Irreducibility of the Ashtekar – Isham – Lewandowski Representation

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Abstract

Much of the work in loop quantum gravity and quantum geometry rests on a mathematically rigorous integration theory on spaces of distributional connections. Most notably, a diffeomorphism invariant representation of the algebra of basic observables of the theory, the Ashtekar-Isham-Lewandowski representation, has been constructed. Recently, several uniqueness results for this representation have been worked out. In the present article, we contribute to these efforts by showing that the AIL-representation is irreducible, provided it is viewed as the representation of a certain C*-algebra which is very similar to the Weyl algebra used in the canonical quantization of free quantum field theories.

1 Introduction

Canonical, background independent quantum field theories of connections [1] play a fundamental role in the program of canonical quantization of general relativity (including all types of matter), sometimes called loop quantum gravity or quantum general relativity (for a review geared to mathematical physicists see [2]). The classical canonical theory can be formulated in terms of smooth connections A on principal $G-$bundles over a $D-$dimensional spatial manifold Σ for a compact gauge group G and smooth sections of an associated (under the adjoint representation) vector bundle of Lie$(G)-$valued vector densities E of weight one. The pair (A, E) coordinatizes an infinite dimensional symplectic manifold $(\mathcal{M}, \sigma)$ whose (strong) symplectic structure s is such that A and E are canonically conjugate.

In order to quantize $(\mathcal{M}, s)$, it is necessary to smear the fields A, E. This has to be done in such a way that the smearing interacts well with two fundamental automorphisms of the principal $G-$ bundle, namely the vertical automorphisms formed by $G-$gauge transformations and the horizontal automorphisms formed by Diff(Σ) diffeomorphisms. These requirements naturally lead to holonomies and electric fluxes, that is, exponentiated (path-ordered) smearings of the connection over $1-$dimensional submanifolds e of Σ as well as smearings of the electric field over $(D-1)-$dimensional submanifolds S,

$$h_e[A] = \mathcal{P} \exp i \int_e A, \qquad E_{S,n}[E] = \int_S *E_i n^i.$$

These functions on $\mathcal{M}$ generate a closed Poisson $*$-algebra $\mathcal{P}$ and separate the points of $\mathcal{M}$. They do not depend on a choice of coordinates nor on a background metric. Therefore, diffeomorphisms and gauge transformations act on these variables in a remarkably simple way.

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Quantization now means to promote $\mathcal{P}$ to an abstract $*$-algebra $\mathfrak{A}$ and to look for its representations. Definitions of $\mathfrak{A}$ have been given in [9, 10, 11], and we will review what we need in the next section. Remarkably, $\mathfrak{A}$ admits a simple and mathematically elegant representation, the Ashtekar-Isham-Lewandowski representation $(\mathcal{H}_0, \pi_0)$ [6, 7]. An important feature of this representation is that it contains a cyclic vector that is invariant under the action of both diffeomorphisms and gauge transformations. Therefore it is a good starting point to tackle the implementation of the Gauss- and the diffeomorphism constraint of gravity [1].

In the present work we will add another item to the list of desirable properties that the AIL representation possesses: We will show that it is irreducible in a specific sense. The first thing we have to point out in this context is that $\mathfrak{A}$ is not an algebra of bounded operators. Consequently, to define a notion of irreducibility one has to worry about domain questions, and definitions as well as proofs become rather cumbersome. Maybe it is due to these technical difficulties that up to now, little has been said concerning irreducibility of the AIL representation. To the best of our knowledge, the only result in this direction is that the algebra $\mathfrak{A}$ allows us to map between any two vectors $f, f' \in \mathcal{D}$ where $\mathcal{D}$ is a dense subset of $\mathcal{H}_0$ [2].

In this situation, it is worthwhile to note that there is a strong analogy between the AIL representation of $\mathfrak{A}$ and the Schrödinger representation of the Heisenberg algebra in quantum mechanics. In both representations, the representation spaces are roughly speaking L_2 spaces over the configuration space, the configuration variables act by multiplication and the momenta by derivations. The Heisenberg algebra is again an algebra of unbounded operators, which makes the definition of irreducibility difficult. Moreover it is dubious that its Schrödinger representation can be irreducible in any sense, since for example the subspaces generated by functions which vanish on fixed open sets are invariant under action with multiplication operators and differentiation (if defined). However, the Schrödinger representation of the Heisenberg algebra can be obtained from the Schrödinger representation of the corresponding Weyl algebra, and it is *this* representation that is irreducible. In fact, von Neumann's famous uniqueness result states that it is the *only* irreducible, strongly continuous representation of the Weyl algebra.

In the light of this analogy, it seems worthwhile to investigate whether the AIL representation derives from a representation of an algebra of bounded operators which is irreducible. In fact, in [10] we introduced an algebra $\mathfrak{W}$ that contains the unitary one-parameter groups generated by the fluxes $E_{S,f}$ instead of the fluxes themselves. This algebra of *bounded* operators turns out to be closely analogous to the Weyl algebra used in the quantization of free fields, and the AIL representation of $\mathfrak{A}$ can be recovered from a strongly continuous representation of $\mathfrak{W}$ (which, in the following, will also be called "AIL representation"). The main result of [10] was that requiring diffeomorphism invariance, strong continuity, and certain technical conditions uniquely singles out the AIL-representation, very much in analogy to von Neumann's theorem. In the present article, we will carry this analogy further by showing that the AIL representation of $\mathfrak{W}$ is indeed irreducible.

The article is organized as follows:

In section 2 we recall from [1, 6, 7, 8] the essentials of the classical formulation of canonical, background independent theories of connections, that is, the symplectic manifold $(\mathcal{M}, \sigma)$ and the classical Poisson $*$-algebra generated by holonomies and electric fluxes. We then recall from [9, 10] the definition of the corresponding abstract algebra $\mathfrak{A}$ and its companion $\mathfrak{W}$.
In section 3 we prove irreducibility of the AIL representation of $\mathfrak{W}$.
In section 4 we finish with some conclusions.

2 Preliminaries

This section serves to review the definitions of the algebras $\mathfrak{A}$ and $\mathfrak{W}$. As most of this has been treated in detail elsewhere, we will just give an overview and refer to the appropriate literature for details.

Let G be a a compact, connected Lie group. For convenience, fix a basis τ_i of the corresponding Lie algebra. The resulting indices are dragged with the Cartan-Killing metric on G, although for simplicity we will not write this explicitely. Let Σ be an analytic, connected and orientable $D-$dimensional and P a principal G-bundle over Σ. The smooth connections in P form the classical configuration space $\mathcal{A}$, the corresponding momenta can be identified with sections in a vector bundle E_P associated to P under the adjoint representation, whose typical fiber is a Lie$(G)-$valued $(D-1)-$form on P. In a local trivialization, the Poisson brackets read

$$\{A_a^i(x), A_{a'}^{i'}(x')\} = \{E_i^a(x), E_{i'}^{a'}(x')\} = 0, \qquad \{A_a^i(x), E_{i'}^{a'}(x')\} = \delta_a^{a'}\delta_{i'}^i\delta(x, x'). \tag{2.1}$$

In a concrete gauge field theory the right hand side of the last equation will be multiplied by a constant which depends on the coupling constant of the theory. In order not to clutter our formulae we will assume that A and E respectively have dimension cm^{-1} and cm$^{-(D-1)}$ respectively.

As pointed out in the introduction, it turned out to be very fruitful to go over to certain functionals of A and E. For the connection, the functionals are chosen to be the parallel transports along analytical paths e in Σ,

$$h_e[A] = \mathcal{P}\exp\left[i\int_e A_a ds^a\right].$$

In fact, it turns out to be convenient to consider functions of A which are slightly more general.

Definition 2.1. *A graph in Σ is a collection of analytic, oriented curves in Σ which intersect each other at most in their endpoints. Given a graph γ, the set of its constituting curves ("edges") will be denoted by $E(\gamma)$.*

A function f depending on connections A on Σ just in terms of their holonomies along the edges of a graph, i.e.

$$f[A] \equiv f(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_n}[A]), \qquad e_1, e_2, \ldots, e_n \quad \text{edges of some } \gamma,$$

where $f(g_1, \ldots, e_n)$ viewed as a function on $\mathrm{SU}(2)^n$ *is continuous, will be called cylindrical.*

It turns out that the set of cylindrical functions can be equipped with a norm (essentially the sup-norm for functions on $\mathrm{SU}(2)^n$) such that its closure (denoted by Cyl) with respect to that norm is a commutative C*-algebra. We will not spell out the details of this construction but refer the reader to the presentations [7, 8]. We note furthermore that by changing the word "continuous" in the above definition to "n times differentiable", we can define subsets Cyl^n of Cyl and, most importantly for us,

$$\mathrm{Cyl}^\infty := \bigcap_n \mathrm{Cyl}^n,$$

the space of smooth cylindrical functions.

By Gelfand's theory, Cyl can be identified with the continuous function on a compact Hausdorff space $\overline{\mathcal{A}}$. The inclusion of the functions on $\mathcal{A}$ defined in Definition 2.1 into the continuous functions on $\overline{\mathcal{A}}$ is afforded by the fact that $\overline{\mathcal{A}}$ is a projective limit: Each graph γ defines a projection $p_\gamma : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}_\gamma$, where $\overline{\mathcal{A}}_\gamma$ is diffeomorphic to G^N, N being the number of edges of γ.[1]

The density weight of E on the other hand is such that, using an additional real (co-)vector field n^i, it can be naturally integrated over oriented surfaces $D-1$ dimensional submanifolds S to form a quantity

$$E_{S,n} = \int_S E_i^a n^i \epsilon_{abc}\, dx^b\, dx^c$$

analogous to the electric flux through S. In the following we will only consider analytic submanifolds S to avoid certain pathologies in the algebra relations to be defined below.

[1] The diffeomorphism is not unique but roughly speaking depends on the choice of a gauge. But we can ignore this fact since the algebraic structures one obtains in the end are not affected by this choice.

It has been shown in [17] that the algebra generated by Cyl and the fluxes $\{E_{S,n}\}$ can be given the structure of a Lie algebra which derives in a precise sense from the Poisson relations (2.1). To spell out this structure, we will first define the action of certain derivations $Y_{S,n}$ on Cyl, where again S is an analytical surface and n a vectorfield in $E_P|_S$. Let f be a smooth function cylindrical on γ and assume without loss of generality that all transversal intersections of γ and S are in vertices of γ. Then one defines

$$Y_{S,n}[f] := \frac{1}{2} \sum_{p \in S \cap \gamma} \sum_{e_p} \sigma(e_p, S) n_i(p) Y^i_{e_p}[f],$$

where the second sum is over the edges of γ adjacent to p,

$$\sigma(e_p, S) = \begin{cases} 1 & \text{if } e_p \text{ lies above } S \\ 0 & \text{if } e_p \cap S = \emptyset \text{ or } e_p \cap S = e_p \\ -1 & \text{if } e_p \text{ lies below } S \end{cases},$$

and $Y^i_{e_p}$ is the ith left-invariant (right-invariant) vector field on SU(2) acting on the argument of f corresponding to the holonomy h_{e_p} if e_p is pointing away from (towards) S.

A Lie product between elements of Cyl^∞ and fluxes $\{E_{S,n}\}$ can now be defined as

$$\{f, E_{S,n}\} = Y_{S,n}[f]. \tag{2.2}$$

A bit surprisingly at first sight, it turns out that Lie products between elements of $\{E_{S,n}\}$ can not, in general vanish. However, their Lie products with cylindrical functions are completely determined by (2.2) together with the requirement that the Jacobi identity has to hold.[2] For example one finds

$$\{f, \{E_{S,n}, E_{S',n'}\}\} = \left[X_{S',n'}\,,\, X_{S,n}\right][f].$$

We are now in a position to define the algebra $\mathfrak{A}$:

Definition 2.2. *Let $\mathfrak{A}$ be the algebra generated by* Cyl *together with symbols $\{E_{S,n}\}$, divided by the commutation relations*

$$[E_{S,n}\,,\, f] = \frac{1}{i} Y_{S,n}[f].$$

Equip $\mathfrak{A}$ with an involution by defining

$$f^* := \overline{f}, \qquad E^*_{S,n} := E_{S,n}.$$

The general representation theory of the algebra $\mathfrak{A}$ gets complicated due to the fact that the $\{E_{S,n}\}$ will be represented by unbounded operators, so that domain questions will arise. In [10] we proposed to circumvent these difficulties by passing to exponentials of i times the fluxes, thereby obtaining an algebra $\mathfrak{W}$ analogous to the Weyl algebra used in the quantization of free field theories:

Definition 2.3.
Let $\mathfrak{W}$ be the algebra generated by Cyl *together with symbols $W^f_t(S)$, $t \in \mathbb{R}$, divided by the relations*

$$W^n_{t_1+t_2}(S) = W^n_{t_1}(S) W^n_{t_2}(S), \qquad W^n_0(S) = \mathbf{1},$$
$$W^n_t(S)\, f(h_{e e \in E(\gamma)})\, W^n_{-t}(S) = f(\{e^{t\sigma(S,e) n^j(b(e))\tau_j} h_e\}_{e \in E(\gamma)})$$

where $f \in$ Cyl is supposed to be cylindrical on γ, and $b(e)$ denotes the starting point of e. Equip $\mathfrak{W}$ with an involution by defining

$$f^* := \overline{f}, \qquad (W^n_t(S))^* := W^n_{-t}(S).$$

[2] The fact that the fluxes among themselves do not commute can be traced back to the fact that going over from (2.1) to (2.2) involves a nontrivial limiting procedure. For more information on this point see [17].

We note that these relations follow from formally identifying $W_t^n(S)$ with $\exp(itE_{S,n})$ and using the relations (2.2).

Finally, we want to describe the AIL representation of $\mathfrak{A}$, show that it defines a representation of $\mathfrak{W}$ as well, and notice that it can consequently be used to define a C*-norm on $\mathfrak{W}$.

The representation space of the AIL representation is given by $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$ where $\overline{\mathcal{A}}$ is the spectrum of the C^*-subalgebra of $\mathfrak{A}$ (and $\mathfrak{W}$) given by Cyl and μ_0 is a regular Borel probability measure on $\overline{\mathcal{A}}$ consistently defined by

$$\mu_0(p_\gamma^* f_\gamma) = \int_{G^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e\in E(\gamma)}) \tag{2.3}$$

for measurable f_γ and extended by $\sigma-$additivity. Then

$$\pi_0(f)\psi[A] = f[A]\psi[A], \qquad \pi_0(E_{S,n})f = \frac{1}{i} Y_{S,n}[f] \tag{2.4}$$

(where $\psi \in \mathcal{H}_0$, $f \in \mathrm{Cyl}^\infty \hookrightarrow \mathcal{H}_0$), define a representation of $\mathfrak{A}$. Note especially that the $\pi_0(E_{S,n})$ as defined above can be closed to selfadjoint operators. This representation of $\mathfrak{A}$ also defines a representation of $\mathfrak{W}$, which, in abuse of notation, we will also denote by π_0, via

$$\pi_0(W_t^n(S)) = e^{it\pi_0(E_{S,n})} = e^{tY_{S,n}}.$$

It follows from the left invariance of the Haar measure that $\pi_0(W_t^f(S))$ are unitary operators as they should be.

Note also that the continuous functions $C^0(\overline{\mathcal{A}})(\simeq \mathrm{Cyl})$ are dense in $\mathcal{H}_0$ because $\mathcal{H}_0$ is the GNS Hilbert space induced by the positive linear functional ω_0 on Cyl defined by $\omega_0(f) = \mu_0(f)$ where $\Omega_0 = 1$ is the cyclic GNS vector.

Finally we equip the algebra $\mathfrak{W}$ with a C* structure. Recall [18] that if a Banach algebra admits a C* norm at all then it is unique and determined purely algebraically by $||a|| = \sqrt{a^*a}$ where ρ denotes the spectral radius of $a \in \mathfrak{W}$. Since the operator norm in a representation π of $\mathfrak{W}$ on a Hilbert space $\mathcal{H}$ *does* define a C*$-$ norm through $||a|| := ||\pi(a)||_{\mathcal{H}}$ we just need to find a representation of $\mathfrak{W}$ (and complete it in the corresponding operator norm). However, the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_0$ *is* a representation space for a representation π_0, hence a C*$-$norm exists. Let us compute it explicitely: As remarked earlier, the $\pi_0(W_t^n(S)$ are unitary operators, thus

$$\begin{aligned} ||f||_{\mathfrak{W}} &= ||\pi_0(f)||_{\mathcal{B}(\mathcal{H}_0)} = \sup_{||\psi||=1} ||f\psi||_{\mathcal{H}_0} = \sup_{a\in\mathcal{A}} |f(A)| \\ ||W_t^n(S)||_{\mathfrak{W}} &= ||\pi_0(W_t^n(S))||_{\mathcal{B}(\mathcal{H}_0)} = \sup_{||\psi||=1} ||W_t^n(S)\psi||_{\mathcal{H}_0} = 1 \end{aligned} \tag{2.5}$$

where $\mathcal{B}$ denotes the bounded operators on a Hilbert space. The C^*-norm of any other element of $\mathfrak{W}$ can be computed by using the commutation relations and the inner product on $\mathcal{H}_0$.

Certainly other completions of $\mathfrak{W}$ might exist, but this is of no concern here as we will not make essential use of the C*$-$norm in the present paper.

Finally, let us recall that the Hilbert space $\mathcal{H}_0$ has an orthonormal basis given by spin network functions [19]. These are particular cylindrical functions labeled by a spin network $s = (\gamma, \{\pi_e\}, \{m_e\}, \{n_e\})_{e\in E(\gamma)}$ defined by

$$T_s(A) = \prod_{e\in E(\gamma)} \{\sqrt{d_{\pi_e}}\, [\pi_e(h_e)]_{m_e n_e}\} \tag{2.6}$$

where γ denotes a graph, π denotes an irreducible representation of G (one fixed representative from each equivalence class), d_π its dimension and $[\pi(h)]_{mn},\ m, n = 1, .., d_\pi$ denote the matrix elements of $\pi(h)$ for $h \in G$. We write $\gamma(s)$ when $s = (\gamma, ., ., .)$.

This concludes our exposition about the C*$-$algebra $\mathfrak{W}$ and its representation π_0 on $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$.

3 Irreducibility Proof

Theorem 3.1.
The Ashtekar – Isham – Lewandowski representation π_0 of the algebra $\mathfrak{W}$ on $\mathcal{H}_0$ is irreducible.

Before we prove the theorem, we first need two preparational results. Let γ be a graph. Split each edge $e \in E(\gamma)$ into two halves $e = e'_1 \circ (e'_2)^{-1}$ and replace the e's by the e'_1, e'_2. This leaves the range of γ invariant but changes the set of edges in such a way that each edge is outgoing from the vertex $b(e') = v \in V(\gamma)$ (notice that by a vertex we mean a point in γ which is not the interior point of an analytic curve so that the break points $e'_1 \cap e'_2$ do not count as vertices). We call a graph refined in this way a *standard graph*. Every cylindrical function over a graph is also cylindrical over its associated standard graph so there is no loss of generality in sticking with standard graphs in what follows.

With this understanding, the following statement holds.

Lemma 3.1.
Let γ be a standard graph. Assign to each $e \in E(\gamma)$ a vector $t_e = (t^j_e)_{j=1}^{\dim(G)}$ and collect them into a label $t_\gamma = (t_e)_{e \in E(\gamma)}$.

Then there exists a vector field $Y(t_\gamma, \gamma)$ in the Lie algebra of the flux vector fields $Y_{S,f}$ such that for any cylindrical function $f = p^*_\gamma f_\gamma$ over γ we have

$$Y_\gamma(t_\gamma) p^*_\gamma f_\gamma = p^*_\gamma \sum_{e \in E(\gamma)} t^j_e R^e_j f_\gamma \tag{3.1}$$

Proof of lemma 3.1:
Any compact connected Lie group G has the structure $G/Z = A \times S$ where Z is a discrete central subgroup, A is an Abelean Lie group group and S is a semisimple Lie group.
We will first construct an appropriate vector field Y^j_e for each j and each $e \in E(\gamma)$. The construction is somewhat different for the Abelean and non-Abelean generators respectively so that we distinguish the two cases.
Abelean Factor
Let j label only Abelean generators for this paragraph. Consider any $e \in E(\gamma)$ and take any surface S_e which intersects γ only in an interior point of e and such that the orientation of S_e agrees with that of e_2 where $e = e_1 \circ e_2,\ e_1 \cap e_2 = S_e \cap \gamma$. Then for any cylindrical function $f = p^*_\gamma f_\gamma$ we have

$$Y_j(S_e) p^*_\gamma f_\gamma = p^*_\gamma [R^j_{e_2} - R^j_{e_1}] f_\gamma \tag{3.2}$$

Due to gauge invariance $[R^j_{e_1} + R^j_{e_2}] f_\gamma = 0$, thus

$$Y^j_e p^*_\gamma f_\gamma = \frac{1}{2} Y_j(S_e) p^*_\gamma f_\gamma \tag{3.3}$$

is an appropriate choice.

Non-Abelean Factor
Let j label only non-Abelean generators for this paragraph. Given γ select a vertex v and one $e \in E(\gamma)$ with $b(e) = v$. We claim that there exists an analytic surface $S_{v,e}$ through v such that $s_e \subset S_{v,e} = \gamma \cap S_{v,e}$ for some beginning segment s_e of e but such that any other $e' \in E(\gamma)$ is transversal to $S_{v,e}$. The analytic surface S is completely determined by its germ $[S]_v$, that is, the Taylor coefficients in the expansion of its parameterization

$$S(u, v) = \sum_{m,n=0}^{\infty} \frac{u^m\ v^n}{m!\ n!} S^{(m,n)}(0, 0) \tag{3.4}$$

Likewise, consider the germ $[e]_v$ of e

$$e(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}e^{(n)}(0) \tag{3.5}$$

In order that $s_e e \subset S_{v,e}$ we just need to choose a parametrization of S such that, say, $S(t,0)=e(t)$ which fixes the Taylor coefficients

$$S^{(m,0)}(0,0)=e^{(m)}(0) \tag{3.6}$$

for any m. By choosing the range of t,u,v sufficiently small we can arrange that $s_e \subset S$.

We now choose the freedom in the remaining coefficients to satisfy the additional requirements. We must avoid that for finitely many, say N, edges $e'_1,..,e'_n$ that there is any beginning segment s_k of e'_k with $s_k \subset S$. If s_k would be contained in S then there would exist an analytic function $t \mapsto v_k(t)$, such that $s_k(t)=S(t,v_k(t))$. Notice that v_k must be different from the zero function in a sufficiently small neighborhood around $t=0$ as otherwise we would have $s_k=s_e$ which is not the case. For each k let $n_k>0$ be the first derivative such that $v_k^{(n_k)}(0)\neq 0$. By relabeling the edges we may arrange that $n_1 \le n_2 \le .. \le n_N$. Consider $k=1$ and take the n_1-th derivative at $t=0$. We find

$$s_1^{(n_1)}(0)=S^{(n_1,0)}(0,0)+S^{(0,1)}(0,0)v_1^{(n_1)}(0) \tag{3.7}$$

Since $v_1^{(n_1)}(0)\neq 0$ we can use the freedom in $S^{(0,1)}(0,0)$ in order to violate this equation. Now consider $k=2$ and take the $(n_2+1)-$th derivative. We find

$$s_2^{(n_2+1)}(0)=S^{(n_2+1,0)}(0,0)+2S^{(1,1)}(0,0)v_2^{(n_2)}(0)+S^{(0,1)}(0,0)v_2^{(n_2+1)}(0) \tag{3.8}$$

Since $v_2^{(n_2)}(0)\neq 0$ we can use the freedom in $S^{(1,1)}$ in order to violate this equation. Proceeding this way we see that we can use the coefficients $S^{(k-1,1)}(0,0)$ in order to violate $s_k(t)=S(t,v_k(t))$ for $k=1,..,N$.

Having constructed the surfaces $S_{v,e}$ we can compute the associated vector field applied to a cylindrical function over γ

$$Y_j(S_{v,e})p^*_\gamma f_\gamma = p^*_\gamma \sum_{e'\in E(\gamma)-\{e\},b(e')=v} \sigma(S_{v,e},e')R^j_{e'}f_\gamma \tag{3.9}$$

where by construction $|\sigma(S_{v,e},e')|=1$ for any $e'\neq e,\ b(e)=v$. Taking the commutator

$$[Y_j(S_{v,e}),Y_k(S_{v,e})]p^*_\gamma f_\gamma = f_{jkl}p^*_\gamma \sum_{e'\in E(\gamma)-\{e\},b(e')=v} R^j_{e'}f_\gamma \tag{3.10}$$

Using the Cartan Killing metric normalization for the totally skew structure constants $f_{jkl}f_{lmj}=-\delta_{km}$ and writing

$$R^j_v := \sum_{e'\in E(\gamma),\ b(e')=v} R^j_{e'} \tag{3.11}$$

we get

$$f_{jkl}[Y_k(S_{v,e}),Y_l(S_{v,e})]p^*_\gamma f_\gamma = p^*_\gamma [R^j_v - R^j_e]f_\gamma \tag{3.12}$$

Thus, if $n_v=|\{e\in E(\gamma);\ b(e)=v\}|$ denotes the valence of v

$$\begin{aligned} Y^j_e p^*_\gamma f_\gamma &:= \{-f_{jkl}[Y_k(S_{v,e}),Y_l(S_{v,e})]+\frac{1}{n_v-1}\sum_{e\in E(\gamma)}(f_{jkl}[Y_k(S_{v,e}),Y_l(S_{v,e})])\}p^*_\gamma f_\gamma \\ &= p^*_\gamma R^j_e f_\gamma \end{aligned} \tag{3.13}$$

Collecting the vector fields Y^j_e for the Abelean and non-Abelean labels j respectively and contracting them with t^j_e and summing over $e\in E(\gamma)$ yields an appropriate vector field

$$Y_\gamma(t_\gamma)=\sum_{e\in E(\gamma)} t^e_j Y^j_e \tag{3.14}$$

□

Lemma 3.1 has the following important implication: The algebra $\mathfrak{A}$ also contains the vector field $Y_\gamma(t_\gamma)$ and therefore $\mathfrak{W}$ contains the corresponding Weyl element $W_\gamma(t_\gamma)$. Also, let us write $I_\gamma = (\{\pi_e\}, \{m_e\}, \{n_e\})_{e\in E(\gamma)}$ for a spin network $s = (\gamma, I_\gamma)$ over γ. Denoting by $T_s = T_{\gamma,I_\gamma}$ the corresponding spin network function (where we also allow trivial π_e for any e) we define for any two $\psi, \psi' \in \mathcal{H}_0$ the function

$$(t_\gamma, I_\gamma) \mapsto M_{\psi,\psi'}(t_\gamma, I_\gamma) := \langle \psi \, , \, T_{\gamma,I_\gamma} \; W_\gamma(t_\gamma)\psi' \rangle_{\mathcal{H}_0} \tag{3.15}$$

We now exploit that for a compact connected Lie group the exponential map is onto. Thus, there exists a region $D_G \subset \mathbb{R}^{\dim(G)}$ such that $\exp: \; D_G \to G;\; t \mapsto \exp(t^j\tau_j)$ is a bijection. Consider the measure μ on D_G defined by $d\mu(t) = d\mu_H(\exp(t^j\tau_j))$ where μ_H is the Haar measure on G. Finally, let $D_\gamma = \prod_{e\in E(\gamma)} D_G$ and let L_γ be the space of the I_γ. We now define an inner product on the functions of the type (3.15) by

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma := \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \overline{M_{\psi_1,\psi_1'}(t_\gamma, I_\gamma)} \;\; M_{\psi_2,\psi_2'}(t_\gamma, I_\gamma) \tag{3.16}$$

where $d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e)$.

The inner product of the type (3.16) is a crucial ingredient in an elementary irreducibility proof of the Schrödinger representation of ordinary quantum mechanics (see for example [20]) and we can essentially copy the corresponding argument. Of course, we must extend the proof somewhat in order to be able to deal with an infinite number of degrees of freedom. The following result prepares for that.

Lemma 3.2.
i) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ we have

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \le ||\psi_1|| \;\; ||\psi_1'|| \;\; ||\psi_2|| \;\; ||\psi_2'||. \tag{3.17}$$

ii) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_{0,\gamma}$ we have

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma = \langle \psi_2 \, , \, \psi_1 \rangle_{\mathcal{H}_0} \;\; \langle \psi_1' \, , \, \psi_2' \rangle_{\mathcal{H}_0} \tag{3.18}$$

where $\mathcal{H}_{0,\gamma}$ denotes the closure of the cylindrical functions over γ.

Proof of lemma 3.2:
We simply compute

$$\begin{aligned}
& (M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma \\
= & \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') \overline{T_{\gamma,I_\gamma}(A)} T_{\gamma,I_\gamma}(A') \psi_1(A) \overline{[W_\gamma(t_\gamma)\psi_1'](A)\psi_2(A')} [W_\gamma(t_\gamma)\psi_2'](A') \\
= & \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') [\sum_{I_\gamma} \overline{T_{\gamma,I_\gamma}(A)} T_{\gamma,I_\gamma}(A')] \psi_1(A) \overline{[W_\gamma(t_\gamma)\psi_1'](A)\psi_2(A')} [W_\gamma(t_\gamma)\psi_2'](A') \\
= & \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') \int_{D_\gamma} d\mu(t_\gamma) \delta_\gamma(A,A') \psi_1(A) \overline{[W_\gamma(t_\gamma)\psi_1'](A)\psi_2(A')} [W_\gamma(t_\gamma)\psi_2'](A')
\end{aligned} \tag{3.19}$$

where we have defined the cylindrical $\delta-$distribution

$$\delta_\gamma(A, A') = \prod_{e\in E(\gamma)} \delta_{\mu_H}(h_e[A], h_e[A']) \tag{3.20}$$

which arises due to the Plancherel formula

$$\delta_{\mu_H}(g, g') = \sum_{\pi,m,n} \overline{T_{\pi,m,n}(g)} \; T_{\pi,m,n}(g') \tag{3.21}$$

The interchange of integrals over $\overline{\mathcal{A}}\times\overline{\mathcal{A}}$ and the sum over L_γ in (3.21) is justified by the Plancherel theorem.
i)
In order to evaluate the cylindrical $\delta-$distribution in (3.19) we subdivide the degrees of freedom $A \in \overline{\mathcal{A}}$ into the set $\overline{\mathcal{A}}_\gamma = \overline{\mathcal{A}}_{|\gamma}$ and the complement $\overline{\mathcal{A}}_{\bar\gamma} = \overline{\mathcal{A}} - \overline{\mathcal{A}}_\gamma$ in the following sense: Each of the functions f_1, f_1', f_2, f_2' is a countable linear combination of spin network functions T_s, each of which is cylindrical over some graph $\gamma(s)$. We may consider those functions as cylindrical over the graph $\gamma\cup\gamma(s)$ and since the edges $e \in E(\gamma)$ are holonomically independent, we can express each edge $\tilde e \in E(\gamma(s))$ as a finite composition of the edges of $E(\gamma)$ and some other edges e' of $\gamma(s)\cup\gamma$ such that no segment of any of the e' is a beginning segment of one of the e. Thus, each $T_s(A)$ depends on the $h_e,\ e \in E(\gamma)$ and some other $h_{e'}$ which are not finite compositions of the h_e. We can thus write symbolically for any $f \in \mathcal{H}_0$

$$f(A) = F(A_{|\bar\gamma}, A_{|\gamma}) \tag{3.22}$$

where the separation of the degrees of freedom is to be understood in the sense just discussed, that is, $A_{|\gamma} \in \overline{\mathcal{A}}_\gamma,\ A_{\bar\gamma} \in \overline{\mathcal{A}}_{\bar\gamma}$. It just means that when expanding out inner products of L_2 functions into those of spin network functions, that one can perform the integrals over the degrees of freedom in $\overline{\mathcal{A}}_\gamma$ and in $\overline{\mathcal{A}}_{\bar\gamma}$ independently. Given a function of the type (3.22) we define the measure on $\overline{\mathcal{A}}_\gamma$ by $\mu_{0\gamma} = \mu_0 \circ p_\gamma^{-1}$ and the (effective) measure on $\overline{\mathcal{A}}_{\bar\gamma}$ by

$$\begin{aligned}&\int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})[\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})F(A_{|\bar\gamma}, A_{|\gamma})] \cdot \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})[\int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})F(A_{|\bar\gamma}, A_{|\gamma})] \\ &\quad := \int_{\overline{\mathcal{A}}} d\mu_0(A) f(A)\end{aligned} \tag{3.23}$$

In order to perform concrete integrals of $f \in L_1(\overline{\mathcal{A}}, d\mu_0)$ over either $\overline{\mathcal{A}}_\gamma$ or $\overline{\mathcal{A}}_{\bar\gamma}$ we notice that all our occurring f are countable linear combinations of spin network functions. Thus either integral can be written as a countable linear combination of integrals over spin-network functions T_s and then the prescription is to integrate only either over the degrees of freedom $A(e),\ e \in E(\gamma)$ or $A(e'),\ e' \in E(\gamma(s)\cup\gamma) - E(\gamma)$ for each individual integral with the corresponding product Haar measure. It follows that $\mu_0 = \mu_{0\bar\gamma} \otimes \mu_{0\gamma}$ is a product measure.

We may therefore neatly split (3.19) as

$$\begin{aligned}(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma = &\int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma}) \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A'_{|\bar\gamma}) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\times \\ &\times \Psi_1(A_{|\bar\gamma}, A_\gamma)\overline{[W_\gamma(t_\gamma)\Psi_1'](A_{|\bar\gamma}, A_\gamma)\Psi_2(A'_{|\bar\gamma}, A_\gamma)}[W_\gamma(t_\gamma)\Psi_2'](A'_{|\bar\gamma}, A_\gamma)\end{aligned} \tag{3.24}$$

In order to evaluate the Weyl operators, consider a spin network function T_s cylindrical over $\gamma(s)$ which we write in the form

$$T_s(A) = F(\{h_{e'}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)}, \{h_e\}_{e\in E(\gamma)}) \tag{3.25}$$

Our concrete vector field $Y_\gamma(t_\gamma)$ involves a finite collection of surfaces to which the edges $e \in E(\gamma)$ are already adapted in the sense that they are all of a definite type ("in", "out", "up" or "down") and we may w.l.g. assume that the same is true for the e'. Then it is easy to see that the action of $Y_\gamma(t_\gamma)$ on T_s is given by

$$Y_\gamma(t_\gamma)T_s = p^*_{\gamma(s)\cup\gamma}[\sum_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)} t_j^{e'}(t_\gamma)R^j_{e'} + \sum_{e\in E(\gamma)} t_j^e R^j_e]F \tag{3.26}$$

where $t_j^{e'}(t_\gamma)$ is a certain linear combination of the t_j^e depending on e' and the concrete surfaces $S_e, S_{v,e}$ used in the construction of $Y_\gamma(t_\gamma)$. Since the beginning segments of the e', e are mutually independent, the corresponding vector fields commute and it follows that

$$\begin{aligned}(W_\gamma(t_\gamma)T_s)(A) &= F(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)}, \{e^{t_j^e\tau_j}h_e\}_{e\in E(\gamma)}) \\ &= F(\{W_\gamma(t_\gamma)h_{e'}W_\gamma(t_\gamma)^{-1}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)}, \{W_\gamma(t_\gamma)h_e W_\gamma(t_\gamma)^{-1}\}_{e\in E(\gamma)})\end{aligned} \tag{3.27}$$

Consider now any L_2 function ψ. Since it is a countable linear combination of spin network functions we can generalize (3.27) to

$$(W_\gamma(t_\gamma)\psi)(A) = \psi(W_\gamma(t_\gamma)A_{|\bar\gamma}W_\gamma(t_\gamma)^{-1}, W_\gamma(t_\gamma)A_{|\gamma}W_\gamma(t_\gamma)^{-1}) \tag{3.28}$$

where the crucial point is that for each $t_\gamma \in D_\gamma$ the map $\alpha_{t_\gamma} : \overline{\mathcal{A}} \to \overline{\mathcal{A}};\ A \mapsto W_\gamma(t_\gamma)AW_\gamma(t_\gamma)$ is just some right or left translation. We can thus estimate (notice that we can interchange the sequence of integration w.r.t. the factors of a product measure)

$$\begin{aligned}
|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \le & \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\times \\
& \times [\int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})|\Psi_1(A_{|\bar\gamma}, A_\gamma)|\ \ |\Psi_1'(\alpha_{t_\gamma}(A_{|\bar\gamma}), \alpha_{t_\gamma}(A_\gamma))|\times \\
& \times [\int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A'_{|\bar\gamma})|\Psi_2(A'_{|\bar\gamma}, A_\gamma)|\ \ |\Psi_2'(\alpha_{t_\gamma}(A'_{|\bar\gamma}), \alpha_{t_\gamma}(A_\gamma))| \\
\le & \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})||\Psi_1(A_\gamma)||_{|\bar\gamma}\ \ ||\Psi_1'(\alpha_{t_\gamma}(A_\gamma))||_{\bar\gamma}||\Psi_2(A_\gamma)||_{\bar\gamma}\ \ ||\Psi_2'(\alpha_{t_\gamma}(A_\gamma))||_{\bar\gamma}
\end{aligned}$$

where we have used the Cauchy Schwarz inequality applied to functions such as $\Psi_1(A_\gamma)$ on $L_2(\overline{\mathcal{A}}_{\bar\gamma}, d\mu_{0\bar\gamma})$ defined by $[\Psi_1(A_\gamma)](A_{|\bar\gamma}) = \Psi_1(A_{|\bar\gamma}, A_\gamma)$. Here it was crucial to note that due to the bi-invariance of the measure $\mu_{0\bar\gamma}$ we have e.g.

$$\int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})|\Psi_1'(\alpha_{t_\gamma}(A_{|\bar\gamma}), \alpha_{t_\gamma}(A_\gamma))|^2 = \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})|\Psi_1'(A_{|\bar\gamma}, \alpha_{t_\gamma}(A_\gamma))|^2 = ||\Psi_1'(\alpha_{t_\gamma}(A_\gamma))||^2_{\bar\gamma}$$

To see this, expand ψ_1' into spin-network functions. Then the integral is of the form

$$\begin{aligned}
& \sum_{m,n=1}^\infty \bar z_m z_n \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})\overline{T_{s_m}(\alpha_{t_\gamma}(A))}\, T_{s_n}(\alpha_{t_\gamma}(A)) \\
&= \sum_{m,n=1}^\infty \bar z_m z_n \int_{G^{|E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma)-E(\gamma)|}} [\prod_{e'\in E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma)-E(\gamma)} d\mu_H(h_{e'})]\ \times \\
&\qquad \times \overline{T_{s_m}(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\}, \{e^{t_j^e\tau_j}h_e\})}\ \ T_{s_n}(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\}, \{e^{t_j^e\tau_j}h_e\}) \\
&= \sum_{m,n=1}^\infty \bar z_m z_n \int_{G^{|E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma)-E(\gamma)|}} [\prod_{e'\in E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma)-E(\gamma)} d\mu_H(h_{e'})]\times \\
&\qquad \times \overline{T_{s_m}(\{h_{e'}\}, \{e^{t_j^e\tau_j}h_e\})}\ \ T_{s_n}(\{h_{e'}\}, \{e^{t_j^e\tau_j}h_e\}) \\
&= \sum_{m,n=1}^\infty \bar z_m z_n \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})\overline{T_{s_m}(A_{|\bar\gamma}, \alpha_{t_\gamma}(A_{|\gamma}))}\ \ T_{s_m}(A_{|\bar\gamma}, \alpha_{t_\gamma}(A_{|\gamma})) \\
&= \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})|\Psi_1'(A_{|\bar\gamma}, \alpha_{t_\gamma}(A_{|\gamma}))|^2
\end{aligned}$$

We now exploit that

$$\alpha_{t_\gamma}(A_{|\gamma}) = \{e^{t_j^e\tau_j}h_e\}_{e\in E(\gamma)} \tag{3.29}$$

and introduce new integration variables $h_e' := g(t_e)h_e$ where $g(t_e) = \exp(t_j^e\tau_j)$. Since by definition

$$d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e) = \prod_{e\in E(\gamma)} d\mu_H(g(t_e)) \tag{3.30}$$

we can estimate further

$$
\begin{aligned}
|(M_{\psi_1,\psi'_1}, M_{\psi_2,\psi'_2})_\gamma| \leq & \int_{G^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(g_e) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\times \\
& \times ||\Psi_1(A_{|\gamma})||_{|\bar\gamma} \;\; ||\Psi'_1(\{g_e A(e)\}_{e\in E(\gamma)})||_{\bar\gamma} ||\Psi_2(A_{|\gamma})||_{\bar\gamma} \;\; ||\Psi'_2(\{g_e A(e)\}_{e\in E(\gamma)})||_{\bar\gamma} \\
= & [\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})||\Psi_1(A_{|\gamma})||_{|\bar\gamma} \;\; ||\Psi_2(A_{|\gamma})||_{\bar\gamma}][\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma})||\Psi'_1(A'_{|\gamma})||_{\bar\gamma} \;\; ||\Psi'_2(A'_{|\gamma})||_{\bar\gamma}] \\
\leq & || \;\; ||\Psi_1||_{\bar\gamma} \;\; ||_\gamma \;\; || \;\; ||\Psi'_1||_{\bar\gamma} \;\; ||_\gamma \;\; || \;\; ||\Psi_2||_{\bar\gamma} \;\; ||_\gamma \;\; || \;\; ||\Psi'_2||_{\bar\gamma} \;\; ||_\gamma
\end{aligned}
$$

where we have used the Fubini theorem and have again applied the Cauchy Schwarz inequality to functions in $L_2(\overline{\mathcal{A}}_\gamma, d\mu_{0\gamma})$. But

$$
\begin{aligned}
|| \;\; ||\Psi_1||_{\bar\gamma} \;\; ||^2_\gamma = & \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})| \;\; ||\Psi_1(A_{|\gamma})||_{\bar\gamma} \;\; |^2 \\
= & \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{\mathcal{A}}_{\bar\gamma}} d\mu_{0\bar\gamma}(A_{|\bar\gamma})|\Psi_1(A_{|\bar\gamma}, A_{|\gamma})|^2 = \int_{\overline{\mathcal{A}}} d\mu_0(A)|\psi_1(A)|^2 = ||\psi_1||^2_{\mathcal{H}_0}
\end{aligned}
$$

so we get (3.17).

ii)
If all functions in question are cylindrical L_2-functions over γ then the integrals over $\overline{\mathcal{A}}_{|\bar\gamma}$ are trivial and (3.24) simplifies to

$$
\begin{aligned}
(M_{\psi_1,\psi'_1}, M_{\psi_2,\psi'_2})_\gamma = & \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\Psi_1(A_\gamma)\overline{[W_\gamma(t_\gamma)\Psi'_1](A_\gamma)\Psi_2(A_\gamma)}[W_\gamma(t_\gamma)\Psi'_2](A_\gamma) \\
= & \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma})\Psi_1(A_\gamma)\overline{\Psi'_1(A'_\gamma)\Psi_2(A_\gamma)}\Psi'_2(A'_\gamma) \\
= & [\int_{\overline{\mathcal{A}}} d\mu_0(A)\overline{\psi_2(A)}\psi_1(A)] \;\; [\int_{\overline{\mathcal{A}}} d\mu_0(A')\overline{\psi'_1(A')}\psi'_2(A')] \\
= & \langle\psi_2\, , \, \psi_1\rangle_{\mathcal{H}_0} \;\; \langle\psi'_1\, , \, \psi'_2\rangle_{\mathcal{H}_0}
\end{aligned}
$$

that is, (3.18).
□

Proof of theorem 3.1:
Suppose that the representation π_0 of $\mathfrak{W}$ is not irreducible, that is, not every vector iscyclic. Thus, we find non zero vectors $\psi, \psi' \in \mathcal{H}_0$ such that

$$\langle\psi\, , \, a\psi'\rangle = 0 \;\; \forall \;\; a \in \mathfrak{W} \tag{3.31}$$

Since the cylindrical functions lie dense in $\mathcal{H}_0$, for any $\epsilon > 0$ we find a graph γ and functions f, f' cylindrical over γ such that

$$||\psi - f|| < \epsilon, \;\; ||\psi' - f'|| < \epsilon \tag{3.32}$$

From (3.31) we have in particular that $M_{\psi,\psi'}(t_\gamma, I_\gamma) = 0$ for all $t_\gamma \in D_\gamma, \; I_\gamma \in L_\gamma$, hence

$$
\begin{aligned}
0 \;\; = \;\; & (M_{\psi,\psi'}, M_{\psi,\psi'})_\gamma \qquad (3.33) \\
= \;\; & (M_{\psi-f,\psi'}, M_{\psi,\psi'})_\gamma + (M_{f,\psi'-f'}, M_{\psi,\psi'})_\gamma + (M_{f,f'}, M_{\psi-f,\psi'})_\gamma + (M_{f,f'}, M_{f,\psi'-f'})_\gamma + (M_{f,f'}, M_{f,f'})_\gamma \\
= \;\; & (M_{\psi-f,\psi'}, M_{\psi,\psi'})_\gamma + (M_{f,\psi'-f'}, M_{\psi,\psi'})_\gamma + (M_{f,f'}, M_{\psi-f,\psi'})_\gamma + (M_{f,f'}, M_{f,\psi'-f'})_\gamma + ||f||^2 \;\; ||f'||^2
\end{aligned}
$$

where (3.18) has been used. Exploiting $\psi, \psi' \neq 0$ we may choose $\epsilon < ||\psi||, ||\psi'||$ and using (3.32) and (3.17) we have

$$\begin{aligned}
& (||\psi|| - \epsilon)^2 \ (||\psi'|| - \epsilon)^2 \\
\leq \ & ||f||^2 \ \ ||f'||^2 \\
\leq \ & ||\psi - f|| \ \ ||\psi'|| \ \ ||\psi|| \ \ ||\psi'|| + ||f|| \ \ ||\psi' - f'|| \ ||\psi|| \ \ ||\psi'|| \\
& + ||f|| \ \ ||f'|| \ ||\psi - f|| \ \ ||\psi'|| + ||f|| \ \ ||f'|| \ \ ||f|| \ \ ||\psi' - f'|| \\
\leq \ & \epsilon\{||\psi'||^2 \ \ ||\psi|| + (||\psi|| + \epsilon) \ ||\psi|| \ \ ||\psi'|| + (||\psi|| + \epsilon) \ \ (||\psi'|| + \epsilon) \ \ ||\psi'|| + (||\psi|| + \epsilon)^2 \ \ (||\psi'|| + \epsilon)\}
\end{aligned} \tag{3.34}$$

Since this inequality holds for all ϵ we can take $\epsilon \to 0$ and find

$$||\psi||^2 \ \ ||\psi'||^2 = 0 \tag{3.35}$$

that is, either $\psi = 0$ or $\psi' = 0$ in contradiction to our assumption. Hence π_0 is irreducible. □

4 Conclusions

In this paper we have shown that the Ashtekar – Isham – Lewandowski representation π_0 of the Weyl algebra $\mathfrak{W}$ underlying diffeomorphism invariant quantum gauge field theories for compact gauge groups is irreducible. While this has been common belief, this is, to the best of our knowledge, the first time that this has been shown rigorously.

From a mathematical point of view it can be considered as an extension of the known irreducibility proofs for the Weyl algebras of quantum scalar field theories to the non-Abelean context.

From a physical point of view, irreducibility is an important concept because it makes the superselection structure of the theory trivial: There are no distinguished sectors in the Hilbert space that are left invariant and one does not need to worry about the charges that distinguishes those sectors. It has been known that the vector $1 \in \mathcal{H}_0$ is cyclic already for the subalgebra Cyl of $\mathfrak{W}$ consisting of cylindrical functions. However, that does not imply that the representation is irreducible because if there would be a non-trivial, closed, invariant subspace V then, since $\mathfrak{W}$ is closed under involution, its orthogonal complement $V^\perp$ is also closed and invariant and all we knew is that 1 could be uniquely decomposed as $1 = P_V \cdot 1 \oplus P_{V^\perp} \cdot 1$ where $P_V, \ P_{V^\perp}$ are the associated orthogonal projections.

If a theory has non-trivial closed invariant subspaces then this is typically a sign for the fact that either the algebra $\mathfrak{W}$ is too small (it has no elements that map between the sectors) or that the representation space $\mathcal{H}_0$ is too large because interesting physics can already be captured by one of its invariant subspaces [21]. In this paper we have shown that this is not the case for π_0, thus giving yet one more piece of evidence for the physical assumption that it is a suitable kinematical starting point for the quantization of diffeomorphism invariant gauge field theories, which seems to be suitable in order to define the quantum (constraint) dynamics of Quantum General Relativity (coupled to all known matter) [22].

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