space-independent, a similar higher dimensional origin of the plane-wave matrix theory has remained obscure. One aim of this paper is to fill in this structural gap in the understanding of supersymmetric gauge quantum mechanics: We shall point out that the plane-wave matrix theory arises by considering $\mathcal{N} = 4$, $D = 4$ Super Yang-Mills compactified on a three-sphere and performing a consistent truncation of the resulting Kaluza-Klein spectrum. It turns out that such a truncation to a finite number of fields is only possible if one drops half of the vector and fermion zero modes, as we describe in detail. Upon performing this truncation we obtain a relation between the four-dimensional Yang-Mills coupling constant g_{YM} and the mass parameter m of the matrix model

$$\left(\frac{m}{3}\right)^3 = \frac{32\pi^2}{g_{\text{YM}}^2} \,. \tag{3}$$

Furthermore as the radial quantization of $\mathcal{N} = 4$ Super Yang-Mills on $\mathbb{R} \times S^3$ relates the energy to the dilatation operator of the conformal field theory, one might speculate on a relation of the spectrum of the truncated model to the scaling dimensions of composite operators in the full Super Yang-Mills field theory. We shall show that indeed the *full* one loop scaling dimensions of scalar operators in the four-dimensional gauge field theory are reproduced by the massive gauge quantum mechanics (2) in leading order perturbation theory. This analogy, however, breaks down at the two-loop level.

Having established the relation between the plane-wave matrix model and $\mathcal{N} = 4$, $D = 4$ Super Yang-Mills, it is natural to seek for its possible implications on the AdS/CFT correspondence. The study of pure scalar operators is particularly relevant to the conjecture of Berenstein, Maldacena and Nastase (BMN) [4] which involves operators whose conformal dimensions and U(1) R-charges are taken to infinity. Based on holographic arguments applied to the dual IIB plane-wave superstring it was claimed [13] that the gauge field theory should have an effective one-dimensional description in the BMN limit. In fact it was proposed that this effective quantum mechanical model arises as a Kaluza-Klein reduction of Super Yang-Mills on $\mathbb{R} \times S^3$, which is precisely what we are interested in. The result of this paper shows that the plane-wave matrix model serves this proposal at one-loop level, but not at higher loops. Put differently and slightly amusingly: The effective one-dimensional description of BMN gauge theory is the BMN matrix model in the BMN limit – at one-loop level and in the pure scalar sector. Whether next-to-leading order corrections can be also succinctly translated into the framework of matrix quantum mechanics is a very interesting problem which lies beyond our scope in this paper.

The paper is organized as follows. In the next section we consider the Kaluza-Klein reduction of Super Yang-Mills on a three-sphere, and illustrate how it is related to the plane-wave matrix model. The effective vertices for pure SO(6) bosonic excitations are constructed perturbatively in section 3 up to next-to-leading order in perturbation theory. Our notation and some details of the computations are given in the appendices.

2 $\mathcal{N} = 4$ Super Yang-Mills on $\mathbb{R} \times S^3$

The field content of $D = 4$, $\mathcal{N} = 4$ superconformal Yang-Mills theory consists of a vector field A_μ, six real scalars ϕ_i as well as four Weyl spinors $\lambda_{\alpha A}$ all in the adjoint representation of the gauge group. The action of the theory on Minkowski space-time may be obtained through a trivial dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in $D = 10$ dimensions on a six-torus. When the theory is formulated on a curved background, superconformal symmetry is retained if the background admits (conformal) Killing vectors and spinors which generate the superalgebra $\mathrm{SU}(2, 2|4)$. In addition there are deformations of the flat space action and the supersymmetry transformation rules induced by the curved background, as discussed by [14, 15] and [16]. Most notably in a curved background the scalars become massive through a coupling to the Ricci scalar, reflecting the fact that in curved spaces the d'Alembertian operator alone is not Weyl invariant[1]. In a notation where the $\mathrm{SO}(1, 3) \otimes \mathrm{SO}(6)$ split of the $D = 10$ Dirac matrices has been performed, the action reads

$$\begin{aligned} S = \frac{2}{g^2_{\mathrm{YM}}} \int d^4x \, \sqrt{|g|} \, \mathrm{tr} \Big[& - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{2} D^\mu \phi_i D_\mu \phi_i - \tfrac{\mathcal{R}}{12} \phi_i^2 + \tfrac{1}{4} [\phi_i, \phi_j]^2 - 2i \lambda^\dagger_A \sigma^\mu D_\mu \lambda^A \\ & + (\rho_i)^{AB} \lambda^\dagger_A i\sigma^2 [\phi_i, \lambda^*_B] - (\rho^\dagger_i)_{AB} (\lambda^A)^\top i\sigma^2 [\phi_i, \lambda^B] \Big] \, . \end{aligned} \tag{4}$$

We use coordinates $x^\mu = (t, \theta, \psi, \chi)$ labeled by $\mu, \nu, \ldots = 0, 1, 2, 3$. When referring to spatial coordinates $\mathbf{x}^a = (\theta, \psi, \chi)$ only, we use the (curved) indices $a, b, \ldots = 1, 2, 3$. The gauge covariant derivative is defined as $D_\mu = \nabla_\mu - i[A_\mu, \ \ ]$, where ∇_μ denotes the space-time covariant derivative. Moreover the field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. We use the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + R^2 (d\theta^2 + \sin^2\theta \, d\psi^2 + \sin^2\theta \, \sin^2\psi \, d\chi^2) \tag{5}$$

on $\mathbb{R} \times S^3$, where the radius of the three-sphere R is kept as a free parameter. For this background the Ricci scalar is $\mathcal{R} = 6/R^2$. In (4) we have introduced $\sigma^\mu := (\mathbf{1}, \sigma^a)$ where

[1]In d-dimensions the conformally invariant wave operator is $\Box - \frac{d-2}{4(d-1)} \mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar.

σ^a are the usual Pauli matrices pulled back onto the S^3, and Clebsch-Gordan coefficients $(\rho_i)^{AB}$ of SU(4) which relate two $\mathbf{4}$'s with one $\mathbf{6}$ (cf. appendix A for our conventions and how the $(\rho_i)^{AB}$ are related to the $\mathrm{SO}(1,9)$ Dirac matrices).

The action (4) is invariant under the following modified supersymmetry transformations

$$\delta A_\mu = 2i(\lambda^\dagger_A\sigma_\mu\eta^A - \eta^\dagger_A\sigma_\mu\lambda^A) \,, \tag{6}$$

$$\delta\phi_i = -2i(\lambda^\dagger_A i\sigma^2(\rho_i)^{AB}\eta^*_B - (\lambda^A)^\top i\sigma^2(\rho^\dagger_i)_{AB}\eta^B) \,, \tag{7}$$

$$\begin{aligned}\delta\lambda^A = {}& \tfrac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}\eta^A - D_\mu\phi_i\bar\sigma^\mu i\sigma^2(\rho_i)^{AB}\eta^*_B - \tfrac{i}{2}[\phi_i,\phi_j](\rho_i\rho^\dagger_j)^A{}_B\eta^B \\ & -\tfrac{1}{2}\phi_i\bar\sigma^\mu i\sigma^2(\rho_i)^{AB}\nabla_\mu\eta^*_B \,,\end{aligned} \tag{8}$$

where $\bar\sigma^\mu := (-\mathbf{1},\sigma^a)$ and $\sigma^{\mu\nu} := \frac{1}{2}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)$. The last term in (8) represents the modification with respect to the flat space transformation law. The supersymmetry parameters $\eta_{\alpha A}$ are four Weyl spinors that satisfy either of the two conformal Killing spinors equations:

$$\nabla_\mu\eta = \pm\frac{i}{2R}\,\sigma_\mu\,\eta \,. \tag{9}$$

Counting the components of the Killing spinors and taking into account the degeneracy of the solutions of (9) as well as the two signs yields the number of 32 independent real supersymmetry parameters. The conserved charges corresponding to η_+ and η_- are denoted by Q_L and Q_R respectively. We finally note that the conformal Killing spinors can also be obtained from the Killing spinors of AdS_5 restricted on the boundary $\mathbb{R}\times S^3$.

2.1 Harmonic expansion on S^3

In order to perform the dimensional reduction we expand the four-dimensional fields in terms of the spherical harmonics on S^3, which come in irreducible representations $(\mathbf{m_L},\mathbf{m_R})$ of the isometry group $\mathrm{SO}(4)\cong\mathrm{SU}(2)_L\otimes\mathrm{SU}(2)_R$. The set of harmonics that appear in the expansion of a particular field depend on its spin and are listed in table 1 on page 5. See also [17] for a useful discussion on this topic.

We work in Coulomb gauge on S^3, $\nabla_a A^a = 0$, and have the following mode expansions

$$\phi_i(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)^2}\phi_i^{kI}(t)Y^{kI}_{(0)}(\mathbf{x}) \,, \tag{10a}$$

$$\lambda^A_\alpha(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)(k+2)}\sum_{\pm}\lambda^{A,kI\pm}(t)Y^{kI\pm}_{(1/2)\,\alpha}(\mathbf{x}) \,, \tag{10b}$$

$$A_0(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)^2}\omega^{kI}(t)Y^{kI}_{(0)}(\mathbf{x}) \,, \tag{10c}$$

$$A_a(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)(k+3)}\sum_{\pm}A^{kI\pm}(t)Y^{kI\pm}_{(1)\,a}(\mathbf{x}) \,. \tag{10d}$$

Spin	Harmonical functions		Irreps	Masses
0	Scalar spherical harmonics:	$Y^{kI}_{(0)}$	$(\mathbf{k+1}, \mathbf{k+1})$	$(k+1)/R$
1/2	Spinor spherical harmonics:	$Y^{kI+}_{(1/2)}$	$(\mathbf{k+2}, \mathbf{k+1})$	$(k+\frac{3}{2})/R$
		$Y^{kI-}_{(1/2)}$	$(\mathbf{k+1}, \mathbf{k+2})$	
1	Vector spherical harmonics:	$Y^{kI+}_{(1)}$	$(\mathbf{k+3}, \mathbf{k+1})$	$(k+2)/R$
		$Y^{kI-}_{(1)}$	$(\mathbf{k+1}, \mathbf{k+3})$	

Table 1: Spherical harmonics on S^3 that appear in the expansion of spin-0, spin-1/2 and spin-1 fields. Here $k = 0, 1, 2, \ldots$ labels different irreducible representations of SU(2). The index I enumerates the elements of a particular representation and therefore takes values from 1 to the dimension of the representation.

Note that the spinor spherical harmonics $Y^{kI\pm}_{(1/2)\,\alpha}$ are two-dimensional commuting Weyl spinors. Upon inserting the above harmonic expansions into the action (4) and carrying out the integration over the three-sphere one obtains a one-dimensional theory with an infinite number of fields. In order to determine the mass spectrum of these excitations we shall perform this integration for the quadratic terms of the action (4). For this we need only a few properties of the spherical harmonics. They are orthonormalized to

$$\int_{S^3} Y^{kI}_{(0)}\, Y^{lJ}_{(0)} = \delta^{kl}\delta^{IJ} \,, \int_{S^3} \left(Y^{kI\pm}_{(1/2)\,\alpha}\right)^* Y^{lJ\pm}_{(1/2)\,\alpha} = \delta^{kl}\delta^{IJ} \,, \int_{S^3} Y^{kI\pm}_{(1)\,a}\, Y^{lJ\pm\,a}_{(1)} = \delta^{kl}\delta^{IJ} \tag{11}$$

and their eigenvalues of the Laplace-Beltrami operators are

$$\nabla^2 Y^{kI}_{(0)} = -\tfrac{1}{R^2} k(k+2) Y^{kI}_{(0)} \,, \tag{12a}$$

$$\nabla\!\!\!/\, Y^{kI\pm}_{(1/2)} = \pm\tfrac{i}{R}(k+\tfrac{3}{2}) Y^{kI\pm}_{(1/2)} \,, \tag{12b}$$

$$\nabla^2 Y^{kI\pm}_{(1/2)\,\alpha} = -\tfrac{1}{R^2}\left[k(k+3)+\tfrac{3}{4}\right] Y^{kI\pm}_{(1/2)\,\alpha} \,, \tag{12c}$$

$$\nabla^2 Y^{kI\pm}_{(1)\,a} = -\tfrac{1}{R^2}\left[k(k+4)+2\right] Y^{kI\pm}_{(1)\,a} \,. \tag{12d}$$

The above operators include, of course, only the spatial parts $\nabla\!\!\!/ := \sigma^a \nabla_a$ and $\nabla^2 := \nabla^a \nabla_a$.

Upon inserting the mode expansions (10) into the quadratic part of the action (4), one obtains

$$S_{\text{quadratic}} = \frac{4\pi^2 R^3}{g^2_{\text{YM}}} \int dt \left[\sum_{k,I} \left\{ \frac{1}{2} \operatorname{tr} \dot{\phi}_i^{kI} \dot{\phi}_i^{kI} - \frac{(k+1)^2}{2R^2} \operatorname{tr} \phi_i^{kI} \phi_i^{kI} \right\} \right.$$
$$+ \sum_{k,I,\pm} \left\{ \frac{1}{2} \operatorname{tr} \dot{A}^{kI\pm} \dot{A}^{kI\pm} - \frac{(k+2)^2}{2R^2} \operatorname{tr} A^{kI\pm} A^{kI\pm} \right\} \tag{13}$$
$$\left. -i \sum_{k,I,\pm} \left\{ \operatorname{tr} \lambda_A^{kI\pm\dagger} \dot{\lambda}^{A,kI\pm} + \frac{k+\frac{3}{2}}{R} \operatorname{tr} \lambda_A^{kI\pm\,\dagger} \lambda^{A,kI\pm} \right\} \right]$$

where we have made use of the orthonormality conditions, integration by parts and the properties (12). In the computation of the masses for the vector modes one also needs the transversality of the vector spherical harmonics, $\nabla^a Y^{kJ\pm}_{(1)\,a} = 0$, and the identity

$$\nabla_a \nabla_b Y^{lJ\pm\,a}_{(1)} = [\nabla_a, \nabla_b] Y^{lJ\pm\,a}_{(1)} = R^a{}_{cab} Y^{lJ\pm\,c}_{(1)} = \mathcal{R}_{ab} Y^{lJ\pm\,a}_{(1)} = \frac{2}{R^2} Y^{lJ\pm}_{(1)\,b} \tag{14}$$

using $\mathcal{R}_{ab} = \frac{2}{R^2} g_{ab}$ of S^3. The obtained mass spectrum is summarized in table 1.

In figure 1 on page 7 we present the resulting Kaluza-Klein mass tower up to mass $\frac{3}{R}$. One may climb up the various states of the tower by acting with the two supercharges $Q_L = (\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}})$ (to the upper-left) and $Q_R = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ (to the upper-right). Note that unlike the trivial dimensional truncation on flat spaces, where superpartners have the same masses, the superconformal transformations in our curved background geometry relate the entire tower of Kaluza-Klein modes. In other words we find that the infinite tower of Kaluza-Klein modes is not decomposed into finite dimensional irreducible representations of the superconformal algebra. Instead, the entire tower itself is a single irreducible representation.

On the other hand, if we consider only half of the supercharges, say Q_L, which together with the bosonic symmetries generate the subalgebra SU(2|4), the Kaluza-Klein modes are decomposed in terms of finite dimensional irreducible representations. We see that $(\mathbf{1}, \mathbf{1}, \mathbf{6}) + (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{3}, \mathbf{1}, \mathbf{1})$ (encircled in figure 1) is the lowest such supermultiplet. Higher ones in general branch into five irreducible representations under the bosonic subalgebra $\mathrm{SU}(2) \otimes \mathrm{SU}(4)$, implying that they are all short supermultiplets of SU(2|4). In particular the lowest lying multiplet consisting of three floors is 1/2 BPS and the following multiplet comprised of four floors is 1/4 BPS, as may be deduced from the results of [10]. An important result of the supersymmetry algebra is that the ground state energy given as the sum of zero point energies should vanish, which we can easily verify. For instance, for the lowest lying supermultiplet $(\mathbf{1}, \mathbf{1}, \mathbf{6}) + (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{3}, \mathbf{1}, \mathbf{1})$ the summation goes as follows

$$6 \cdot \frac{1}{R} - 8 \cdot \frac{3}{2R} + 3 \cdot \frac{2}{R} = 0\,. \tag{15}$$

The truncation we are about to perform consists precisely in the restriction to this lowest lying multiplet.

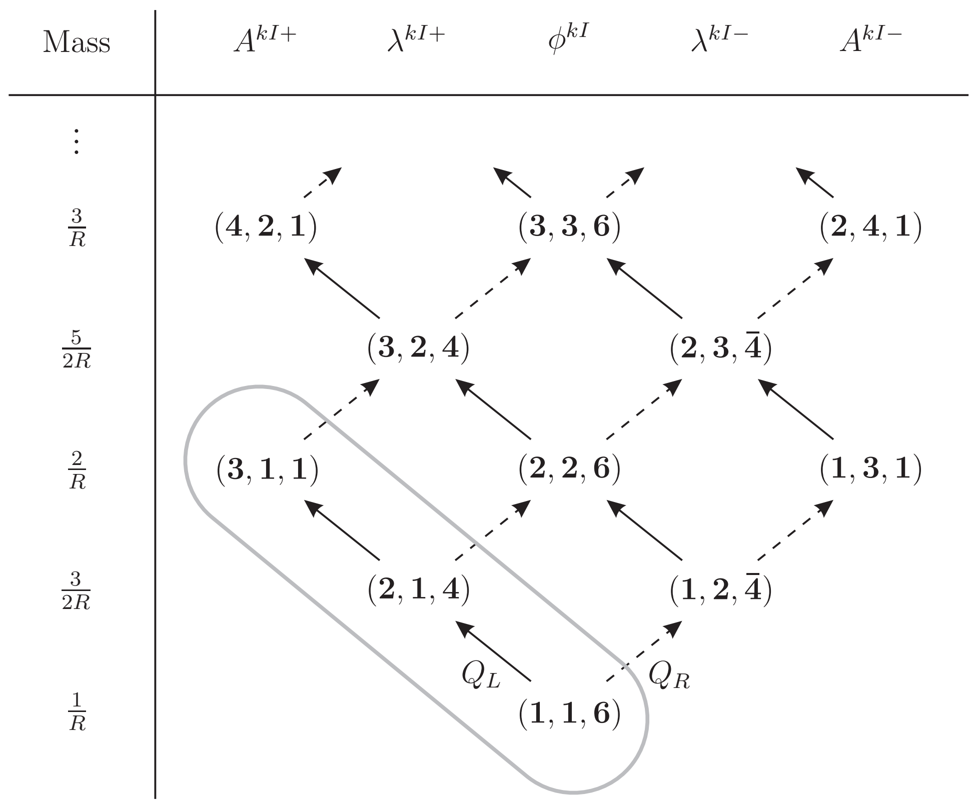

Figure 1: Kaluza-Klein particle spectrum of $\mathcal{N}=4$ Super Yang-Mills on $\mathbb{R}\times S^3$. The states are labeled by representations of $\mathrm{SU}(2)_L\otimes\mathrm{SU}(2)_R\otimes\mathrm{SU}(4)$. One climbs upward to the left by acting with the supercharge $Q_L=(\mathbf{2},\mathbf{1},\bar{\mathbf{4}})$ and upward to the right with $Q_R=(\mathbf{1},\mathbf{2},\mathbf{4})$. The encircled states comprise the consistent truncation to the plane-wave matrix theory.

2.2 Derivation of Plane-wave matrix theory

We now aim to show that the infinite tower of Kaluza-Klein states can be consistently truncated to the lowest lying supermultiplet, and that the one-dimensional action, which governs its dynamics, is precisely the plane-wave matrix model. Hence we restrict the expansion (10) to the zero modes which are $\mathrm{SU}(2)_R$ singlets. For these modes the harmonic functions are given by a constant, two Killing spinors $S_\alpha^{\hat\alpha+}$, and three Killing vectors $V_a^{\hat a+}$. The two Killing spinors obey the defining equation

$$\nabla_a S_\alpha^{\hat\alpha+}=\frac{i}{2R}\,(\sigma_a)_{\alpha\beta}\,S_\beta^{\hat\alpha+},\qquad \hat\alpha=1,2 \tag{16}$$

and lead to the three positive chirality Killing vectors $V_a^{\hat{a}+}$ via[2]

$$S^{\hat{\alpha}+\,\dagger}\sigma_a S^{\hat{\beta}+} = (\sigma_{\hat{a}})^{\hat{\alpha}\hat{\beta}}\, V_a^{\hat{a}+} \; . \tag{17}$$

Their explicit form along with further useful properties may be found in appendix B. The truncated mode expansion then takes the form

$$\phi_i(x) = X_i(t) \; , \tag{18a}$$

$$\lambda_\alpha^A(x) = \sum_{\hat{\alpha}=1}^{2} \theta_{\hat{\alpha}}^A(t) S_\alpha^{\hat{\alpha}+}(\mathbf{x}) \; , \tag{18b}$$

$$A_0(x) = \omega(t) \; , \tag{18c}$$

$$A_a(x) = \sum_{\hat{a}=1}^{3} X_{\hat{a}}(t) V_a^{\hat{a}+}(\mathbf{x}) \; . \tag{18d}$$

In order to prove that the above ansatz is a consistent truncation of the spectrum, we have to insert this ansatz into the four-dimensional equations of motion and show that this leads to a set of equations without contradictions. The equations of motions which follow from the four-dimensional action (4) are given by

$$D^\nu F_{\nu\mu} + i[\phi_i, D_\mu \phi_i] + 2\{\lambda^\dagger, \sigma_\mu \lambda\} = 0 \; , \tag{19a}$$

$$D^\mu D_\mu \phi_i - \tfrac{1}{R^2}\phi_i + [\phi_j, [\phi_i, \phi_j]] - \{\lambda^\dagger, i\sigma^2 \rho_i \lambda^*\} + \{\lambda^\top, i\sigma^2 \rho_i^\dagger \lambda\} = 0 \; , \tag{19b}$$

$$i\sigma^\mu D_\mu \lambda - i\sigma^2 \rho_i [\phi_i, \lambda^*] = 0 \; . \tag{19c}$$

where the anticommutator of two fermions is defined as $\{\psi^\top, \chi\}^A := i f^{ABC} (\psi^\top)^B \chi^C$. Inserting the ansatz (18) is a straight-forward computation which we comment on in appendix C. One finds that the S^3 coordinate dependences of all terms in each equation exactly matches, and that the $D = 4$ equations of motion are satisfied provided that the one-dimensional fields $X^I(t), \omega(t)$ and $\theta(t)$ obey

$$[X_I, iD_t X_I] - 2\{\theta^\dagger, \theta\} = 0 \; , \tag{20a}$$

$$D_t^2 X_{\hat{a}} + \tfrac{4}{R^2} X_{\hat{a}} - \tfrac{6i}{R} \varepsilon_{\hat{a}\hat{b}\hat{c}} X_{\hat{b}} X_{\hat{c}} - [X_I, [X_{\hat{a}}, X_I]] - 2\{\theta^\dagger, \sigma_{\hat{a}} \theta\} = 0 \; , \tag{20b}$$

$$D_t^2 X_i + \tfrac{1}{R^2} X_i - [X_I, [X_i, X_I]] + \{\theta^\dagger, i\sigma^2 \rho_i \theta^*\} - \{\theta^\top, i\sigma_2 \rho_i^\dagger \theta\} = 0 \; , \tag{20c}$$

$$iD_t \theta - \tfrac{3}{2R}\theta + [X_{\hat{a}}, \sigma_{\hat{a}} \theta] - [X_i, i\sigma_2 \rho_i \theta^*] = 0 \; . \tag{20d}$$

Note that the label $\hat{\alpha}$ of the fermion zero modes $\theta_{\hat{\alpha}}^A$ which used to account for their degeneracy has turned into a proper (Weyl) spinor index. Having found the equations of motion (20) for the zero modes, one may try to find a quantum mechanical action which leads to

[2]The Killing vectors of opposite chirality $V_a^{\hat{a}-}$ are induced from the opposite chirality Killing spinor bilinear $S^{\hat{\alpha}-\,\dagger}\sigma_a S^{\hat{\beta}-}$ which obey (16) with an opposite sign.

these equations of motion. It can be checked easily that these equations are derived from the following one-dimensional Lagrangian

$$
\begin{aligned}
L = \operatorname{tr}\Big[& \tfrac{1}{2}(D_t X_I)^2 - \tfrac{1}{2}(\tfrac{m}{3})^2 X_a^2 - \tfrac{1}{2}(\tfrac{m}{6})^2 X_i^2 + \tfrac{m}{3} i\varepsilon_{abc} X_a X_b X_c + \tfrac{1}{4}[X_I, X_J]^2 \\
& - 2i\theta^\dagger D_t \theta + \tfrac{m}{2}\theta^\dagger\theta - 2\theta^\dagger\sigma^a[X_a,\theta] + \theta^\dagger i\sigma^2 \rho_i [X_i, \theta^*] - \theta^\top i\sigma^2 \rho_i^\dagger [X_i,\theta]\Big]
\end{aligned}
\tag{21}
$$

where the radius of the three-sphere R has been traded for a mass parameter $m = 6/R$. This lagrangian is in fact nothing but the plane-wave matrix theory (2) written in terms of $\mathrm{SU}(2) \otimes \mathrm{SU}(4)$ spinor variables. This dimensional split was performed for instance in [12], in appendix A we repeat this split using our conventions and notation for easier reference.

This result also induces a relation of the four-dimensional Yang-Mills coupling constant g_{YM} to the mass parameter m of the plane-wave matrix model. It is found by requiring the prefactor of the reduced Super Yang-Mills action (13) to match the unit prefactor of the matrix model action (2). Taking into account that $m = 6/R$ one finds

$$
\left(\frac{m}{3}\right)^3 = \frac{32\pi^2}{g_{\text{YM}}^2}\,. \tag{22}
$$

3 Perturbation Theory

Having established this connection of $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills to the plane-wave matrix model one might wonder what additional structures of the field theory carry over to the matrix quantum mechanics. By virtue of the state-operator map of conformal field theories one might expect a connection of scaling dimensions of Super Yang-Mills operators on $\mathbb{R}^4$ to the energy of corresponding states in the plane-wave matrix model. As noted in [6, 9] one such connection is already manifest: The protected multiplets of chiral primary operators in the gauge theory, their lightest representative being the pure multi-scalar operators $\mathcal{O}_{CPO} = c_{i_1\ldots i_n}\,\operatorname{tr}(\phi_{i_1}\ldots\phi_{i_n})$, with $c_{i_1\ldots i_n}$ symmetric and traceless, have vanishing anomalous dimensions, i.e. their scaling dimension is exactly given by their free field value $\Delta_{\mathcal{O}_{CPO}} = n$. This property is paralleled by the existence of protected multiplets in the plane-wave matrix theory, with their lightest member again being the symmetric traceless excitations $c_{i_1\ldots i_n}\,\operatorname{tr}(a^\dagger_{i_1}\ldots a^\dagger_{i_n})|0\rangle$ in the SO(6) sector of the matrix model (2) [6, 9]. The energy of these states is similarly not corrected by interactions and simply given by the free theory value $E = n \cdot \frac{m}{6}$ to all orders in perturbation theory[3]. So here a complete analogy exists and it is natural to ask whether it extends to non-protected states in the SO(6) sector as well. That is, is there a general relation between the scaling dimension in 4d and the quantum mechanical energy? As we will show in the following this analogy extends to the one loop level in the SO(6) sector, but breaks down at two loops. Technically what we

[3]Note that only the energy eigenvalue is protected, the eigenstates are renormalized in perturbation theory.

demonstrate is the precise matching of the effective quantum mechanical interaction vertex in first order perturbation theory with the one loop piece of the dilatation operator of $D=4$, $\mathcal{N}=4$ Super Yang-Mills computed in [18, 19]. Comparing our second-order perturbation theory result to the recently established two-loop piece of the dilatation operator [21] we find a discrepancy, although the same structure of terms does appear. One might still hope that this discrepancy disappears once one considers the BMN limit of the gauge theory, however, this turns out to not be the case.

Let us then reconsider the perturbative evaluation of energy shifts in the SU(N) plane-wave matrix model discussed in [8, 6]. The hamiltonian associated to (2) may be split into a free and an interaction piece. Working in the conventions of [6] the free piece reads ($a=1,2,3$; $i=1,\dots,6$; $\alpha=1,\dots,16$)

$$H_0 := \operatorname{tr}\Big[\frac{m}{6}a_i^\dagger a_i + \frac{m}{3}a_a^\dagger a_a + \frac{m}{2}\theta^+_\alpha\theta^-_\alpha\Big] \tag{23}$$

with the matrix oscillators

$$\begin{aligned} a_i &:= \sqrt{\tfrac{3}{m}}\left(P_i - i\tfrac{m}{6}X_i\right) & \theta^\pm_\alpha &:= \Pi_{\alpha\beta}\,\theta_\beta \\ a_a &:= \sqrt{\tfrac{3}{2m}}\left(P_a - i\tfrac{m}{3}X_a\right) & \Pi^\pm_{\alpha\beta} &:= \tfrac{1}{2}(\mathbf{1}\pm i\Gamma_{123})_{\alpha\beta} \end{aligned} \tag{24}$$

obeying

$$\begin{aligned} [(a_i)_{rs},(a_j^\dagger)_{tu}] &= \delta_{ij}\left(\delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu}\right) \\ [(a_a)_{rs},(a_b^\dagger)_{tu}] &= \delta_{ab}\left(\delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu}\right) \\ \{(\theta^-_\alpha)_{rs},(\theta^+_\alpha)_{tu}\} &= \tfrac{1}{2}(\Pi^-)_{\alpha\beta}\left(\delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu}\right)\ . \end{aligned} \tag{25}$$

The interacting piece of the hamiltonian is comprised of cubic and quartic terms in the fields

$$H_{\text{int}} := V_1 + V_2 \tag{26}$$

where

$$\begin{aligned} V_1 &= -\frac{m}{3}i\varepsilon_{abc}\operatorname{tr}X_aX_bX_c - \operatorname{tr}\theta\gamma_I[X_I,\theta] \\ V_2 &= -\frac{1}{4}\operatorname{tr}[X_I,X_J][X_I,X_J] \end{aligned}$$

and I is a joint index $I=(a,i)$. For $m\gg 1$ the interaction piece H_{int} may be treated perturbatively[4]. The first order energy shift of a state $\big|\phi_0\big\rangle$ reads

$$\delta E_1 = \big\langle\phi_0\big|V^{(1)}_{\text{eff}}\big|\phi_0\big\rangle \qquad \text{with} \quad V^{(1)}_{\text{eff}} = V_1\Delta V_1 + V_2 \qquad \text{and} \quad \Delta = \frac{\mathbf{1}-\big|\phi_0\big\rangle\big\langle\phi_0\big|}{E_0 - H_0} \tag{27}$$

[4]This is easily seen by performing a rescaling $X\to X/m$ and $\theta\to\theta/\sqrt{m}$, rendering the hamiltonian into the form $H=\tilde{H}_0 + \frac{1}{m^2}\tilde{V}_1 + \frac{1}{m^4}\tilde{V}_2$ with no m dependence in $\tilde{H}_0$, $\tilde{V}_1$ and $\tilde{V}_2$.

where we assume $\langle\phi_0|\phi_0\rangle = 1$ and a non-degenerate free energy E_0. As we shall only investigate the energy shifts of pure SO(6) excitations we will normal order $V^{(1)}_{\text{eff}}$ and drop all normal ordered terms containing fermions or SO(3) oscillators, as these would annihilate a pure SO(6) excitation state. We indicate this simplification by an arrow. Using `Mathematica` and `Form` [20] we find

$$\begin{aligned} V_1\Delta V_1 \to\; & -\frac{99}{4M^2}(N^3-N) - \frac{12N}{M^2} : \operatorname{tr} a_i^\dagger a_i : \qquad (28) \\ V_2 \to\; & \frac{99}{4M^2}(N^3-N) + \frac{13N}{M^2} : \operatorname{tr} a_i^\dagger a_i : + \frac{1}{2M^2} : \operatorname{tr}[a_i^\dagger, a_i][a_j^\dagger, a_j] : \\ & - \frac{1}{2M^2} : \operatorname{tr}[a_i^\dagger, a_j][a_i^\dagger, a_j] : - \frac{1}{M^2} : \operatorname{tr}[a_i^\dagger, a_j^\dagger][a_i, a_j] : \qquad (29) \end{aligned}$$

introducing the shorthand $M := m/3$ which controls the perturbative expansion. When adding these contributions all constants sum to zero

$$V^{(1)}_{\text{eff}} \to \frac{1}{M^2}\Big(N : \operatorname{tr} a_i^\dagger a_i : + \frac{1}{2} : \operatorname{tr}[a_i^\dagger, a_i][a_j^\dagger, a_j] : - \frac{1}{2} : \operatorname{tr}[a_i^\dagger, a_j][a_i^\dagger, a_j] : - : \operatorname{tr}[a_i^\dagger, a_j^\dagger][a_i, a_j] :\Big) \tag{30}$$

a manifestation of the vanishing energy shift of the groundstate $|0\rangle$. This expression can be simplified further as the state $|\phi_0\rangle$ acting upon it is gauge invariant, e.g. some multi-trace excitation. The simplification concerns all terms with a commutator of contracted creation and annihilation operators: $[a_i^\dagger, a_i]$, i.e. the second term in the above. One splits off this commutator from the rest and removes the normal ordering between the two factors:

$$\begin{aligned} \frac{1}{2} : \operatorname{tr}[a_i^\dagger, a_i][a_j^\dagger, a_j] : &= \frac{1}{2} : \operatorname{tr}[a_i^\dagger, a_i]T^A \operatorname{tr} T^A[a_j^\dagger, a_j] : \\ &= \frac{1}{2}\Big(: \operatorname{tr}[a_i^\dagger, a_i]T^A :\Big)\Big(: \operatorname{tr} T^A[a_j^\dagger, a_j] :\Big) - N : \operatorname{tr} a_i^\dagger a_i : \end{aligned} \tag{31}$$

where T^A denote the generators of SU(N) and we note that

$$\operatorname{tr}(T^A A T^A B) = \operatorname{tr} A \operatorname{tr} B - \tfrac{1}{N}\operatorname{tr}(AB)\;, \tag{32}$$

$$\operatorname{tr}(T^A A)\operatorname{tr}(T^A B) = \operatorname{tr}(AB) - \tfrac{1}{N}\operatorname{tr} A \operatorname{tr} B\;. \tag{33}$$

The first term in (31) can be dropped for reasons given in the appendix D and the second term cancels the two point interaction that was already present. Hence we arrive at the compact result

$$V^{(1)}_{\text{eff}} \to \frac{1}{M^2}\Big(-\frac{1}{2} : \operatorname{tr}[a_i^\dagger, a_j][a_i^\dagger, a_j] : - : \operatorname{tr}[a_i^\dagger, a_j^\dagger][a_i, a_j] :\Big) \tag{34}$$

We want to compare this 1-loop effective vertex, which determines the first order energy shifts of states in the plane-wave matrix model, to the 1-loop dilatation operator of $\mathcal{N} = 4$

Super Yang-Mills, which determines the anomalous dimensions of operators. We take the dilatation operator from [21]. The 1-loop contribution is given in Eq. (1.6):

$$D_2 = \frac{g^2_{\mathrm{YM}}}{16\pi^2}\left(-\frac{1}{2} : \mathrm{tr}[\phi_i, \check{\phi}_j][\phi_i, \check{\phi}_j] : - : \mathrm{tr}[\phi_i, \phi_j][\check{\phi}_i, \check{\phi}_j] :\right) , \tag{35}$$

where $\check{\phi}_i := \frac{d}{d\phi_i}$. This shows that by identifying the scalars of both theories

$$a_i^\dagger \hat{=} \phi_i \qquad , \qquad a_i \hat{=} \check{\phi}_i , \tag{36}$$

we have an exact agreement between energy shifts and anomalous dimensions for all pure SO(6) states/operators at one loop level!

This result is intriguing in view of the fact [22] that the 1-loop dilatation operator D_2 of $\mathcal{N} = 4$ Super Yang-Mills in the strict $N \to \infty$ limit may be viewed as the hamiltonian of an integrable SO(6) spin chain model, hinting at the integrability of $\mathcal{N} = 4$ Super Yang-Mills. This structure thus carries over to the plane-wave matrix model. The higher-loop corrections to the Super Yang-Mills dilatation operator represent non-standard deformations of this spin chain model as discussed in [21]. The fact that the 2-loop effective vertex in the plane-wave matrix quantum mechanics to be discussed below departs from the Super Yang-Mills dilatation operator indicates that one is facing an alternative deformation of the spin chain in the plane-wave matrix theory, iff the integrability is indeed conserved at higher loop level. Including the fermionic and SO(3) excitations one might expect to uncover an integrable spin chain structure of plane-wave matrix theory based on the underlying supergroup SU(2|4). It is certainly interesting to pursue these issues further.

Moving on to the 2-loop energy shift we are faced with the evaluation of the effective vertex $V^{(2)}_{\mathrm{eff}}$

$$\delta E_2 = \big\langle \phi_0 \big| V^{(2)}_{\mathrm{eff}} \big| \phi_0 \big\rangle \tag{37}$$

with

$$\begin{aligned} V^{(2)}_{\mathrm{eff}} = {} & V_1 \Delta V_1 \Delta V_1 \Delta V_1 \\ & + V_1 \Delta V_1 \Delta V_2 + V_1 \Delta V_2 \Delta V_1 + V_2 \Delta V_1 \Delta V_1 \\ & + V_2 \Delta V_2 - V_1 \Delta \Delta V_1 P V_1 \Delta V_1 - V_1 \Delta \Delta V_1 P V_2 \end{aligned} \tag{38}$$

where

$$\Delta = \frac{\mathbf{1} - \big|\phi_0\big\rangle\big\langle\phi_0\big|}{E_0 - H_0} , \qquad\qquad P = \big|\phi_0\big\rangle\big\langle\phi_0\big| .$$

Here some information on the particular forms of V_1 and V_2 entered in the derivation of (38): All terms have been omitted that contain an odd number of fields of one kind, as they do not contribute in the computation of expectation values. We again project onto pure, gauge invariant SO(6) excitations acting on $V^{(2)}_{\mathrm{eff}}$. The resulting expression is, however, still rather lengthy and has been relegated to appendix E eq. (89). Let us then exemplify our claim of the disagreement at 2-loop level by considering the energy shifts of an explicit state.

Consider the "Konishi" state in the plane-wave matrix theory

$$\big|K\big\rangle := \frac{1}{\sqrt{12(N^2-1)}}\,\mathrm{tr}\, a_i^\dagger a_i^\dagger \big|0\big\rangle\,. \tag{39}$$

We then find

$$E_0 = \big\langle K\big|H_0\big|K\big\rangle = M \tag{40}$$

$$\delta E_1 = \big\langle K\big|V_{\text{eff}}^{(1)}\big|K\big\rangle = \frac{12N}{M^2} \tag{41}$$

$$\delta E_2 = \big\langle K\big|V_{\text{eff}}^{(2)}\big|K\big\rangle = -\frac{228N^2}{M^5} \tag{42}$$

This is to be compared to the scaling dimension of the Konishi operator $K = \mathrm{tr}(\phi_i\phi_i)$ of $\mathcal{N}=4$ Super Yang-Mills [23]

$$\Delta_K = 2 + \frac{3g_{\text{YM}}^2 N}{4\pi^2} - \frac{3g_{\text{YM}}^4 N^2}{16\pi^4} + \dots \tag{43}$$

As the overall constant of the hamiltonian is not fixed we can only compare the ratios of the energy shifts δE_i to the free energy E_0 with the ratios of the anomalous dimensions $\delta\Delta_i$ to the engineering dimension Δ_0.

$$E_{|K\rangle} = \frac{M}{2}\left(2 + \frac{24\,N}{M^3} - \frac{456\,N^2}{M^6} + \dots\right) \tag{44}$$

Inserting the uncovered relation between the plane-wave matrix theory mass parameter $M = \frac{m}{3}$ and the gauge theory coupling constant g_{YM} of (3) into (43)

$$\frac{1}{M^3} = \frac{g_{\text{YM}}^2}{32\pi^2} \tag{45}$$

the 1-loop correction indeed agrees[5]. Using this relation in the 2-loop energy shift we would predict the following 2-loop anomalous dimension for the Konishi field from the matrix quantum mechanics

$$\frac{19}{8}\cdot\frac{-3g_{\text{YM}}^4 N^2}{16\pi^4} \tag{46}$$

which is off by a factor of 19/8.

[5] An alert reader might think that in choosing the relation (45) one does not have an independent check of the 1-loop agreement. This is of course true. The point here is that after choosing (45) *all* further 1-loop corrections for higher excited states are fixed and do agree with the corresponding 1-loop Super Yang-Mills scaling dimensions. This is a consequence of the matching of (34) with (35).

3.1 The Berenstein-Maldacena-Nastase limit

Recently there has been considerable interest in a novel double scaling limit of SU(N) $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills following the work of Berenstein, Maldacena and Nastase [4] where one takes

$$N \to \infty, \quad J \to \infty \quad \text{with} \quad \frac{J^2}{N} \quad \text{and} \quad g_{\text{YM}} \quad \text{fixed.} \tag{47}$$

Here J is the U(1) R-charge associated to the complex combination of two of the six scalar fields ϕ_i, e.g.

$$Z = \frac{1}{\sqrt{2}} \left(\phi_5 + i\,\phi_6\right) . \tag{48}$$

This limit leads to a dual gauge theory description of the IIB plane-wave superstring. It represents an extreme reduction of the field theory: Only two- and three-point functions of so-called BMN operators, having large scaling dimensions and U(1) R-charges $\Delta_0 > J \gg 1$, exist [19]. It also appears at this stage, that the only sensible observable of BMN gauge theory is the dilatation operator. Moreover, holographic arguments indicate that the gauge theory dual of the IIB plane-wave superstring should be given by a one-dimensional model and it was proposed [13] that this effective quantum mechanical model indeed arises as the Kaluza-Klein reduction of $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills on $\mathbb{R} \times S^3$, which we have considered in section 2. As long as one restricts one's attention to the pure scalar SO(6) sector we have seen that the plane-wave matrix theory serves this purpose at the one-loop level. In principle there is the logical possibility that although this match of energies and scaling dimensions ceases to exist at higher loop level for the full Super Yang-Mills theory, it does hold for the BMN sector of the gauge theory at higher loops.

For convenience we further reduce the SO(6) excitations under consideration in the plane-wave matrix model, to solely two complex combinations

$$Z := \tfrac{1}{\sqrt{2}} \left(a_5^\dagger + i a_6^\dagger\right) \qquad \bar{Z} := \tfrac{1}{\sqrt{2}} \left(a_5 - i a_6\right) \tag{49}$$

$$\phi := \tfrac{1}{\sqrt{2}} \left(a_3^\dagger + i a_4^\dagger\right) \qquad \bar{\phi} := \tfrac{1}{\sqrt{2}} \left(a_3 - i a_4\right) \tag{50}$$

which will make our formulas more transparent. The energy shifts of excitations made entirely from the creation operators Z and ϕ are then governed by the simple effective vertices

$$H_0 \to \frac{M}{2} \; : \mathrm{tr}(Z\bar{Z} + \phi\bar{\phi}) : \tag{51}$$

$$V_{\text{eff}}^{(1)} \to -\frac{2}{M^2} : \mathrm{tr}[Z,\phi][\bar{Z},\bar{\phi}] : \tag{52}$$

$$\begin{aligned} V_{\text{eff}}^{(2)} \to & \frac{22N}{M^5} : \mathrm{tr}[Z,\phi][\bar{Z},\bar{\phi}] : \\ & + \frac{4}{M^5}\Big(: \mathrm{tr}[Z,\phi][\bar{Z},[Z,[\bar{Z},\bar{\phi}]]] : + : \mathrm{tr}[Z,\phi][\bar{\phi},[\phi,[\bar{Z},\bar{\phi}]]] : \Big) . \end{aligned} \tag{53}$$

Let us stress that no limit has been performed yet, eq. (53) follows directly from (89), upon restricting its action to states given by excitations in gauge-invariant words made of Z's and ϕ's only.

This result may be directly compared to the two-loop structure of the $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills dilatation operator in the analog scalar sector, which has been computed in [21] eq. (5.5)

$$D_0 =: \mathrm{tr}(Z\check{Z} + \phi\check{\phi}) : \tag{54}$$

$$D_2 = \frac{g_{\mathrm{YM}}^2}{16\pi^2}\left(-2 : \mathrm{tr}[Z,\phi][\check{Z},\check{\phi}] :\right) \tag{55}$$

$$\begin{aligned} D_4 = \frac{g_{\mathrm{YM}}^4}{(16\pi^2)^2}\Big(&4N : \mathrm{tr}[Z,\phi][\check{Z},\check{\phi}] : \\ &+ 2 : \mathrm{tr}[Z,\phi][\check{Z},[Z,[\check{Z},\check{\phi}]]] : +2 : \mathrm{tr}[Z,\phi][\check{\phi},[\phi,[\check{Z},\check{\phi}]]] :\Big) \end{aligned} \tag{56}$$

The agreement of the one-loop terms was already observed for general SO(6) excitations in the last subsection. However, the closeness of the two-loop result D_4 to $V_{\mathrm{eff}}^{(2)}$ is striking: The same structure of terms appears, only one relative factor is wrong! Remarkably, not all terms of (56) are indeed relevant in the BMN limit (47), as shown in [21]. However, it is precisely the last term in (56) which is irrelevant in the BMN limit when acting on states (operators) of the form $\mathrm{tr}(Z^p\phi Z^{J-p}\phi)$ in the BMN limit (47).

So even if one considers the BMN limit (47) of the next-to-leading order effective vertex of the plane-wave matrix model (53), the two-loop discrepancy to the scaling dimensions of the four-dimensional gauge theory persists.

The observed two-loop discrepancy of plane-wave matrix theory and $\mathcal{N} = 4$ Super Yang-Mills should be understandable in the framework of Wilsonian effective quantum field theory, where one integrates out all non-SU$(2)_R$-singlets in a perturbative fashion. The resulting effective action of the $SU(2)_R$ singlet modes will start out with the plane-wave matrix model, but the inclusion of the non-$SU(2)_R$-singlet modes in loops will lead to a renormalization of the matrix model mass parameter M as well as higher order interaction terms in the plane-wave matrix model. As is evident from our results at one-loop order these corrections are not yet effective, but apparently start to contribute beyond this order. It would be very interesting to study this in detail, in particular what restrictions on these higher order interaction terms arise from the underlying supersymmetry.

4 Discussion

In this paper we have shown how the mass deformed gauge quantum mechanics of plane-wave matrix theory arises from the Kaluza-Klein reduction of $\mathcal{N} = 4$, $D = 4$ Super Yang-Mills theory on $\mathbb{R}\times S^3$. Whereas the ordinary gauge quantum mechanics is obtained through a trivial dimensional reduction of *any* higher dimensional Yang-Mills theory with maximal supersymmetry, the mass deformed supersymmetric quantum mechanics at hand is intrinsically related to four-dimensions, as the symmetry algebra SU(2|4) suggests.

We went on to explore the relation of the spectrum of the plane-wave matrix model to the scaling dimensions of $\mathcal{N} = 4$, $D = 4$ Super Yang-Mills operators. At the free theory level the two are equivalent, provided one considers Yang-Mills operators without higher space derivatives. Once interactions are turned on one would in general expect a different behavior of the two quantities. However, for the special class of operators which are protected from perturbative corrections the analogy persist: The two theories share the same 1/2-BPS multiplets, whose primaries are given as totally symmetric traceless $SO(6)$ tensors. We have seen in this paper that the analogy also extends to the one-loop level for pure scalar operators and their supersymmetry descendants, but ceases to exist beyond that. It is natural to ask whether this leading-order agreement will turn out to be true for the entire spectrum of the massive gauge quantum mechanics as well, and we hope to address this issue in the future.

The discussions presented in section 2 were purely classical, which implies that we can in fact start with any supersymmetric field theory with classical superconformal invariance and consider Kaluza-Klein reductions on $\mathbb{R}\times S^3$ to obtain a mass-deformed supersymmetric gauge quantum mechanics. Some of them might also be given a M-theory interpretation. Let us take the example of pure $\mathcal{N} = 2$ Yang-Mills theory, whose trivial dimensional reduction gives a gauge quantum mechanics with 8 supersymmetries and SO(5) symmetry. Put on $\mathbb{R} \times S^3$, the mass deformation will induce the symmetry breaking $\mathrm{SO}(5) \to \mathrm{SO}(3) \times \mathrm{SO}(2)$. If we add one hypermultiplet in the adjoint representation one recovers the plane-wave matrix model. If one adds further hypermultiplets in the fundamental representation one obtains the matrix model of supersymmetric M5-branes in a plane-wave background, which are extended along four of the SO(6) directions as well as two light-cone directions. The Supersymmetry of the relevant M5-brane configurations in the plane-wave background is elucidated in [24]. As an alternative avenue for generalization one can also consider $\mathcal{N} = 1$ field theories with nontrivial target spaces for chiral multiplets, i.e. gauged nonlinear sigma models. The relevant massive quantum mechanical models could give matrix models of eleven dimensional pp-waves with curved transverse spaces, which are analogs of ten dimensional solutions studied in [25, 26].

In this paper we made use of the specific formulation of $\mathcal{N} = 4$, $D = 4$ Super Yang-Mills on $\mathbb{R} \times S^3$, but it is certainly quite desirable to carry out a systematic study of supersymmetric field theories on curved backgrounds in various dimensions, which do not allow parallel spinors. This has been initiated by Blau in [16], where pure $\mathcal{N} = 1$ Super Yang-Mills theories in $D = 10, 6, 4, 3$ and their trivial dimensional reductions to intermediate lower dimensions are considered. He finds that by adding mass and Chern-Simons like terms the actions can be made supersymmetric with respect to generalized Killing spinor equations on Einstein manifolds.

Finally let us stress once more the emergence of an integrable SO(6) spin chain structure at leading order perturbation theory of the plane-wave matrix model, in analogy to the situation found at one-loop in $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills by Minahan and Zarembo [22]. It is certainly worthwhile to explore the origin and implications of this symmetry further, possibly leading to the integrability of the complete model. One would expect this program to be simpler than in the full four-dimensional case, that is, if it is performable at all.

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A Dimensional split of Clifford algebras

In this appendix we give the various forms of the fermionic part of the action of maximally supersymmetric Yang-Mills theory. In 1+9 dimensions we use G^M-matrices[6] satisfying the Clifford algebra $\{G^M, G^N\} = 2\eta^{MN}\mathbf{1}_{32}$ with $\eta^{MN} = \mathrm{diag}(-, +^9)$. The charge conjugation matrix C_{10} is symmetric and acts as $C_{10}G^M C_{10}^{-1} = G^{M\top}$. In ten dimensions the fermionic field content of $\mathcal{N} = 1$ Super Yang-Mills theory is a single 32-component complex spinor Λ_α subject to the Majorana condition $\bar{\Lambda} := \Lambda^\dagger G^0 \overset{!}{=} \Lambda^\top C_{10}$ as well as the Weyl condition $G_{11}\Lambda \overset{!}{=} \Lambda$, where $G_{11} := G^0 \cdots G^9$ is the chirality matrix. With these definitions the fermionic part of the $D = 4$, $\mathcal{N} = 4$ theory can be written concisely as

$$\mathcal{L}_{\text{ferm}} = \tfrac{i}{2}\bar{\Lambda}G^\mu D_\mu \Lambda + \tfrac{1}{2}\bar{\Lambda}G^i[\phi_i, \Lambda] , \tag{57}$$

where $\mu = 0, 1, 2, 3$ and $i = 4, \ldots, 9$. We now split the G^M-matrices into Γ^I-matrices of SO(9) and a 2×2 matrix factor according to

$$G^0 = i\sigma^2 \otimes \mathbf{1}_{16} \tag{58}$$

$$G^I = \sigma^1 \otimes \Gamma^I \qquad \text{with } I = 1, \ldots, 9 . \tag{59}$$

The charge conjugation matrix decomposes as $C_{10} = \sigma^1 \otimes C_9$ and leads to a symmetric charge conjugation matrix in nine dimensions which satisfies $C_9\Gamma^I C_9^{-1} = \Gamma^{I\top}$. Due to the Majorana-Weyl condition the spinor can be written as

$$\Lambda = \sqrt{2}\begin{pmatrix} L \\ 0 \end{pmatrix} \qquad \text{with} \quad L^\dagger = L^\top C_9 \tag{60}$$

and the lagrangian takes the form

$$\mathcal{L}_{\text{ferm}} = -iL^\dagger D_t L + iL^\dagger \Gamma^a D_a L + L^\dagger \Gamma^i [\phi_i, L] , \tag{61}$$

[6]Throughout this appendix all indices are understood as frame indices.

where $a = 1, 2, 3$. We further split the Γ^I-matrices to account for SO(3) $\otimes$ SO(6):

$$\Gamma^a = \begin{pmatrix} -\sigma^a \otimes \mathbf{1}_4 & 0 \\ 0 & \sigma^a \otimes \mathbf{1}_4 \end{pmatrix} \qquad \Gamma^i = \begin{pmatrix} 0 & \mathbf{1}_2 \otimes \rho^i \\ \mathbf{1}_2 \otimes \rho^{i\dagger} & 0 \end{pmatrix} , \tag{62}$$

where σ^a are the three Pauli matrices and the 4×4 matrices ρ^i satisfy

$$\rho^i \rho^{j\dagger} + \rho^j \rho^{i\dagger} = \rho^{i\dagger} \rho^j + \rho^{j\dagger} \rho^i = 2\delta^{ij} \mathbf{1}_4 . \tag{63}$$

The charge conjugation matrix in this representation is given by

$$C_9 = \begin{pmatrix} 0 & -i\sigma^2 \otimes \mathbf{1}_4 \\ i\sigma^2 \otimes \mathbf{1}_4 & 0 \end{pmatrix} , \tag{64}$$

allowing one to write the spinor as

$$L = \begin{pmatrix} \lambda^{\alpha A} \\ i(\sigma^2)^{\alpha\beta} \lambda^*_{\beta A} \end{pmatrix} , \qquad \alpha = 1, 2 , \quad A = 1, \ldots, 4 , \tag{65}$$

where $\lambda^{\alpha A}$ are now four 2-component Weyl spinors. In this notation the lagrangian reads

$$\mathcal{L}_{\text{ferm}} = -2i\lambda^\dagger_A D_t \lambda^A - 2i\lambda^\dagger_A \sigma^a D_a \lambda^A + \lambda^\dagger_A i\sigma^2 (\rho_i)^{AB} [\phi_i, \lambda^*_B] - (\lambda^A)^\top i\sigma^2 (\rho^\dagger_i)_{AB} [\phi_i, \lambda^B] . \tag{66}$$

If one finally adds the unit matrix to the set of Pauli matrices by the definition $\sigma^\mu = (\mathbf{1}, \sigma^a)$, this is exactly the form (4) given in the main text.

When we reduce the field theory to the zero mode excitation in section 2.2 we end up with the plane-wave matrix model where the fermions were written in SU(2) $\otimes$ SU(4) notation. This version of the matrix model can easily be obtained from the original form (2) when the above realization of Γ-matrices (62) is used and the matrix model fermions are written in terms of Weyl spinors

$$\theta = \begin{pmatrix} \theta^{\alpha A} \\ i(\sigma^2)^{\alpha\beta} \theta^*_{\beta A} \end{pmatrix} . \tag{67}$$

Note that the charge conjugation matrix C_9 is no longer unity, which has been used in (2). We particularly note that

$$\Gamma^{123} = \begin{pmatrix} -i\mathbf{1}_2 \otimes \mathbf{1}_4 & 0 \\ 0 & i\mathbf{1}_2 \otimes \mathbf{1}_4 \end{pmatrix} \tag{68}$$

which simplifies the fermionic mass term to

$$\tfrac{m}{4} i\theta^\dagger \Gamma_{123} \theta = \tfrac{m}{2} \theta^\dagger_A \theta^A . \tag{69}$$

B Killing spinors and Killing vectors

The Killing spinor equation is given by

$$\nabla_a S^{\hat{\alpha}\pm} = \pm \tfrac{i}{2R}\sigma_a S^{\hat{\alpha}\pm} \; . \tag{70}$$

The hatted index $\hat{\alpha} = 1, 2$ represents the degeneracy of the solution. When the different signs are also taken into account the four solutions give the lowest spinor spherical harmonics as $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ of $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$. In the following we will often suppress the $\pm$ index, when the given equations hold for both signs. The solutions of (70) can be orthonormalized to

$$S^{\hat{\alpha}\dagger} S^{\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}} \tag{71}$$

and form a complete basis

$$\sum_{\hat{\alpha}} S^{\hat{\alpha}}_{\alpha} S^{\hat{\alpha}*}_{\beta} = \delta_{\alpha\beta} \; . \tag{72}$$

Moreover they satisfy

$$S^{\hat{\alpha}\top} i\sigma^2 S^{\hat{\beta}} = k(i\sigma^2)^{\hat{\alpha}\hat{\beta}} \quad \text{with} \quad k \in \mathbb{C}, \; |k| = 1 \tag{73}$$

and we are free to define them such that $k = 1$.

The Killing vectors can be obtained as bilinears $S^{\hat{\alpha}\pm\dagger}\sigma_a S^{\hat{\beta}\pm}$ of (commuting) Killing spinors. One can show that these bilinears, considered as 2×2 matrices in the indices $\hat{\alpha}$ and $\hat{\beta}$ are hermitian and traceless. Therefore they may be expanded into Pauli matrices:

$$S^{\hat{\alpha}\pm\dagger}\sigma_a S^{\hat{\beta}\pm} = (\sigma_{\hat{a}})^{\hat{\alpha}\hat{\beta}} V^{\hat{a}\pm}_a \; , \tag{74}$$

where $\hat{a} = 1, 2, 3$ is a flat index. The Killing spinor equation (70) for $S^{\hat{\alpha}}$ implies the Killing vector equation

$$\nabla_a V^{\hat{a}\pm}_b + \nabla_b V^{\hat{a}\pm}_a = 0 \tag{75}$$

for the coefficients of this expansion. Hence $V^{\hat{a}\pm}$ give the six Killing vectors $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$ which are also the lowest vector spherical harmonics of S^3. Their orthonormality

$$V^{\hat{a}a} V^{\hat{b}}_a = \delta^{\hat{a}\hat{b}} \tag{76}$$

can be shown from the orthonormality of the Killing spinors by making use of the Fierz identity. As an immediate consequence of (75) we have the divergencelessness of the Killing vectors

$$\nabla^a V^{\hat{a}}_a = 0 \; . \tag{77}$$

It can be also shown that

$$\nabla_a V^{\hat{a}\pm}_b = \pm \tfrac{1}{2R} \varepsilon^{\hat{a}\hat{b}\hat{c}} V^{\hat{b}\pm}_a V^{\hat{c}\pm}_b \tag{78}$$

as well as

$$\nabla_a V^{\hat{a}\pm}_b = \pm \tfrac{1}{R} \varepsilon_{abc} V^{a\hat{a}\pm} \; . \tag{79}$$

Let us also give an explicit realization of the Killing spinors and vectors. However, we emphasize that the derivation of the matrix model can be entirely performed without making use of them. With the obvious choice of vielbein

$$e^1 = d\theta, \quad e^2 = \sin\theta d\psi, \quad e^3 = \sin\theta \sin\psi d\chi, \tag{80}$$

it is a simple matter to find the solutions

$$S^{\pm} = e^{\pm i\frac{\theta}{2}\sigma^1} e^{i\frac{\psi}{2}\sigma^3} e^{i\frac{\chi}{2}\sigma^1} S_0^{\pm} \tag{81}$$

and

$$\begin{aligned} V_a^{\hat{1}} d\mathbf{x}^a &= \cos\psi e^1 - \cos\theta \sin\psi e^2 \pm \sin\theta \sin\psi e^3 \ , \\ V_a^{\hat{2}} d\mathbf{x}^a &= \sin\psi \cos\chi e^1 + (\cos\theta \cos\psi \cos\chi \mp \sin\theta \sin\chi) e^2 \\ &\quad - (\cos\theta \sin\chi \pm \sin\theta \cos\psi \cos\chi) e^3 \ , \\ V_a^{\hat{3}} d\mathbf{x}^a &= \sin\psi \sin\chi e^1 + (\cos\theta \cos\psi \sin\chi \pm \sin\theta \cos\chi) e^2 \\ &\quad + (\cos\theta \cos\chi \mp \sin\theta \cos\psi \sin\chi) e^3 \ . \end{aligned} \tag{82}$$

C Reduction of equation of motion

In this appendix we give some details of the derivation of the equations (20) for the zero modes from the 4-dimensional equations of motion (19).

The first equation (19a) for $\mu = 0$ can be written as

$$D_t(\nabla_a A^a) - i[A_a, D_t A^a] - i[\phi_i, D_t \phi_i] + 2\{\lambda^\dagger, \lambda\} = 0 \ . \tag{83}$$

When the ansatz (18) is inserted one immediately finds (20a) using the orthogonality of Killing spinors (71) and Killing vectors (76) as well as the divergencelessness of the Killing vectors (77).

For the spatial components of the same equation (19a) one finds

$$\begin{aligned} &D_t^2 A_a - \nabla^2 A_a + \mathcal{R}_a{}^b A_b + D_a(\nabla_b A^b) + i[A^b, 2\nabla_b A_a - \nabla_a A_b] \\ &- [A^b, [A_a, A_b]] - i[\phi_i, \nabla_a \phi_i] - [\phi_i, [A_a, \phi_i]] - 2\{\lambda^\dagger, \sigma_a \lambda\} = 0 \ , \end{aligned} \tag{84}$$

where $\mathcal{R}_{ab} = \frac{2}{R^2} g_{ab}$ is the Ricci tensor of S^3. After inserting (18) we project onto the Killing vectors. Then we use the identities

- $V^{\hat{a}a} V^{\hat{b}b} \nabla_a V_b^{\hat{c}} = \frac{1}{R} \varepsilon_{abc} V^{\hat{a}a} V^{\hat{b}b} V^{\hat{c}c} = \frac{1}{R} \varepsilon^{\hat{a}\hat{b}\hat{c}} \det(V) = \frac{1}{R} \varepsilon^{\hat{a}\hat{b}\hat{c}}$
- $S^{\hat{\alpha}\dagger} \sigma_a S^{\hat{\beta}} V^{\hat{a}a} = (\sigma_{\hat{b}})^{\hat{\alpha}\hat{\beta}} V_a^{\hat{b}} V^{\hat{a}a} = (\sigma^{\hat{a}})^{\hat{\alpha}\hat{\beta}}$
- $\nabla^2 V_a^{\hat{a}} = -\frac{2}{R^2} V_a^{\hat{a}}$

and find (20b).

The equation of motion for the scalar field (19b) becomes

$$
\begin{aligned}
&D_t^2\phi_i - \nabla^2\phi_i + \tfrac{1}{R^2}\phi_i - i[\phi_i, \nabla_a A^a] + 2i[A^a, \nabla_a \phi_i] - [A^a, [\phi_i, A_a]] \\
&- [\phi_j, [\phi_i, \phi_j]] + \{\lambda^\dagger, i\sigma^2\rho_i\lambda^*\} - \{\lambda^\top, i\sigma^2\rho_i^\dagger\lambda\} = 0
\end{aligned}
\tag{85}
$$

Here one has to make use of property (73), then (20c) immediately follows.

The fermion equation of motion (19c) is split up into

$$
iD_t\lambda - i\sigma^a\nabla_a\lambda + \sigma^a[A_a, \lambda] - i\sigma^2\rho_i[\phi_i, \lambda^*] = 0 \tag{86}
$$

and projected onto the Killing spinors. Using similar identities as above one obtains (20d).

D A useful identity for gauge invariant states

The operator $:\operatorname{tr} T^a[a_i^\dagger, a_i]:$ annihilates any n-trace state

$$
\operatorname{tr}(a^\dagger_{j_{1,1}} \cdots a^\dagger_{j_{1,k_1}}) \operatorname{tr}(a^\dagger_{j_{2,1}} \cdots a^\dagger_{j_{2,k_2}}) \cdots \operatorname{tr}(a^\dagger_{j_{n,1}} \cdots a^\dagger_{j_{n,k_n}}) \big|0\big\rangle \,. \tag{87}
$$

This follows from the facts that it commutes with a trace of creation operators and annihilates the vacuum. It commutes since the sum of all Wick contractions vanishes

$$
\begin{aligned}
:\operatorname{tr} T^a[a_i^\dagger, a_i] : \operatorname{tr}(a^\dagger_{j_1}\cdots a^\dagger_{j_k}) &= :\operatorname{tr}[T^a, a_i^\dagger] a_i : \operatorname{tr}(a^\dagger_{j_1}\cdots a^\dagger_{j_k}) \\
&= \operatorname{tr}([T^a, a^\dagger_{j_1}] a^\dagger_{j_2} \cdots a^\dagger_{j_k}) + \operatorname{tr}(a^\dagger_{j_1}[T^a, a^\dagger_{j_2}] \cdots a^\dagger_{j_k}) \\
&\quad + \operatorname{tr}(a^\dagger_{j_1} a^\dagger_{j_2} \cdots [T^a, a^\dagger_{j_k}]) \\
&= \operatorname{tr}(T^a a^\dagger_{j_1} a^\dagger_{j_2} \cdots a^\dagger_{j_k}) - \operatorname{tr}(a^\dagger_{j_1} a^\dagger_{j_2} \cdots a^\dagger_{j_k} T^a) = 0 \,.
\end{aligned}
$$

E Effective 2-loop vertex

Summing all 2-loop contributions, we find

$$
\begin{aligned}
V_{\text{eff}}^{(2)} \to\ & -\frac{51N^2}{4M^5} : \operatorname{tr} a_i^\dagger a_i : \\
& -\frac{2}{M^5}\Big(: \operatorname{tr} a_i a_j \operatorname{tr} a_i^\dagger a_j^\dagger : + : \operatorname{tr} a_i a_i^\dagger \operatorname{tr} a_j a_j^\dagger : + : \operatorname{tr} a_i a_j^\dagger \operatorname{tr} a_i a_j^\dagger : \\
& \qquad + : \operatorname{tr} a_i a_i \operatorname{tr} a_j^\dagger a_j^\dagger : +2 : \operatorname{tr} a_i a_j^\dagger \operatorname{tr} a_i a_j^\dagger : \Big) \\
& -\frac{N}{M^5}\Big(-30 : \operatorname{tr} a_i a_j a_i^\dagger a_j^\dagger : +\tfrac{67}{8} : \operatorname{tr} a_i a_i^\dagger a_j a_j^\dagger : +\tfrac{67}{8} : \operatorname{tr} a_i a_j^\dagger a_j a_i^\dagger : +\tfrac{61}{4} : \operatorname{tr} a_i a_j a_j^\dagger a_i^\dagger : \\
& \qquad +17 : \operatorname{tr} a_i a_i a_j^\dagger a_j^\dagger : -15 : \operatorname{tr} a_i a_j^\dagger a_i a_j^\dagger : \Big) \\
& -\frac{1}{M^5}\Big(: \operatorname{tr}[a_j,[a_i,a_j^\dagger]][a_k,[a_i^\dagger,a_k^\dagger]] : + : \operatorname{tr}[a_j^\dagger,[a_i,a_j]][a_k,[a_i^\dagger,a_k^\dagger]] : \\
& \qquad + : \operatorname{tr}[a_j,[a_i,a_j^\dagger]][a_k^\dagger,[a_i^\dagger,a_k]] : + : \operatorname{tr}[a_j^\dagger,[a_i,a_j]][a_k^\dagger,[a_i^\dagger,a_k]] : \\
& \qquad + : \operatorname{tr}[a_j,[a_i^\dagger,a_j^\dagger]][a_k,[a_i^\dagger,a_k]] : + : \operatorname{tr}[a_j^\dagger,[a_i^\dagger,a_j]][a_k,[a_i^\dagger,a_k]] : \\
& \qquad + : \operatorname{tr}[a_j,[a_i,a_j^\dagger]][a_k^\dagger,[a_i,a_k^\dagger]] : + : \operatorname{tr}[a_j^\dagger,[a_i,a_j]][a_k^\dagger,[a_i,a_k^\dagger]] : \\
& \qquad + : \operatorname{tr}[a_j,[a_i^\dagger,a_j]][a_k^\dagger,[a_i,a_k^\dagger]] : -\tfrac{1}{2} : \operatorname{tr}[a_j,[a_i,a_j]][a_k^\dagger,[a_i^\dagger,a_k^\dagger]] : \Big)
\end{aligned}
$$

This can be simplified using Jacobi identities and a reasoning similar to that at one loop, e. g. we have

$$
\begin{aligned}
: \operatorname{tr}[a_j,[a_i,a_j^\dagger]][a_k,[a_i^\dagger,a_k^\dagger]] : = & : \operatorname{tr}[a_j^\dagger,[a_i,a_j]][a_k,[a_i^\dagger,a_k^\dagger]] : +N : \operatorname{tr}[a_i,a_j][a_i^\dagger,a_j^\dagger] : \\
& + : \operatorname{tr}[a_i,[a_n,[a_i^\dagger,a_n^\dagger]]]T^a : : \operatorname{tr} T^a[a_j^\dagger,a_j] :
\end{aligned} \tag{88}
$$

The last line may be dropped by arguments given in appendix D.

Then we find the somewhat more compact final result

$$
\begin{aligned}
V_{\text{eff}}^{(2)} \to\ & -\frac{2}{M^5}\Big(: \operatorname{tr} a_i a_i \operatorname{tr} a_j^\dagger a_j^\dagger : +2 : \operatorname{tr} a_i a_j^\dagger \operatorname{tr} a_i a_j^\dagger : \Big) \\
& -\frac{N}{M^5}\Big(-11 : \operatorname{tr}[a_i^\dagger,a_j^\dagger][a_i,a_j] : +17 : \operatorname{tr} a_i a_i a_j^\dagger a_j^\dagger : -15 : \operatorname{tr} a_i a_j^\dagger a_i a_j^\dagger : \Big) \\
& -\frac{4}{M^5} : \operatorname{tr}[a_j^\dagger,[a_i,a_j]][a_k,[a_i^\dagger,a_k^\dagger]] : \\
& -\frac{1}{M^5}\Big(: \operatorname{tr}[a_j,[a_i^\dagger,a_j^\dagger]][a_k,[a_i^\dagger,a_k]] : + : \operatorname{tr}[a_j^\dagger,[a_i^\dagger,a_j]][a_k,[a_i^\dagger,a_k]] : \\
& \qquad + : \operatorname{tr}[a_j,[a_i,a_j^\dagger]][a_k^\dagger,[a_i,a_k^\dagger]] : + : \operatorname{tr}[a_j^\dagger,[a_i,a_j]][a_k^\dagger,[a_i,a_k^\dagger]] : \\
& \qquad + : \operatorname{tr}[a_j,[a_i^\dagger,a_j]][a_k^\dagger,[a_i,a_k^\dagger]] : -\tfrac{1}{2} : \operatorname{tr}[a_j,[a_i,a_j]][a_k^\dagger,[a_i^\dagger,a_k^\dagger]] : \Big)
\end{aligned} \tag{89}
$$

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