Global existence problem in T^3-Gowdy symmetric IIB superstring cosmology

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Abstract

We show global existence theorems for Gowdy symmetric spacetimes with type IIB stringy matter. The areal and constant mean curvature time coordinates are used. Before coming to that, it is shown that a wave map describes evolution of this system.

1 Introduction

The strong cosmic censorship (SCC) conjecture, which states that generic initial data sets have maximal Cauchy developments which are locally inextendible as Lorentz manifolds [K, KN], is the most important and unsolved problem in classical general relativity. In order to prove this conjecture two steps are needed. The first step is: Solve global Cauchy problem of the Einstein-matter equations in a suitable time coordinate. The second is : Show inextendibility of the maximal Cauchy development. This article concerns the first step.

The SCC conjecture is entirely out of reach at the present time. A reason is that the Einstein-matter equations are nonlinear hyperbolic partial differential equations (PDEs) in some gauge and then, it is very hard to solve global Cauchy problem of them without any restrictions and/or simplifications. Such restricted and simpler systems can be obtained by looking for solutions invariant under a smooth, effective action of isometry groups. In this paper we shall consider Gowdy symmetric spacetimes which have $U(1) \times U(1)$ isometry group on compact spacelike hypersurfaces. The resulting system of PDEs is one of $(1 + 1)$ semi-linear wave equations.

It is natural to expect that global structure of cosmological spacetimes must be ultimately found on superstring/M-theory which is the most promising candidate for the unified theory including general relativity. Then, it is of prime importance to study the possible applications of superstring/M-theory to global existence problems.

In our previous work, the Einstein-Maxwell-dilaton-axion (EMDA) system, which comes from the low energy effective heterotic superstring theory, have been considered [N]. An important feature of the heterotic theory is that all terms of the fields in the effective action couple directly to the dilaton field, i.e., they consist of Neveu-Schwarz−Neveu-Schwarz (NS-NS) sector and there is no Ramond−Ramond (R-R) sector. In the recent research, it is suggested that the existence of the R-R fields influences dynamics of spacetimes [LWC]. Therefore, we would like to consider type IIB superstring theory, whose all bosonic degree of freedom consist of both NS-NS and R-R sectors. This is one motivation to consider the IIB theory.

Another motivation why we consider IIB theory is that the theory has a non-perturbative duality, that is, self-S-duality. As general assumptions, one ignores higher derivative terms (i.e. strong curvature effects) in the effective action and/or string-loop (i.e. quantum) corrections when the global structure of spacetimes is considered in superstring/M-theory, since nobody know the *complete* theory. There is no much sense in including some of higher derivative terms at the present time. Contrarily, string-loop corrections can been considered by using S-duality. It has been conjectured that the type IIB superstring theory has self-S-duality, that is, the IIB theory at coupling constant $g_s^{(\text{IIB})}$, which controls the importance

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of string-loop corrections, is equivalent to the IIB theory at coupling constant $1/g_s^{(\text{IIB})}$ in ten-dimensions [T]. This means that the IIB theory is valid in the region of spacetimes where quantum corrections for fundamental fields including gravity are needed. Then, the self-S-duality of the IIB theory is appropriate for our purpose that is to analyze spacetimes globally, i. e. from the vicinity of the singularity to the future.

In this paper global existence of solutions of the Gowdy symmetric spacetimes with type IIB stringy matter will be proved for arbitrary large initial data in the areal coordinate. The essential ingredients of the proof are light cone argument [MV] and Christodoulou-Tahvildar-Zadeh's identity [CTZ1]. Before coming to that, we need to show that a wave map is equivalent to the system of evolution equations of the Einstein-matter equations. Furthermore, we will obtain results that the spacetimes have crushing singularities into the past and the mean curvature of the foliation tends uniformly to zero into the future. Thus, global existence of constant mean curvature (CMC) foliations is shown by established arguments.

2   Gowdy symmetric spacetimes in IIB superstring theory

The low-energy limit of the type IIB superstring is given by $N = 2$, $D = 10$ chiral supergravity. The massless bosonic fields in the type IIB superstring theory come from both NS-NS and R-R sectors. The NS-NS sector has the graviton g_{MN}, an anti-symmetric tensor field $B_{MN}^{(1)}$ and the dilaton field Φ, where M, N run from 0 to 9. The R-R sector has the axion field χ, another anti-symmetric tensor field $B_{MN}^{(2)}$ and a rank four anti-symmetric tensor field D_{MNPQ} whose field strength is self-dual. Although there is no action in ten dimensions giving us the full field equations, one can write down an action giving rise to the field equations if $D_{MNPQ} \equiv 0$ [BHO]. Then we assume this condition. The action is

$$S_{IIB} = \int d^{10}x \sqrt{-\,^{10}g} \left(e^\Phi \left[-\,^{10}R - (D\Phi)^2 + \frac{1}{12}(H^{(1)})^2 \right] + \frac{1}{2}(D\chi)^2 + \frac{1}{12}(H^{(1)}\chi + H^{(2)})^2 \right), \quad (1)$$

where ^{10}R is the ten-dimensional Ricci scalar, $^{10}g := \det g_{MN}$, D is the covariant derivative of g_{MN} and $H^{(i)}$ are the field strengths of the two-form fields $B^{(i)}$. This ten-dimensional effective action can be expressed in the manifestly self-S-duality covariant form in the Einstein frame [H]. The toroidal compactification of the action to lower-dimensions, preserving self-S-duality, has been obtained [MJ]. Thus, we will suppose such compactification. Furthermore, some restrictions are imposed for simplicity. In summarize,

Assumptions

1. *The simplest toroidal ansatz*

$$ds_{10} = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + e^{\beta(x)/\sqrt{3}}\delta_{ab}dX^a dX^b,$$

   where β is a moduli field describing the volume of the internal space, μ, ν run from 0 to 3 and a, b run from 4 to 9.

2. *All fields are independent of the internal coordinate X^a.*

3. *There are no vector or moduli fields coming from the compactification of the metric or the three-forms.* □

Under the above assumptions, the reduced four-dimensional dual action in the Einstein frame

$$g_{\mu\nu} = e^{-\phi}\tilde{g}_{\mu\nu}$$

is given by [LWC]

$$S_4 = \int d^4x \left[\mathcal{L}_G + \mathcal{L}_M \right] \tag{2}$$

$$\mathcal{L}_G = -\sqrt{-g}\,^4R, \tag{3}$$

$$\mathcal{L}_M = \frac{\sqrt{-g}}{2} \left[(\nabla\phi)^2 + (\nabla\beta)^2 + \frac{1}{6}e^{-2\phi}(H^{(1)})^2 + e^{\sqrt{3}\beta+\phi}(\nabla\chi)^2 + \frac{1}{6}e^{\sqrt{3}\beta-\phi}(H^{(1)}\chi + H^{(2)})^2 \right], \qquad (4)$$

where ^4R is Ricci scalar with respect to four dimensional Lorentzian metric $g_{\mu\nu}$, $g = \det g_{\mu\nu}$, ∇ is the covariant derivative with respect to $g_{\mu\nu}$ and $\phi := \Phi - \sqrt{3}\beta$ is the effective dilaton field.

In four-dimensions, we can define two pseudo-scalar fields σ_i dual to the NS-NS and R-R three-form field strength $H^{(i)}$, respectively.

$$H^{(1)\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\kappa}e^{2\phi}(\nabla_\kappa\sigma_1 - \chi\nabla_\kappa\sigma_2), \qquad (5)$$

$$H^{(2)\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\kappa}[e^{\phi-\sqrt{3}\beta}\nabla_\kappa\sigma_2 - \chi e^{2\phi}(\nabla_\kappa\sigma_1 - \chi\nabla_\kappa\sigma_2)]. \qquad (6)$$

In the dual formulation, the effective action for matter fields is as follows.

$$\mathcal{L}_M = \frac{\sqrt{-g}}{2} \left[(\nabla\phi)^2 + (\nabla\beta)^2 + e^{\sqrt{3}\beta+\phi}(\nabla\chi)^2 + e^{-\sqrt{3}\beta+\phi}(\nabla\sigma_2)^2 + e^{2\phi}(\nabla\sigma_1 - \chi\nabla\sigma_2)^2 \right]. \qquad (7)$$

As mentioned earlier, this action has self-S-dual symmetry (see appendix A).

Varying the action (2) with respect to the functions, we have the following field equations.

$$\begin{aligned}
G_{\mu\nu} &= T_{\mu\nu} \\
&= \frac{1}{2}[\nabla_\mu\phi\nabla_\nu\phi + \nabla_\mu\beta\nabla_\nu\beta + e^{\sqrt{3}\beta+\phi}\nabla_\mu\chi\nabla_\nu\chi + e^{-\sqrt{3}\beta+\phi}\nabla_\mu\sigma_2\nabla_\nu\sigma_2 \\
&\quad + e^{2\phi}(\nabla_\mu\sigma_1 - \chi\nabla_\mu\sigma_2)(\nabla_\nu\sigma_1 - \chi\nabla_\nu\sigma_2)] \\
&\quad - \frac{1}{4}g_{\mu\nu}[(\nabla\phi)^2 + (\nabla\beta)^2 + e^{\sqrt{3}\beta+\phi}(\nabla\chi)^2 + e^{-\sqrt{3}\beta+\phi}(\nabla\sigma_2)^2 + e^{2\phi}(\nabla\sigma_1 - \chi\nabla\sigma_2)^2],
\end{aligned} \qquad (8)$$

$$\nabla_\mu\nabla^\mu\phi - \frac{1}{2}e^{\sqrt{3}\beta+\phi}(\nabla\chi)^2 - \frac{1}{2}e^{-\sqrt{3}\beta+\phi}(\nabla\sigma_2)^2 - e^{2\phi}(\nabla\sigma_1 - \chi\nabla\sigma_2)^2 = 0, \qquad (9)$$

$$\nabla_\mu\nabla^\mu\beta - \frac{\sqrt{3}}{2}e^{\sqrt{3}\beta+\phi}(\nabla\chi)^2 + \frac{\sqrt{3}}{2}e^{-\sqrt{3}\beta+\phi}(\nabla\sigma_2)^2 = 0, \qquad (10)$$

$$\nabla_\mu\nabla^\mu\chi + (\sqrt{3}\nabla_\mu\beta + \nabla_\mu\phi)\nabla^\mu\chi + e^{-\sqrt{3}\beta+\phi}(\nabla^\mu\sigma_1 - \chi\nabla^\mu\sigma_2)\nabla_\mu\sigma_2 = 0, \qquad (11)$$

$$\begin{aligned}
\nabla_\mu\nabla^\mu\sigma_1 &- \nabla_\mu\chi\nabla^\mu\sigma_2 + 2\nabla_\mu\phi(\nabla^\mu\sigma_1 - \chi\nabla^\mu\sigma_2) \\
&+ \chi[\nabla^\mu\sigma_2(-\sqrt{3}\nabla_\mu\beta + \nabla_\mu\phi) - e^{\sqrt{3}\beta+\phi}\nabla_\mu\chi(\nabla^\mu\sigma_1 - \chi\nabla^\mu\sigma_2)] = 0,
\end{aligned} \qquad (12)$$

$$\nabla_\mu\nabla^\mu\sigma_2 + (-\sqrt{3}\nabla_\mu\beta + \nabla_\mu\phi)\nabla^\mu\sigma_2 - e^{\sqrt{3}\beta+\phi}[\nabla_\mu\chi(\nabla^\mu\sigma_1 - \chi\nabla^\mu\sigma_2)] = 0. \qquad (13)$$

Now, let us fix spacetimes we consider as follows.

Definition 1 [GR, CP] *Spacetimes are T^3-Gowdy symmetric spacetimes if the following conditions hold.*

1. *Compact, connected, orientable Cauchy surfaces exist.*

2. *There is an effective action of $U(1) \times U(1)$ isometry group on the Cauchy surfaces.*

3. *The twist constants associated to the $U(1) \times U(1)$ isometry group vanish.*

4. *Topology of the Cauchy surfaces is T^3.*

A metric of Gowdy symmetric spacetimes can be written by

$$ds = e^{\lambda/2} t^{-1/2}(-dt^2 + d\theta^2) + R[e^{-Z}(dy + Xdz)^2 + e^Z dz^2]. \tag{14}$$

$\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are spacelike Killing vectors and each direction of space is periodic. Then, the metric functions λ, R, Z, X and functions describing matter fields ϕ, β, χ, σ_i depend on time coordinate t and space coordinate θ and they are periodic in θ.

Under the metric (14), we have the following Lagrangean density for type IIB superstring theory.

$$\mathcal{L}_G = \frac{1}{2R}\left[\dot{R}^2 - R'^2 + 4R(\ddot{R} - R'')\right] - \frac{R}{2}\left[\frac{1}{t^2} + \ddot{\lambda} - \lambda'' + \dot{Z}^2 - Z'^2 + e^{-2Z}(\dot{X}^2 - X'^2)\right], \tag{15}$$

and

$$\mathcal{L}_M = -\frac{R}{2}(\dot{\phi}^2 - \phi'^2 + \dot{\beta}^2 - \beta'^2 \quad + \quad e^{\sqrt{3}\beta+\phi}(\dot{\chi}^2 - \chi'^2)$$
$$+ \quad e^{-\sqrt{3}\beta+\phi}(\dot{\sigma_2}^2 - \sigma_2'^2) + e^{2\phi}\{(\dot{\sigma_1} - \chi\dot{\sigma_2})^2 - (\sigma_1' - \chi\sigma_2')^2\}), \tag{16}$$

where dot and prime denote partial derivative with respect to t and θ, respectively.

Now, we must fix *time*, which should be chosen geometrically. From the Einstein-matter equations, $G_{tt} - G_{\theta\theta} = T_{tt} - T_{\theta\theta}$, we have the following one.

$$\ddot{R} - R'' = 0. \tag{17}$$

Then, we can take the *areal coordinate* globally [GR, MV], that is $R = t$, since we assume that topology of space is T^3. As one can see in the metric (14), geometrical meaning of R is the area function of the orbit of the isometry group. Under this time coordinate, the Einstein-matter equations are as follows.

Constraint equations

$$\dot{\lambda} = t[\dot{Z}^2 + Z'^2 + e^{-2Z}(\dot{X}^2 + X'^2) + 4T_{tt}], \tag{18}$$

where

$$4T_{tt} = \dot{\phi}^2 + \phi'^2 + \dot{\beta}^2 + \beta'^2 \quad + \quad e^{\sqrt{3}\beta+\phi}(\dot{\chi}^2 + \chi'^2) + e^{-\sqrt{3}\beta+\phi}(\dot{\sigma_2}^2 + \sigma_2'^2)$$
$$+ \quad e^{2\phi}[(\dot{\sigma_1} - \chi\dot{\sigma_2})^2 + (\sigma_1' - \chi\sigma_2')^2], \tag{19}$$

and

$$\lambda' = 2t[\dot{Z}Z' + e^{-2Z}\dot{X}X' + 2T_{t\theta}], \tag{20}$$

where

$$2T_{t\theta} = \dot{\phi}\phi' + \dot{\beta}\beta' + e^{\sqrt{3}\beta+\phi}\dot{\chi}\chi' + e^{-\sqrt{3}\beta+\phi}\dot{\sigma_2}\sigma_2' + e^{2\phi}(\dot{\sigma_1} - \chi\dot{\sigma_2})(\sigma_1' - \chi\sigma_2'). \tag{21}$$

Evolution equations

$$\ddot{\lambda} - \lambda'' \quad + \quad \dot{Z}^2 - Z'^2 + e^{-2Z}(\dot{X}^2 - X'^2) + \dot{\phi}^2 - \phi'^2 + \dot{\beta}^2 - \beta'^2$$
$$+ \quad e^{\sqrt{3}\beta+\phi}(\dot{\chi}^2 - \chi'^2) + e^{-\sqrt{3}\beta+\phi}(\dot{\sigma_2}^2 - \sigma_2'^2) + e^{2\phi}[(\dot{\sigma_1} - \chi\dot{\sigma_2})^2 - (\sigma_1' - \chi\sigma_2')^2] = 0, \tag{22}$$

$$\Box_G Z = -e^{-2Z}(\dot{X}^2 - X'^2), \tag{23}$$

$$\Box_G X = 2(\dot{Z}\dot{X} - Z'X'), \tag{24}$$

$$\Box_G \phi = \frac{1}{2}\left[e^{\sqrt{3}\beta+\phi}(\dot{\chi}^2 - \chi'^2) + e^{-\sqrt{3}\beta+\phi}(\dot{\sigma_2}^2 - \sigma_2'^2) + 2e^{2\phi}\{(\dot{\sigma_1} - \chi\dot{\sigma_2})^2 - (\sigma_1' - \chi\sigma_2')^2\}\right], \tag{25}$$

$$\Box_G \beta = \frac{\sqrt{3}}{2} \left[e^{\sqrt{3}\beta+\phi}(\dot{\chi}^2 - \chi'^2) - e^{-\sqrt{3}\beta+\phi}(\dot{\sigma_2}^2 - \sigma_2'^2) \right], \tag{26}$$

$$\begin{aligned}
\Box_G \sigma_1 &= (\dot{\chi}\dot{\sigma_2} - \chi'\sigma_2') - 2\{(\dot{\sigma_1} - \chi\dot{\sigma_2})\dot{\phi} - (\sigma_1' - \chi\sigma_2')\phi'\} \\
&\quad - \chi\left[(-\sqrt{3}\dot{\beta} + \dot{\phi})\dot{\sigma_2} - (-\sqrt{3}\beta' + \phi')\sigma_2' - e^{\sqrt{3}\beta+\phi}\{(\dot{\sigma_1} - \chi\dot{\sigma_2})\dot{\chi} - (\sigma_1' - \chi\sigma_2')\chi'\}\right], \tag{27}
\end{aligned}$$

$$\Box_G \sigma_2 = -(-\sqrt{3}\dot{\beta} + \dot{\phi})\dot{\sigma_2} + (-\sqrt{3}\beta' + \phi')\sigma_2' + e^{\sqrt{3}\beta+\phi}\left[(\dot{\sigma_1} - \chi\dot{\sigma_2})\dot{\chi} - (\sigma_1' - \chi\sigma_2')\chi'\right], \tag{28}$$

$$\Box_G \chi = -(\sqrt{3}\dot{\beta} + \dot{\phi})\dot{\chi} + (\sqrt{3}\beta' + \phi')\chi' - e^{-\sqrt{3}\beta+\phi}\left[(\dot{\sigma_1} - \chi\dot{\sigma_2})\dot{\sigma_2} - (\sigma_1' - \chi\sigma_2')\sigma_2'\right], \tag{29}$$

where

$$\Box_G := \frac{\partial^2}{\partial t^2} + \frac{1}{t}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \theta^2}.$$

We call the above system consisting of equations (18)-(29) T^3-*Gowdy symmetric IIB system* and T^3-*Gowdy symmetric IIB spacetimes* is defined as solutions to T^3-Gowdy symmetric IIB system.

Equation (22) can be obtained from other equations. Thus, the metric function λ is decoupled with other functions. Therefore, λ is evaluated by the constraint equations (18) and (20) after obtaining estimates of other functions.

3 Global existence theorem in the areal time

We now state a global existence theorem as follows.

Theorem 1 *Let* $(\mathcal{M}, g_{\mu\nu}, \phi, \beta, \chi, \sigma_1, \sigma_2)$ *be the maximal Cauchy development of* C^∞ *initial data for the* T^3-*Gowdy symmetric IIB system. Then* $\mathcal{M}$ *can be covered by the areal coordinate with* $t \in (0, \infty)$.

The system of evolution equations (23)-(29) is one of semi-linear wave equations. Therefore, the local existence theorem is shown by standard arguments. In order to show that solution to T^3-Gowdy symmetric IIB system can be extended to $t \to +\infty$ all that needs to be done for all finite time.

Definition 2 *Let* M *and* N *be manifolds with Lorentzian metric* ^Bg *and Riemannian metric* ^Tg, *respectively. For a map* $\Psi : M \to N$ *we define the Lagrangian*

$$L[\Psi] = \frac{1}{2}\int_M \langle d\Psi, d\Psi \rangle = \frac{1}{2}\int_M {}^Bg^{\alpha\beta}\,{}^Tg_{ab}\partial_\alpha \Psi^a \partial_\beta \Psi^b d\nu_M. \tag{30}$$

The resulting Euler-Lagrange equations will be hyperbolic and the solutions are called wave maps. *In local coordinates, the equation for* Ψ *is*

$$\Box_B \Psi^a + \Gamma^a_{bc}(\Psi)\partial_\alpha \Psi^b \partial^\alpha \Psi^c = 0, \tag{31}$$

where $\Box_B$ *is the d'Alembertian with respect to* ^Bg *and* Γ *is the Christoffel symbol of the target manifold* N.

Lemma 1 *Let* $M = \mathrm{R} \times \mathrm{T}^2$ *and* $N = \mathrm{R}^7$ *be manifolds with Lorentzian metric* ^Bg *and Riemannian metric* ^Tg, *respectively. The wave map* $\Psi : M \to N$ *is equivalent to the evolution equations (23)-(29) of the* T^3-*Gowdy symmetric IIB system. Here,*

$$^Bg = -dt^2 + d\theta^2 + t^2 d\psi^2, \quad 0 \le t, \quad 0 \le \theta, \psi \le 2\pi, \tag{32}$$

$$^Tg = {}^Tg_H \oplus {}^Tg_{HS}, \tag{33}$$

where

$$^Tg_H := dZ^2 + e^{-2Z}dX^2, \tag{34}$$

$$^Tg_{HS} := d\phi^2 + d\beta^2 + e^{2\phi}(d\sigma_1 - \chi d\sigma_2)^2 + e^{\sqrt{3}\beta+\phi}d\chi^2 + e^{-\sqrt{3}\beta+\phi}d\sigma_2^2, \tag{35}$$

and Ψ *is independent of* ψ.

Proof of lemma 1: We can verify the above lemma by direct calculation. See appendix B. □

Remark: The target manifold is direct product of a hyperbolic space, whose metric is ${}^T g_H$, and a Riemannian space, whose metric is ${}^T g_{HS}$. If one consider vacuum Gowdy spacetimes, the former hyperbolic space is used. The latter target space represents the SL(3,R)/SO(3) coset corresponding to a homogeneous symmetric space [LWC]. Note that the action (7) is invariant under global SL(3,R) transformations. This symmetry causes the target space with the metric ${}^T g_{HS}$. □

Thus, our problem has come to one of the wave map. Note that Shatah's elegant method for a wave map from a $(1+1)$-Minkowski spacetime to a Riemannian manifold [SJ] can not be applied to our problem, since our wave map is not so, although it is a $(1+1)$ PDE system. For global existence and regularity of wave maps for smooth data in $(2+1)$-Minkowski spacetime to a Riemannian manifold, general results have not been known yet [SS]. Here, a global existence theorem for T^3-Gowdy symmetric IIB spacetimes is shown by using light cone argument and a differential identity for the energy momentum tensor of wave maps.

The energy-momentum tensor $\mathcal{T}_{\alpha\beta}$ associated with the Lagrangian (30) has the form

$$\mathcal{T}_{\alpha\beta} = {}^T g_{ab}(\partial_\alpha \Psi^a \partial_\beta \Psi^b - \frac{1}{2} {}^B g_{\alpha\beta} \partial_\alpha \Psi^a \partial^\alpha \Psi^b), \tag{36}$$

where the Greek indices denote t, θ, ψ and the Roman indices denote Z, X, ϕ, β, σ_1, σ_2, χ. For wave maps, $\mathcal{T}_{\alpha\beta}$ satisfied the local conservation laws

$$ {}^B\nabla^\alpha \mathcal{T}_{\alpha\beta} = 0, \tag{37}$$

where ${}^B\nabla$ is covariant differentiation with respect to the metric ${}^B g$.

In our case, nonzero components of $\mathcal{T}_{\alpha\beta}$ are as follows, with $\langle,\rangle$ and $\|\bullet\|$ denoting the inner product and the norm with respect to ${}^T g$.

$$\mathcal{T}_{tt} = \frac{1}{2}(\|\dot\Psi\|^2 + \|\Psi'\|^2) =: \mathcal{E}, \tag{38}$$

$$\mathcal{T}_{t\theta} = \langle\dot\Psi, \Psi'\rangle =: \mathcal{F}, \tag{39}$$

$$\mathcal{T}_{\theta\theta} = \frac{1}{2}(\|\dot\Psi\|^2 + \|\Psi'\|^2) = \mathcal{T}_{tt}, \tag{40}$$

and

$$\mathcal{T}_{\psi\psi} = -\frac{t^2}{2}(-\|\dot\Psi\|^2 + \|\Psi'\|^2) =: -t^2\mathcal{G}. \tag{41}$$

Clearly, the following inequality holds:

$$|\mathcal{G}| \le \mathcal{E}. \tag{42}$$

Now let $Y = f(t,\theta)\partial_t + g(t,\theta)\partial_\theta$ be a smooth vector field on the base manifold M. Using the conservation laws (37) we obtain a differential identity [CTZ1]

$$ {}^B\nabla_\alpha(\mathcal{T}^\alpha_\beta Y^\beta) = \frac{1}{2}\mathcal{T}^{\alpha\beta}\pi_{\alpha\beta}, \tag{43}$$

where $\pi_{\alpha\beta} = \mathcal{L}_Y \, {}^B g_{\alpha\beta}$ and its nonzero components are as follows:

$$\pi_{tt} = -2\dot f, \tag{44}$$

$$\pi_{t\theta} = -f' + \dot g, \tag{45}$$

$$\pi_{\theta\theta} = 2g', \tag{46}$$

and

$$\pi_{\psi\psi} = 2tf. \tag{47}$$

Taking $f = 1$ and $g = 0$ into the equation (43), we have

$$\partial_t(t\mathcal{E}) - \partial_\theta(t\mathcal{F}) = \mathcal{G}. \tag{48}$$

Similarly, by taking $f = 0$ and $g = 1$ we obtain

$$\partial_t(t\mathcal{F}) - \partial_\theta(t\mathcal{E}) = 0. \tag{49}$$

Now we can show the following lemma.

Lemma 2 *There is a positive constant C such that*

$$\mathcal{E} \leq C(1 + 1/t^2), \quad t \in (0, \infty), \tag{50}$$

where C depends only on the initial data at $t = t_0$.

Remark: This result holds for any Riemannian target manifolds since the following proof is independent of geometry of them. □

Proof of lemma 2: Let us define
$$\tilde{\mathcal{E}} := t\mathcal{E}, \quad \tilde{\mathcal{F}} := t\mathcal{F}.$$

From equations (48) and (49)

$$\partial_\xi(\tilde{\mathcal{E}} - \tilde{\mathcal{F}}) = \mathcal{G}, \tag{51}$$

and

$$\partial_\eta(\tilde{\mathcal{E}} + \tilde{\mathcal{F}}) = \mathcal{G}, \tag{52}$$

where $\partial_\xi := \partial_t + \partial_\theta$ and $\partial_\eta := \partial_t - \partial_\theta$.

Now, we will use light cone argument for our problem. Let us consider the future direction. Integrating equations (51) and (52) along null paths starting at $(\hat{t}, \hat{\theta})$ and ending at the initial t_0-Cauchy surface, where $0 < t_0 \leq \hat{t}$, and adding these equations, we have

$$2\tilde{\mathcal{E}}(\hat{t}, \hat{\theta}) = \quad [\tilde{\mathcal{E}}(t_0, \hat{\theta} + \hat{t} - t_0) + \tilde{\mathcal{E}}(t_0, \hat{\theta} - \hat{t} + t_0) + \tilde{\mathcal{F}}(t_0, \hat{\theta} - \hat{t} + t_0) - \tilde{\mathcal{F}}(t_0, \hat{\theta} + \hat{t} - t_0)]$$
$$+ \quad \int_{t_0}^{\hat{t}} \mathcal{G}(s, \hat{\theta} + s - t_0) + \mathcal{G}(s, \hat{\theta} - s + t_0) ds. \tag{53}$$

Taking supremums over all values of θ on the both side of equation (53), we obtain

$$\sup_\theta \tilde{\mathcal{E}}(\hat{t}, \theta) \leq \sup_\theta \tilde{\mathcal{E}}(t_0, \theta) + \sup_\theta \tilde{\mathcal{F}}(t_0, \theta) + \int_{t_0}^{\hat{t}} \frac{1}{s} \sup_\theta \tilde{\mathcal{E}}(s, \theta) ds, \tag{54}$$

where inequality (42) was used. From Gronwall's lemma [SC] it follows that

$$\sup_\theta \tilde{\mathcal{E}}(\hat{t}, \theta) \leq [\sup_\theta \tilde{\mathcal{E}}(t_0, \theta) + \sup_\theta \tilde{\mathcal{F}}(t_0, \theta)] \exp\left[\int_{t_0}^{\hat{t}} \frac{1}{s} ds\right] = [\sup_\theta \tilde{\mathcal{E}}(t_0, \theta) + \sup_\theta \tilde{\mathcal{F}}(t_0, \theta)] \frac{\hat{t}}{t_0}. \tag{55}$$

Then,

$$\sup_\theta \mathcal{E}(\hat{t}, \theta) \leq C_1, \quad \forall \hat{t} \in [t_0, \infty), \tag{56}$$

where C_1 is a positive constant depending on only the initial data $t = t_0$.

We can use similar argument into the past direction. Indeed, integrating equations (51) and (52) along null paths starting at $(\hat{t}, \hat{\theta})$ and ending at the initial t_0-Cauchy surface and repeating the previous argument, we have

$$\sup_{\theta} \tilde{\mathcal{E}}(\hat{t}, \theta) \leq [\sup_{\theta} \tilde{\mathcal{E}}(t_0, \theta) + \sup_{\theta} \tilde{\mathcal{F}}(t_0, \theta)] \exp\left[\int_{\hat{t}}^{t_0} \frac{1}{s} ds\right] = [\sup_{\theta} \tilde{\mathcal{E}}(t_0, \theta) + \sup_{\theta} \tilde{\mathcal{F}}(t_0, \theta)] \frac{t_0}{\hat{t}}, \qquad (57)$$

where $\hat{t} \in (0, t_0]$. Then,

$$\sup_{\theta} \mathcal{E}(\hat{t}, \theta) \leq \frac{C_2}{\hat{t}^2}, \qquad \forall \hat{t} \in (0, t_0]. \qquad (58)$$

Thus, we have the estimate (50) and $\mathcal{E}$ is bounded on every compact interval of $(0, \infty)$. □

Proof of theorem 1: From lemma 2, we have the desired bounds on $|\dot{Z}|$, $|Z'|$, $|e^{-Z}\dot{X}|$, $|e^{-Z}X'|$, $|\dot{\phi}|$, $|\phi'|$, $|\dot{\beta}|$, $|\beta'|$, $|e^{\frac{\sqrt{3}\beta+\phi}{2}}\dot{\chi}|$, $|e^{\frac{\sqrt{3}\beta+\phi}{2}}\chi'|$, $|e^{\frac{-\sqrt{3}\beta+\phi}{2}}\dot{\sigma}_2|$, $|e^{\frac{-\sqrt{3}\beta+\phi}{2}}\sigma'_2|$, $|e^{\phi}(\dot{\sigma}_1 - \chi\dot{\sigma}_2)|$, $|e^{\phi}(\sigma'_1 - \chi\sigma'_2)|$ for all $t \in (0, \infty)$. Once we have bounds on the first derivatives of Z, ϕ and β, it follows that Z, ϕ and β are bound for all $t \in (0, \infty)$. Then, we have bounds on $\dot{X}$, X', $\dot{\chi}$, χ', $\dot{\sigma}_2$, σ'_2, $\dot{\sigma}_1 - \chi\dot{\sigma}_2$ and $\sigma'_1 - \chi\sigma'_2$. Consequently, X, χ, σ_2, $\dot{\sigma}_1$ and σ'_1 are bounded. Finally, we have boundedness on σ_1.

Next, we must show bounds on the second derivatives of the functions. There is a well-known general fact that, in order to ensure the continued existence of a solution of a system of semi-linear wave equations, it is enough to bound the first derivative pointwise. Then, we have boundedness of the higher derivatives.

By the constraint equations (18) and (20), boundedness for the function λ is also shown. Indeed, we have the following equations:

$$\dot{\lambda} = 2t\mathcal{E} \qquad \text{and} \qquad \lambda' = 2t\mathcal{F}. \qquad (59)$$

Since $\mathcal{E}$ and $\mathcal{F}$ are bounded as we have already seen, the first derivatives of λ must be bounded uniformly for all $0 < t < \infty$. Consequently, λ itself also is bounded. Concerning the second derivative of λ, we can apply the previous argument again. Then, we have uniform C^2 bounds on all of the functions of T^3-Gowdy symmetric IIB system for all $t \in (0, \infty)$.

Finally, we must demand that the function λ is compatible with the periodicity in θ. This is true if $\lambda(t, -\pi) = \lambda(t, \pi)$ over the interval of existence. Integrating equation (20) for the interval $\theta \in [-\pi, \pi]$, we have a constraint

$$\int_{-\pi}^{\pi} d\theta \lambda' = 2 \int_{-\pi}^{\pi} d\theta t\mathcal{F} = 0. \qquad (60)$$

This constraint condition need only be imposed on the initial Cauchy surface since this integral is conservation on any time interval if all other functions satisfy the periodicity condition. This fact follows from the constraint equations (59):

$$\partial_t(t\mathcal{F}) = \partial_\theta(t\mathcal{E}), \qquad (61)$$

where $\mathcal{E}$ consists of periodic functions. Thus, we have completed the proof of Theorem 1. □

4 Global existence theorem in the CMC time

It is known that global existence of CMC foliations would be shown for some Gowdy symmetric spacetimes with appropriate matter if one can get global foliations in the areal time for such spacetimes [ARR]. Indeed, we can show the existence of global CMC foliations for T^3-Gowdy symmetric IIB spacetimes. An essential point of the arguments is the existence of a crushing singularity.

Definition 3 *The spacetime singularity approached as t tends to zero is called a crushing singularity if the mean curvature of spacelike hypersurfaces blows up uniformly as $t \to 0$.*

Proposition 1 T^3-*Gowdy symmetric IIB spacetimes have crushing singularities.*

Proof of proposition 1: For the Gowdy metric (14) we can compute that

$$trK(t) = -e^{-\frac{\lambda}{4}}t^{\frac{1}{4}}\left[\frac{1}{4}\dot{\lambda} + \frac{3}{4t}\right], \tag{62}$$

where $trK(t)$ is the mean curvature of a Cauchy hypersurface Σ_t at the areal time t. The Hamiltonian constraint equation (18) is $\dot{\lambda} = 2\tilde{\mathcal{E}}$. Then, the equation (62) is

$$trK(t) = -e^{-\frac{\lambda}{4}}t^{\frac{1}{4}}\mathcal{B}_t, \tag{63}$$

where $\mathcal{B}_t := \frac{3}{4t} + \frac{1}{2}\tilde{\mathcal{E}}$. Note that

$$\inf_{\Sigma_t} \mathcal{B}_t \geq \frac{3}{4t}. \tag{64}$$

Let us assume $0 < t \leq t_0$. From the Hamiltonian constraint equation (18)

$$\int_t^{t_0} \dot{\lambda} ds = 2\int_t^{t_0} \tilde{\mathcal{E}} ds. \tag{65}$$

Then, we have

$$\exp\left[-\frac{\lambda(t)}{4}\right] = \exp\left[-\frac{\lambda(t_0)}{4} + \frac{1}{2}\int_t^{t_0} \tilde{\mathcal{E}} ds\right] \geq \exp\left[-\frac{\lambda(t_0)}{4}\right]. \tag{66}$$

Thus, we obtain the following inequality for $t \in (0, t_0]$:

$$trK(t) \leq -C_3 t^{\frac{1}{4}} \mathcal{B}_t, \tag{67}$$

where $C_3 > 0$ is a constant. From equations (64) and (67) it is shown that T^3-Gowdy symmetric IIB spacetime has a crushing singularity into the past because of

$$\sup_{\Sigma_t} trK(t) \leq -C_4 t^{\frac{1}{4}}t^{-1} = -C_4 t^{-\frac{3}{4}} \to -\infty \quad \text{as} \quad t \to 0. \tag{68}$$

$$\square$$

Theorem 2 *Let $(\mathcal{M}, g_{\mu\nu}, \phi, \beta, \chi, \sigma_1, \sigma_2)$ be the maximal Cauchy development of C^∞ initial data for the T^3-Gowdy symmetric IIB system. Then $\mathcal{M}$ can be covered by the constant mean curvature hypersurfaces with $\tau \in (-\infty, 0)$.*

Sketch of the proof of theorem 2: It was shown by Gerhardt [GC] that a neighborhood of a crushing singularity can be foliated by CMC hypersurfaces. As we have already seen by proposition 1, since the spacetime has a crushing singularity as the initial singularity, there is a CMC foliation in $D^-(\Sigma_{t_0})$. Once it is shown that the existence of a CMC hypersurface, say Σ_{τ_0}, one can prove that $D^+(\Sigma_{\tau_0})$ admits a unique, monotonic CMC foliation which covers $D^+(\Sigma_{\tau_0})$ by Lemma 2 of the previous paper of the author [N] [2]. Thus, the global existence result of CMC foliations is obtained and CMC time τ takes all values in the range $(-\infty, 0)$. $\square$

5 Comments

We have not discussed on the second step to prove the SCC, that is, inextendibility of maximal Cauchy development. We would like to comment on it. Into the future direction, we must show future completeness of any causal geodesic. The light cone argument and the CTZ's identity used to prove our

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[2]In the lemma, the EMDA system is considered. As getting a global existence theorem in the areal coordinate, the same result can be also obtained in IIB system since non-negativity of T_{tt} is used as a property of matter fields.

theorems have many applicabilities. The tools would be able to use for investigations of asymptotic behavior into the future of the spacetimes [CTZ2]. Into the past direction, we have to prove blowup of the Kretschemann invariant $\mathcal{K} := R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ as $t \to 0$. Unfortunately, there is no idea to estimate $\mathcal{K}$ at the present time. After solving these problem, we can conclude about inextendibility and then, would prove the (in)validity of the SCC conjecture for Gowdy symmetric IIB spacetimes.

Acknowledgments

A   Self-S-duality

Put $u := \frac{\phi}{2} + \frac{\sqrt{3}\beta}{2}(= \frac{\Phi}{2})$ and $v := \frac{\sqrt{3}\phi}{2} - \frac{\beta}{2}$. The action (3) takes the form

$$S_4 = \int d^4x\sqrt{-g}\left(-\,{}^4R + \frac{1}{2}\left[(\nabla u)^2 + e^{2u}(\nabla\chi)^2 + (\nabla v)^2 + e^{\sqrt{3}v}\left[e^{-u}(\nabla\sigma_2)^2 + e^u(\nabla\sigma_1 - \chi\nabla\sigma_2)^2\right]\right]\right). \tag{69}$$

Now taking

$$w := \chi + ie^{-u}, \tag{70}$$

$$\mathcal{A}_{ij} := \frac{1}{\text{Im } w}\left(\begin{array}{cc} |w|^2 & -\text{Re } w \\ -\text{Re } w & 1 \end{array}\right), \tag{71}$$

$$\mathcal{B}^i := \left(\begin{array}{c} \nabla\sigma_2 \\ \nabla\sigma_1 \end{array}\right). \tag{72}$$

Then, the action (69) is

$$S_4 = \int d^4x\sqrt{-g}\left(-\,{}^4R + \frac{1}{2}\left[\frac{|\nabla w|^2}{(\text{Im } w)^2} + e^{\sqrt{3}v}\mathcal{B}^i\mathcal{A}_{ij}\mathcal{B}^j + (\nabla v)^2\right]\right). \tag{73}$$

This action is invariant under the following SL(2,R) transformation:

$$\bar{w} = \frac{aw + b}{cw + d}, \quad \bar{g}_{\mu\nu} = g_{\mu\nu}, \quad \bar{v} = v, \quad \bar{\mathcal{B}}^i = \Lambda^i_j\mathcal{B}^j, \quad \Lambda^i_j = \left(\begin{array}{cc} d & c \\ b & a \end{array}\right), \quad ad - bc = 1. \tag{74}$$

Indeed, we can show

$$\nabla\bar{w} = \frac{\nabla w}{(cw + d)^2}, \quad \text{Im } \bar{w} = \frac{\text{Im } w}{|cw + d|^2}, \quad \text{Re } \bar{w} = \frac{ac|w|^2 + (ad + bc)\text{Re } w + bd}{|cw + d|^2}, \tag{75}$$

and

$$\mathcal{A}(\bar{w}) = (\Lambda^{-1})^T\mathcal{A}(w)\Lambda^{-1}. \tag{76}$$

In particular, when $\chi = 0$ for $a = d = 0$ and $b = -c = -1$, we have

$$\bar{\Phi} = -\Phi, \quad \bar{g}_{\mu\nu} = g_{\mu\nu}, \quad \bar{v} = v, \quad \bar{\sigma}_2 = \sigma_1, \quad \bar{\sigma}_1 = -\sigma_2. \tag{77}$$

As e^Φ implies the string coupling, the above transformation means that the strong coupling regime of the four-dimensional IIB theory is mapped onto the weak coupling one and vice versa, that is, *self-S-duality.*

B   Proof of lemma 1

It is easy to verify $\Box_B = \Box_G$. Nonzero components of the Christoffel symbol Γ of N are as follows.

$$\Gamma^Z_{XX} = e^{2Z}, \quad \Gamma^X_{ZX} = -1 = -\Gamma^{\sigma_1}_{\sigma_1\phi}, \quad \Gamma^\phi_{\sigma_1\sigma_1} = -e^{2\phi}, \quad \Gamma^\phi_{\sigma_1\sigma_2} = \chi e^{2\phi}, \quad \Gamma^\phi_{\chi\chi} = -\frac{1}{2}e^{\sqrt{3}\beta+\phi} = \Gamma^{\sigma_2}_{\sigma_1\chi},$$

$$\Gamma^\phi_{\sigma_2\sigma_2} = -\frac{1}{2}(2\chi^2 e^{2\phi} + e^{-\sqrt{3}\beta+\phi}), \quad \Gamma^\beta_{\sigma_2\sigma_2} = \frac{\sqrt{3}}{2}e^{-\sqrt{3}\beta+\phi}, \quad \Gamma^\beta_{\chi\chi} = -\frac{\sqrt{3}}{2}e^{\sqrt{3}\beta+\phi}, \quad \Gamma^{\sigma_1}_{\sigma_1\chi} = -\frac{1}{2}e^{\sqrt{3}\beta+\phi},$$

$$\Gamma^{\sigma_1}_{\sigma_2\phi} = -\frac{1}{2}\chi, \quad \Gamma^{\sigma_1}_{\sigma_2\beta} = -\frac{\sqrt{3}}{2}\chi, \quad \Gamma^{\sigma_1}_{\sigma_2\chi} = \frac{1}{2}(\chi^2 e^{\sqrt{3}\beta+\phi} - 1), \quad \Gamma^{\sigma_2}_{\sigma_2\phi} = \frac{1}{2} = \Gamma^\chi_{\chi\phi},$$

$$\Gamma^{\sigma_2}_{\sigma_2\beta} = -\frac{\sqrt{3}}{2} = -\Gamma^\chi_{\chi\beta}, \quad \Gamma^{\sigma_2}_{\sigma_2\chi} = \frac{1}{2}\chi e^{\sqrt{3}\beta+\phi}, \quad \Gamma^\chi_{\sigma_1\sigma_2} = \frac{1}{2}e^{-\sqrt{3}\beta+\phi}, \quad \Gamma^\chi_{\sigma_2\sigma_2} = -\chi e^{-\sqrt{3}\beta+\phi}.$$

From these, we can get the fact that equation (31) under conditions of lemma 1 is equivalent with the system of PDEs (23)-(29). $\Box$

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