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Kaluza-Klein supergravity on $\mathrm{AdS}_3 \times S^3$

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Abstract

We construct a Chern-Simons type gauged $N = 8$ supergravity in three space-time dimensions with gauge group $\mathrm{SO}(4) \times T_\infty$ over the infinite dimensional coset space $\mathrm{SO}(8, \infty)/\,(\mathrm{SO}(8) \times \mathrm{SO}(\infty))$, where T_∞ is an infinite dimensional translation subgroup of $\mathrm{SO}(8, \infty)$. This theory describes the effective interactions of the (infinitely many) supermultiplets contained in the two spin-1 Kaluza-Klein towers arising in the compactification of $N = (2, 0)$ supergravity in six dimensions on $\mathrm{AdS}_3 \times S^3$ with the massless supergravity multiplet. After the elimination of the gauge fields associated with T_∞, one is left with an Yang Mills type gauged supergravity with gauge group $SO(4)$, and in the vacuum the symmetry is broken to the (super-)isometry group of $\mathrm{AdS}_3 \times S^3$, with infinitely many fields acquiring masses by a variant of the Brout-Englert-Higgs effect.

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1 Introduction

One of the best studied examples of the celebrated AdS/CFT duality conjecture [1,2] is the D1-D5 system, which relates IIB string theory on $\mathrm{AdS}_3 \times S^3 \times M_4$ to a two-dimensional $N = (4,4)$ supersymmetric conformal field theory (CFT) living on the boundary of AdS_3 [3, 4,5,6,7]. The latter is believed to be described by a non-linear σ-model whose target space is a deformation of the symmetric orbifold $(M_4)^n/S_n$ [8,9,10]. In the supergravity limit, the Kaluza-Klein (KK) modes on the $\mathrm{AdS}_3 \times S^3 \times M_4$ near horizon geometry of the D1-D5 system are dual to chiral primary operators in the conformal field theory. Although CFT calculations have been mainly performed at the 'orbifold point' [11,12], where the supergravity approximation breaks down, nontrivial tests of the correspondence are possible for quantities protected by non-renormalization theorems; in particular, BPS spectra and elliptic genera were matched successfully [13,14,15,16].

The computation of higher point correlation functions requires the evaluation of higher order supergravity couplings which have been extensively studied for the compactification of IIB supergravity on $\mathrm{AdS}_5 \times S^5$ [17,18,19,20] and for $\mathrm{AdS}_3 \times S^3$ [21,22]. With increasing order of interactions, however, this program becomes stymied by a plethora of non-linear field redefinitions. Nevertheless, for the $\mathrm{AdS}_5 \times S^5$ compactification, the effective low energy theory for the lowest (massless) supermultiplet at the bottom of the KK tower is believed to be known to *all* orders and to coincide with the maximal $D = 5$, $N = 8$ gauged supergravity with gauge group SO(6) [23].[1] In particular, the scalar potential of this theory carries information about the deformations of the dual CFT by relevant operators and the corresponding renormalization group flows [27,28,29]. It would clearly be desirable to have an effective theory describing the full non-linear couplings of the higher KK supermultiplets as well, but this does not appear possible in five dimensions due to the impossibility of consistently coupling a finite number of massive spin-2 fields. In three dimensions, the situation is different. The $\mathrm{AdS}_3 \times S^3$ background is only half maximally supersymmetric, and instead of a single tower of supermultiplets as for $\mathrm{AdS}_5 \times S^5$, there are *three* different KK towers in the reduction. One of these contains the massive spin-2 supermultiplets (with the massless $N = 8$ supergravity multiplet at the bottom), while the other two consist of spin-1 supermultiplets [14,30]. In view of the duality between vector gauge fields and scalar fields in three dimensions it is therefore plausible that there should exist a unified description at least of the two spin-1 towers in terms of an infinite number of $N = 8$ supermultiplets coupled to the massless (nonpropagating) $N = 8$ supergravity

[1] To be sure, the consistency of this truncation has never been fully established despite much supporting evidence (see e.g. [24] and references therein), unlike for the $\mathrm{AdS}_4 \times S^7$ [25] and $\mathrm{AdS}_7 \times S^4$ [26] truncations of $D = 11$ supergravity.

multiplet.

In this paper, we will demonstrate that such a construction is indeed possible, and present an effective three-dimensional theory that describes the massless $N = 8$ supergravity multiplet and the entire two infinite spin-1 towers and their interactions in terms of a gauged supergravity over a *single* irreducible coset space. Furthermore, we will show that the spin-1 towers can be consistently truncated to any finite subset of spin-1 multiplets. Our construction exploits the special properties of gauged supergravities in three dimensions [31,32,33,34,35,36,37], which have no analog in dimensions $D \geq 4$, and makes essential use of the results of [34] establishing the link between Yang-Mills (YM) and Chern-Simons (CS) type gauged supergravities in three dimensions. It follows from these results that all the relevant information about the effective $D = 3$ theory is encoded in the infinite-dimensional coset space $\mathrm{G/H} = \mathrm{SO}(8, \infty)/(\mathrm{SO}(8) \times \mathrm{SO}(\infty))$, or more precisely,

$$\mathrm{G/H} \;=\; \mathrm{SO}\Big(8\,,\, \sum_{k\geq 2} k^2 + n \sum_{l\geq 1} l^2\Big) \Big/ \Big(\mathrm{SO}(8) \times \mathrm{SO}\Big(\sum_{k\geq 2} k^2 + n \sum_{l\geq 1} l^2\Big)\Big)\,. \tag{1.1}$$

The parameter n denotes the number of tensor multiplets in the six-dimensional theory which is $n = 5$, and $n = 21$ for $M_4 = T^4$ and $M_4 = K_3$, respectively. Instead of the infinite sums, one may for definiteness consider any finite truncation which yields a consistent and supersymmetric theory, coupling a finite but arbitrarily large number of supermultiplets to the basic $N = 8$ supergravity Lagrangian. In particular, we give the scalar potential as a function on the coset manifold (1.1) which yields the KK scalar and vector masses by virtue of a three-dimensional variant of the Brout-Englert-Higgs mechanism on the infinite-dimensional space (1.1). The fact that this agreement extends to the complete self-interactions induced by the KK compactification is a consequence of the uniqueness of the effective locally $N = 8$ supersymmetric theory. The extension of these results to the spin-2 tower, and hence to the *full* KK theory, remains an open problem for the time being; see, however, the comments in section 5. We emphasize that we do not wish to address here the issue of consistency of the KK truncation from six dimensions, but at this stage focus on the consistent three-dimensional theory. [2] Settling this issue will presumably require the inclusion of the spin-2 tower into the analysis.

We note that, already some time ago and in a different context, the idea of describing the effective interactions of an infinite number of fields in terms of a gauged supergravity was proposed in an attempt to describe the effective interactions of the massive scalar

[2]So far, consistency of truncations has been shown for the lowest supermultiplets in the $N = (1, 0)$ six-dimensional theory by explicitly constructing the non-linear ansatz in the higher-dimensional theory [33, 35]. An order by order analysis of the consistency for the higher modes was initiated in [22].

string modes arising in the compactification to four dimensions in terms of an $N = 4$ gauged supergravity over the coset $\mathrm{SO}(6,\infty)/(\mathrm{SO}(6) \times \mathrm{SO}(\infty))$ [38]. There the relevant gauge group, which must be a subgroup of $\mathrm{SO}(6,\infty)$, is based on an indefinite lattice algebra of string vertex operators.

The paper is organized as follows. In section 2 we briefly review the KK spectrum of six-dimensional $N = (2,0)$ supergravity on $\mathrm{AdS}_3 \times S^3$. In particular, we discuss the lowest floors of the three KK towers, given by the massless (nonpropagating) supergravity multiplet, a short spin-$\frac{1}{2}$ matter multiplet, and the massive spin-1 multiplet containing the YM vector fields. In section 3, we present the effective three-dimensional theory which describes the coupling of these three multiplets alternatively as an SO(4) YM theory, or as a CS theory with gauge group $\mathrm{SO}(4) \ltimes \mathrm{T}_6$. The theory is extended in section 4 to include massive spin-1 multiplets of arbitrarily high KK level. The construction is based on the infinite-dimensional coset space (1.1) while the CS gauge group is enlarged to $\mathrm{SO}(4) \ltimes \mathrm{T}_\infty$, where T_∞ denotes an infinite translational subgroup of H. We close in section 5 with a few comments on the possible inclusion of the spin-2 tower.

2 Spectrum of supergravity on $\mathrm{AdS}_3 \times S^3$

The mass spectrum of six-dimensional $N = (2,0)$ supergravity on $\mathrm{AdS}_3 \times S^3$ has been computed in [30] by linearizing the equations of motion around the AdS background, and in [14] by group theoretical arguments in terms of unitary irreducible representations of the supergroup $\mathrm{SU}(2|1,1)_L \times \mathrm{SU}(2|1,1)_R$. Here, we briefly review these results to the extent needed in the following. The field content of the six-dimensional theory [39] comprises the supergravity multiplet with graviton, gravitini and five self-dual tensor fields, and n tensor multiplets, each containing an anti-selfdual tensor field, four fermions and five scalars. The scalar sector forms a coset space σ-model $\mathrm{SO}(5,n)/(\mathrm{SO}(5)\times\mathrm{SO}(n))$. The $\mathrm{AdS}_3 \times S^3$ background endows one of the five tensor fields of the supergravity multiplet with a vacuum expectation value

$$B^5_{\mu\nu\rho} = f\varepsilon_{\mu\nu\rho} \;\; , \quad B^5_{mnp} = f\varepsilon_{mnp} \; , \qquad B^{\tilde\imath}_{\mu\nu\rho} = 0 = B^{\tilde\imath}_{mnp} \; , \quad \tilde\imath = 1, \ldots, 4 \; , \tag{2.1}$$

where f is the Freund-Rubin parameter. For $f \neq 0$, the R-symmetry group is broken from SO(5) down to SO(4). Together with the SO(n) rotating the tensor multiplets, this group survives as a global symmetry of the three-dimensional effective theory.

The spectrum of the three-dimensional theory is hence organized under the AdS_3 supergroup $\mathrm{SU}(2|1,1)_L\times\mathrm{SU}(2|1,1)_R$ whose bosonic extension $\mathrm{SO}(3)_L\times\mathrm{SO}(3)_R \equiv \mathrm{SO}(4)_{\mathrm{gauge}}$

Δ	s_0	$\mathrm{SO}(4)_{\mathrm{gauge}}$	$\mathrm{SO}(4)_{\mathrm{glob}}$	# dof
k	0	$\left[\frac{k}{2},\frac{k}{2}\right]$	$[0,0]$	$(k+1)^2$
$k+\frac{1}{2}$	$\frac{1}{2}$	$\left[\frac{k}{2},\frac{k-1}{2}\right]$	$\left[0,\frac{1}{2}\right]$	$2k(k+1)$
$k+\frac{1}{2}$	$-\frac{1}{2}$	$\left[\frac{k-1}{2},\frac{k}{2}\right]$	$\left[\frac{1}{2},0\right]$	$2k(k+1)$
$k+1$	0	$\left[\frac{k-1}{2},\frac{k-1}{2}\right]$	$\left[\frac{1}{2},\frac{1}{2}\right]$	$4k^2$
$k+1$	1	$\left[\frac{k}{2},\frac{k-2}{2}\right]$	$[0,0]$	k^2-1
$k+1$	-1	$\left[\frac{k-2}{2},\frac{k}{2}\right]$	$[0,0]$	k^2-1
$k+\frac{3}{2}$	$\frac{1}{2}$	$\left[\frac{k-1}{2},\frac{k-2}{2}\right]$	$\left[\frac{1}{2},0\right]$	$2k(k-1)$
$k+\frac{3}{2}$	$-\frac{1}{2}$	$\left[\frac{k-2}{2},\frac{k-1}{2}\right]$	$\left[0,\frac{1}{2}\right]$	$2k(k-1)$
$k+2$	0	$\left[\frac{k-2}{2},\frac{k-2}{2}\right]$	$[0,0]$	$(k-1)^2$

Table I: Spin-1 multiplet $(\mathbf{k{+}1},\mathbf{k{+}1})_S$ of $\mathrm{SU}(2|1,1)_L \times \mathrm{SU}(2|1,1)_R$.

corresponds to the isometry group of the three-sphere S^3, and a global $\mathrm{SO}(4)\times\mathrm{SO}(n)$. It consists of three Kaluza-Klein (KK) towers: a spin-2 tower, and two spin-1 towers transforming as vector and singlet under $\mathrm{SO}(n)$, respectively. For later use, we give the generic spin-1 multiplet in table I. It contains $16k^2$ degrees of freedom, and following [14] we will designate it by $(\mathbf{k{+}1},\mathbf{k{+}1})_S$. The $\mathrm{SO}(4)$ representations are labeled by their spins $[j_1, j_2]$ under $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_R$ while the numbers (Δ, s_0) label the representations of the AdS group $\mathrm{SO}(2,2)$.

Let us describe in a little more detail the lowest levels of the three KK towers which we have collected in table II. For details on the higher dimensional origin of the modes, indicated in the last column, we refer to [30]. Recall that the six-dimensional theory does not admit a Lagrangian formulation due to the self-duality constraint obeyed by the two-forms B_{MN}. Upon compactification on S^3, these self-duality equations are used to eliminate the components $B_{\mu\nu}$ from the theory. The spin-2 tower starts from the massless supergravity multiplet, which in three dimensions does not carry propagating degrees of freedom. It comprises the metric, gravitinos and pure gauge modes of the $\mathrm{SO}(4)$ vectors, see below. The lowest level of the spin-1 $\mathrm{SO}(n)$-vector tower is occupied by the degenerate short (spin-$\frac{1}{2}$) multiplet $(\mathbf{2},\mathbf{2})_S$ of table I that contains 8 scalars and 8 fermions, all transforming in the vector representation of $\mathrm{SO}(n)$, labeled by the index $\tilde{r}$. By contrast, the spin-1 $\mathrm{SO}(n)$-singlet tower starts from the generic multiplet $(\mathbf{3},\mathbf{3})_S$ of table I whose bosonic content is given by 26 scalars and 6 propagating vector fields.

The six-dimensional origin of the vector fields in these multiplets is somewhat subtle due to the mixing between the KK vectors $g_{\mu m}$ and the tensor components $B^5_{\mu m}$. Parametrizing the lowest order fluctuations of the metric and the distinguished tensor

Δ	s_0	$\mathrm{SO}(4)_{\mathrm{gauge}}$	$\mathrm{SO}(4)_{\mathrm{glob}}$	$\mathrm{SO}(n)_{\mathrm{glob}}$	# dof	6d origin
Nonpropagating gravity multiplet $(\mathbf{3},\mathbf{1})_S+(\mathbf{1},\mathbf{3})_S$						
2	2	$[0,0]$	$[0,0]$	1	0	$g_{\mu\nu}$
2	-2	$[0,0]$	$[0,0]$	1	0	$g_{\mu\nu}$
$\frac{3}{2}$	$\frac{3}{2}$	$[0,\frac{1}{2}]$	$[0,\frac{1}{2}]$	1	0	ψ_μ
$\frac{3}{2}$	$-\frac{3}{2}$	$[\frac{1}{2},0]$	$[\frac{1}{2},0]$	1	0	ψ_μ
1	1	$[0,1]$	$[0,0]$	1	0	$g_{\mu m}, B^5_{\mu m}$
1	-1	$[1,0]$	$[0,0]$	1	0	$g_{\mu m}, B^5_{\mu m}$
Spin-$\frac{1}{2}$ hypermultiplet $(\mathbf{2},\mathbf{2})_S$						
1	0	$[\frac{1}{2},\frac{1}{2}]$	$[0,0]$	n	$4n$	$\phi^{5\tilde{r}}, B^{\tilde{r}}_{mn}$
$\frac{3}{2}$	$\frac{1}{2}$	$[\frac{1}{2},0]$	$[0,\frac{1}{2}]$	n	$4n$	$\chi^{\tilde{r}}$
$\frac{3}{2}$	$-\frac{1}{2}$	$[0,\frac{1}{2}]$	$[\frac{1}{2},0]$	n	$4n$	$\chi^{\tilde{r}}$
2	0	$[0,0]$	$[\frac{1}{2},\frac{1}{2}]$	n	$4n$	$\phi^{\tilde{i}\tilde{r}}$
Spin-1 multiplet $(\mathbf{3},\mathbf{3})_S$						
2	0	$[1,1]$	$[0,0]$	1	9	$B^5_{mn}, g_m{}^m, g_\mu{}^\mu$
$\frac{5}{2}$	$\frac{1}{2}$	$[1,\frac{1}{2}]$	$[0,\frac{1}{2}]$	1	12	ψ_m
$\frac{5}{2}$	$-\frac{1}{2}$	$[\frac{1}{2},1]$	$[\frac{1}{2},0]$	1	12	ψ_m
3	0	$[\frac{1}{2},\frac{1}{2}]$	$[\frac{1}{2},\frac{1}{2}]$	1	16	$B^{\tilde{i}}_{mn}$
3	1	$[1,0]$	$[0,0]$	1	3	$g_{\mu m}, B^5_{\mu m}$
3	-1	$[0,1]$	$[0,0]$	1	3	$g_{\mu m}, B^5_{\mu m}$
$\frac{7}{2}$	$\frac{1}{2}$	$[\frac{1}{2},0]$	$[\frac{1}{2},0]$	1	4	ψ_m
$\frac{7}{2}$	$-\frac{1}{2}$	$[0,\frac{1}{2}]$	$[0,\frac{1}{2}]$	1	4	ψ_m
4	0	$[0,0]$	$[0,0]$	1	1	$g_m{}^m, g_\mu{}^\mu$

Table II: Lowest mass spectrum on $AdS_3\times S^3$.

field (2.1) as $g_{\mu m}\equiv K^\pm_\mu Y^{(1,\pm1)}_m, B^5_{\mu m}\equiv Z^\pm_\mu Y^{(1,\pm1)}_m$ with the lowest S^3 vector harmonics $Y^{(1,\pm1)}_m$, one arrives at the following linearized coupled system of YM and CS equations

$$\nabla^\nu K^\pm_{\mu\nu}-\frac{4}{L_0}\,\epsilon_{\mu\nu\rho}Z^{\pm\,\nu\rho} \;=\; 0\;, \qquad \epsilon_{\mu\nu\rho}Z^{\pm\,\nu\rho}+\frac{2}{L_0}\left(K^\pm_\mu\pm 2Z^\pm_\mu\right) \;=\; 0\;, \tag{2.2}$$

where $K_{\mu\nu}\equiv\partial_\mu K_\nu-\partial_\nu K_\mu$ and $Z_{\mu\nu}\equiv\partial_\mu Z_\nu-\partial_\nu Z_\mu$, with remaining gauge freedom $\delta K^\pm_\mu=\partial_\mu\Lambda^\pm$, $\delta Z^\pm_\mu=\mp\frac{1}{2}\partial_\mu\Lambda^\pm$ and the AdS length L_0. The system (2.2) has three eigenmodes

$$K^\pm_\mu=\mp 2Z^\pm_\mu\;,\qquad K^\pm_\mu=\mp 4Z^\pm_\mu\;,\qquad K^\pm_\mu=\pm 2Z^\pm_\mu\;. \tag{2.3}$$

The first mode preserves gauge invariance and leads to the pure gauge states of the gravity multiplet in table II. The second mode in (2.3) yields the propagating vectors of the spin-1 multiplet in table II, carrying one degree of freedom and satisfying a massive CS equation $\epsilon_{\mu\nu\rho}K^{\pm\nu\rho} = \pm(4/L_0)K^{\pm}_{\mu}$. Alternatively, one may assemble this mode together with the non-propagating gauge mode above into a vector field satisfying the gauge covariant YM equation $\nabla^{\nu}K^{\pm}_{\mu\nu} \pm (1/L_0)\epsilon_{\mu\nu\rho}K^{\pm\,\nu\rho} = 0$ with topological mass term. Finally, the third eigenmode in (2.3) gives rise to massive CS vectors that are located in the lowest massive multiplet of the spin-2 tower [30].

In the next section we will present a three-dimensional supergravity theory with local SO(4) symmetry that combines all the lowest level multiplets given in table II and admits an $N = (4,4)$ supersymmetric AdS_3 groundstate. From the above discussion, we expect the vector fields of this theory to be given by either 12 CS fields whose equations of motion linearized around the groundstate take the form

$$\epsilon_{\mu\nu\rho}K^{\pm\nu\rho} \;=\; \pm\frac{4}{L_0}\,K^{\pm}_{\mu}\;, \qquad \text{and} \qquad \epsilon_{\mu\nu\rho}\hat{K}^{\pm\nu\rho} \;=\; 0\;, \qquad \text{respectively}\;, \tag{2.4}$$

or by a set of 6 YM vector fields, satisfying equations

$$\nabla^{\nu}K^{\pm}_{\mu\nu} \pm \frac{1}{L_0}\,\epsilon_{\mu\nu\rho}K^{\pm\,\nu\rho} \;=\; 0\;. \tag{2.5}$$

In [34] we have established the equivalence of YM gaugings with CS gaugings in three dimensions, in the sense that a YM gauged supergravity with gauge group G_0 is equivalent on shell to a CS gauged supergravity with gauge group $G_0 \ltimes T$, where T is a translation group containing a subgroup transforming in the adjoint of G_0. [3] In the following we shall exploit this result in order to construct a three-dimensional theory whose two equivalent formulations give rise to vector equations of the form (2.4) and (2.5), respectively.

3 The three-dimensional theory with YM multiplet

In this section we present the construction of the three-dimensional theory that describes the coupling of the lowest level multiplets of the three KK towers, collected in table II. The supergravity multiplet contains the non-propagating fields in three dimensions. The corresponding topological supergravity theory has been given in [40,41] as a Chern-Simons

[3] Although fermionic terms were not explicitly considered in [34], it is straightforward to determine the modifications of the supersymmetry variations and the YM type Lagrangian coming from the elimination of the translational CS vector fields by retaining the fermionic bilinears in their equations of motion. However, the resulting YM type Lagrangian has many more terms than the original CS type Lagrangian, which makes the comparison with a direct construction of the YM type theory somewhat cumbersome.

theory based on the supergroup $\mathrm{SU}(2|1,1)_L \times \mathrm{SU}(2|1,1)_R$. The coupling of this theory to propagating matter in the n spin-$\frac{1}{2}$ hypermultiplets $(\mathbf{2},\mathbf{2})_S$ has been constructed in [32]. It comes as an $\mathrm{SO}(8,n)/(\mathrm{SO}(8)\times\mathrm{SO}(n))$ coset space model with SO(4) gauge group and non-propagating CS vector fields; its scalar potential has been studied in the context of holographic RG flows in [42].

We shall now extend this construction to include the coupling to the spin-1, $\mathrm{SO}(n)$ singlet, supermultiplet $(\mathbf{3},\mathbf{3})_S$ of table II which contains the SO(4) Yang-Mills gauge vectors, in order to describe the full lowest mass spectrum of supergravity on $\mathrm{AdS}_3\times S^3$. The construction follows the strategy outlined in [34] yielding a CS gauged supergravity with the particular non-semisimple gauge group that allows for an on shell equivalent formulation with propagating YM gauge fields.

First of all, counting of degrees of freedom we see that after dualizing all degrees of freedom into the scalar sector, the spectrum of table II consists of $32+8n$ bosonic and the same number of fermionic degrees of freedom. The theory describing this field content should thus be obtainable as a CS gauging of the $N=8$ theory with coset space

$$\mathrm{G}/\mathrm{H} \;=\; \mathrm{SO}(8,4+n)/(\mathrm{SO}(8)\times\mathrm{SO}(4+n))\;. \tag{3.1}$$

The next step is the identification of the gauge group within G. According to [34], an SO(4) YM gauging is equivalent on shell to a CS gauging with gauge group

$$\mathrm{SO}(4)_{\mathrm{gauge}} \ltimes \mathrm{T}\;, \tag{3.2}$$

where $\mathrm{T}\equiv\mathrm{T}_6$ denotes an abelian group of six translations that transform in the adjoint representation of $\mathrm{SO}(4)_{\mathrm{gauge}}$. In addition, the precise embedding of this group into G is constrained by the group-theoretical algebraic constraints on its embedding tensor [32,36], see (3.11) below.

To start with, we embed the SO(4) subgroup, which will be identified with the YM gauge group, into the compact subgroup $\mathrm{H}\subset\mathrm{G}$ according to

$$\begin{aligned}\mathrm{SO}(4)_{\mathrm{gauge}} \;&\subset\; \underbrace{\mathrm{SO}(4)_+\times\mathrm{SO}(4)_-}\times\underbrace{\mathrm{SO}(4)_2\times\mathrm{SO}(n)}\\ &\subset\qquad\quad \mathrm{SO}(8)\qquad\;\;\times\qquad \mathrm{SO}(4+n)\qquad\quad,\end{aligned} \tag{3.3}$$

where

$$\mathrm{SO}(4)_{\mathrm{gauge}} \equiv \mathrm{diag}\,\Big(\mathrm{SO}(4)_+\times\mathrm{SO}(4)_2\Big)\;, \tag{3.4}$$

denotes the diagonal subgroup of $\mathrm{SO}(4)_+$ and $\mathrm{SO}(4)_2$. The $32{+}8n$ scalars that parametrize the coset space (3.1) transform as a bivector $(8_v, 4+n)$ under H. Under $\mathrm{SO}(4)_{\mathrm{gauge}}$ they decompose into

$$\begin{aligned}\left([\tfrac{1}{2},\tfrac{1}{2}]+4\cdot[0,0]\right)\times\left([\tfrac{1}{2},\tfrac{1}{2}]+n\cdot[0,0]\right)\\ = \; n\cdot[\tfrac{1}{2},\tfrac{1}{2}]+4n\cdot[0,0]+[0,0]+4\cdot[\tfrac{1}{2},\tfrac{1}{2}]+[0,1]+[1,0]+[1,1]\;. \end{aligned}\tag{3.5}$$

We observe that these are precisely the bosonic representations appearing in table II. Recalling that the fermions transform as $(8_c, 4+n)$ under H it is straightforward to verify that the fermionic spectrum also comes out correctly:

$$\begin{aligned}\left(2\cdot[\tfrac{1}{2},0]+2\cdot[0,\tfrac{1}{2}]\right)\times\left([\tfrac{1}{2},\tfrac{1}{2}]+n\cdot[0,0]\right)\\ = \; 2\cdot[1,\tfrac{1}{2}]+2\cdot[0,\tfrac{1}{2}]+2\cdot[\tfrac{1}{2},1]+2\cdot[\tfrac{1}{2},0]+2n\cdot[\tfrac{1}{2},0]+2n\cdot[0,\tfrac{1}{2}]\;. \end{aligned}\tag{3.6}$$

Moreover, the factors $\mathrm{SO}(4)_-$ and $\mathrm{SO}(n)$ in (3.3) commute with $\mathrm{SO}(4)_{\mathrm{gauge}}$ and thus represent global symmetries of the gauged theory. Further identifying

$$\mathrm{SO}(4)_{\mathrm{glob}} \;\equiv\; \mathrm{SO}(4)_-\;, \qquad \mathrm{SO}(n)_{\mathrm{glob}} \;\equiv\; \mathrm{SO}(n)\;, \tag{3.7}$$

the decomposition according to (3.3) precisely reproduces the spectrum of representations of table II. Note however, that the vector degrees of freedom of table II appear among the scalars in (3.5). These are the Goldstone bosons which give mass to the associated CS gauge vectors. Accordingly, the gauge group (3.4) is enlarged to (3.2) by the essentially unique set of six nilpotent abelian translations $\mathrm{T}\subset\mathrm{G}$ transforming in the adjoint representation of $\mathrm{SO}(4)_{\mathrm{gauge}}$. This part of the gauge group is broken at the groundstate in order to account for the massive vectors. We have thus identified the group (3.2) within G.

The Lagrangian and supersymmetry variations of the most general $D=3$, $N=8$ gauged supergravity have been given in [32]. As shown there, the theory is completely specified by the coset space (3.1) and the symmetric embedding tensor $\Theta_{\mathcal{MN}}$, which encodes the minimal coupling of vector fields to scalars according to

$$D_\mu\mathcal{S} \;\equiv\; \left(\partial_\mu+\Theta_{\mathcal{MN}}\,B_\mu^{\mathcal{M}}\,t^{\mathcal{N}}\right)\mathcal{S}\;. \tag{3.8}$$

The matrix $\mathcal{S}\in\mathrm{G}=\mathrm{SO}(8,4+n)$ here contains the scalar fields of the theory; by $t^{\mathcal{M}}$ we denote the generators of $\mathfrak{g}=\mathrm{Lie}\,\mathrm{G}$ acting by left multiplication, with the curly indices $\mathcal{M}$, $\mathcal{N}$ referring to the adjoint representation of $\mathfrak{g}$. The number of vector fields involved in

(3.8) is equal to the rank of $\Theta_{\mathcal{MN}}$. Because the embedding tensor characterizes the theory completely, the task can be reduced to the identification of the tensor $\Theta_{\mathcal{MN}}$ that correctly reproduces the gauge group (3.2) shown above, and at the same time is compatible with the algebraic constraints (3.11) below, imposed by supersymmetry. It turns out that there is a unique $\Theta_{\mathcal{MN}}$ that fits all the requirements.

We denote by indices $I, J, \ldots$ and indices $r, s, \ldots$ the vector representations of SO(8) and SO$(4+n)$, respectively. The generators $\{t^{\mathcal{M}}\}$ of $\mathfrak{g}$ split into the compact generators $\{X^{[IJ]}, X^{[rs]}\}$, and the noncompact generators $\{Y^{Ir}\}$ with commutation relations[4]

$$
\begin{aligned}
[X^{IJ}, X^{KL}] &= 2\,(\delta^{I[K}\,X^{L]J} - \delta^{J[K}\,X^{L]I})\,, \qquad & [X^{IJ}, Y^{Kr}] &= -\,2\delta^{K[I}\,Y^{J]r}\,,\\
[X^{rs}, X^{uv}] &= 2\,(\delta^{r[u}\,X^{v]s} - \delta^{s[u}\,X^{v]r})\,, \qquad & [X^{rs}, Y^{Ku}] &= -\,2\delta^{u[r}\,Y^{Ks]}\,,\\
[Y^{Ir}, Y^{Js}] &= \delta^{IJ}\,X^{rs} + \delta^{rs}\,X^{IJ}\,. & &
\end{aligned}
\tag{3.9}
$$

In particular, the current (3.8) decomposes into

$$
\mathcal{S}^{-1}D_\mu\mathcal{S} \;\equiv\; \tfrac{1}{2}\mathcal{Q}_\mu^{IJ}X^{IJ} + \tfrac{1}{2}\mathcal{Q}_\mu^{rs}X^{rs} + \mathcal{P}_\mu^{Ir}Y^{Ir}\,. \tag{3.10}
$$

In this basis, the algebraic constraints imposed by supersymmetry on the embedding tensor $\Theta_{\mathcal{MN}}$ from (3.8) read

$$
\begin{aligned}
\Theta_{IJ,KL} &= \Theta_{[IJ,KL]}\,, \qquad & \Theta_{IJ,rs} &= \Theta_{Ir,Js}\,, \qquad & \Theta_{rs,uv} &= \Theta_{[rs,uv]}\,,\\
\Theta_{IJ,Kr} &= \Theta_{[IJ,K]r}\,, \qquad & \Theta_{Kr,su} &= \Theta_{K[r,su]}\,. & &
\end{aligned}
\tag{3.11}
$$

Let us mention that supersymmetry actually implies a slightly weaker set of constraints, e.g. it also allows for a trace part in Θ, see [32,36] for details. For our purpose, however, the constraints (3.11) are sufficient to determine the consistent embedding tensor.

In order to describe the embedding according to (3.3), we further need to split these indices into $I = (i, \tilde{\imath})$ and $r = (\hat{\imath}, \tilde{r})$ with

$$
\begin{aligned}
I, J, \ldots &: \qquad i, j, \ldots \in \{1,2,3,4\}\,, \quad \tilde{\imath}, \tilde{\jmath}, \ldots \in \{5,6,7,8\}\,,\\
r, s, \ldots &: \qquad \hat{\imath}, \hat{\jmath}, \ldots \in \{1,2,3,4\}\,, \quad \tilde{r}, \tilde{s}, \ldots \in \{1,2,\ldots,n\}\,.
\end{aligned}
\tag{3.12}
$$

[4]Upon redefining $X^{rs} \to -X^{rs}$, the algebra (3.9) may be written in the more familiar form $[X^{\mathcal{IJ}}, X^{\mathcal{KL}}] = 2\,(\eta^{\mathcal{I}[\mathcal{K}}\,X^{\mathcal{L}]\mathcal{J}} - \eta^{\mathcal{J}[\mathcal{K}}\,X^{\mathcal{L}]\mathcal{I}})$ with SO$(8,4+n)$ vector indices $\mathcal{I} = (I, r)$, and the metric $\eta^{\mathcal{IJ}} \equiv (\delta^{IJ}, -\delta^{rs})$. The supersymmetry constraints (3.11) are then equivalent to the total antisymmetry of the embedding tensor $\Theta_{\mathcal{IJ},\mathcal{KL}}$ in the SO$(8,4+n)$ indices $[\mathcal{IJKL}]$.

The indices $\tilde{\imath}, \tilde{\jmath}, \ldots$ are the same as in (2.1), and the indices $\tilde{r}, \tilde{s}, \ldots$ the same as in table II. The generators $\{t^{\mathcal{M}}\}$ accordingly decompose into

$$\mathfrak{g} \;=\; \left\{X^{[ij]}, X^{i\tilde{\jmath}}, X^{[\tilde{\imath}\tilde{\jmath}]}, X^{[\hat{\imath}\hat{\jmath}]}, X^{\hat{\imath}\tilde{r}}, X^{[\tilde{r}\tilde{s}]}\right\} \oplus \left\{Y^{i\hat{\jmath}}, Y^{i\tilde{r}}, Y^{\tilde{\imath}\hat{\jmath}}, Y^{\tilde{\imath}\tilde{r}}\right\} \;. \tag{3.13}$$

From these we may explicitly build the generators of $\mathfrak{so}(4)_{\rm gauge}$ and the six abelian nilpotent translations $\mathfrak{t}$ as

$$\begin{aligned} \mathfrak{so}(4)_{\rm gauge} &\equiv \left\{\mathcal{J}^{[ij]} \equiv X^{[ij]} + X^{[\hat{\imath}\hat{\jmath}]}\right\} \;, \\ \mathfrak{t} &\equiv \left\{\mathcal{T}^{[ij]} \equiv X^{[ij]} - X^{[\hat{\imath}\hat{\jmath}]} + Y^{i\hat{\jmath}} - Y^{j\hat{\imath}}\right\} \;. \end{aligned} \tag{3.14}$$

It is straightforward to verify that the $\mathcal{J}^{[ij]}$ close into the SO(4) algebra (3.4) while the mutually commuting generators $\mathcal{T}^{[ij]}$ transform in the adjoint representation under $\mathcal{J}^{[ij]}$. This is the Lie algebra underlying (3.2).

Similarly defining vector fields

$$C^{[ij]} \;\equiv\; B^{[ij]} + B^{[\hat{\imath}\hat{\jmath}]} \;, \qquad A^{[ij]} \;\equiv\; B^{[ij]} - B^{[\hat{\imath}\hat{\jmath}]} + B^{i\hat{\jmath}} - B^{j\hat{\imath}} \;, \tag{3.15}$$

we start from the following ansatz for the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ [34]

$$\begin{aligned} \Theta_{\mathcal{M}\mathcal{N}}\, B_\mu^{\mathcal{M}} t^{\mathcal{N}} &= \tfrac{1}{2} g_1 \left(C_\mu^{+[ij]}\mathcal{T}^{+[ij]} - C_\mu^{-[ij]}\mathcal{T}^{-[ij]} + A_\mu^{+[ij]}\mathcal{J}^{+[ij]} - A_\mu^{-[ij]}\mathcal{J}^{-[ij]}\right) \\ &\quad + \tfrac{1}{2} g_2 \left(A_\mu^{+[ij]}\mathcal{T}^{+[ij]} - A_\mu^{-[ij]}\mathcal{T}^{-[ij]}\right) \;, \end{aligned} \tag{3.16}$$

with real constants g_1, g_2, and where $A^{\pm[ij]}$ denote the selfdual and anti-selfdual part of $A^{[ij]}$, respectively, etc. Translating (3.16) back into the basis (3.9), (3.13), this embedding tensor takes the form

$$\begin{aligned} \Theta_{ij,kl} &= (g_2 + 2g_1)\,\epsilon_{ijkl} \;, & \Theta_{ij,\hat{k}\hat{l}} &= -\, g_2\,\epsilon_{ijkl} \;, & \Theta_{ij,k\hat{l}} &= (g_1 + g_2)\,\epsilon_{ijkl} \;, \\ \Theta_{\hat{\imath}\hat{\jmath},\hat{k}\hat{l}} &= (g_2 - 2g_1)\,\epsilon_{ijkl} \;, & \Theta_{i\hat{k},j\hat{l}} &= -\, g_2\,\epsilon_{ijkl} \;, & \Theta_{\hat{\imath}\hat{\jmath},k\hat{l}} &= (g_1 - g_2)\,\epsilon_{ijkl} \;. \end{aligned} \tag{3.17}$$

The choice of a relative minus sign between selfdual and anti-selfdual components in (3.16), or equivalently the relative coupling constant (-1) between the two SO(3) factors in SO(4) is necessary to ensure that terms proportional to δ_{ij}^{kl} drop out in (3.17), such that the supersymmetry constraints (3.11) are satisfied for any choice of g_1 and g_2. That is, at this stage we still have a class of physically distinct theories for different choices of g_1, g_2. We further emphasize that these constraints harmonize beautifully with the particular non-semisimple type of gauge group (3.2). Indeed, coupling the diagonal SO(4) of (3.4) requires a nonvanishing contribution in $\Theta_{IJ,rs}$. By means of (3.11) this induces a nonvanishing $\Theta_{Ir,Js}$ which in turn precisely corresponds to coupling the nilpotent contributions of (3.14). In other words, the diagonal $\mathrm{SO}(4)_{\rm gauge}$ from (3.4) alone is not a consistent CS gauge group; supersymmetry requires its non-semisimple extension to (3.2).[5]

We may now state the complete bosonic Lagrangian of the three-dimensional theory given as a gravity coupled CS gauged G/H coset space σ-model

$$e^{-1}\mathcal{L} \;=\; -\tfrac{1}{4}R + \tfrac{1}{4}g^{\mu\nu}\,\mathcal{P}^{Ir}_\mu\mathcal{P}^{Ir}_\nu - e^{-1}\mathcal{L}_{\rm CS} - W \;. \tag{3.18}$$

The kinetic scalar term is obtained from putting together (3.8), (3.10), and (3.17), while the CS term is

$$\mathcal{L}_{\rm CS} \;=\; \tfrac{1}{4}\varepsilon^{\mu\nu\rho}B^{\mathcal{M}}_\mu\,\Theta_{\mathcal{M}\mathcal{N}}\,\left(\partial_\nu B^{\mathcal{N}}_\rho + \tfrac{1}{3}\,f^{\mathcal{N}\mathcal{P}}{}_{\mathcal{L}}\,\Theta_{\mathcal{P}\mathcal{K}}\,B^{\mathcal{K}}_\nu B^{\mathcal{L}}_\rho\right)\;, \tag{3.19}$$

with $\Theta_{\mathcal{M}\mathcal{N}}$ from (3.17) and the $\mathrm{SO}(8,4+n)$ structure constants from (3.9). The potential W is given as a function of the scalar fields as

$$W \;=\; -\tfrac{1}{48}\left(T_{[IJ,KL]}T_{[IJ,KL]} + \tfrac{1}{4!}\,\epsilon^{IJKLMNPQ}\,T_{IJ,KL}T_{MN,PQ} - 2\,T_{IJ,Kr}T_{IJ,Kr}\right)\;, \tag{3.20}$$

in terms of the so-called T-tensor

$$T_{IJ,KL} \;=\; \mathcal{V}^{\mathcal{M}}{}_{IJ}\mathcal{V}^{\mathcal{N}}{}_{KL}\,\Theta_{\mathcal{M}\mathcal{N}}\;, \qquad T_{IJ,Kr} \;=\; \mathcal{V}^{\mathcal{M}}{}_{IJ}\mathcal{V}^{\mathcal{N}}{}_{Kr}\,\Theta_{\mathcal{M}\mathcal{N}}\;, \tag{3.21}$$

where $\mathcal{V}$ defines the group matrix $\mathcal{S}$ in the adjoint representation:

$$\mathcal{S}^{-1}t^{\mathcal{M}}\mathcal{S} \;\equiv\; \tfrac{1}{2}\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\,X^{IJ} + \tfrac{1}{2}\,\mathcal{V}^{\mathcal{M}}{}_{rs}\,t^{rs} + \mathcal{V}^{\mathcal{M}}{}_{Ir}\,Y^{Ir}\;. \tag{3.22}$$

Hence, like all other terms in (3.18), the scalar potential W depends crucially on the precise form of the embedding tensor (3.17). For the fermionic contributions and full supersymmetry transformations we refer to [32]. Here, we just quote the variations of the gravitinos ψ^A_μ and fermion fields $\chi^{\dot{A}r}$ (neglecting cubic spinor terms)

$$\begin{aligned} \delta\psi^A_\mu &= D_\mu\epsilon^A - \tfrac{\mathrm{i}}{48}\Gamma^{IJKL}_{AB}T_{IJ,KL}\gamma_\mu\epsilon^B\;,\\ \delta\chi^{\dot{A}r} &= \left(\tfrac{\mathrm{i}}{2}\Gamma^I_{A\dot{A}}\,\not{\mathcal{P}}^{Ir} - \tfrac{1}{12}\Gamma^{IJK}_{A\dot{A}}T_{IJ,Kr}\right)\epsilon^A\;, \end{aligned} \tag{3.23}$$

[5]However, a consistent gauging with gauge group SO(4) is still possible with a *different* embedding, corresponding to only one of the factors in (3.4), cf. eq. (19) of [32].

with SO(8) Γ-matrices $\Gamma^I_{A\dot{A}}$. These variations are likewise expressed in terms of the T-tensor from (3.21) and may serve as BPS equations for bosonic solutions. In particular, they show that an AdS ground state preserving all supersymmetries requires $T_{IJ,Kr} = 0$. Recall that we seek that theory whose groundstate at the origin $\mathcal{S} = \mathbb{I}$ precisely corresponds to the six-dimensional $\mathrm{AdS}_3 \times S^3$ background with full $N = (4,4)$ supersymmetry. Since T at this point reduces to the embedding tensor Θ, together this imposes a nontrivial relation between the constants g_1, g_2 in (3.17)

$$\Theta_{IJ,Kr} \;=\; 0 \qquad \Longrightarrow \qquad \Theta_{ij,k\hat{l}} \;=\; 0 \qquad \Longrightarrow \qquad g_2 = -g_1\;. \tag{3.24}$$

That is, existence of a maximally supersymmetric AdS groundstate eventually fixes the ratio g_1/g_2, such that the final theory is completely determined up to an overall coupling constant which may be expressed in terms of the AdS length L_0 at the origin as $g_1 = -g_2 = 1/L_0$. At this point, the gauge group (3.2) breaks down to its compact part $\mathrm{SO}(4)_{\mathrm{gauge}}$; i.e. the background isometry group which organizes the spectrum of fluctuations around this point is the desired $\mathrm{SU}(2|1,1)_L \times \mathrm{SU}(2|1,1)_R$. The vector fields corresponding to the translational part of (3.2) acquire a mass in accordance with table II.

We have thus succeeded in constructing a three-dimensional $N = 8$ supersymmetric theory with an $N = (4,4)$ supersymmetric AdS_3 ground state at the origin of the scalar potential which reproduces the correct symmetries and the field content of table II. As a further check, one may explicitly compute quadratic fluctuations of the potential (3.20) around the ground state $\mathcal{S} = \mathbb{I}$ to obtain the scalar masses. Indeed, after some calculation, one confirms the scalar mass spectrum obtained from table II via the standard relation

$$m^2 L_0^2 \;=\; \Delta(\Delta-2)\;, \tag{3.25}$$

together with vanishing masses for the Goldstone bosons. Furthermore, half of the 12 CS vector fields acquire mass in a three-dimensional variant of the Brout-Englert-Higgs effect: linearizing their first order field equations around the origin of the scalar potential leads to

$$\epsilon_{\mu\nu\rho}\mathcal{A}^{\pm[ij]\,\nu\rho} \;=\; \mp 4 g_1\,\left(A_\mu^{\pm[ij]} - C_\mu^{\pm[ij]}\right)\;, \qquad \mathcal{C}^{\pm[ij]}_{\mu\nu} \;=\; 0\;. \tag{3.26}$$

After the obvious redefinitions, these equations are identical with (2.4). Alternatively, we recall from [43] that the vector mass matrix around the origin may be directly obtained from the projection of the embedding tensor $\Theta_{Ir,Js}$ onto its noncompact directions, i.e. here from the components $\Theta_{i\hat{k},j\hat{l}}$ in (3.17). Via the mass-dimension relation $\Delta = 1 + |mL_0|$

for three-dimensional vectors, we find precise agreement with the masses of table II. We may finally employ the results of [34] to obtain the on-shell equivalent version of the theory (3.18) which describes an SO(4) YM gauging. To this end, we split off the scalars ϕ_{ij} corresponding to the six nilpotent directions of T_6 from the group matrix $\mathcal{S}$ according to

$$\mathcal{S} \;\equiv\; e^{\phi_{ij}\mathcal{T}^{[ij]}}\,\tilde{\mathcal{S}}\;, \tag{3.27}$$

where $\tilde{\mathcal{S}}$ now combines the remaining $26+8n$ scalar fields. Correspondingly, we define its currents

$$\tilde{\mathcal{Q}}_\mu + \tilde{\mathcal{P}}_\mu \;\equiv\; \tilde{\mathcal{S}}^{-1}\left(\partial_\mu \;+\; \tfrac{1}{2}g_1(A_\mu^{+[ij]}\mathcal{J}^{+[ij]} - A_\mu^{-[ij]}\mathcal{J}^{-[ij]})\right)\tilde{\mathcal{S}}\;, \tag{3.28}$$

and the matrix $\tilde{\mathcal{V}}^{\underline{[ij]}}{}_{\mathcal{M}}$ as

$$\tilde{\mathcal{S}}^{-1}\mathcal{T}^{[ij]}\tilde{\mathcal{S}} \;\equiv\; \tfrac{1}{2}\,\tilde{\mathcal{V}}^{\underline{[ij]}}{}_{IJ}\,X^{IJ} + \tfrac{1}{2}\,\tilde{\mathcal{V}}^{\underline{[ij]}}{}_{rs}\,X^{rs} + \tilde{\mathcal{V}}^{\underline{[ij]}}{}_{Ir}\,Y^{Ir}\;. \tag{3.29}$$

Eliminating the scalar fields ϕ_{ij} together with the CS vector fields $C_\mu^{[ij]}$ from the above Lagrangian (3.18) as described in [34], turns this theory into a YM type gauged supergravity with SO(4) YM vector fields $A_\mu^{[ij]}$ and the Lagrangian

$$\begin{aligned} e^{-1}\tilde{\mathcal{L}} \;&=\; -\tfrac{1}{4}R + \tfrac{1}{4}g^{\mu\nu}\,G_{Ir,Js}\,\tilde{\mathcal{P}}_\mu^{Ir}\tilde{\mathcal{P}}_\nu^{Js} + \tfrac{1}{16}\,M_{[ij],[kl]}\,\mathcal{A}^{[ij]\mu\nu}\mathcal{A}_{\mu\nu}^{[kl]} - e^{-1}g_1\Big(\tilde{\mathcal{L}}_{\rm CS}^{(+)} - \tilde{\mathcal{L}}_{\rm CS}^{(-)}\Big) \\ &\quad + \tfrac{1}{2}e^{-1}\,\varepsilon^{\mu\nu\rho}\,M_{[ij],[kl]}\,\tilde{\mathcal{V}}^{\underline{[kl]}}{}_{Ir}\,\mathcal{A}_{\mu\nu}^{[ij]}\,\tilde{\mathcal{P}}_\rho^{Ir} - W\;, \end{aligned} \tag{3.30}$$

with

$$\begin{aligned} G_{Ir,Js} \;&\equiv\; \delta_{IJ}\delta_{rs} - \tilde{\mathcal{V}}^{\underline{[ij]}}{}_{Ir}M_{[ij],[kl]}\tilde{\mathcal{V}}^{\underline{[kl]}}{}_{Js}\;, \qquad M_{[ij],[kl]} \;\equiv\; (\tilde{\mathcal{V}}^{\underline{[ij]}}{}_{Ir}\tilde{\mathcal{V}}^{\underline{[kl]}}{}_{Ir})^{-1}\;, \\ \tilde{\mathcal{L}}_{\rm CS}^{(\pm)} \;&=\; \tfrac{1}{8}\,\varepsilon^{\mu\nu\rho}A_\mu^{\pm[ij]}\,\left(\partial_\nu A_\rho^{\pm[ij]} \pm \tfrac{1}{3}g_1A_\nu^{\pm[ik]}A_\rho^{\pm[jk]}\right)\;. \end{aligned}$$

Notably, the scalar potential W of (3.30) coincides with the one derived above (3.20). In particular, the scalar mass spectrum around the origin $\tilde{\mathcal{S}}=\mathbb{I}$ coincides with the one of (3.18) and thus with table II. On the other hand, the vector field equations obtained from linearizing (3.30) around $\tilde{\mathcal{S}}=\mathbb{I}$ give

$$\nabla^\nu\mathcal{A}_{\mu\nu}^{\pm[ij]} \;=\; \pm g_1\epsilon_{\mu\nu\rho}\,\mathcal{A}^{\pm[ij]\nu\rho}\;, \tag{3.31}$$

and thus precisely reproduce (2.5). Summarizing, with (3.18) and (3.30) we have constructed the two equivalent versions of the three-dimensional theory that yield the full nonlinear extension of the linearized field equations (2.4) and (2.5), respectively.

4 Coupling the KK towers

In the previous section we have constructed a theory coupling the two supermultiplets

$$1 \cdot (\mathbf{3},\mathbf{3})_S + n \cdot (\mathbf{2},\mathbf{2})_S \,, \tag{4.1}$$

to the massless supergravity multiplet, and hence succeeded in combining the lowest levels of the three KK towers of six-dimensional $N=(2,0)$ supergravity on $\mathrm{AdS}_3 \times S^3$. Apart from the spin-2 tower, the entire KK spectrum is given by two infinite towers of spin-1 multiplets

$$\mathcal{H}_{\text{spin}-1} \;=\; \sum_{k\geq 2} (\mathbf{k+1},\mathbf{k+1})_S \;+\; n \cdot \sum_{l\geq 1} (\mathbf{l+1},\mathbf{l+1})_S \,, \tag{4.2}$$

transforming in the singlet and the vector representation of $\mathrm{SO}(n)$, respectively. In this section, we will show that extending the above construction we may in fact construct a three-dimensional theory that comprises the entire spectrum (4.2) of spin-1 multiplets. It is obtained by gauging the $N=8$ theory with the coset space given in (1.1). If one wishes, one can "regularize" the infinite component theory by considering only a finite but arbitrarily large number of multiplets.

Counting the number of bosonic degrees of freedom in (4.2) via table I, indeed reproduces the infinite dimensional coset (1.1), i.e. gives agreement for each value of k and l separately. It remains to determine the CS gauge group and its embedding. According to [34], this group, which we denote by $\mathrm{SO}(4)_{\text{gauge}} \ltimes (\hat{\mathrm{T}}_\infty, \mathrm{T}_6)$, is obtained from exponentiating an extension of the non-semisimple algebra (3.14) described above, by a set of additional nilpotent generators $\hat{\mathfrak{t}}_\infty$ corresponding to the massive vector fields appearing in (4.2). More precisely, the generators of $\hat{\mathfrak{t}}_\infty$ should transform like the massive vector fields in the

$$\sum_{k\geq 3} \left(\left[\tfrac{k-2}{2}, \tfrac{k}{2}\right] + \left[\tfrac{k}{2}, \tfrac{k-2}{2}\right] \right) \;+\; \sum_{l\geq 2} \left(n \cdot \left[\tfrac{l-2}{2}, \tfrac{l}{2}\right] + n \cdot \left[\tfrac{l}{2}, \tfrac{l-2}{2}\right] \right) \,, \tag{4.3}$$

under $\mathrm{SO}(4)_{\text{gauge}}$ (cf. table I) and close into $\mathfrak{t}$. To illustrate the embedding of the gauge group and the global symmetries in the group G, let us first consider the decomposition of its maximal compact subgroup H into

$$\begin{array}{c}
\underbrace{\mathrm{SO}(4)_+ \times \mathrm{SO}(4)_-}_{\mathrm{SO}(8)} \times \underbrace{\mathrm{SO}(4)_2}_{\mathrm{SO}(4)} \times \ldots \times \underbrace{\mathrm{SO}(4)_k}_{\mathrm{SO}(k^2)} \times \ldots \times \underbrace{\mathrm{SO}(n)_1}_{\mathrm{SO}(n)} \times \underbrace{\mathrm{SO}(n)_2 \times \mathrm{SO}(4)'_2}_{\mathrm{SO}(4n)} \times \ldots \times \underbrace{\mathrm{SO}(n)_l \times \mathrm{SO}(4)'_l}_{\mathrm{SO}(nl^2)} \times \ldots \\
\mathrm{SO}(8) \times \underbrace{\mathrm{SO}(4) \times \ldots \times \mathrm{SO}(k^2) \times \ldots}_{\mathrm{SO}(\sum_{k\geq 2} k^2)} \times \underbrace{\mathrm{SO}(n) \times \mathrm{SO}(4n) \times \ldots \times \mathrm{SO}(nl^2) \times \ldots}_{\mathrm{SO}(n \sum_{l\geq 1} l^2)} \\
\mathrm{SO}(8) \times \mathrm{SO}(\textstyle\sum_{k\geq 2} k^2) \times \mathrm{SO}(n \sum_{l\geq 1} l^2)
\end{array}$$

$$(4.4)$$

with the specific embeddings

$$\begin{aligned} \mathrm{SO}(4)_k &\subset \mathrm{SO}(k^2) &: \quad & k^2 \to \left[\tfrac{k-1}{2}, \tfrac{k-1}{2}\right] \;, \\ \mathrm{SO}(n)_l \times \mathrm{SO}(4)'_l &\subset \mathrm{SO}(nl^2) &: \quad & nl^2 \to \left[n, \tfrac{l-1}{2}, \tfrac{l-1}{2}\right] \;. \end{aligned} \tag{4.5}$$

We then define the group $\mathrm{SO}(4)_{\mathrm{gauge}}$ and the global symmetry groups as the diagonal subgroups

$$\begin{aligned} \mathrm{SO}(4)_{\mathrm{gauge}} &\equiv \mathrm{diag}\left(\mathrm{SO}(4)_+ \times \mathrm{SO}(4)_2 \times \mathrm{SO}(4)_3 \times \ldots \times \mathrm{SO}(4)'_2 \times \mathrm{SO}(4)'_3 \times \ldots\right) \;, \\ \mathrm{SO}(4)_{\mathrm{glob}} &\equiv \mathrm{SO}(4)_- \;, \qquad \mathrm{SO}(n)_{\mathrm{glob}} \equiv \mathrm{diag}\left(\mathrm{SO}(n)_1 \times \mathrm{SO}(n)_2 \times \ldots\right) \;. \end{aligned} \tag{4.6}$$

Working out the products of the relevant representations, viz.

$$\begin{array}{ll} \left(1\cdot\left[\tfrac12,\tfrac12\right] + 4\cdot[0,0]\right) \times \left[\tfrac{k-1}{2},\tfrac{k-1}{2}\right] & \qquad (\text{bosons}) \;, \\ \left(2\cdot\left[\tfrac12,0\right] + 2\cdot[0,\tfrac12]\right) \times \left[\tfrac{k-1}{2},\tfrac{k-1}{2}\right] & \qquad (\text{fermions}) \;, \end{array}$$

it is straightforward to verify that under these subgroups the fields reproduce the correct representation content as given by (4.2) together with table I. Again, the vector degrees of freedom show up through the associated Goldstone scalars, transforming in the same representations as the massive vector fields (4.3).

For transparency, we will now describe in detail the theory which couples the distinguished lowest spin-1 multiplet $(\mathbf{3},\mathbf{3})_S$ to a single additional higher spin-1 multiplet, say, $(\mathbf{k{+}1},\mathbf{k{+}1})_S$ from the $\mathrm{SO}(n)$ singlet tower. The extension to an arbitrary number of these multiplets and multiplets from the $\mathrm{SO}(n)$ vector tower is straightforward.

Let us therefore consider the coset space

$$\mathrm{G/H} \;=\; \mathrm{SO}(8\,,\,4+k^2)/\left(\mathrm{SO}(8)\times\mathrm{SO}(4+k^2)\right) \;, \tag{4.7}$$

with indices split according to $I=(i,\tilde\imath)$ and $r=(\hat\imath,ab)$ with $a,b,\ldots=1,\ldots,k$, extending (3.12). The generators $\{t^{\mathcal{M}}\}$ of $\mathrm{SO}(8,4+k^2)$ accordingly decompose as

$$\mathfrak{g} \;=\; \left\{X^{[ij]}, X^{i\tilde\jmath}, X^{[\tilde\imath\tilde\jmath]}, X^{[\hat\imath\hat\jmath]}, X^{\hat\imath,ab}, X^{ab,cd}\right\} \oplus \left\{Y^{i\hat\jmath}, Y^{i,ab}, Y^{\tilde\imath\hat\jmath}, Y^{\tilde\imath,ab}\right\} \;. \tag{4.8}$$

The commutation relations can be read off from (3.9). For the $\mathrm{SO}(k^2)$ subgroup, we thus have

$$[X^{ab,cd}, X^{ef,gh}] = \eta^{cd,ef}X^{ab,gh} - \eta^{cd,gh}X^{ab,ef} - \eta^{ab,ef}X^{cd,gh} + \eta^{ab,gh}X^{cd,ef} \;, \tag{4.9}$$

where the metric $\eta_{ab,cd} \equiv \eta_{ac}\eta_{bd}$ serves to lower and raise indices, such that

$$\eta_{ab,ef}\eta^{ef,cd} = \delta_a^c \delta_b^d \;, \qquad (4.10)$$

and the tensor η_{ab} is the quadratic invariant of the k-dimensional representation of SO(3); it is symmetric for bosonic representations, i.e. odd k, and skew-symmetric for fermionic representations, i.e. even k.

In line with our general arguments above we now seek a theory with CS gauge group

$$\mathrm{SO}(4)_{\rm gauge} \ltimes \left(\hat{\mathrm{T}}^{(k)}, \mathrm{T}_6\right) \subset \mathrm{SO}(8, 4+k^2) \;, \qquad (4.11)$$

with $\mathrm{SO}(4)_{\rm gauge}$ from (4.6), extending (3.2) by $2(k^2-1)$ generators transforming in the $\left[\frac{k-2}{2}, \frac{k}{2}\right] + \left[\frac{k}{2}, \frac{k-2}{2}\right]$ under $\mathrm{SO}(4)_{\rm gauge}$ and closing into T_6, in order to correctly describe a theory with SO(4) YM gauging and $2(k^2-1)$ massive vector fields. The generators of $\mathfrak{so}(4)_{\rm gauge}$ are defined as the extension of the previous SO(4) to the new diagonal SO(4) subgroup embedded into $\mathrm{SO}(8, 4+k^2)$ according to (4.6), viz.

$$\mathfrak{so}(4)_{\rm gauge} = \left\{ \mathcal{J}^{[ij]} \equiv X^{[ij]} + X^{[\hat{\imath}\hat{\jmath}]} + \tfrac{1}{2}\, \zeta^{+(k)}_{ij\;ac} \eta_{bd}\, X^{ab,cd} + \tfrac{1}{2}\, \zeta^{-(k)}_{ij\;bd} \eta_{ac}\, X^{ab,cd} \right\} \;. \qquad (4.12)$$

By $\zeta^{\pm(k)}_{ij}$, we here denote the generators of SO(4) in the spin $\left[\frac{k-1}{2}, 0\right]$ and spin $\left[0, \frac{k-1}{2}\right]$ representation, respectively. Accordingly, $\zeta^{\pm(k)}_{ij\;ab}$ is symmetric in ab for fermionic (even k) and skew-symmetric for bosonic (odd k) representations. Explicit expressions for these generators can be constructed in terms of Clebsch-Gordan coefficients. The $\mathrm{SO}(3)^{\pm}$ commutation relations in this representation are

$$\begin{aligned} \left[\, \zeta^{\pm(k)}_{ij}, \zeta^{\pm(k)}_{mn} \,\right] &= 2 \left(\delta_{i[m} \zeta^{\pm(k)}_{n]j} - \delta_{j[m} \zeta^{\pm(k)}_{n]i} \right) \;, \qquad \zeta^{\pm(k)}_{ij} = \pm\, \tfrac{1}{2}\, \epsilon^{ijmn}\, \zeta^{\pm(k)}_{mn} \;, \\ \mathrm{tr}\left(\zeta^{(\pm k)}_{ij}\, \zeta^{(\pm k)}_{ji} \right) &= k\,(k^2-1) \;, \end{aligned} \qquad (4.13)$$

in obvious matrix notation. Let us also record the relation

$$\zeta^{\pm(k)}_{im\,ac}\, \zeta^{\pm(k)}_{mj\,cb} = \tfrac{1}{4}(k^2-1)\delta_{ij}\eta_{ab} + \zeta^{\pm(k)}_{ij\,ab} \;, \qquad (4.14)$$

which follows from (4.13) and the (anti-)selfduality of the ζ's.

The generators of the translation subgroup $(\hat{T}^{(k)}, T_6)$ of (4.11) are given by

$$\mathfrak{t} \equiv \left\{ \mathcal{T}^{[ij]} \equiv X^{[ij]} - X^{[\hat{\imath}\hat{\jmath}]} + Y^{i\hat{\jmath}} - Y^{j\hat{\imath}} \right\} \;, \qquad (4.15)$$

which is the same as before, and by

$$\hat{\mathfrak{t}}^{(k)} = \left\{ \left(\mathbb{P}_{\left[\frac{k-2}{2},\frac{k}{2}\right]} \right)_{i\,ab,j\,cd} \left(X^{\hat{j}\,cd} - Y^{j\,cd} \right) \right\} \oplus \left\{ \left(\mathbb{P}_{\left[\frac{k}{2},\frac{k-2}{2}\right]} \right)_{i\,ab,j\,cd} \left(X^{\hat{j}\,cd} - Y^{j\,cd} \right) \right\} \;, \quad (4.16)$$

where

$$\begin{aligned}
\left(\mathbb{P}_{\left[\frac{k-2}{2},\frac{k}{2}\right]} \right)_{i\,ab,j\,cd} &= \tfrac{k^2-1}{4k^2}\, \eta_{ac}\eta_{bd}\, \delta_{ij} + \tfrac{k+1}{2k^2}\, \zeta^{+(k)}_{ij\;ac}\, \eta_{bd} - \tfrac{k-1}{2k^2}\, \eta_{ac}\, \zeta^{-(k)}_{ij\;bd} - \tfrac{1}{k^2}\, \zeta^{+(k)}_{im\;ac}\, \zeta^{-(k)}_{mj\;bd} \;, \\
\left(\mathbb{P}_{\left[\frac{k}{2},\frac{k-2}{2}\right]} \right)_{i\,ab,j\,cd} &= \tfrac{k^2-1}{4k^2}\, \eta_{ac}\eta_{bd}\, \delta_{ij} - \tfrac{k-1}{2k^2}\, \zeta^{+(k)}_{ij\;ac}\, \eta_{bd} + \tfrac{k+1}{2k^2}\, \eta_{ac}\, \zeta^{-(k)}_{ij\;bd} - \tfrac{1}{k^2}\, \zeta^{+(k)}_{im\;ac}\, \zeta^{-(k)}_{mj\;bd} \;,
\end{aligned}$$

are the projectors onto the $\left[\frac{k-2}{2},\frac{k}{2}\right]$ and $\left[\frac{k}{2},\frac{k-2}{2}\right]$ representations, respectively, in the tensor product $\left[\frac{1}{2},\frac{1}{2}\right] \times \left[\frac{k-1}{2},\frac{k-1}{2}\right]$ of $\mathrm{SO}(4)_{\mathrm{gauge}}$. The projector properties are most easily verified by writing the above projectors as products of the corresponding operators for the chiral product $\left[\frac{1}{2}\right] \times \left[\frac{k-1}{2}\right] = \left[\frac{k}{2}\right] \oplus \left[\frac{k-2}{2}\right]$

$$\left(\mathbb{P}_{\left[\frac{k}{2}\right]} \right)_{ij,ab} \equiv \tfrac{k+1}{2k}\, \delta_{ij}\eta_{ab} - \tfrac{1}{k}\, \zeta^{(k)}_{ij\;ab} \;, \qquad \left(\mathbb{P}_{\left[\frac{k-2}{2}\right]} \right)_{ij,ab} \equiv \tfrac{k-1}{2k} \delta_{ij}\eta_{ab} + \tfrac{1}{k} \zeta^{(k)}_{ij\;ab} \;,$$

and by use of (4.14). With the relation

$$\left[X^{\hat{\imath}\,ab} - Y^{i\,ab}, X^{\hat{j}\,cd} - Y^{j\,cd} \right] = \eta^{ab,cd} \left(X^{ij} - X^{\hat{\imath}\hat{j}} + Y^{i\hat{j}} - Y^{j\hat{\imath}} \right) \;,$$

it is easy to show that

$$[\, \hat{\mathfrak{t}}^{(k)}, \hat{\mathfrak{t}}^{(k)} \,] \subset \mathfrak{t} \;, \qquad [\, \hat{\mathfrak{t}}^{(k)}, \hat{\mathfrak{t}}^{(l)} \,] = 0 \quad \text{for } \; k \neq l \;, \qquad [\, \hat{\mathfrak{t}}^{(k)}, \mathfrak{t} \,] = 0 \;. \quad (4.17)$$

Thus, even taking into account an infinite number of translation subalgebras, the full gauge algebra still has a rather simple structure.

The embedding tensor (3.17) acquires the additional components

$$\begin{aligned}
\Theta^{(k)}_{ij,ab\,cd} &= -\Theta^{(k)}_{\hat{\imath}\hat{j},ab\,cd} = \Theta^{(k)}_{i\hat{j},ab\,cd} = g_1\, \zeta^{+(k)}_{ij\;ac}\, \eta_{bd} - g_1\, \zeta^{-(k)}_{ij\;bd}\, \eta_{ac} \;, \\
\Theta^{(k)}_{i\,ab,j\,cd} &= \Theta^{(k)}_{\hat{\imath}\,ab,\hat{j}\,cd} = -\,\Theta^{(k)}_{i\,ab,\hat{j}\,cd} = g_1\, \zeta^{+(k)}_{ij\;ac}\, \eta_{bd} - g_1\, \zeta^{-(k)}_{ij\;bd}\, \eta_{ac} \;, \qquad (4.18)
\end{aligned}$$

which are obviously compatible with the algebraic constraints (3.11) imposed by supersymmetry (with antisymmetry under interchange of the indices i, j and the pairs ab and cd, each pair being regarded as a single $\mathrm{SO}(k^2)$ index). Moreover, they do not obstruct the existence of an $N = (4,4)$ supersymmetric AdS groundstate (3.24) if we keep $g_1 = -g_2$ as we did in (3.17). The components in the first line of (4.18) can be read off directly from (4.12) (keeping in mind the relative factor (-1) between the two SO(3) factors in SO(4)), and give rise to the generalization of $\mathcal{J}^{[ij]}$ from (4.12). The remaining components, i.e. the second line in (4.18) lead to the additional contribution in (3.16)

$$\Theta^{(k)}_{\mathcal{MN}}\, B^{\mathcal{M}}_{\mu} t^{\mathcal{N}} \;\;=\;\; \Theta^{(k)}_{i\,ab,j\,cd}\left(B^{\hat{\imath}\,ab}-B^{i\,ab}\right)\left(X^{\hat{\jmath}\,cd}-Y^{j\,cd}\right)\;. \tag{4.19}$$

These components can be determined in two *a priori* different ways. On the one hand, they are related to the first line of (4.18) by supersymmetry (3.11), i.e. complete antisymmetry in the $\mathrm{SO}(8,4+k^2)$ indices, implying e.g. $\Theta^{(k)}_{i\,ab,j\,cd}=\Theta^{(k)}_{ij,ab\,cd}$. On the other hand, their values are proportional to the difference between the two projectors from (4.16)

$$\Theta^{(k)}_{i\,ab,j\,cd} \;\;=\;\; g_1 k\left(\mathbb{P}_{\left[\frac{k}{2},\frac{k-2}{2}\right]}-\mathbb{P}_{\left[\frac{k-2}{2},\frac{k}{2}\right]}\right)_{i\,ab,j\,cd}\;, \tag{4.20}$$

again featuring the relative factor of (-1) between the different "chiralities" that we saw already in (3.16). This remarkable coincidence guarantees that (4.19) picks out precisely $2(k^2{-}1)$ vector fields from the a priori $4k^2$ fields $(B^{\hat{\imath}\,ab}-B^{i\,ab})$. Likewise, the combinations

$$\hat{\mathcal{T}}_{i\,ab} \;\;\equiv\;\; \Theta^{(k)}_{i\,ab,j\,cd}\,(X^{\hat{\jmath}\,cd}-Y^{j\,cd})\;, \tag{4.21}$$

appearing in (4.19) correspond to a projection of the $4k^2$ nilpotent generators $(X^{\hat{\jmath}\,cd}{-}Y^{j\,cd})$ onto a subset of $2(k^2-1)$ generators, which span (4.16). Had supersymmetry (3.11) imposed another value for $\Theta^{(k)}_{i\,ab,j\,cd}$, the minimal coupling (4.19) would have involved too many vector fields and generators. The CS gauge group identified by (4.18) precisely realizes the desired algebra (4.11). Again supersymmetry matches beautifully with the algebraic structure.

The Lagrangian of the theory coupling the higher spin-1 multiplet $(\mathbf{k{+}1},\mathbf{k{+}1})_S$ is then given by (3.18)–(3.22) with the coset space G/H from (4.7) and the embedding tensor $\Theta_{\mathcal{MN}}$ by (3.17), (4.18). As a first non-trivial check one may compute the vector mass spectrum around the origin, encoded in the eigenvalues of $\Theta^{(k)}_{i\,ab,j\,cd}$, see [43] and the discussion after (3.26) above. Indeed, from (4.20) one finds the masses $mL_0=\pm k$, reproducing the spectrum $\Delta=1+|mL_0|$ of table I. Finally, upon eliminating the scalar fields corresponding to $\mathcal{T}^{[ij]}$, $\hat{\mathcal{T}}_{i\,ab}$ following [34], one obtains the equivalent formulation of this theory as an SO(4) YM gauge theory with the vector fields $A^{[ij]}$ promoted to propagating YM vector fields, and the massive CS vector fields $\Theta^{(k)}_{i\,ab,j\,cd}\,(B^{\hat{\jmath}\,cd}-B^{j\,cd})$.

The theory describing the entire spectrum (4.2) is straightforwardly constructed starting from the coset (1.1) and summing over the additional contributions (4.18) of the embedding tensor Θ for the different k. Note that all $\Theta^{(k)}$ in (4.18) act in different sectors;

consequently, there are no divergent infinite sums of any kind in the limit of infinitely many multiplets. Similar comments apply to the multiplets from the spin-1 tower in the vector representation of SO(n).

5 Conclusions

It is rather striking that the complete Lagrangian coupling an arbitrary number of multiplets from the spin-1 KK towers can be cast into the simple form of (3.18)

$$e^{-1}\mathcal{L} \;=\; -\tfrac{1}{4}R + \tfrac{1}{4}g^{\mu\nu}\,\mathcal{P}_\mu^{Ir}\mathcal{P}_\nu^{Ir} - e^{-1}\mathcal{L}_{\rm CS} - W + \mathcal{L}_{\rm ferm}\;, \tag{5.1}$$

with all the complexity encoded in the coset space structure of (1.1) and the precise form of the embedding tensor (3.17), (4.18). The complete fermionic Lagrangian as well as the supersymmetry transformation rules are obtained from [32] upon using this explicit form of the embedding tensor.

It remains an open problem whether the KK tower of massive spin-2 supermultiplets can be incorporated in the effective three-dimensional theory in a similar fashion. This would amount to casting the *full* Kaluza-Klein theory into a single $D = 3$ supergravity with an infinite dimensional irreducible coset space and in particular allow us to address the issue of consistent truncations to finite subsectors directly within the three-dimensional theory.

We conclude with some intriguing hints that an extension to the full KK theory actually exists. The massive spin-2 KK tower contains the representations

$$\mathcal{H}_{\rm spin-2} \;=\; \sum_{p\geq 2}(\mathbf{p},\mathbf{p+2})_S \;+\; \sum_{p\geq 2}(\mathbf{p+2},\mathbf{p})_S\;, \tag{5.2}$$

with the massive spin-2 multiplet $(\mathbf{p},\mathbf{p}{+}\mathbf{2})_S$ given in table III. The numbers of bosonic and fermionic degrees of freedom are separately equal $8(p^2-1)$. This suggests that we further enlarge the coset (1.1) to a coset space G/H with the group

$$\mathrm{G} \;=\; \mathrm{SO}\Big(8\;,\sum_{k\geq 2}k^2 + n\sum_{l\geq 1}l^2 + 2\sum_{p\geq 2}(p^2-1)\Big)\;, \tag{5.3}$$

and H its maximal compact subgroup. With the specific embeddings

$$\mathrm{SO}(4)_p \;\subset\; \mathrm{SO}(2(p^2{-}1)) \quad:\qquad 2\,(p^2-1) \rightarrow \left[\tfrac{p-2}{2},\tfrac{p}{2}\right] + \left[\tfrac{p}{2},\tfrac{p-2}{2}\right]\;, \tag{5.4}$$

Δ	s_0	$\mathrm{SO}(4)_{\mathrm{gauge}}$	$\mathrm{SO}(4)_{\mathrm{glob}}$	# dof
p	1	$\left[\frac{p-1}{2}, \frac{p+1}{2}\right]$	$[0,0]$	$p(p+2)$
$p+\frac{1}{2}$	$\frac{3}{2}$	$\left[\frac{p-1}{2}, \frac{p}{2}\right]$	$\left[0, \frac{1}{2}\right]$	$2p(p+1)$
$p+\frac{1}{2}$	$\frac{1}{2}$	$\left[\frac{p-2}{2}, \frac{p+1}{2}\right]$	$\left[\frac{1}{2}, 0\right]$	$2(p-1)(p+2)$
$p+1$	1	$\left[\frac{p-2}{2}, \frac{p}{2}\right]$	$\left[\frac{1}{2}, \frac{1}{2}\right]$	$4(p^2-1)$
$p+1$	2	$\left[\frac{p-1}{2}, \frac{p-1}{2}\right]$	$[0,0]$	p^2
$p+1$	0	$\left[\frac{p-3}{2}, \frac{p+1}{2}\right]$	$[0,0]$	p^2-4
$p+\frac{3}{2}$	$\frac{3}{2}$	$\left[\frac{p-2}{2}, \frac{p-1}{2}\right]$	$\left[\frac{1}{2}, 0\right]$	$2p(p-1)$
$p+\frac{3}{2}$	$\frac{1}{2}$	$\left[\frac{p-3}{2}, \frac{p}{2}\right]$	$\left[0, \frac{1}{2}\right]$	$2(p+1)(p-2)$
$p+2$	1	$\left[\frac{p-3}{2}, \frac{p-1}{2}\right]$	$[0,0]$	$p(p-2)$

Table III: Spin-2 multiplet $(\mathbf{p}, \mathbf{p+2})_S$ of $\mathrm{SU}(2|1,1)_L \times \mathrm{SU}(2|1,1)_R$. The conjugate multiplet $(\mathbf{p+2}, \mathbf{p})_S$ is obtained by $s_0 \to -s_0$, and $[j_1, j_2] \to [j_2, j_1]$ under $\mathrm{SO}(4)_{\mathrm{gauge}}$ and $\mathrm{SO}(4)_{\mathrm{glob}}$.

one may define the group $\mathrm{SO}(4)_{\mathrm{gauge}}$ as the diagonal of (4.6) and the additional $\mathrm{SO}(4)_p$ groups. It is then straightforward to verify that the representation content of (5.3) indeed reproduces (5.2) with table III in the fermionic and the scalar sectors. The construction of a consistent gauged supergravity with this gauge group, however, is less obvious than the one presented above. In particular, it remains an open question if the spin-2 fields may be described by a coset space theory (5.3) and acquire masses by some as yet undiscovered version of the Brout-Englert-Higgs mechanism, or if this theory requires explicit extra couplings (which would vitiate the economy and beauty of the present scheme to some extent). These issues in turn hinge on the question whether there exists a novel type of duality between symmetric tensors and scalars in three dimensions which would generalize the well-known scalar vector duality. We hope to come back to these questions in the near future.

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