Lectures on Branes in Curved Backgrounds

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Abstract

These lectures provide an introduction to the microscopic description of branes in curved backgrounds. After a brief reminder of the flat space theory, the basic principles and techniques of (rational) boundary conformal field theory are presented in the second lecture. The general formalism is then illustrated through a detailed discussion of branes on compact group manifolds. In the final lecture, many more recent developments are reviewed, including some results for non-compact target spaces.

Contents

1 Introduction

During the last years the study of branes has been a powerful tool to gain new insights into various string-dualities and thereby into the non-perturbative aspects of string theory. As long as the string length is much smaller than any other scale in a given problem, one can replace genuine string theory computations by a supergravity analysis. But the current problems in field and string theory challenge us more and more to go beyond this supergravity approximation. For example, *string effects* are very relevant for all realistic string compactifications with N=1 supersymmetry on the brane. When the compactification scale gets small, even simple information such as the spectrum of BPS-branes is not protected by supersymmetry and can deviate drastically (see e.g. [1, 2]) from the gravity 'predictions'. Through the AdS/CFT correspondence string effects also have direct implications on non-perturbative aspects of gauge theories with finite (small) 't Hooft coupling (see e.g. [3]) since the latter is tied up with the curvature radius of the AdS space.

String corrections to supergravity can be studied with methods of 2-dimensional conformal field theory which provide an exact construction of the string perturbation expansion. When D-branes are present, the world-surface of strings can end on them [4]. Hence, the strings' parametrization fields X live on a 2D surface Σ with boundaries. The choice of the boundary condition encodes the geometry of the brane. If we consider e.g. branes in a D-dimensional background with metric $g_{\mu\nu}(X)$, B-field $B_{\mu\nu}(X)$ and constant dilaton, the associated 2D field theory is the non-linear σ-model

$$S[X] \;=\; \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 z \, (g_{\mu\nu}(X) + B_{\mu\nu}(X)) \, \partial X^{\mu} \bar{\partial} X^{\nu} + \dots$$

where the dots stand for contributions from world-sheet fermions. This action has to be supplemented with appropriate boundary conditions on $X = X(z, \bar{z})$.

The simplest situation occurs when both g and B are constant. Then the action S is quadratic and hence we are dealing with a 2D free field theory which can be solved easily. But when the background is curved, the metric cannot be constant and we are suddenly facing the problem of constructing a 2D interacting model. This is where the powerful techniques of (boundary) conformal field theory step in.

For closed strings, much of the relevant technology had been developed more than 10 years ago (see e.g. [5, 6, 7]) following the seminal paper by Belavin, Polyakov and

Zamolodchikov [8]. World-sheet theories with boundaries, however, received only very limited attention, with a few notable exceptions [9, 10, 11, 12, 13, 14, 15, 16, 17]. The situation changed in '96 when Polchinski demonstrated that understanding branes requires to study open strings with non-trivial boundary conditions [4]. This discovery prompted rapid and beautiful new developments in boundary conformal field theory and its applications to string theory. While most of the initial work focused on branes in toroidal compactifications and orbifolds thereof, it soon became clear that dealing with less trivial Calabi-Yau compactifications would require much more sophisticated methods (see [18, 19]).

These lectures intend to provide a self-contained introduction to some of the most important ideas and results in boundary conformal field theory. We shall begin with an extensive discussion of branes in flat backgrounds, including a derivation of brane non-commutativity. This part is technically quite transparent and it will serve as a guide line during our ascend through the general structure of boundary conformal field theory. Once we have left flat space, we shall explain notions related to the *bulk theory*, analyze the importance of one-point functions and show how they can be encoded in *boundary states*. Furthermore, we derive the famous *Cardy constraint* and two important *sewing relations* for the bulk one-point functions and the boundary operator product expansions. The second part concludes by presenting a generic family of exact solutions which not only applies to a very large class of backgrounds but also turns out to be fundamental for many of the more recent generalizations. Along the way we fill in some material that helps to bridge back to flat space. In the third lecture, the whole technology is then applied to the construction of branes on group manifolds. In this example, geometric ideas match nicely with the algebraic approach of boundary conformal field theory. We will close with a short guide to further developments and to some of the existing literature.

Assuming that the reader is familiar with some basic concepts from open string theory, we will switch rather freely between string and conformal field theory terminology. Particular aspects of world-sheet or space-time supersymmetry are mostly ignored. For much of what we are about to explain, they are either irrelevant or can easily be incorporated using only standard ingredients (see e.g. [20, 21, 22, 23]). Let us also mention that there exist several nice and rather complementary lecture notes on boundary conformal

field theory [24] including applications to condensed matter theory [25] and open strings [26, 27].

Before concluding this introduction, I would like to thank the the organizers of the winter school in Utrecht for a very enjoyable meeting and the participants for their encouraging feedback and interesting questions. These lectures grew out of several other courses, e.g. at the Erwin Schrödinger Institute in Vienna and at the 2nd Lisbon School on Superstrings. I am grateful for many useful comments during those earlier events that helped to improve the presentation. Finally, I also thank Stefan Fredenhagen, Thomas Quella and Andreas Recknagel for reading these notes carefully and for all their constructive remarks.

2   Flat Backgrounds and Free Field Theory

Our aim in the first lecture is to review the microscopic theory of branes in flat backgrounds. The corresponding world-sheet theory can be solved with elementary methods (see Section 2.1). We shall use it to compute the coupling of closed strings to the brane (Section 2.2) and the scattering amplitudes of open string modes (Section 2.3). In a certain limit, the latter give rise to non-commutative Yang-Mills theory. The presentation is tailored to prepare for the construction of branes in curved backgrounds.

2.1   Solution of the world-sheet theory

The problem.   We want to study the motion of open strings in a D-dimensional Euclidean background $\mathbb{R}^D$ that is equipped with a metric $g_{\mu\nu}$ and an anti-symmetric field $B_{\mu\nu}$. For the moment we shall assume that both background fields are constant. The world-surface of an open string is parametrized by a field $X : \Sigma \to \mathbb{R}^D$. Here, Σ is the world-sheet of the string, i.e. the strip $[0, \pi] \times \mathbb{R}$ or, equivalently, the upper half plane

$$\Sigma \; = \; \{\, z \in \mathbb{C} \mid \mathrm{Im} z \geq 0 \,\} \;\; .$$

These two realizations of the world-sheet are related by the exponential map. The motion of open strings in the given background geometry (g, B) is controlled by the following quadratic action functional,

$$S(X) \; = \; \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 z \; (g_{\mu\nu} + B_{\mu\nu}) \, \partial X^\mu \bar{\partial} X^\nu \;\; . \tag{2.1}$$

It is important to notice that for constant B the world-sheet action can be re-written in the form,

$$S(X) \; = \; \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 z \, g_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu \; + \; \frac{1}{2\pi\alpha'} \int_{\mathbb{R}} du \, B_{\mu\nu} X^\mu \partial_u X^\nu$$

where the second term involving the B-field is a pure boundary term and we have used the decomposition $z = u + iv$, i.e. the coordinate u parametrizes the boundary of Σ. Hence, the B-field does not affect the dynamics in the interior of Σ, but it provides a linear background $A_\mu(X) = B_{\mu\nu} X^\nu$ to which the end-points of the open strings couple as if they were charged particles in a magnetic background. The complete description of the

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system requires to specify boundary conditions for the parametrization field X. To this end, we single out some d-dimensional hyper-plane V in $\mathbb{R}^D$ and demand

$$(\partial_u X^\mu(z,\bar{z}))^\perp_{z=\bar{z}} \;\; = \;\; 0 \tag{2.2}$$

$$(g_{\mu\nu}\partial_v X^\nu(z,\bar{z}))^\parallel_{z=\bar{z}} \;\; = \;\; (i B_{\mu\nu}\partial_u X^\nu(z,\bar{z}))^\parallel_{z=\bar{z}} \;\; . \tag{2.3}$$

The symbols $\perp$ ($\parallel$) refer to directions perpendicular (parallel) to the hyper-surface V. With the first relation, we impose Dirichlet boundary conditions in the directions perpendicular to V, thereby restricting the endpoints of open strings to move along V. In other words, with our boundary conditions we have placed a D-brane along V. For $B = 0$, the second condition reduces to usual Neumann boundary condition along V. Hence, a non-vanishing B-fields gives rise to a deformation of Neumann boundary conditions. This has some interesting effects which we shall address below.

Without restriction we can assume that the d-dimensional plane V is stretched out along the plane defined by the equations $x^a = x_0^a$, $a = d+1, \ldots, D$, where the parameters x_0^a describe the brane's transverse location. Dirichlet boundary conditions are then imposed for X^a with $a = d+1, \ldots, D$.

Solution transverse to the brane. For the transverse directions $a = d+1, \ldots, D$, the fields X^a are constructed through the following formula for the general solution of the 2-dimensional Laplace equation $\partial\bar{\partial}X^a(z,\bar{z}) = 0$ with Dirichlet boundary conditions,

$$X^a(z,\bar{z}) \;\; = \;\; x_0^a + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^a}{n} \left(z^{-n} - \bar{z}^{-n} \right) \;\; . \tag{2.4}$$

In the quantum theory, the objects α_n^a become operators obeying the following relations

$$[\,\alpha_n^a, \alpha_m^b\,] \;\; = \;\; n\, g^{ab}\, \delta_{n,-m} \;\;\; , \;\;\;\; (\alpha_n^a)^* \;\; = \;\; \alpha_{-n}^a \;\; . \tag{2.5}$$

The commutation relations for α_n^a ensure that the field X^a and its time derivative possess the usual canonical commutator. Reality of the bosonic field X^a is encoded in the behavior of α_n^a under conjugation.

The operators $\alpha_n^a, n \neq 0$, act as creation and annihilation operators on the Fock space $\mathcal{H}_0^a = \mathcal{V}_0$ which is generated by $\alpha_n^a, n < 0$, from a unique ground state $|0\rangle$ subject to the

conditions

$$\alpha_n^a \, |0\rangle \; = \; 0 \quad \text{for} \quad n > 0 \; .$$

This construction of the state space $\mathcal{H}_0^a$ along with the formula (2.4) provides the complete solution for any direction transverse to our brane V.

Before we turn to the directions along the brane, let us briefly remark that the operators α_n^a can be obtained as the Fourier modes

$$\alpha_n^a \; = \; \frac{1}{2\pi i} \int_C z^n \, J^a(z) \, dz - \frac{1}{2\pi i} \int_C \bar{z}^n \, \bar{J}^a(\bar{z}) \, d\bar{z} \; . \tag{2.6}$$

Here, C is a semi-circle in Σ centered around the point $z = 0$ and $J^a, \bar{J}^a$ denote the usual chiral currents

$$J^a(z) \; = \; i\partial X^a(z, \bar{z}) \; = \; \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^a \, z^{-n-1} \; ,$$

$$\bar{J}^a(\bar{z}) \; = \; i\bar{\partial} X^a(z, \bar{z}) \; = \; -\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^a \, \bar{z}^{-n-1} \; .$$

From these explicit formulas we read off that the currents obey

$$J^a(z) = -\bar{J}^a(\bar{z}) \tag{2.7}$$

all along the real line $z = \bar{z}$. This relation is equivalent to the Dirichlet boundary condition and it tells us that the two sets of conserved currents in our theory are identified along the real line. Hence, there is only a single set of currents living on the boundary, while there are two sets throughout the bulk of the world-sheet.

Solution along the brane. Let us now repeat the above free field theory analysis for the directions along the branes which are subject to the boundary condition (2.3). The fields $X^i, i = 1, \ldots, d$, are once more constructed using the general solution of the wave equation

$$X^i(z, \bar{z}) \; = \; \hat{x}^i - i\sqrt{\frac{\alpha'}{2}} \, \alpha_0^i \ln z\bar{z} - i\sqrt{\frac{\alpha'}{2}} \, B^i{}_j \, \alpha_0^j \ln \frac{z}{\bar{z}} +$$

$$+i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^i}{n} \left(z^{-n} + \bar{z}^{-n} \right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{B^i{}_j \alpha_n^j}{n} \left(z^{-n} - \bar{z}^{-n} \right) \tag{2.8}$$

where summation over $j = 1, \ldots, d$, is understood. In passing to the quantum theory, $\hat{x}^i, \alpha_n^i$ become operators satisfying

$$[\alpha_n^i, \alpha_m^j] = n\, G^{ij}\, \delta_{n,-m} \qquad , \qquad [\hat{x}^i, \alpha_n^j] = i\sqrt{\alpha'}\, G^{ij}\, \delta_{0,n} \quad , \tag{2.9}$$

$$[\hat{x}^i, \hat{x}^j] = i\, \Theta^{ij} \tag{2.10}$$

Furthermore, they obey the reality properties $(\hat{x}^i)^* = \hat{x}^i$ and $(a_n^i)^* = a_{-n}^i$. The commutation relations involve new structure constants G^{ij} and Θ^{ij} which are obtained from the background fields through

$$G^{ij} = \left(\frac{1}{g+B}\right)^{ij}_{\mathrm{S}} \quad , \quad \Theta^{ij} = \left(\frac{\alpha'}{g+B}\right)^{ij}_{\mathrm{A}} . \tag{2.11}$$

Here, S or A mean that the expression in brackets gets symmetrized or anti-symmetrized, respectively. Note that the matrix Θ vanishes if and only if the B-field vanishes. A non-zero Θ causes the center of mass coordinates $\hat{x}^i$ of the open string to be quantized. We shall see below that this has very interesting consequences.

Once more, the operators α_n^i act as creation and annihilation operators but now there exists a d-parameter family of ground states $|k\rangle$ which are parametrized by a momentum $k = (k_i)_{i=1,\ldots,d}$,

$$\alpha_0^i\, |k\rangle = \sqrt{\alpha'}\, G^{ij}\, k_j\, |k\rangle \ .$$

If we denote the associated Fock spaces by $\mathcal{V}_k$, the state space $\mathcal{H}^B$ for the directions along the brane can be written as a direct integral $\mathcal{H}^B = \int_k d^d k\, \mathcal{V}_k$. On this state space we can also represent the position operators $\hat{x}^i$ as simple shifts of the momentum,

$$\exp(ik_i'\hat{x}^i)\, |k\rangle = e^{\frac{i}{2}k \times k'}\, |k+k'\rangle$$

where the vector product $\times$ is defined through $k \times k' = k_i \Theta^{ij} k_j'$. The fields we shall consider below involve only exponentials of $\hat{x}^i$ and not $\hat{x}^i$ itself.

From X^i we obtain the chiral currents J^i and $\bar{J}^i$ in the same way as above

$$J^i(z) = i\partial X^i(z,\bar{z}) = \sqrt{\frac{\alpha'}{2}} \sum_n (1+B)^i{}_j \alpha_n^j z^{-n-1} \quad , \tag{2.12}$$

$$\bar{J}^i(\bar{z}) = i\bar{\partial}X^i(z,\bar{z}) = \sqrt{\frac{\alpha'}{2}} \sum_n (1-B)^i{}_j \alpha_n^j \bar{z}^{-n-1} \quad . \tag{2.13}$$

These currents obey a linear boundary condition all along the real line $z = \bar{z}$ which is equivalent to the condition (2.3),

$$J^i(z) \;=\; \left(\frac{1+B}{1-B}\right)^i_{\;j} \bar{J}^j(\bar{z}) \;=:\; \left(\Omega^B \bar{J}\right)^i (\bar{z}) \;. \tag{2.14}$$

In this case, there appears a non-trivial map Ω^B that rotates the anti-holomorphic fields before they are identified with their holomorphic counterparts. It replaces the simple sign that we found in eq. (2.7) for the directions transverse to the brane. The map Ω^B also shows up in the formula

$$\alpha^i_n \;=\; \frac{1}{2\pi i} \int_C z^n \, J^i(z) \, dz + \frac{1}{2\pi i} \int_C \bar{z}^n \, \left(\Omega^B \bar{J}\right)^i (\bar{z}) \, d\bar{z} \tag{2.15}$$

which is used to obtain the oscillators α^i_n from the local fields J^i and $\bar{J}^i$. These remarks complete our solution of the world-sheet theory.

2.2   The closed string sector

We are now prepared to discuss some of the bulk fields and their properties. After a few brief remarks on the Virasoro fields, we shall explain how to obtain the vertex operators for closed string tachyons and compute their couplings to the brane along V. These couplings encode all information about the density distribution of the brane.

The Virasoro field.   Along with the chiral currents J^μ and $\bar{J}_\mu$, there exists another very important pair of chiral fields, namely the Virasoro fields $T, \overline{T}$. The holomorphic field T is obtained from the chiral currents J^μ (the index μ is taken from $\mu = 1, \ldots, D$) through the prescription

$$T(z) \;:=\; \frac{1}{\alpha'} \sum g_{\mu\nu} :J^\mu J^\nu : (z) \;:=\; \frac{1}{\alpha'} \lim_{w \to z} \left(g_{\mu\nu} J^\mu(w) J^\nu(z) - \frac{\alpha'}{2} \frac{D}{(w-z)^2} \right) \;.$$

Here, we use the limiting procedure on the right hand side to define the conformal normal ordering $: \cdot :$. For the anti-holomorphic partner $\overline{T}$ we employ the same construction with currents $\bar{J}^\mu$ instead of J^μ. The boundary conditions for the chiral currents (2.7),(2.14) imply that the two Virasoro fields coincide along the boundary $z = \bar{z}$,

$$T(z) \;=\; \overline{T}(\bar{z}) \;. \tag{2.16}$$

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Such a relation can be seen to prevent world-sheet momentum from leaking out across the boundary of Σ. Technically, it allows us to construct the following modes

$$L_n := \frac{1}{2\pi i} \int_C z^{n+1} T(z) \, dz - \frac{1}{2\pi i} \int_C \bar{z}^{n+1} \overline{T}(\bar{z}) \, d\bar{z} \quad . \tag{2.17}$$

The elements L_n generate one copy of the Virasoro algebra with central charge $c = D$. They can also be expressed through the oscillators α_n^μ,

$$L_n = \frac{1}{2} \sum_{n=-\infty}^{\infty} G_{ij} \, {}^{\circ}_{\circ} \alpha_{m-n}^i \alpha_n^j \, {}^{\circ}_{\circ} + \frac{1}{2} \sum_{n=-\infty}^{\infty} g_{ab} \, {}^{\circ}_{\circ} \alpha_{m-n}^a \alpha_n^b \, {}^{\circ}_{\circ} \tag{2.18}$$

where G with lower indices denotes the inverse of (G^{ij}) and it appears in the expression for L_n because of the factors $(1 \pm B)$ in eqs. (2.12),(2.13). The symbol ${}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}$ stands for operator normal ordering, i.e. it instructs us to move all the annihilation operators $\alpha_n^\mu, n \geq 0$, to the right of the creation operators $\alpha_n^\mu, n < 0$.

One-point functions. Let us now turn to the family of bulk fields that are associated with closed string tachyons. These are defined by the expression

$$\phi_{k,k}(z, \bar{z}) := \, :e^{ikX(z,\bar{z})}: = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \, :X^n: (z, \bar{z}) \quad . \tag{2.19}$$

Here $k = (k_\mu)$ and we have suppressed all indices on the fields X and the momenta k. Furthermore, we extended the prescription $: \cdot :$ for the conformal normal ordering to arbitrary powers of the bosonic field X. The nth-order normal ordered product is defined recursively by

$$:X^{\mu_1}(z_1, \bar{z}_1) \cdots X^{\mu_n}(z_n, \bar{z}_n): = \, X^{\mu_1}(z_1, \bar{z}_1) \cdots X^{\mu_n}(z_n, \bar{z}_n) + \sum \text{subtractions}$$

where the sum runs over all ways of choosing pairs of fields from the product and replacing them by (-1) times the free propagator, i.e.

$$X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) \longrightarrow \, -\langle X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) \rangle = -\alpha' \, g^{\mu\nu} \ln |z_1 - z_2| \quad .$$

For more details see e.g. [22]. Note that the normal ordering prescription for bulk fields is the same as on the full complex plane because it uses the propagator of the free bosonic

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field on the full complex plane. This differs from the propagator on the upper half plane

$$
\begin{aligned}
\langle X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) \rangle_B \;=\; & -\alpha'\, g^{\mu\nu} \ln|z_1 - z_2| + \alpha'\, g^{\mu\nu} \ln|z_1 - \bar{z}_2| \\
& -\alpha'\, G^{\mu\nu} \ln|z_1 - \bar{z}_2|^2 - \frac{\Theta^{\mu\nu}}{2\pi} \ln \frac{z_1 - \bar{z}_2}{\bar{z}_1 - z_2}
\end{aligned}
\tag{2.20}
$$

by terms which are regular in the upper half plane $\mathrm{Im}\, z > 0$ and become singular only along the boundary. In writing down eq. (2.20), we promoted G and Θ to $D \times D$-matrices such that all new elements vanish.

After these remarks it is easy to rewrite the bulk field $\phi_{k,k}(z, \bar{z})$ in terms of operator normal ordering, i.e. such that annihilation operators stand to the right of the creation operators,

$$
\phi_{k,k}(z, \bar{z}) \;:=\; \frac{1}{|z - \bar{z}|^{\alpha'(k^2 - 2k \cdot k)}} \; {}^{\circ}_{\circ} e^{ikX(z,\bar{z})} {}^{\circ}_{\circ} \;.
\tag{2.21}
$$

In the exponent we use $k \cdot k = G^{ij} k_i k_j$ and $k^2 = g^{\mu\nu} k_\nu k_\mu$. The singularity is related to the second and third term in the propagator (2.20). In the operator normal ordering we agree to treat $\hat{x}^\mu$ as a creation operator, i.e. we move it to the left. Now that the tachyon vertex operators are well defined and conveniently expressed through the operator normal ordering, we can compute all their correlation functions and in particular the one-point function on the upper half plane. With our brane localized along $V = \{x^a = x_0^a\}$ these one-point functions are given by

$$
\langle \phi_{k,k}(z, \bar{z}) \rangle_{x_0} \;=\; \delta^{(d)}(k_i)\, \frac{e^{ik_a x_0^a}}{|z - \bar{z}|^{\alpha' k^2}} \;.
\tag{2.22}
$$

From the set of all these one-point functions we can recover the parameters x_0^a, i.e. the brane's transverse position is completely specified by the one-point function of the bulk tachyon vertex operators. We shall see that a similar statement remains true for branes in curved backgrounds.

It will be useful to understand how a density distribution of the brane can be read off from the one-point functions. In string theory, one-point functions of bulk fields describe how closed string modes couple to the D-brane. Our formula (2.22) implies that the coupling is completely delocalized in the directions of momentum space that are normal to the brane. Hence, after Fourier transformation, closed strings are seen to couple to

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some object that it localized along $x^a = x_0^a$ in position space. This is just the location of our brane in the background. In formulas we find

$$\lim_{\alpha' \to 0} \langle \phi_{k,k}(z, \bar{z}) \rangle_{x_0} = \delta^{(d)}(k_i)\, e^{ik_a x_0^a} \sim \int d^D x\, \delta^{(D-d)}(x^a - x_0^a)\, \phi_{k,k}(x) \qquad (2.23)$$

where $\phi_{k,k}(x) = \exp(ik_\mu x^\mu)$ is the wave function of a scalar particle moving in $\mathbb{R}^D$ with momentum k. The first factor $\delta^{(D-d)}(x^a - x_0^a)$ in the integrand is interpreted as the density distribution of the brane.

2.3   The open string sector

After our discussion of bulk fields we now turn to a new set of fields which can be inserted at points on the boundary of the world-sheet. Such boundary fields are associated to the modes of open strings on the brane. We will briefly talk about their spectrum before we compute correlators of the tachyon vertex operators. The results of these computations can be expressed with the help of the non-commutative Moyal-Weyl product. We review the latter to make this presentation self-contained. Finally, we shall argue that - in a certain decoupling limit - the scattering amplitudes of massless open string modes can be reproduced by the so-called non-commutative Yang-Mills theory.

Spectrum of boundary fields.   Boundary fields are in one-to-one correspondence with states of the boundary theory. The space of these states was constructed explicitly when we solved the model in the first subsection. We remind the reader that it is given by

$$\mathcal{H}_{(B,d)} = \int d^d k\, \mathcal{V}_k \otimes \mathcal{V}_0^{\otimes D-d} \quad .$$

Here the integral over momenta came with the directions along the brane while the $D - d$ factors $\mathcal{V}_0$ are associated with the transverse space.

$\mathcal{H}_{(B,d)}$ is the space on which all our bulk and boundary fields act. In particular, through eq. (2.17), it carries an action of the Virasoro algebra. Among the Virasoro modes, L_0 is distinguished because it agrees with the Hamiltonian of the world-sheet theory up to a simple shift, i.e. $H = L_0 - D/24$. Using the explicit formula (2.18) for L_0 it is rather easy to calculate the partition function of the theory,

$$Z_{(B,d)}(q) := \operatorname{tr}_\mathcal{H}\left(q^H\right) = \frac{1}{\eta^D(q)} \int d^d k\, q^{\frac{\alpha' k \cdot k}{2}} \qquad (2.24)$$

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where $\eta(q) = q^{1/24} \prod(1 - q^n)$ is Dedekind's η-function. The factor $1/\eta^D(q)$ is associated with the oscillations of the bosonic string in the D-dimensional flat background. In addition, open strings can move along the brane and this motion gives rise to the integral over the d-dimensional center of mass momentum $k^\parallel$. The term $\alpha' k \cdot k/2$ in the exponent is the kinetic energy of a particle moving in d-dimensional flat space with metric G. Our discussion here shows that the partition function Z is an important quantity containing quite detailed information about the boundary condition.

The Weyl product. Before we start discussing open string scattering amplitudes, we want to recall some elementary mathematical results on the quantization of a very simple classical system. It consists of a d-dimensional linear space V along with a constant antisymmetric $d \times d$ matrix Θ^{ij}. The latter defines a Poisson bracket for functions on V. When evaluated on the coordinate functions $x^j, j = 1, \ldots, d$, the Poisson structure reads,

$$\{\, x^i \,,\, x^j \,\} \;=\; \Theta^{ij} \;\;. \tag{2.25}$$

Quantization means to associate a self-adjoint operator $\hat{x}^j : \mathcal{H} \to \mathcal{H}$ on some state space $\mathcal{H}$ to each coordinate function such that

$$[\, \hat{x}^i \,,\, \hat{x}^j \,] \;=\; i\, \Theta^{ij} \;\;. \tag{2.26}$$

More generally, one would like to associate a self-adjoint operator $\mathrm{F} \;=\; Q(f)$ to any real valued function f on V such the commutator $[\mathrm{F}_1, \mathrm{F}_2]$ is approximated by the Poisson bracket $\{f_1, f_2\}$ in a sense that we shall make more precise below. An appropriate mapping $f \to \mathrm{F} = Q(f)$ was suggested by Weyl [28],

$$\mathrm{F} \;=\; Q(f) = \int d^d k \, \hat{f}(k) \; \exp(ik_j \hat{x}^j)$$

where $\hat{f}(k)$ denotes the Fourier transform of f. The operator F is trace class, if f is smooth and decreases, together with all its derivatives, faster than the reciprocal of any polynomial at infinity. A detailed discussion of appropriate spaces of functions can be found e.g. in [29].

We want to compute the product of any two operators $Q(f)$ and $Q(g)$ and compare this to the operator which Weyl's formula assigns to the Poisson bracket of the two functions

f and g. Using the famous Baker-Campbell-Hausdorff formula one finds the following auxiliary result for the product of two exponentials

$$\exp(ik_i\hat{x}^i)\,\exp(ik_j'\hat{x}^j) \;=\; \exp(-\frac{i}{2}k_i\Theta^{ij}k_j')\,\exp(i(k+k')_i\hat{x}^i) \quad. \tag{2.27}$$

As one can show by a short computation, this formula implies that the product of two operators $Q(f)$ and $Q(g)$ is given by

$$Q(f)\,Q(g) \;=\; Q(f*g) \qquad \text{where}$$

$$f*g\,(x) \;=\; \exp(-\frac{i}{2}\Theta^{\mu\nu}\partial_\mu\tilde{\partial}_\nu)\,f(x)g(\tilde{x})|_{\tilde{x}=x} \quad. \tag{2.28}$$

The multiplication $*$ defined in the second row is known as the *Moyal product* [30] associated with the constant anti-symmetric matrix Θ. It is an associative and non-commutative product for functions on the d-dimensional space V. Moreover, to leading order in the number of derivatives, one finds

$$[f\stackrel{*}{,}g] \;:=\; f*g - g*f \;=\; -i\Theta^{\mu\nu}\,\partial_\mu f\partial_\nu g + \dots \;=\; -i\{f,g\} + \dots \quad.$$

Hence, the Moyal-commutator of the functions f and g is approximated by the Poisson bracket of these functions. In the same sense, the commutator of the operators $Q(f)$ and $Q(g)$ is approximated by the operator $Q(\{f,g\})$.

Correlation functions. Following the standard wisdom of conformal field theory, there is a boundary field associated with each state in the space $\mathcal{H}_{(B,d)}$. For the ground states $|k\rangle$, the corresponding fields are the 'open string tachyon vertex operators',

$$\psi_k(u) \;:=\; {}^\circ_\circ e^{ik_iX^i(u)}{}^\circ_\circ \;=\; e^{ik_iX^i_<(u)}\,e^{ik_iX^i_>(u)}$$

where

$$X^\mu_>(u) \;=\; -i\sqrt{2\alpha'}\,\alpha_0^\mu\,\ln u + i\sqrt{2\alpha'}\sum_{n>0}\frac{\alpha_n^\mu}{n}u^{-n} \quad, \tag{2.29}$$

$$X^\mu_<(u) \;=\; \hat{x}^\mu + i\sqrt{2\alpha'}\sum_{n<0}\frac{\alpha_n^\mu}{n}u^{-n} \quad. \tag{2.30}$$

From our exact construction of the theory it is rather straightforward now to compute all the correlation functions of these tachyonic vertex operators. Before we present the result

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of this computation, let us introduce the *decoupling limit* of a functional $F(\alpha'; g, B)$. It is defined by [31]

$$F^{DL}(g, B) = \lim_{\epsilon \to 0} F(\epsilon; g\epsilon^2, B\epsilon) .$$

We shall explain the idea behind this limit at the end of the section. For the moment, let us return to the correlators we were about to compute. They can be evaluated easily with the help of the Baker-Campbell-Hausdorff formula,

$$\langle \psi_{k_1}(u_1) \cdots \psi_{k_n}(u_n) \rangle = \prod_{r<s} e^{-\frac{i}{2} k_r \times k_s} \frac{\delta(\sum_r k_r)}{|u_r - u_s|^{\alpha' k_r \cdot k_s}}$$

$$\overset{DL}{\longrightarrow} \prod_{r<s} e^{-\frac{i}{2} k_r \times k_s} \delta(\sum_r k_r) . \tag{2.31}$$

Here, $r, s = 1, \ldots, n$, and we have used the notation $k \times k' = k_i \Theta^{ij} k'_j$, as before. Note that the phase factors that appear when we evaluate the correlation function of the exponential field are identical to the phase factors we encountered in multiplying two exponential functions using the Moyal-Weyl product (see eq. (2.27)). In the decoupling limit, $\alpha' G^{ij}$ vanishes and hence we are left with the phase factors and a δ function that enforces momentum conservation. Hence, in this limit of the theory, the correlation functions are determined entirely by the Moyal-Weyl product, i.e.

$$\langle \psi_{k_1}(u_1) \cdots \psi_{k_n}(u_n) \rangle^{DL} = \int_V d^d x \, e_{k_1} * \cdots * e_{k_n}$$

where $e_k = \exp(ik_j x^j)$ is the exponential function. If we introduce the fields $\psi[f](u)$ by

$$\psi[f](u) = \int d^d k \, \hat{f}(k) \, \psi_k(u)$$

then the result can be restated as follows

$$\langle \psi[f_1](u_1) \cdots \psi[f_n](u_n) \rangle^{DL} = \int_V d^d x \, f_1 * \cdots * f_n . \tag{2.32}$$

In conclusion, the decoupling limit of all boundary correlation 'functions' is independent of the world-sheet coordinates u_r and its value can be computed from the non-commutative Moyal-Weyl product associated with the anti-symmetric tensor Θ. A related observation was made by several authors [32, 33, 34, 35, 36]. The formulation we have presented here was found in [36].

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Our results have a rather nice physical explanation. Recall that the action functional (2.1) for open strings has two terms: to begin with there is a boundary term which describes the motion of the charged open string ends in a magnetic field. It is well known that the coordinates of charged particles in a magnetic background have a non-vanishing Poisson bracket and hence they do not commute after quantization. To reach such a particle limit, we have to suppress the string oscillations, i.e we have to send α' to zero. The B-field should be scaled down at the same rate so that B/α' remains constant. But even in this limiting regime the resulting theory for the string endpoints does not approach the theory for charged particles in a magnetic field because of the bulk term in the action (2.1) for open strings. This term makes the open string ends remember that they are attached to a string which becomes very stiff as we try to turn off the oscillations. Consequently, the open string ends dissipate energy into these tails and the strength of this dissipation is given by g/α' (see also [37]). If we want to suppress this effect, the closed string metric g (more precisely its components g_{ij} along the brane) has to vanish at a faster rate than α'. All this in achieved by the decoupling limit we defined above. It also ensures that the open string metric G remains finite.

Non-commutative Yang-Mills theory. In a supersymmetric string vacuum, the scalar tachyon for which our previous discussion was most relevant does not arise but there appears a massless vector field associated with the 2D boundary fields

$$\substack{\circ\\\circ} J^\mu(u) \exp(ik_i X^i(u)) \substack{\circ\\\circ} \;=\; J^\mu_<(u) \substack{\circ\\\circ} \exp(ik_i X^i(u)) \substack{\circ\\\circ} + \substack{\circ\\\circ} \exp(ik_i X^i(u)) \substack{\circ\\\circ} J^\mu_>(u) \;\;. \qquad (2.33)$$

Once more, normal ordering for these boundary operator means to move all the annihilation operators to the right of the creation operators. Consequently, $J_>(u)$ is defined by

$$J^\mu_>(u) \;=\; \sum_{n>-1} \alpha^\mu_n u^{-n-1} \quad \text{and} \quad J^\mu_< \;=\; J^\mu - J^\mu_> \;=\; \sum_{n\leq-1} \alpha^\mu_n u^{-n-1} \;\;.$$

To compute the correlation functions of the above fields in the decoupling limit, we proceed in two steps. The first one is to remove all the currents from the correlators with the help of Ward identities. Once we are left with correlation functions of the exponential fields, we can then use the results from the discussion above. Since we are only interested in

the decoupling limit of the correlation function, we can drop sub-leading terms whenever they arise in the computation.

Let us discuss this in some more detail. Using a bit of algebra, it is not difficult to derive that

$$(J^\mu(u_1)J^\nu(u_2))_{\text{sing}} \ := \ [\,J^\mu_>(u_1), J^\nu(u_2)\,] \ = \ \frac{\alpha'}{2}\frac{G^{\mu\nu}}{(u_1 - u_2)^2} \tag{2.34}$$

$$(J^\mu(u_1)\psi_k(u_2))_{\text{sing}} \ := \ [\,J^\mu_>(u_1), \psi_k(u_2)\,] \ = \ \frac{\alpha'}{2}\frac{G^{\mu i}k_i}{u_1 - u_2}\psi_k(u_2) \ . \tag{2.35}$$

The commutators we have listed here compute the singular part of the corresponding operator product expansions. This is indicated in the notation on the left hand side. The two commutation relations along with the properties $J^\mu_>(u)|0\rangle = 0$ and $\langle 0|J^\mu_<(u) = 0$ of the vacuum can be used to remove all currents from an arbitrary correlator. In order to do so, we commute the lowering term $J^\mu_>$ of each current insertion to the right until it meets the vacuum and similarly the raising terms are all pushed to the left. The commutation relations give rise to two different contributions. If one currents hits another, both of them disappear from the correlator and we obtain a factor of $\alpha'G^{\mu\nu}$. Commuting a current through an exponential, on the other hand, removes only one current insertion and furnishes a factor $\alpha'G^{\mu i}k_i$ instead. Since both factors are of the same order in α', the leading contribution to an n-point function is obtained when we contract as many pairs of currents as possible, i.e. $n/2$ for even n and $(n-1)/2$ for odd n. In the latter case, the last current will necessarily lead to a linear dependence on the momentum k.

It follows from these general remarks on the evaluation of n-point functions that the three- and four-point functions both contain terms which are second order in α'. All higher correlators are subleading. While the dominant contributions to the three-point function contain a factor linear in the momenta k, the corresponding factor for the four-point function is independent of k. Both these factors are finally multiplied with correlators (2.31) of the exponential fields. The latter certainly introduce a strong k dependence whenever there is a non-vanishing B-field.

Our final task now is to interpret the resulting expressions for correlators as vertices of some effective low-energy field theory on the brane. The structure of the terms we have just outlined shares all the essential features with the vertices in Yang-Mills theory. For

instance, the three gluon vertex is linear in the external momenta just as the factor we have talked about in our evaluation of the three-point function. The non-vanishing B-field, however, causes all the vertices to be multiplied by momentum dependent phase factors. Hence, after Fourier transformation, we expect the fields in the effective field theory to be multiplied with the Moyal-product rather than the ordinary point-wise multiplication.

Though our arguments have been a bit sketchy, all our conclusions can be confirmed by an exact computation. Taking the physical state conditions into account, the effective action for a stack of M branes is indeed given by the Yang-Mills action for fields $A_\mu \in \mathrm{Mat}_M(Fun(\mathbb{R}^D))$ on a non-commutative $\mathbb{R}^D$ [31],

$$\mathcal{S}_N(A) \;=\; \frac{1}{4} \int d^D x \; \mathrm{tr} \; (F_{\mu\nu} * F^{\mu\nu})$$

$$\text{where} \quad F_{\mu\nu}(A) \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu \overset{*}{,} A_\nu]$$

and with $*$ being the Moyal product as before. The integration extends over the world-volume of the brane. In the formula we suppressed fermionic contributions (which appear for superstring theories) and we assumed that $D = d$. The action for lower dimensional branes is obtained by dimensional reduction. One can easily see that this non-commutative Yang-Mills theory is invariant under the following gauge transformations

$$A_\mu \;\longrightarrow\; A_\mu + \partial_\mu \lambda + i[A^\mu \overset{*}{,} \lambda]$$

for $\lambda \in \mathrm{Mat}_M(Fun(\mathbb{R}^D))$. The relation between branes in flat space and non-commutative geometry has been the main motivation during the last years to study non-commutative field theories. In particular, there has been significant progress in constructing their classical solutions (see e.g. [38, 39, 40, 41, 42, 43]). The latter allow for an interpretation as condensates on branes. We shall come back to related issues in the third lecture when we analyse branes on a 3-sphere. For an overview over many of the recent developments in this field and other aspects that we have not touched, we recommend e.g. [31, 44, 29, 45, 46] and references therein.

3 2D Boundary conformal field theory

We now want to extend the microscopic formalism to branes in general backgrounds. The extension relies heavily on methods and ideas from conformal field theory. After a brief review of some basic concepts from bulk conformal field theory, we explain how branes can be described through boundary conformal field theory. In particular we shall argue that they are uniquely characterized by the way in which they couple to closed string modes or, in terms of the world-sheet theory, by the one-point functions of bulk fields. Using world-sheet duality it is then possible to determine the corresponding open string spectra. Along the way we shall derive a number of algebraic relations for the couplings of closed strings to the brane and the interaction of open strings. Universal solutions of these relations are the subject of the last subsection.

3.1 Some background from CFT

In an attempt to make these lectures self-contained, we shall begin our discussion with some more or less well known material on conformal field theory (CFT). Readers who have been exposed to CFT may skip most of this subsection where we present some notations and the basic data of the bulk theory. These include the space of bulk fields, the bulk partition function and the operator product expansion. In the second subsection we collect some background material on chiral algebras which arise as symmetries of 2D bulk and boundary CFTs.

3.1.1 The bulk theory.

Bulk conformal field theories, i.e. 2D CFTs defined on the full complex plane, appear in the world-sheet desription of closed strings. Their state spaces $\mathcal{H}^{(P)}$ contain all the closed string modes and the coefficients $C = C^{(P)}$ of their operator product expansions encode closed string interactions. There exist many non-trivial examples of 2D conformal field theories, and hence of exactly solvable string backgrounds. These are constructed with the help of certain infinite dimensional symmetries known as chiral or W-algebras.

Bulk fields and bulk OPE. All constructions of boundary conformal field theories start from the data of a usual conformal field theory on the complex plane which we shall

refer to as *bulk theory*. It consists of a space $\mathcal{H}^{(P)}$ of states equipped with the action of a Hamiltonian $H^{(P)}$ and of field operators $\varphi(z, \bar{z})$. According to the famous state-field correspondence, the latter can be labeled by elements in the state space $\mathcal{H}^{(P)}$,

$$\varphi(z, \bar{z}) \;=\; \Phi^{(P)}(|\varphi\rangle; z, \bar{z}) \quad \text{for all} \quad |\varphi\rangle \in \mathcal{H}^{(P)} \;. \tag{3.1}$$

The reverse relation is given by $\varphi(0,0)|0\rangle = |\varphi\rangle$ where $|0\rangle$ denotes the unique vacuum state in the state space $\mathcal{H}^{(P)}$ of the bulk theory.

Among the fields of a CFT one distinguishes so-called *chiral fields* which depend on only one of the coordinates z or $\bar{z}$ so that they are either holomorphic, $W = W(z)$, or anti-holomorphic, $\overline{W} = \overline{W}(\bar{z})$. The most important of these chiral fields, the Virasoro fields $T(z)$ and $\overline{T}(z)$, come with the stress tensor and hence they are present in any CFT. But in most models there exist further (anti-)holomorphic fields whose Laurent modes W_n and $\overline{W}_n$ defined through

$$W(z) \;=\; \sum W_n \, z^{-n-h} \quad , \quad \overline{W}(\bar{z}) \;=\; \sum \overline{W}_n \, \bar{z}^{-n-\bar{h}} \quad , \tag{3.2}$$

generate two commuting *chiral algebras*, $\mathcal{W}$ and $\overline{\mathcal{W}}$. The numbers h and $\bar{h}$ are the (half-)integer conformal weights (scaling dimension) of W and $\overline{W}$. Throughout this text we shall assume the two chiral algebras $\mathcal{W}$ and $\overline{\mathcal{W}}$ to be isomorphic. We encountered an example of such a chiral algebra in the first lecture. There it was generated by the Laurent modes α_n^μ and $\bar{\alpha}_n^\mu$ of the currents $J(z)$ and $\bar{J}(\bar{z})$.

In general, the state space $\mathcal{H}^{(P)}$ of the bulk theory admits a decomposition into irreducible representations $\mathcal{V}_i$ and $\overline{\mathcal{V}}_{\bar{\imath}}$ of the two commuting chiral algebras,

$$\mathcal{H}^{(P)} \;=\; \bigoplus_{i,\bar{\imath}} n_{i\bar{\imath}}^{(P)} \, \mathcal{V}_i \otimes \overline{\mathcal{V}}_{\bar{\imath}} \;. \tag{3.3}$$

For simplicity we shall assume that the multiplicities satisfy $n_{i\bar{\imath}}^{(P)} \in \{0,1\}$. Higher multiplicities can easily be incorporated but they would require additional indices. We reserve the label $i = 0$ for the vacuum representation $\mathcal{V}_0$ of the chiral algebra. It is mapped to $\mathcal{W}$ via the state-field correspondence $\Phi^{(P)}$. As suggested by our notation, the set of representations often turns out to be discrete. This is in contrast to the situation we met in our discussion of strings in $\mathbb{R}^D$ where i ran over the continuum of closed string momenta.

The discrete sum in eq. (3.3) signals that our analysis focuses primarily on compact backgrounds, even though some of the general ideas apply to non-compact situations as well (see last Section).

Each irreducible representation $\mathcal{V}_i$ of $\mathcal{W}$ acquires an integer grading under the action of the Virasoro mode L_0 and hence it may be decomposed as $\mathcal{V}_i = \bigoplus_{n \geq 0} V_i^n$. The subspace V_i^0 of ground states in $\mathcal{V}_i$ carries an irreducible action of all the zero modes W_0. We will denote the corresponding linear maps by X_W^i,

$$X_W^i \ := \ W_0 \mid_{V_i^0} : V_i^0 \ \longrightarrow \ V_i^0 \quad \text{for all chiral fields } W \ . \tag{3.4}$$

The whole irreducible representation $\mathcal{V}^i$ may be recovered from the elements of the finite-dimensional subspace V_i^0 by acting with W_n, $n < 0$.

Using the state-field correspondence $\Phi^{(P)}$, we can assign fields to all states in $V_i^0 \otimes \bar{V}_{\bar{i}}^0$. We shall assemble them into a single object which one can regard as a matrix of fields after choosing some basis $|e_i^a\rangle \otimes |\bar{e}_{\bar{i}}^b\rangle \in V_i^0 \otimes \bar{V}_{\bar{i}}^0$,

$$\varphi_{i,\bar{i}}(z, \bar{z}) \ := \ \left(\Phi^{(P)}(|e_i^a\rangle \otimes |\bar{e}_{\bar{i}}^b\rangle; z, \bar{z}) \right) \ . \tag{3.5}$$

The matrix elements are labeled by $a = 1, \ldots, \dim V_i^0$ and $b = 1, \ldots, \dim \bar{V}_{\bar{i}}^0$. We shall refer to these field multiplets as closed string vertex operators or *primary fields*. All other fields in the theory can be obtained by multiplying with chiral fields and their derivatives.

So far, we have merely talked about the space of bulk fields. But more data are needed to characterize a closed string background. These are encoded in the short distance singularities of correlation functions or, equivalently, in the structure constants of the operator product expansions

$$\varphi_{i,\bar{i}}(z_1, \bar{z}_1) \varphi_{j,\bar{j}}(z_2, \bar{z}_2) \ = \ \sum_{n\bar{n}} C_{i\bar{i};j\bar{j}}^{n\bar{n}} z_{12}^{h_n - h_i - h_j} \bar{z}_{12}^{\bar{h}_n - \bar{h}_i - \bar{h}_j} \varphi_{n,\bar{n}}(z_2, \bar{z}_2) + \ldots \ . \tag{3.6}$$

Here, $z_{12} = z_1 - z_2$ and $h_i, \bar{h}_{\bar{i}}$ denote the conformal weights of the field $\varphi_{i,\bar{i}}$, i.e. the values of L_0 and $\bar{L}_0$ on $V_i^0 \otimes \bar{V}_i^0$. The numbers C describe the scattering amplitude for two closed string modes combining into a single one ("pant diagram").

Together, the state space (3.3) and the set of couplings C in rel. (3.6) can be shown to specify the bulk theory completely. We shall assume that these data are given to us, i.e. that the closed string background has been solved already.

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Example: The free boson. Our use of the state field correspondence Φ may seem a bit formal at first, but it can clarify some conceptual issues and even simplifies many equations later on. Since it certainly takes time to get used to this rather abstract formalism, let us pause for a moment and illustrate the general concepts in the example of a single free boson. In this case, the state space of the bulk theory is given by

$$\mathcal{H}^{(P)} = \int dk \, \mathcal{V}_k \otimes \bar{\mathcal{V}}_k \quad . \tag{3.7}$$

As long as we do not compactify the theory, there is a continuum of sectors parametrized by $i = k = \bar{\imath}$. In the first lectures we have discussed how chiral currents $J(z)$ and the Virasoro field act $T(z)$ act on $\mathcal{V}_k$. The same discussion applies to their anti-holomorphic partners. Hence, the formula (3.7) provides a decomposition of the space of bulk fields into irreducible representations of the chiral algebra that is generated by the modes α_n and $\bar{\alpha}_n$. States in $\mathcal{V}_k \otimes \bar{\mathcal{V}}_k$ are used to describe all the modes of a closed string that moves with center of mass momentum k through the flat space.

The ground states $|k\rangle \otimes |k\rangle$ are non-degenerate in this case and hence they give rise to a single bulk field $\varphi_{k,k}(z, \bar{z})$ for each momentum k. These fields are the familiar exponential fields,

$$\varphi_{k,k}(z, \bar{z}) = \Phi^{(P)}(|k\rangle \otimes |k\rangle; z, \bar{z}) = : \exp(ikX(z, \bar{z})) : \quad .$$

Their correlation functions are rather easy to compute (see e.g. [22]). From such expressions one can read off the following short distance expansion

$$\varphi_{k_1,k_1}(z_1, \bar{z}_1)\varphi_{k_2,k_2}(z_2, \bar{z}_2) \sim \int dk \, \delta(k_1 + k_2 - k) \, |z_1 - z_2|^{\alpha'(k_1^2 + k_2^2 - k^2)} \varphi_{k,k}(z_2, \bar{z}_2) + \dots \quad .$$

Comparison with our general form (3.6) of the operator products shows that the coefficients C are simply given by the δ function which expresses momentum conservation. Note that the exponent and the coefficient of the short distance singularity are a direct consequence of the equation of motion $\Delta X(z, \bar{z}) = 0$ for the free bosonic field. In fact, the equation implies that correlators of X itself possess the usual logarithmic singularity when two coordinates approach each other. After exponentiation, this gives rise to the leading term in the operator product expansion of the fields $\varphi_{k,k}$. In this sense, the short distance singularity encodes the dynamics of the bulk field and hence characterizes the background of the model.

3.1.2 Chiral algebras

Chiral algebras can be considered as symmetries of 2D conformal field theory. Since they play such a crucial role for all exact solutions, we shall briefly go through the most important notions in the representation theory of chiral algebras. These include the set $\mathcal{J}$ of representations, the modular S-matrix S, the fusion rules N and the fusing matrix F. The general concepts are illustrated in the case of the U(1)-current algebra.

Representation theory. Chiral- or W-algebras are generated by the modes of a finite set of chiral fields W_n^ν. These algebras mimic the role played by Lie algebras in atomic physics. Recall that transition amplitudes in atomic physics can be expressed as products of Clebsch-Gordan coefficients and so-called reduced matrix elements. While the former are purely representation theoretic data which depend only on the symmetry of the theory, the latter contain all the information about the physics of the specific system. Similarly, amplitudes in conformal field theory are built from representation theoretic data of W-algebras along with structure constants of the various operator product expansions, the latter being the reduced matrix elements of conformal field theory. In the (rational) conformal bootstrap, the structure constants are determined as solutions of certain algebraic equations which arise as factorization constraints and we will have to say a lot more about such equations as we proceed. Constructing the representation theoretic data, on the other hand, is a mathematical problem which is the same for all models that possess the same W-symmetry. Throughout most of the following text we shall not be concerned with this part of the analysis and simply use the known results. But we decided to include at least a short general review on representation theory of W-algebras.

We consider a finite number of bosonic chiral fields $W^\nu(z)$ with positive integer conformal dimension h_ν and require that there is one distinguished chiral field $T(z)$ of conformal dimension $h = 2$ whose modes L_n satisfy the usual Virasoro relations for central charge c. Their commutation relations with the Laurent modes W_n^ν of $W^\nu(z)$ are assumed to be of the form

$$[L_n, W_m^\nu] \; = \; \big(n(h_\nu - 1) - m\big) W_{n+m}^\nu \; . \tag{3.8}$$

In addition, the modes of the generating chiral fields also possess commutation relations among each other which need not be linear in the fields. The algebra generated by the

modes W_n^ν is the chiral or W-algebra $\mathcal{W}$ (for a precise definition and examples see [47] and in particular [48]). We shall also demand that $\mathcal{W}$ comes equipped with a $*$-operation.

Sectors $\mathcal{V}_i$ of the chiral algebra are irreducible (unitary) representations of $\mathcal{W}$ for which the spectrum of L_0 is bounded from below. Our requirement on the spectrum of L_0 along with the commutation relations (3.8) implies that any $\mathcal{V}_i$ contains a sub-space V_i^0 of ground states which are annihilated by all modes W_n^ν such that $n > 0$. The spaces V_i^0 carry an irreducible representation of the zero mode algebra $\mathcal{W}^0$, i.e. of the algebra that is generated by the zero modes W_0^ν, and we can use the operators $W_n^\nu, n < 0$, to create the whole sector $\mathcal{V}_i$ out of states in V_i^0. Unitarity of the sectors means that the space $\mathcal{V}_i$ may be equipped with a non-negative bi-linear form which is compatible with the $*$-operation on $\mathcal{W}$. This requirement imposes a constraint on the allowed representations of the zero mode algebra on ground states. Hence, one can associate a representation V_i^0 of the zero mode algebra to every sector $\mathcal{V}_i$, but for most chiral algebras the converse is not true. In other words, the sectors $\mathcal{V}_i$ of $\mathcal{W}$ are labeled by elements i taken from a subset $\mathcal{J}$ within the set of all irreducible (unitary) representations of the zero mode algebra.

For a given sector $\mathcal{V}_i$ let us denote by h_i the lowest eigenvalue of the Virasoro mode L_0. Furthermore, we introduce the *character*

$$\chi_i(q) \;=\; \mathrm{tr}_{\mathcal{V}_i}\left(q^{L_0 - \frac{c}{24}}\right) \;.$$

The full set of these characters $\chi_i, i \in \mathcal{J}$, has the remarkable property to close under modular conjugation, i.e. there exists a complex valued matrix $S = (S_{ij})$ such that

$$\chi_i(\tilde{q}) \;=\; S_{ij}\,\chi_j(q) \tag{3.9}$$

where $\tilde{q} = \exp(-2\pi i/\tau)$ for $q = \exp(2\pi i\tau)$, as before. This *S-matrix* is symmetric and unitary. Once S has been constructed for a given chiral algebra $\mathcal{W}$, one can introduce the numbers

$$N_{ij}{}^k \;=\; \sum_l \frac{S_{il} S_{jl} S_{kl}^*}{S_{0l}} \;. \tag{3.10}$$

Quite remarkably, they turn out to be non-negative integers. This property, however, possesses a direct explanation in representation theory. In fact, there exists a product

$\circ$ of sectors – known as the fusion product – such that $N_{ij}{}^k$ describe analogues of the Clebsch-Gordan multiplicities for the decomposition of $i \circ j$ into the irreducible sectors k [49]. For this reason, we refer to N as the *fusion rules* of $\mathcal{W}$. The relation (3.10) between the fusion rules N and the matrix S is called the *Verlinde formula*.

The *fusing matrix F* is the last quantity in the representation theory of chiral algebras which plays an important role below. Unfortunately, it is not as easy to describe. To begin with, let us be a bit more explicit about the fusion product. Its definition is based on the following family of homomorphisms (see e.g. [50])

$$
\begin{aligned}
\delta_z(W_n^\nu) \;&:=\; e^{-zL_{-1}} W_n^\nu e^{zL_{-1}} \otimes \mathbf{1} + \mathbf{1} \otimes W_n^\nu \\
&= \sum_{m=0}^{\infty} \binom{h_\nu + n - 1}{m} z^{n+h_\nu-1-m} W_{1+m-h_\nu}^\nu \otimes \mathbf{1} + \mathbf{1} \otimes W_n^\nu \qquad (3.11)
\end{aligned}
$$

which is defined for $n > -h_\nu$. The condition on n guarantees that the sum on the right hand side terminates after a finite number of terms. Suppose now that we are given two sectors $\mathcal{V}_j$ and $\mathcal{V}_i$. With the help of δ_z we define an action of the modes $W_n^\nu, n > -h_\nu$, on their product. This action can be used to search for ground states and hence for sectors k in the fusion product $j \circ i$. To any three such labels j, i, k there is assigned an intertwiner

$$
V(\,{}_k{}^j{}_i\,)(z): \; \mathcal{V}_j \otimes \mathcal{V}_i \; \to \; \mathcal{V}_k
$$

which intertwines between the action δ_z on the product and the usual action on $\mathcal{V}_k$. If we pick an orthonormal basis $\{|j,\nu\rangle\}$ of vectors in $\mathcal{V}_j$ we can represent the intertwiner V as an infinite set of operators

$$
V(\,{}_k{}^{j,\nu}{}_i\,)(z) \;:=\; V(\,{}_k{}^j{}_i\,)[|j,\nu\rangle; \,\cdot\,](z): \; \mathcal{V}_i \; \to \; \mathcal{V}_k \quad .
$$

Up to normalization, these operators are uniquely determined by the intertwining property mentioned above. The latter also restricts their operator product expansions to be of the form

$$
V(\,{}_k{}^{j_1,\mu}{}_r\,)(z_1)\, V(\,{}_r{}^{j_2,\nu}{}_i\,)(z_2) \;=\; \sum_{s,\rho} F_{rs}\left[\begin{smallmatrix} j_2 & j_1 \\ i & k \end{smallmatrix}\right] V(\,{}_k{}^{s,\rho}{}_i\,)(z_2)\, \langle s,\rho| V(\,{}_s{}^{j_2,\nu}{}_{j_1}\,)(z_{12})|j_1,\mu\rangle \quad ,
$$

where $z_{12} = z_1 - z_2$. The coefficients F that appear in this expansion form the *fusing matrix* of the chiral algebra $\mathcal{W}$. Once the operators V have been constructed for all ground

states $|j, \nu\rangle$, the fusing matrix can be read off from the leading terms in the expansion of their products. Explicit formulas can be found in the literature. We also note that the defining relation for the fusing matrix admits a nice pictorial presentation (see Figure 1). It presents the fusing matrix as a close relative of the $6J$-symbols which are known from the representation theory of finite dimensional Lie algebras.
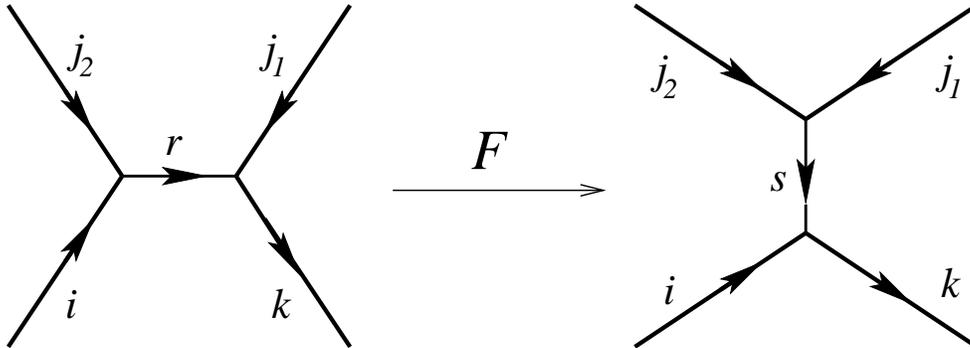
Figure 1: Graphical description of the fusing matrix. All the lines are directed as shown in the picture. Reversal of the orientation can be compensated by conjugation of the label. Note that in our conventions, one of the external legs is oriented outwards. This will simplify some of the formulas below.

Example: the U(1)-theory.   The chiral algebra of a single free bosonic field is known as U(1)-algebra. It is generated by the modes α_n of the current $J(z)$ with the reality condition $\alpha_n^* = \alpha_{-n}$. There is only one real zero mode $\alpha_0 = \alpha_0^*$ so that the zero mode algebra $\mathcal{W}^0$ is abelian. Hence, all its irreducible representations are 1-dimensional and there is one such representation for each real number k. The vector that spans the corresponding 1-dimensional space V_k^0 is denoted by $|k\rangle$, as before. It is easy to see that the space $\mathcal{V}_k$ which we generate out of $|k\rangle$ by the creation operators α_{-n} admits a positive definite bilinear form for any choice of k. Hence, $\mathcal{J} = \mathbb{R}$ coincides with the set of irreducible representations of the zero mode algebra in this special case.

The character χ_k of the sector $\mathcal{V}_k$ with conformal weight $h_k = \alpha' k^2 / 2$ is given by

$$\chi_k(q) \;=\; \frac{1}{\eta(q)} \, q^{\alpha' \frac{k^2}{2}} \;\;.$$

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Along with the well known property $\eta(\tilde{q}) = \sqrt{-i\tau}\eta(q)$, the computation of a simple Gaussian integral shows that

$$\chi_k(\tilde{q}) = \sqrt{\alpha'} \int dk' \, e^{2\pi i \, \alpha' k k'} \, \chi_{k'}(q) =: \sqrt{\alpha'} \int dk' \, S_{kk'} \chi_{k'}(q) \quad . \tag{3.12}$$

This means that the entries of the S-matrix are phases, i.e. $S_{kk'} = \exp(2\pi i \alpha' k k')$. As we have claimed, the S-matrix is unitary and symmetric under exchange of k and k'. When we insert the matrix elements $S_{kk'}$ into the right hand side of Verlinde's formula (3.10) we find

$$\frac{1}{\sqrt{\alpha'}} \delta(k_1 + k_2 - k) = \sqrt{\alpha'} \int dl \, e^{2\pi i \alpha' k_1 l} \, e^{2\pi i \alpha' k_2 l} \, e^{-2\pi i \alpha' k l} \quad . \tag{3.13}$$

This is a continuum version of the Verlinde formula. We want to demonstrate that the left hand side is indeed related to the fusion of representations. In the case of hand, the action of δ_z on the zero mode α_0 is given by

$$\delta_z(\alpha_0) = \alpha_0 \otimes \mathbf{1} + \mathbf{1} \otimes \alpha_0$$

since the current J has conformal weight $h = 1$. This shows that the fusion product amounts to adding the momenta, i.e. $k_1 \circ k_2 = k_1 + k_2$. In other words, the product of two sectors k_1 and k_2 contains a single sector $k_1 + k_2$. This indeed agrees with the left hand side of the Verlinde formula (3.13).

Let us conclude this example with a few comments on the fusing matrix. In this case it is rather easy to write down an explicit formula for the intertwining operators V. Once more, they are given by the normal ordered exponential, restricted to the spaces $\mathcal{V}_k$. When the operator product of two such exponentials with momenta k_1 and k_2 is expanded in the distance $z_1 - z_2$, we find an exponential with momentum $k_1 + k_2$. The coefficient in front of this term is trivial, implying triviality for the fusing matrix.

3.2 Boundary theory - the closed string sector

Our goal now is to place a brane into some given background. We shall argue that such branes are completely characterized by their couplings to closed string modes, i.e. by one-point functions of bulk fields on the upper half plane. The number of couplings one has to specify depends on the exact symmetry the brane preserves. From the so-called cluster property we shall derive a set of quadratic *factorization constraints* on the one-point functions which must be satisfied by any consistent boundary theory.

Branes - the microscopic setup. With some basic notations for the ("parent") bulk theory set up, we can begin our analysis of *associated* boundary theories ("open descendants"). These are conformal field theories on the upper half-plane $\text{Im} z \geq 0$ which, in the interior $\text{Im} z > 0$, are locally equivalent to the given bulk theory: The state space $\mathcal{H}^{(H)}$ of the boundary CFT is equipped with the action of a Hamiltonian $H^{(H)}$ and of bulk fields

$$\phi(z, \bar{z}) \;=\; \Phi^{(H)}(|\varphi\rangle; z, \bar{z})$$

– well-defined for $\text{Im} z > 0$ – which are assigned to the *same* elements $\varphi \in \mathcal{H}^{(P)}$ that were used to label fields in the bulk theory. Note, however, that the associated fields ϕ now act on a different space of states $\mathcal{H}^{(H)}$ and that, for the moment, we do not know any fields that are associated with the elements of $\mathcal{H}^{(H)}$. Furthermore, we demand that all the leading terms in the OPEs of bulk fields coincide with the OPEs (3.6) in the bulk theory, i.e. for the fields $\phi_{i,\bar{\imath}}$ one has

$$\phi_{i,\bar{\imath}}(z_1, \bar{z}_1)\phi_{j,\bar{\jmath}}(z_2, \bar{z}_2) \;=\; \sum_{n\bar{n}} C_{i\bar{\imath};j\bar{\jmath}}{}^{n\bar{n}} \, z_{12}^{h_n - h_i - h_j} \bar{z}_{12}^{\bar{h}_n - \bar{h}_i - \bar{h}_j} \phi_{n,\bar{n}}(z_2, \bar{z}_2) + \dots \qquad (3.14)$$

These relations express our condition that the brane is placed into our given closed string background. At the example of the free bosonic field we have discussed that the structure of the short distance expansion encodes the world-sheet dynamics in the interior of the upper half plane. Having the same singularities as in the bulk theory means that the boundary conditions do not affect the equations of motion in the bulk.

In addition, we must require the boundary theory to be conformal. This is guaranteed if the Virasoro field obeys the following gluing condition

$$T(z) \;=\; \overline{T}(\bar{z}) \quad \text{for} \quad z = \bar{z} \ . \qquad (3.15)$$

In the 2D field theory, this condition guarantees that there is no momentum flow across the boundary. Note that eq. (3.15) is indeed satisfied for the Virasoro fields in the flat space theory (see rel. (2.16)).

Considering *all* possible conformal boundary theories associated to a bulk theory whose chiral algebra is a true extension of the Virasoro algebra is, at present, too difficult a problem to be addressed systematically (see however [51, 52, 53, 54] and remarks in the final section for some recent progress). For the moment, we restrict our considerations to

maximally symmetric boundary theories, i.e. to the class of boundary conditions which leave the whole symmetry algebra $\mathcal{W}$ unbroken. More precisely, we assume that all chiral fields $W(z), \overline{W}(\bar{z})$ can be extended analytically to the real line and that there exists a local automorphism Ω – called the *gluing map* – of the chiral algebra $\mathcal{W}$ such that [18]

$$W(z) = \Omega\overline{W}(\bar{z}) \quad \text{for} \quad z = \bar{z} \ . \tag{3.16}$$

The condition (3.15) is included in equation (3.16) if we require Ω to act trivially on the Virasoro field. Note that the boundary conditions we considered in the first lecture are maximally symmetric since holomorphic and anti-holomorphic currents are glued along the boundary according to eqs. (2.7) and (2.14).

For later use let us remark that the gluing map Ω on the chiral algebra induces a map ω on the set of sectors. In fact, since Ω acts trivially on the Virasoro modes, and in particular on L_0, it may be restricted to an automorphism of the zero modes in the theory. If we pick any representation j of the zero mode algebra we can obtain a new representation $\omega(j)$ by composition with the automorphism Ω. This construction lifts from the representations of $\mathcal{W}^0$ on ground states to the full $\mathcal{W}$-sectors. As a simple example consider the U(1) theory with the Dirichlet gluing map $\Omega(\alpha_n) = -\alpha_n$. We restrict the latter to the zero mode α_0. As we have explained above, different sectors are labeled by the value $\sqrt{\alpha'}k$ of α_0 on the ground state $|k\rangle$. If we compose the action of α_0 with the gluing map Ω, we find $\Omega(\alpha_0)|k\rangle = -\sqrt{\alpha'}k|k\rangle$. This imitates the action of α_0 on $|-k\rangle$. Hence, the map ω is given by $\omega(k) = -k$.

Ward identities. As an aside, we shall discuss some more technical consequences that our assumption on the existence of the gluing map Ω brings about. To begin with, it gives rise to an action of one chiral algebra $\mathcal{W}$ on the state space $\mathcal{H} \equiv \mathcal{H}^{(H)}$ of the boundary theory. Explicitly, the modes $W_n = W_n^{(H)}$ of a chiral field W dimension h are given by

$$W_n := \frac{1}{2\pi i} \int_C z^{n+h-1} W(z) \, dz + \frac{1}{2\pi i} \int_C \bar{z}^{n+h-1} \Omega\overline{W}(\bar{z}) \, d\bar{z}$$

which generalizes the formulas (2.6), (2.15) from the first lecture. The operators W_n on the state space $\mathcal{H}$ are easily seen to obey the defining relations of the chiral algebra $\mathcal{W}$. Note that there is just one such action of $\mathcal{W}$ constructed out of the two chiral bulk fields $W(z)$ and $\Omega\overline{W}(\bar{z})$.

In the usual way, the representation of $\mathcal{W}$ on $\mathcal{H}$ leads to Ward identities for correlation functions of the boundary theory. They follow directly from the singular parts of the operator product expansions of the fields $W, \Omega\overline{W}$ with the bulk fields $\phi(z, \bar{z})$. These expansions are fixed by our requirement of local equivalence between the bulk theory and the bulk of the boundary theory. To make this more precise, we introduce the notation $W_>(z) = \sum_{n>-h} W_n z^{-n-h}$. The singular part of the OPE is then given by

$$(W(w)\,\phi(z,\bar{z}))_{\text{sing}} := [\,W_>(w)\,,\,\Phi(|\varphi\rangle; z, \bar{z})\,] \tag{3.17}$$

$$= \sum_{n>-h} \left(\frac{1}{(w-z)^{n+h}} \Phi\big(W_n^{(P)}|\varphi\rangle; z, \bar{z}\big) + \frac{1}{(w-\bar{z})^{n+h}} \Phi\big(\Omega\overline{W}_n^{(P)}|\varphi\rangle; z, \bar{z}\big) \right) \ .$$

As before, the subscript 'sing' reminds us that we only look at the singular part of the operator product expansion, and we have placed a superscript (P) on the modes $W_n, \overline{W}_n$ to display clearly that they act on the elements $|\varphi\rangle \in \mathcal{H}^{(P)}$ labeling the bulk fields in the theory (superscripts (H), on the other hand, are being dropped for most of our discussion). The sum on the right hand side of eq. (3.17) is always finite because $|\varphi\rangle$ is annihilated by all Laurant modes with sufficiently large n. For $\text{Im}\,w > 0$, only the first terms involving $W_n^{(P)}$ can become singular and the singularities agree with the singular part of the OPE between $W(w)$ and $\phi(z, \bar{z})$ in the bulk theory. Similarly, the singular part of the OPE between $\Omega\overline{W}(w)$ and $\phi(z, \bar{z})$ in the bulk theory is reproduced by the terms which contain $\overline{W}_n^{(P)}$, if $\text{Im}\,w < 0$.

As it stands, the previous formula is rather compressed. So, let us spell out at least one more concrete example in which the chiral field W has dimension $h = 1$ (we shall denote any such chiral currents by the letter J) and where we consider the primaries $\phi_{i,\bar{\imath}}$ in place of ϕ. Since the corresponding ground states are annihilated by all the modes $J_n, \bar{J}_n$ with $n > 0$, equation (3.17) reduces to

$$(J(w)\,\phi_{i,\bar{\imath}}(z,\bar{z}))_{\text{sing}} = \frac{X_J^i}{w-z}\phi_{i,\bar{\imath}}(z,\bar{z}) - \varphi_{i,\bar{\imath}}(z,\bar{z})\frac{X_{\Omega\bar{J}}^{\bar{\imath}}}{w-\bar{z}} \ . \tag{3.18}$$

The linear maps X_J^i and $X_{\Omega\bar{J}}^{\bar{\imath}}$ were introduced in eq. (3.4) above; they act on the primary multiplet $\phi_{i,\bar{\imath}} : V_{\bar{\imath}}^0 \otimes \mathcal{H} \to V_i^0 \otimes \mathcal{H}$ by contraction in the first component V_i^0 resp. $V_{\bar{\imath}}^0$.

Ward identities for arbitrary n-point functions of fields $\phi_{i,\bar{\imath}}$ follow directly from eq. (3.17). They have the same form as those for chiral conformal blocks in a bulk CFT with

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$2n$ insertions of chiral vertex operators with charges $i_1, \ldots, i_n, \omega(\bar{\imath}_1), \ldots, \omega(\bar{\imath}_n)$, see e.g. [9, 10, 18, 19]. Hence, objects familiar from the construction of bulk CFT can be used as building blocks of correlators in the boundary theory ("doubling trick"). Note, however, that the Ward identities depend on the gluing map Ω.

One-point functions. So far we have formalized what it means in world-sheet terms to place a brane in a given background (the principle of 'local equivalence') and how to control its symmetries through gluing conditions (3.16) for chiral fields. Now it is time to derive some consequences and, in particular, to show that a rational boundary theory is fully characterized by just a finite set of numbers.

Using the Ward identities described in the previous paragraph together with the OPE (3.14) in the bulk, we can reduce the computation of correlators involving n bulk fields to the evaluation of one-point functions $\langle \phi_{i,\bar{\imath}} \rangle_\alpha$ for the bulk primaries (see Figure 2). Here, the subscript α has been introduced to label different boundary theories that can appear for given gluing map Ω.
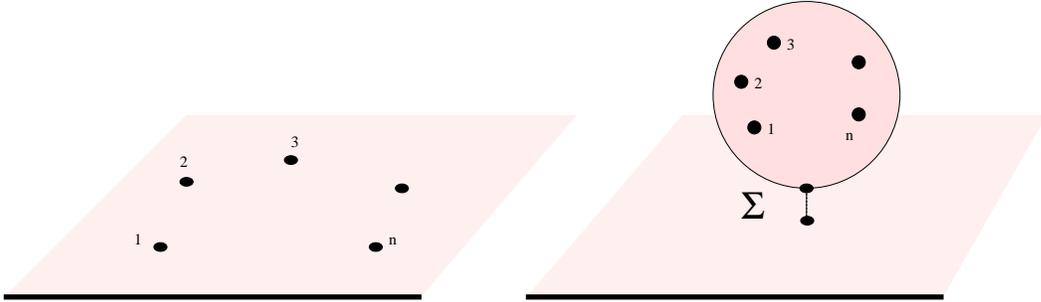
Figure 2: With the help of operator product expansions in the bulk, the computation of n-point functions in a boundary theory can be reduced to computing one-point functions on the half-plane. Consequently, the latter must contain all information about the boundary condition.

To control the remaining freedom, we notice that the transformation properties of $\phi_{i,\bar{\imath}}$ with respect to L_n, $n = 0, \pm 1$, and the zero modes W_0,

$$[W_0, \phi_{i,\bar{\imath}}(z, \bar{z})] = X_W^i \, \phi_{i,\bar{\imath}}(z, \bar{z}) - \phi_{i,\bar{\imath}}(z, \bar{z}) \, X_{\Omega\overline{W}}^{\bar{\imath}} \quad,$$

$$[\, L_n\,,\, \phi_{i,\bar{\imath}}(z,\bar{z})\,] \;\; = \;\; z^n\, (\, z\partial + h_i(n+1)\,)\phi_{i,\bar{\imath}}(z,\bar{z})$$

$$+\, \bar{z}^n\, (\, \bar{z}\bar{\partial} + \bar{h}_{\bar{\imath}}(n+1)\,)\phi_{i,\bar{\imath}}(z,\bar{z})$$

determine the one-point functions up to scalar factors. Indeed, an elementary computation using the invariance of the vacuum state reveals that the vacuum expectation values $\langle \phi_{i,\bar{\imath}} \rangle_\alpha$ must be of the form

$$\langle \phi_{i,\bar{\imath}}(z,\bar{z})\rangle_\alpha \;=\; \frac{\mathcal{A}_{i\bar{\imath}}^\alpha}{|z-\bar{z}|^{h_i+h_{\bar{\imath}}}} \tag{3.19}$$

where

$$\mathcal{A}_{i\bar{\imath}}^\alpha : V_{\bar{\imath}}^0 \to V_i^0 \qquad \text{obeys} \qquad X_W^i\,\mathcal{A}_{i\bar{\imath}}^\alpha \;=\; \mathcal{A}_{i\bar{\imath}}^\alpha\, X_{\Omega\overline{W}}^{\bar{\imath}}\;\;.$$

The intertwining relation in the second line implies $\bar{\imath} = \omega(i^+) \equiv i^\omega$ as a necessary condition for a non-vanishing one-point function (i^+ denotes the representation conjugate to i, i.e. the unique respresentation which obeys $N_{ii^+}^0 = 1$), and since $h_i = h_{i^\omega}$ we can put $h_i + h_{\bar{\imath}} = 2h_i$ in the exponent in eq. (3.19). From the irreducibility of the zero mode representations on the subspaces V_i^0 and Schur's lemma we conclude that each non-zero matrix $\mathcal{A}_{i\bar{\imath}}^\alpha$ is determined up to one scalar factor A_i^α,

$$\mathcal{A}_{i\bar{\imath}}^\alpha \;=\; A_i^\alpha\, \delta_{\bar{\imath},i^\omega}\, \mathcal{U}_{i\bar{\imath}}$$

where $\mathcal{U}_{i\bar{\imath}}$ intertwines between two representations of the zero mode algebra and is normalized by $\mathcal{U}_{i\bar{\imath}}^* \mathcal{U}_{i\bar{\imath}} = \mathbf{1}$. In conclusion, we have argued that boundary conditions associated with the same bulk theory and the same gluing map Ω, can differ only by a set of scalar parameters A_i^α in the one-point functions. Once we know their values, we have specified the boundary theory. This generalizes a similar observation we made for branes in flat backgrounds (see remark after eq. (2.22)) and it also agrees with our intuition that a brane should be completely characterized by its couplings to closed string modes such as the mass and RR charge.

The cluster property. We are certainly not free to choose the remaining parameters A_i^α in the one-point functions arbitrarily. In fact, there exist strong *sewing constraints* on them that have been worked out by several authors [55, 12, 16, 17, 56]. These can be derived from the following *cluster property* of the two-point functions

$$\lim_{a\to\infty} \langle \phi_{i,i^\omega}(z_1,\bar{z}_1)\phi_{j,j^\omega}(z_2+a,\bar{z}_2+a)\rangle \;=\; \langle \phi_{i,i^\omega}(z_1,\bar{z}_1)\rangle\langle \phi_{j,j^\omega}(z_2,\bar{z}_2)\rangle \;\;. \tag{3.20}$$

Here, a is a real parameter, and the field ϕ_{j,j^ω} on the right hand side can be placed at $(z_2, \bar{z}_2)$ since the whole theory is invariant under translations parallel to the boundary.

Let us now see how the cluster property restricts the choice of possible one-point functions. We consider the two-point function of the two bulk fields as in eq. (3.20). There are two different ways to evaluate this function. On the one hand, we can go into a regime where the two bulk fields are very far from each other in the direction along the boundary. By the cluster property, the result can be expressed as a product of two one-point functions and it involves the product of the couplings A_i^α and A_j^α. Alternatively, we can pass into a regime in which the two bulk fields are very close to each other and then employ the operator product (3.14) to reduce their two-point function to a sum over one-point functions. Comparison of the two procedures provides the following important relation,

$$A_i^\alpha \, A_j^\alpha \;=\; \sum_k \Xi_{ij}^k \, A_0^\alpha \, A_k^\alpha \quad . \tag{3.21}$$

It follows from our derivation that the coefficient Ξ_{ij}^k can be expressed as a combination

$$\Xi_{ij}^k \;=\; C_{i\bar{i};j\bar{j}}{}^{k\bar{k}} \; F_{1k}\big[\begin{smallmatrix} i & j \\ \omega(i^+) & \omega(j) \end{smallmatrix}\big] \tag{3.22}$$

of the coefficients C in the bulk OPE and of the fusing matrix. The latter arises when we pass from the regime in which the bulk fields are far apart to the regime in which they are close together (see Figure 3).
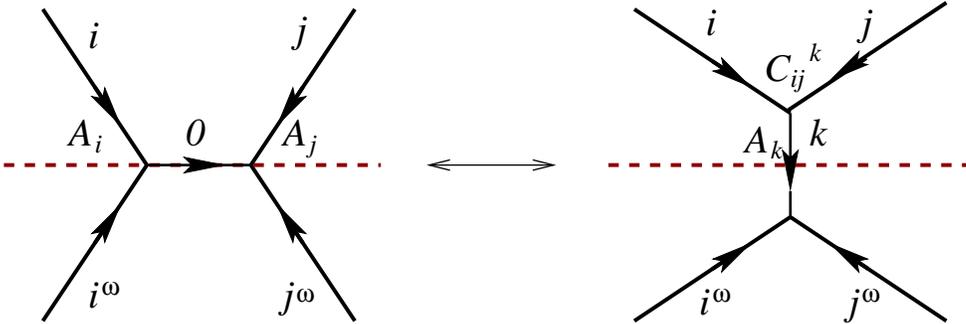
Figure 3: Equations (3.21), (3.22) are derived by comparing two limits of the two-point function. The dashed line represents the boundary of the world-sheet and we have drawn the left moving sector in the lower half-plane (doubling trick).

In some cases, Ξ_{ij}^k has been shown to agree with the fusion multiplicities or some generalizations thereof (see e.g. [16, 57, 56]). The importance of eq. (3.21) for a classification of boundary conformal field theories has been stressed in a number of publications [57, 56, 58] and is further supported by their close relationship with algebraic structures that entered the classification of bulk conformal field theories already some time ago (see e.g. [59, 60, 61]).

The algebraic relations (3.21) typically possess several solutions which are distiguished by our index α. Hence, maximally symmetric boundary conditions are labeled by pairs (Ω, α). The automorphism Ω is used to glue holomorphic and anti-holomorphic fields along the boundary and the consistent choices for Ω are rather easy to classify. Once Ω has been fixed, it determines the set of bulk fields that can have a non-vanishing one-point function and it is also referred to as the 'type' of the boundary theory. For each gluing automorphism Ω, the non-zero one-point functions are constrained by algebraic equations (3.21) with coefficients Ξ which are determined by the closed string background. A complete list of solutions is available in a large number of cases. But before presenting them, we want to show how one can reconstruct other important information on the brane from the couplings A_i^α. In particular, we will be able to recover the open string spectrum. Since the derivation makes use of *boundary states*, we need to introduce this concept first.

Boundary states. It is possible to store all information about the couplings A_i^α of closed strings to a brane in a single object, the so-called *boundary state*. To some extent, such a boundary state can be considered as the wave function of a closed string that is sent off from the brane (Ω, α). It is a special linear combinations of generalized coherent states (the so-called Ishibashi states). The coefficients in this combination are essentially the closed string couplings A_i^α.

One way to introduce boundary states is to equate correlators of bulk fields on the half-plane and on the complement of the unit disk in the plane. With $z, \bar{z}$ as before, we introduce coordinates $\zeta, \bar{\zeta}$ on the complement of the unit disk by

$$\zeta = \frac{1 - iz}{1 + iz} \qquad \text{and} \qquad \bar{\zeta} = \frac{1 + i\bar{z}}{1 - i\bar{z}} \quad . \tag{3.23}$$

If we use $|0\rangle$ to denote the vacuum of the bulk CFT, then the boundary state $|\alpha\rangle = |\alpha\rangle_\Omega$

can be uniquely characterized by [55, 18]

$$\langle\, \Phi^{(H)}(|\varphi\rangle; z, \bar{z})\rangle_\alpha \;\; = \;\; \left(\frac{d\zeta}{dz}\right)^h \left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{h}} \cdot \langle 0|\, \Phi^{(P)}(|\varphi\rangle; \zeta, \bar{\zeta})|\alpha\rangle \tag{3.24}$$

for primaries $|\varphi\rangle$ with conformal weights $(h, \bar{h})$. Note that all quantities on the right hand side are defined in the bulk conformal field theory (super-script P), while objects on the left hand side live on the half-plane (super-script H).

In particular, we can apply the coordinate transformation from $(z, \bar{z})$ to $(\zeta, \bar{\zeta})$ on the gluing condition (3.16) to obtain

$$W(\zeta) \;\; = \;\; (-1)^h\, \bar{\zeta}^{2h} \Omega \overline{W}(\bar{\zeta})$$

along the boundary at $\zeta\bar{\zeta} = 1$. Expanding this into modes, we see that the gluing condition (3.16) for chiral fields translates into the following linear constraints for the boundary state,

$$\left[\, W_n - (-1)^{h_W} \Omega \overline{W}_{-n}\, \right] |\alpha\rangle_\Omega \;\; = \;\; 0 \,. \tag{3.25}$$

These constraints posses a linear space of solutions. It is spanned by generalized coherent (or Ishibashi) states $|i\rangle\!\rangle$. Given the gluing automorphism Ω, there exists one such solution for each pair $(i, \omega(i^+))$ of irreducibles that occur in the bulk Hilbert space [62]. $|i\rangle\!\rangle_\Omega$ is unique up to a scalar factor which can be used to normalize the Ishibashi states such that

$$_\Omega\langle\!\langle j|\, \tilde{q}^{L_0^{(P)} - \frac{c}{24}}\, |i\rangle\!\rangle_\Omega \;\; = \;\; \delta_{i,j}\, \chi_i(\tilde{q}) \;\;. \tag{3.26}$$

Full boundary states $|\alpha\rangle_\Omega \equiv |(\Omega, \alpha)\rangle$ are given as certain linear combinations of Ishibashi states,

$$|\alpha\rangle_\Omega \;\; = \;\; \sum_i B^i_\alpha\, |i\rangle\!\rangle_\Omega \,. \tag{3.27}$$

With the help of (3.24), one can show [55, 18] that the coefficients B^i_α are related to the one-point functions of the boundary theory by

$$A^\alpha_i \;\; = \;\; B^{i^+}_\alpha \;\;. \tag{3.28}$$

The decomposition of a boundary state into Ishibashi states contains the same information as the set of one-point functions and therefore specifies the "descendant" boundary CFT of a given bulk CFT completely.

Following an idea in [62], it is easy to write down an expression for the generalized coherent states (see e.g. [18]), but the formula is fairly abstract. In the case of flat backgrounds, however, their construction can be made very explicit. Let us first discuss this for Dirichlet boundary conditions, i.e. for $\Omega^D \bar{J}^a = -\bar{J}^a$ where $a = d+1, \ldots, D$, labels directions transverse to the brane as in the first lecturs. Since $k_a^+ = -k_a$ (recall that fusion of sectors is given by adding momenta) and $\omega(k_a) = -k_a$, we have $k_a^\omega = k_a$ and so there exists a coherent state for each sector in the bulk theory (3.7). These states are given by

$$|k\rangle\!\rangle_D \;=\; \exp\left(\sum_{n=1}^\infty \frac{g_{ab}}{n}\alpha^a_{-n}\bar{\alpha}^b_{-n}\right) \; |k\rangle \otimes \overline{|k\rangle} \;\;.$$

Using the commutation relation of α^a_n and $\bar{\alpha}^a_n$ it is easy to check that $|k\rangle\!\rangle_D$ is annihilated by $\alpha^a_n - \bar{\alpha}^a_{-n}$ as we required in eq. (3.25). A special case of our formula (2.22) for the one-point function along with the general rule (3.28) lead to the following boundary state for Dirichlet boundary conditions,

$$|x_0\rangle_D \;=\; \int \Pi^D_{a=d+1}(\sqrt{\alpha'}dk_a) \;\; e^{-ik_a x_0^a} \; |k\rangle\!\rangle_D \;\;.$$

For the directions along the brane, the analysis is different. Here we have to use the gluing map Ω^B from eq. (2.14) and a simple computation reveals that the condition $\omega^B(k) = -k$ is only solved by $k = 0$. This means that we can only construct one coherent state,

$$|0\rangle\!\rangle_B \;=\; \exp(-\sum_{n=1}^\infty \frac{G_{ij}}{n}\alpha^i_{-n}\Omega^B\bar{\alpha}^j_{-n}) \; |0\rangle \otimes \overline{|0\rangle} \;\;.$$

According to eqs. (2.22) and (3.28), this coincides with the boundary state $|0\rangle_B = |0\rangle\!\rangle_B$. Hence, the boundary state for the branes discussed in the first lecture is given by the product $|x_0\rangle_{(B,d)} \equiv |0\rangle_B \otimes |x_0\rangle_D$.

3.3 Boundary theory - the open string sector

While the one-point functions (or boundary states) uniquely characterize a boundary conformal field theory, there exist more quantities we are interested in. In particular, we shall now see how the coefficients of the boundary states determine the spectrum of so-called boundary fields which can be inserted along the boundary of the world-sheet. In addition, we derive a set of factorization conditions for the operator product expansions of these new fields.

The boundary spectrum. Our aim is to determine the spectrum of open string modes which can stretch between two branes labeled by α and β, both being of the same type Ω. In world-sheet terms, the quantity we want to compute is the partition function on a strip with boundary conditions α and β imposed along the two sides. This is illustrated on the left hand side of Figure 4. The figure also illustrates the main idea of the calculation. In fact, world-sheet duality allows to exchange space and time and hence to turn the one loop open string diagram on the left hand side into a closed string tree diagram which is depicted on the right hand side. The latter corresponds to a process in which a closed string is created on the brane α and propagates until it gets absorbed by the brane β. Since creation and absorption are controlled by the amplitudes A_i^α and A_j^β, the right hand side - and hence the partition function on the left hand side - is determined by the one-point functions of bulk fields.
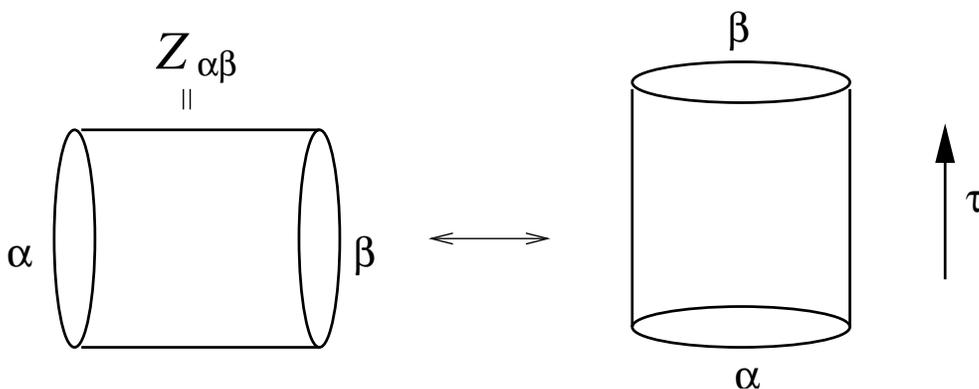
Figure 4: The open string partition function $Z_{\alpha\beta}$ can be computed by world-sheet duality. In the figure, the time runs upwards so that the left hand side is interpreted as an open string 1-loop diagram while the right hand side is a closed string tree diagram.

Let us now become a bit more precise and derive the exact relation between the couplings A and the partition function. Reversing the above sketch of the calculation, we begin on the left hand side of Figure 4 and compute

$$\langle \theta\beta | \tilde{q}^{H^{(P)}} | \alpha \rangle \;=\; \sum_j A_{j^+}^\beta A_j^\alpha \langle\!\langle j^+ | \tilde{q}^{L_0^{(P)} - \frac{c}{24}} | j^+ \rangle\!\rangle \;=\; \sum_j A_{j^+}^\beta A_j^\alpha \chi_{j^+}(\tilde{q}) \quad .$$

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Here we have dropped all subscripts Ω since all the boundary and generalized coherent states are assumed to be of the same type. The symbol θ denotes the world-sheet CPT operator in the bulk theory. It is a anti-linear map which sends sectors to their conjugate, i.e.

$$\theta A^{\beta}_{j+} \, |j\rangle\!\rangle \;=\; \left(A^{\beta}_{j+}\right)^{*} \, |j^{+}\rangle\!\rangle \;\;.$$

Having explained these notations, we can describe the steps we performed in the above short computation. To begin with we inserted the expansion (3.27),(3.28) of the boundary states in terms of Ishibashi states and the formula $H^{(P)} = 1/2(L_0 + \bar{L}_0) - c/24$ for the Hamiltonian on the plane. With the help of the linear relation (3.25) we then traded $\bar{L}_0$ for L_0 before we finally employed the formula (3.26). At this point we need to recall the property (3.9) of characters to arrive at

$$\langle \theta\beta | \tilde{q}^{H^{(P)}} | \alpha \rangle \;=\; \sum_j A^{\beta}_{j+} A^{\alpha}_j S_{j+i} \chi_i(q) \;=:\; Z_{\alpha\beta}(q) \;\;. \tag{3.29}$$

As argued above, the quantity we have computed should be interpreted as a boundary partition function and hence as a trace of the operator $\exp(2\pi i \tau H^{(H)})$ over some space $\mathcal{H}_{\alpha\beta}$ of states for the system on a strip with boundary conditions α and β imposed along the boundaries. Since our boundary conditions preserve the chiral symmetry, the partition function is guaranteed to decompose into a sum of the associated characters. Moreover, the coefficients in this expansion must be integers and so we conclude

$$Z_{\alpha\beta}(q) \;=\; \sum_i n_{\alpha\beta}{}^i \, \chi_i(q) \quad \text{where} \quad n_{\alpha\beta}{}^i \;=\; \sum_j A^{\beta}_{j+} A^{\alpha}_j S_{j+i} \;\in\; \mathbb{N} \;\;. \tag{3.30}$$

This is the desired expression for the partition function in terms of the couplings A^{α}_i. Although there exists no general proof, it is believed that every solution of the factorization constraints (3.21) gives rise to a consistent spectrum with integer coefficients $n_{\alpha\beta}{}^i$. A priori, the integrality of the numbers $n_{\alpha\beta}{}^i$ provides a strong constraint, known as the *Cardy condition*, on the set of boundary states and it has often been used instead of eqs. (3.21) to determine the coefficients A^{α}_i. Note that the Cardy conditions are easier to write down since they only involve the modular S-matrix. To spell out the factorization constraints (3.21), on the other hand, one needs explicit formulas for the fusing matrix and the bulk operator product expansion.

There is one fundamental difference between the Cardy condition (3.30) and the factorization constraints (3.21) that is worth pointing out. Suppose that we are given a set of solutions of the Cardy constraint. Then every non-negative integer linear combination of the corresponding boundary states defines another Cardy-consistent boundary theory. In other words, solutions of the Cardy condition form a cone over the integers. The factorization constraints (3.21) do not share this property. Geometrically, this is easy to understand: we know that it is possible to construct new brane configurations from arbitrary superpositions of branes in the background (though they are often unstable). These brane configurations possess a consistent open string spectrum but they are not elementary. As long as we are solving the Cardy condition, we look for such configurations of branes. The factorization constraints (3.21) were derived from the cluster property which ensures the system to be in a 'pure phase'. Hence, by solving eqs. (3.21) we search systematically for elementary brane configurations that cannot be decomposed any further. Whenever the coefficients Ξ are known, solving the factorization constraints is clearly the preferable strategy, but sometimes the required information is just hard to come by. In such cases, one can still learn a lot about possible brane configurations by studying Cardy's conditions.

Boundary fields and boundary OPE. To the partition function $Z_{\alpha\beta}$ we have computed in the previous paragraph we can associate a state space

$$\mathcal{H}_{\alpha\beta}^{(H)} = \bigoplus_i n_{\alpha\beta}{}^i \mathcal{V}_i .$$

The modes of open strings stretching in between the branes α and β are to be found within this space. Since there should be an open string vertex operator for each such mode, we expect that the elements in the state space $\mathcal{H}_{\alpha\beta}^{(H)}$ correspond to boundary operators which can be inserted at points u of the boundary where the boundary condition jumps from α to β. In other words, we have just argued for a new state-field correspondence $\Psi^{(H)}$ which associates a boundary field with each state in $\mathcal{H}_{\alpha\beta}^{(H)}$,

$$\psi^{\alpha\beta}(u) = \Psi^{(H)}(|\psi\rangle; u) \quad \text{for} \quad |\psi\rangle \in \mathcal{H}_{\alpha\beta}^{(H)} . \tag{3.31}$$

As in the case of bulk fields, we want to introduce a special notation for the boundary fields that come with ground states. Once more we fix a basis $|e_i^a\rangle$ in the space $V_i^0 \subset \mathcal{V}_i$

and introduce the multiplet

$$\psi_i^{\alpha\beta}(x) := \left(\Psi^{(H)}(|e_i^a\rangle; x)\right) \quad \text{with} \quad |e_i^a\rangle \in V_i^0 \subset \mathcal{H}_{\alpha\beta}^{(H)} \quad. \tag{3.32}$$

Elements of this tuple are numbered by $a = 1, \ldots, \dim V_i^0$. We have been a bit sloppy here. In fact, there are certainly cases in which the space $\mathcal{V}_i$ appears with some non-trivial multiplicity $n_{\alpha\beta}{}^i > 1$. If that is the case, the associated boundary fields carry an additional index $r = 1, \ldots, n_{\alpha\beta}{}^i$. For notational reasons we shall omit this extra label, but it is not too difficult to add it back into all the following formulas. The multiplet $\psi_i^{\alpha\beta}$ carries an irreducible representation of the zero mode algebra $\mathcal{W}_0$.

Having introduced boundary fields we are interested in their correlation functions. The latter become singular when their world-sheet arguments come close and the singularity is again encoded in operator product expansions. For the boundary fields $\psi_i^{\alpha\beta}$ these read

$$\psi_i^{\alpha\beta}(u_1)\,\psi_j^{\beta\gamma}(u_2) = \sum_k (u_1 - u_2)^{h_i + h_j - h_k}\,\mathcal{C}_{ij;k}^{\alpha\beta\gamma}\,\psi_k^{\alpha\gamma}(u_2) + \ldots \quad \text{for} \quad u_1 > u_2 \quad. \tag{3.33}$$

The coefficients $\mathcal{C}_{ij;k}^{\alpha\beta\gamma}$ are linear maps which intertwine the action of the zero mode algebra $\mathcal{W}_0$ on the field multiplets. We can split them into a product of a numerical factor $C_{ij;k}^{\alpha\beta\gamma}$ and an intertwiner $\mathcal{U}_{ij;k} : V_k^0 \to V_i^0 \otimes V_j^0$ that does no longer depend on the boundary conditions. The intertwiners $\mathcal{U}_{ij;k}$ are normalized by

$$\mathcal{U}_{ij;l}^*\,\mathcal{U}_{ij;k} = \delta_{l,k}\,\mathbf{1}_{V_k^0} \quad.$$

All dynamical information about the scattering of open strings is encoded in the numerical factors $C_{ij;k}^{\alpha\beta\gamma}$ which can be non-zero only if $N_{ij}{}^k \neq 0$.

Four-point functions of these boundary fields must satisfy certain factorization constraints (see Figure 5) which allow to derive the following constraint on the coefficients of the operator product expansion,

$$C_{ji;r}^{\delta\alpha\beta}\,C_{kr;l}^{\gamma\delta\beta} \sim \sum_s F_{sr}\begin{bmatrix} j & i \\ k & l \end{bmatrix} C_{kj;s}^{\gamma\delta\alpha}\,C_{si;l}^{\gamma\alpha\beta} \quad. \tag{3.34}$$

In writing these equations we have omitted terms associated with boundary two-point functions on both sides. The precise condition can be found e.g. in [63, 64, 65]. Studies of many examples and intuition both suggest that these relations possess a unique solution,

up to some freedom that can be absorbed through the normalization of boundary fields. In this sense, the interactions of open strings are determined by the multiplicities $n_{\alpha\beta}{}^i$ and hence ultimately by the couplings A_i^α of closed string modes to the brane.
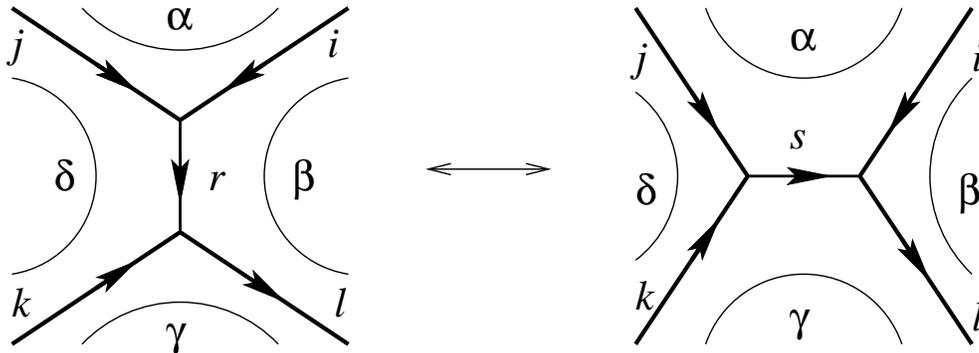
Figure 5: The equation (3.34) is derived by considering scattering amplitudes of four open string modes (i, j, k, l) stretching in between four different boundary conditions $(\alpha, \beta, \gamma, \delta)$.

The Ward identities. We have already discussed Ward identities for correlation functions of bulk fields on the upper half-plane Σ. The extension of such identities to boundary fields is straightforward. Using the same notations as in the corresponding equation (3.17) for bulk fields, the singular part of the operator product expansion between chiral and boundary fields reads

$$(W(w)\,\psi(u))_{\text{sing}} := [\,W_>(w)\,,\,\Psi(|\psi\rangle; u)\,] = \sum_{n>-h} \frac{1}{(w-u)^{n+h}}\,\Psi\big(W_n|\psi\rangle; u\big) \quad . \quad (3.35)$$

These relations agree with the usual Ward identities for chiral vertex operators and so arbitrary correlators in boundary conformal field theory can be constructed out of the known formulas for chiral blocks.

3.4 Solution of the theory

In the last subsections we derived the three conditions, namely the relations (3.21), (3.30) and (3.34), which are fundamental in solving boundary conformal field theory. It is

remarkable that universal solutions exist for a large class of backgrounds. We will now present the basic Cardy solution and then illustrate it in the case a single free bosonic field.

Solution for Cardy case. To begin with, let us formulate the main assumption we have to make. Suppose we are given some rational bulk conformal field theory with a bulk modular invariant partition function of the special form

$$Z(q, \bar{q}) = \sum_j \chi_{Ij}(q) \, \chi_{\bar{j}}(\bar{q}) \ . \tag{3.36}$$

Here, j runs through the set $\mathcal{J}$ of sectors and $\bar{j}$ is some unique sector that is paired with j in the partition function. In the following we describe the solution for all possible possible boundary theories of type Ω, provided that the

$$j^\omega = \omega(j)^+ \stackrel{!}{=} \bar{j} \ . \tag{3.37}$$

Under this condition, Cardy claims that there exist as many boundary theories as there are $\mathcal{W}$-sectors, i.e. the number of boundary theories is equal to the order of $\mathcal{J}$. We shall label these boundary theories by $I, J, K \cdots \in \mathcal{J}$ instead of using $\alpha, \beta, \gamma, \ldots$ to remind us that they run through the same set as the labels $i, j, k, \ldots$.

We have learned that such boundary theories can be characterized by the one-point functions of the primary fields. Cardy proposes that the associated couplings A_j^J are simply given by the modular S-matrix, i.e.

$$\langle \, \phi_{j,j^\omega}(z, \bar{z}) \, \rangle_J = \frac{S_{Jj}}{\sqrt{S_{0j}}} \frac{\mathcal{U}_{jj^\omega}}{|z - \bar{z}|^{2h_j}} \ . \tag{3.38}$$

Note that this formula makes sense since the boundary label J runs through the set of $\mathcal{W}$-sectors. As we explained before, $\mathcal{U}_{jj^\omega}$ is the unitary intertwiner that intertwines between the actions of the zero modes W_0 on the two spaces V_j^0 and $V_{j^\omega}^0$ of ground states.

The spectrum of open strings stretching between the branes that are associated with the labels I and J is encoded in the associated partition functions

$$Z_{IJ}(q) = \sum_j N_{IJ^+}{}^j \, \chi_j(q) \ . \tag{3.39}$$

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Here, J^+ is defined through the conjugation on sectors of the chiral algebra, i.e. through $N_{IJ^+}{}^0 = \delta_{I,J}$. Since fusion rules of the chiral algebra are non-negative integers, the expression (3.39) has the form of a partition function for a $\mathcal{W}$-symmetric system. We will analyze below how equation (3.39) is related to the formula (3.38) for the one-point functions.

The complete solution should include expressions for the boundary operator products. According to the expansion (3.39), there are $N_{IJ^+}{}^j$ boundary fields $\psi_j^{IJ}(u)$ (recall that they are $\mathcal{W}_0$ multiplets whenever dim$V_j^0 > 1$). As in the previous paragraph, we suppress the extra index that labels different fields in cases when $N_{IJ^+}{}^j > 1$. The operator product expansion for two such primary fields is claimed to be of the form

$$\psi_i^{LM}(u_1)\,\psi_j^{MN}(u_2) = \sum_k (u_1 - u_2)^{h_i + h_j - h_k}\,F_{Mk}\big[\begin{smallmatrix} i & j \\ L & N \end{smallmatrix}\big]\mathcal{U}_{ij;k}\,\psi_k^{LN}(u_2) \;\; + \ldots \qquad (3.40)$$

for $u_1 > u_2$. Here F stands for the fusing matrix of the chiral algebra $\mathcal{W}$. It was introduced at the end of the first subsection. The formula (3.40) was originally found for minimal models by Runkel [63] and extended to more general cases in [66, 67, 68].

Let us stress that all the important structure constants of the solution, namely the list of boundary labels, the one-point functions, the partition functions and the boundary operator expansions, have been expressed through representation theoretic data of the underlying symmetry. In fact, we have used the list of sectors, the modular S-matrix, the fusion rules and the fusing matrix to write down the exact solutions. Obviously, the rather simple relation between the set of solutions and purely representation theoretic quantities only appears for very particular choices of bulk modular invariants and gluing maps Ω. This manifests itself in our assumption $j^\omega = \bar{j}$. If the assumptions (3.36), (3.37) are violated, finding solutions is more difficult. We will only provide a brief overview on the current status of this active field of current research (see last section). As restricting as the assumption $j^\omega = \bar{j}$ may appear, it still turns out to apply to a large number of interesting situations.

Testing Cardy's solution. The structure constants $A_j^\alpha, n_{\alpha\beta}{}^i$ and $C_{ij;k}^{\alpha\beta\gamma}$ of any exact solution must satisfy our fundamental algebraic constraints (3.21), (3.30), (3.34). Our aim here is to verify at least two of these conditions, namely the Cardy condition (3.30)

and the factorization constraint (3.34) for the solution we have provided in eqs. (3.38), (3.39) and (3.40). From Cardy's solution we read off that $A_j^J = S_{Jj}/\sqrt{S_{0j}}$. When this is inserted into the formula (3.30) for the integers $n_{IJ}{}^i$ one finds

$$n_{IJ}{}^i = \sum_j \frac{S_{Jj^+} S_{Ij} S_{j+i}}{S_{0j}} = N_{IJ^+}^i \ .$$

In the computation we have used the properties $S_{ij^+} = S_{ij}^* = S_{i^+ j}$ of the modular S-matrix and the Verlinde formula (3.10). In fact, the very close relation between the Verlinde formula and the expressions in eq. (3.30) was the main evidence Cardy relied on to support his solution.

The boundary operator product expansions (3.40) arise from the general expression (3.33) if we equate

$$C_{ij;k}^{IJK} = F_{Jk}\begin{bmatrix} i & j \\ I & K \end{bmatrix} \ .$$

When we plug this expression into eq. (3.34), we end up with the following equation for the fusing matrix

$$F_{Ir}\begin{bmatrix} j & i \\ L & J \end{bmatrix} F_{Ll}\begin{bmatrix} k & r \\ K & J \end{bmatrix} = \sum_s F_{sr}\begin{bmatrix} j & i \\ k & l \end{bmatrix} F_{Ls}\begin{bmatrix} k & j \\ K & I \end{bmatrix} F_{Il}\begin{bmatrix} s & i \\ K & J \end{bmatrix} \ .$$

This is the famous pentagon equation which holds true for the fusing matrix of any chiral algebra (see [50]). The proof of the full factorization constraint is slightly more involved since one has to take the contributions from boundary two-point functions into account. The latter were neglected in our equation (3.34).

Example: D0 branes in flat space. It is nice to see how the general formulas allow to recover the solutions of the boundary conformal field theory describing a point-like brane in a 1-dimensional flat space. The bulk invariant has been spelled out in eq. (3.7). It is diagonal in the sense $k = \bar{k}$ and has the special form (3.36), at least if we close an eye on the fact that the real line is non-compact and hence the theory is not rational. We are interested in boundary theories which obey the Dirichlet gluing condition $J(z) = -\bar{J}(\bar{z})$. As we have explained earlier, this choice of Ω implies $k^\omega = k$ and hence our main assumption (3.37) is satisfied.

Now Cardy assures us that the associated boundary conditions are parametrized by a parameter α which runs through the set $\mathcal{J} = \mathbb{R}$ of sectors in the theory. Hence,

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the parameter α must be associated with the transverse position of our brane in the 1-dimensional background.

The precise relation between α and x_0 can be read off from the one-point functions. If we insert the formula (3.12) for the S-matrix of the U(1) theory into the general expression (3.38) above we obtain

$$\langle \phi_{k,k}(z,\bar{z}) \rangle_\alpha = \frac{e^{2\pi i \alpha' k \alpha}}{|z - \bar{z}|^{\alpha' k^2}} \quad .$$

This indeed agrees with a special case of formula (2.22) for $D = 1, d = 0$ and $x_0 = 2\pi\alpha'\alpha$.

Next we want to look at our formula (3.39) for the partition functions. We determined the fusion rules of the U(1) theory in eq. (3.13) and plugging them into eq. (3.39) gives

$$Z_{x_0 y_0}(q) = \int dk \, \delta(\frac{x_0 - y_0}{2\pi\alpha'} - k) \chi_k(q) = \frac{1}{\eta(q)} \, q^{\frac{(x_0 - y_0)^2}{8\pi^2 \alpha'}} \quad .$$

In the first lecture we only computed the special case $x_0 = y_0$ for which we obtain agreement with eq. (2.24). If x_0 and y_0 are not the same, an open string must stretch over the finite distance $x_0 - y_0$ in between the two branes. The energy of such stretched open strings is proportional to $(x_0 - y_0)^2$ and this explains the exponent of the second factor in the partition function. Needless to say that it can be found directly by solving the Laplace equation in a strip.

The field corresponding to the only ground state in $\mathcal{H}_{x_0 y_0}$ is given by

$$\psi_p^{x_0 y_0}(x) = {}^{\circ}_{\circ} e^{ipX(x)} {}^{\circ}_{\circ} \quad \text{with} \quad p = \frac{x_0 - y_0}{2\pi\alpha'} \quad .$$

Since the fusing matrix of the U(1) theory is trivial, the operator product expansion of these exponential fields is the one predicted by formula (3.40).

Obviously, we did not learn anything new about D0 branes in flat space. What we have seen is some kind of high-tech derivation of the standard results using lots of complicated notions from representation theory of chiral algebras. The point is, however, that now we do have a technology ready to be applied to more complicated interacting theories with non-linear equations of motion for which conventional methods fail. We will illustrate the full power of these developments in the next lecture.

4 Application: Branes on group manifolds

We are now in a position to apply the general techniques of boundary conformal field theory to the construction of branes in curved backgrounds. In many ways, group manifolds provide an ideal area to illustrate the abstract constructions. Since they are homogeneous, they come with a large symmetry that facilitates the exact solution. On the other hand, strings on group manifolds are described by a non-linear 2D field theory and hence are sufficiently non-trivial to demonstrate the full power of boundary conformal field theory. Finally, the involved models are also fundamental for CFT model building.

We shall first approach branes on group manifolds in a more qualitative way, mainly based on our experience from the first lecture. Even though the initial reasoning will hardly go beyond a chain of educated guesses, this can lead us a long way and it will help later to present the exact solution in a new light. Once the branes on group manifolds have been constructed using all the formulas from the previous lectures, we compute their low-energy effective action and discuss several interesting applications to the study of brane dynamics. For simplicity our presentation focuses mainly on the group manifold $SU(2) \cong S^3$, but many aspects generalize directly to other groups (see remarks in the last section).

4.1 The semi-classical geometry.

In the following introductory paragraph we shall see the basic contours of a scenario that we are going to derive later through our microscopic treatment of branes on group manifolds. From the preliminary discussion we will extract certain Poisson spaces which are argued to describe the semi-classical geometry of the relevant branes. Their quantization is the subject of the third paragraph.

Introductory remarks. Group manifolds possess a non-vanishing constant curvature $R = R(g)$ which arises from a non-constant metric g. It is well known that strings are rather picky when it comes to choosing the backgrounds they can propagate on. In fact, to lowest order in α', the background metric $g(x)$ and B-field $B(x)$ of a bosonic string

have to obey the equations

$$R_{\mu\nu}(g) - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha') \;=\; 0 \;\; , \tag{4.1}$$

where $H_{\mu\nu\sigma}$ are the components of the NSNS 3-form $H = dB$ with $B = B_{\mu\nu}(x) dx^\mu \wedge dx^\nu$. In writing eq. (4.1), the dilaton was assumed to be constant. For superstrings the same equations hold as long as we set all the RR background fields to zero. This is the scenario in which the following discussion is placed.

From eq. (4.1) we conclude immediately that a background with non-zero curvature $R_{\mu\nu}$ requires a non-vanishing NSNS field H and hence a non-zero B-field. In our analysis of branes in flat space, such B-fields caused the coordinates along the brane to be quantized and hence they were at the origin of the brane's non-commutative geometry. Although the details will be different in curved spaces, the basic mechanisms are certainly expected to work in the same way. Thus, if we can find branes which extend along some directions of a group manifold, their world-volume geometry is very likely to be quantized.

To see whether we have a chance to construct extended stable branes on group manifolds, let us now restrict to the case of $SU(2) \cong S^3$. This is also of particular interest in string theory because it appears e.g. within the background $\mathbb{R}^{1,5} \times S^3 \times \mathbb{R}_+$ of NS 5-branes [69, 70] or as part of the geometry $AdS_3 \times S^3 \times T^4$. Placing a point-like brane somewhere on the $SU(2)$ is obviously not in conflict with stability. Higher dimensional objects, however, may seem unstable at first sight, since their tension tends to make them collapse. Only a brane wrapping the whole S^3 could be stabilized through the topology, but it is excluded by the presence of the non-vanishing NSNS 3-form [71, 72]. Hence, if the tension would be the only factor contributing to the stability analysis, our story would be rather short and boring. It turns out, however, that the non-vanishing B-field also plays an important role and that it can exert enough pressure on 2-dimensional spherical branes in S^3 to balance the tension [73, 74]. Although the initial arguments for this *flux stabilization* could only be trusted at weak curvature, the statement remains correct even deep in the stringy regime [75]. Thus, we conclude that stable branes on $SU(2)$ can either be point-like or they can wrap a 2-dimensional sphere $S^2 \subset S^3$. From now on we shall consider point-like branes as degenerate 2-spheres so that we do not have to distinguish between the two possibilities.

Spheres S^2 in S^3 are parametrized by the location of their center and by their radius r. Branes wrapping a sphere of radius r carry a non-vanishing B-field which causes their world-volume to be quantized. At a fixed scale of non-commutativity, the area $A(r)$ of the 2-sphere gets tiled by elementary 'Heisenberg cells' each of which contributes a single state to the quantized theory. Since the number of such states must be integer, we conclude that 2-spheres are quantizable only for a discrete set of radii r. Whenever we tune the radius r to one of the allowed values, the space of wave functions on the corresponding quantized sphere is finite dimensional. Coordinates along the brane are observables in the quantum theory and hence they are represented by operators acting on the space of wave functions. If the latter is finite dimensional, then the coordinates become matrices.

This is about as far as our very qualitative discussion can carry. There are three main conclusions that we take along. First of all, stable branes on S^3 are expected to wrap 2-spheres or they can be point-like (see Figure 6). Furthermore, 2-spheres can only be wrapped for a discrete set of radii and finally, the world-volume of such branes has been argued to possess a non-commutative 'matrix geometry'. We will see all these expectations confirmed by the exact treatment.
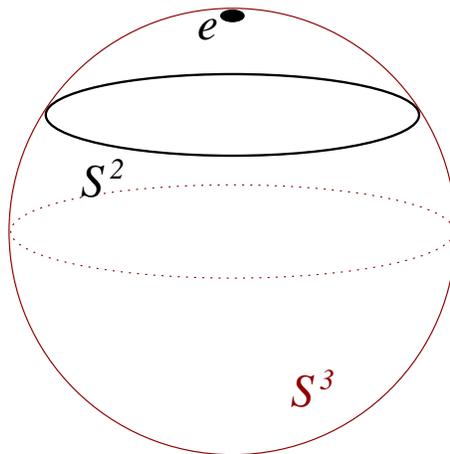
Figure 6: Stable branes on a 3-sphere are either point-like or they wrap a 2-sphere (conjugacy class of $SU(2) \cong S^3$). The 2-dimensional branes are stabilized by the background flux [73, 74].

Gluing condition and brane geometry. Strings moving on a 3-sphere S^3 of radius $R \sim \sqrt{k}$ are described by the SU(2) WZW model at level k. The world-sheet swept out by an open string in S^3 is parametrized by a map $g : \Sigma \to \mathrm{SU}(2)$ which is defined on the upper half-plane Σ as before. Our aim now is to determine the boundary conditions we need to impose on g so that we obtain the desired spherical branes. Following [76], we shall argue that the appropriate choice is given by (see also [77], [78])

$$-(\partial g)g^{-1} = g^{-1}\bar{\partial}g \quad \text{for} \quad z = \bar{z} \ . \tag{4.2}$$

To present the findings of [76], we split $\partial, \bar{\partial}$ into derivatives ∂_u, ∂_v tangential and normal to the boundary and rewrite eq. (4.2) in the form

$$(\mathrm{Ad}(g) - 1)\, g^{-1}\partial_v g = i\, (\mathrm{Ad}(g) + 1)\, g^{-1}\partial_u g \quad \text{for} \quad z = \bar{z} \ . \tag{4.3}$$

Here, $\mathrm{Ad}(g)$ denotes the adjoint action (i.e. action by conjugation) of SU(2) on its Lie algebra su(2). The following analysis requires to decompose the tangent space $T_h\mathrm{SU}(2)$ at each point $h \in \mathrm{SU}(2)$ into a part $T_h^{\parallel}\mathrm{SU}(2)$ tangential to the conjugacy class through h and its orthogonal complement $T_h^{\perp}\mathrm{SU}(2)$ (with respect to the Killing form). Using the simple fact that $\mathrm{Ad}(g)|_{T_g^{\perp}} = 1$ we can now see that with condition (4.3)

1. the endpoints of open strings on SU(2) are forced to move along conjugacy classes, i.e.

$$(g^{-1}\partial_u g)^{\perp} = 0 \ .$$

Except for two degenerate cases, namely the points e and $-e$ on the group manifold, the conjugacy classes are 2-spheres in SU(2).

2. the branes wrapping conjugacy classes of SU(2) carry a B-field which is given by

$$B \sim \frac{\mathrm{Ad}(g) + 1}{\mathrm{Ad}(g) - 1} \ . \tag{4.4}$$

The associated 2-form is obtained as $\mathrm{tr}\,(g^{-1}dg\, B\, g^{-1}dg)$ and a short computation shows that it provides a potential for the NSNS 3-form $H \sim \Omega_3$ where Ω_3 denotes the volume form on S^3.

The second statement follows from eq. (4.3) by comparison with the boundary conditions (2.3) we used in the flat background. Our discussion shows not only that eq. (4.3) is indeed the desired boundary condition for spherical branes on S^3 but also it has left us with an exact formula for the B-field.

We have learned in the first lecture that the relevant object for the brane's non-commutative geometry is not B itself but another anti-symmetric tensor Θ constructed from B through eq. (2.11). Even though this relation was derived for a flat background we may try to apply it naively in the present context. For appropriate choice of the metric g and the Regge slope α', we are then led to the expression

$$\Theta(g) \;=\; \frac{2}{B - B^{-1}} \;=\; \frac{1}{2}\left(\mathrm{Ad}(g^{-1}) - \mathrm{Ad}(g)\right) .$$

This is a rather complicated object, but it simplifies in the limit of large level k where the 3-sphere grows and approaches flat 3-space $\mathbb{R}^3$. One can parametrize points on SU(2) by elements X in the Lie algebra su(2), such that near the group unit $g \approx 1 + X$. Insertion into our formula for Θ gives

$$\Theta \;=\; -\mathrm{ad}(X) \ . \tag{4.5}$$

Here, ad denotes the adjoint action of su(2) on itself. If we expand $X = y^\mu t_\mu$ we can evaluate the matrix elements of Θ more explicitly,

$$\Theta_{\mu\nu} \;=\; -\left(t_\mu ,\, \mathrm{ad}(X) t_\nu \right) \;=\; -y^\rho \left(t_\mu ,\, f_{\rho\nu}{}^\sigma t_\sigma \right) \;=\; f_{\mu\nu\rho}\, y^\rho \ ,$$

where $(\cdot, \cdot)$ denotes the Killing form on su(2), the generators t_μ are normalized such that $(t_\mu, t_\nu) = \delta_{\mu,\nu}$ and $f_{\mu\nu\rho}$ are the structure constants of su(2). It is not difficult to see that the tensor Θ gives rise to a Poisson structure on $\mathbb{R}^3$,

$$\{\, y^\mu ,\, y^\nu \,\} \;=\; \Theta^{\mu\nu}(y) \;=\; f^{\mu\nu}{}_\rho\, y^\rho \ . \tag{4.6}$$

In contrast to the Poisson bracket we met in our discussion of flat branes, Θ has a linear dependence on the coordinates. The Poisson algebra defined by eq. (4.6) possesses a large center. In fact, any function of $c(y) = \sum_\mu (y^\mu)^2$ has vanishing Poisson bracket with any other function on $\mathbb{R}^3$ so that formula (4.6) induces a Poisson structure on the 2-spheres

$$c(y) \;:=\; \sum_\mu (y_\mu)^2 \;\overset{!}{=}\; \mathrm{c} \ . \tag{4.7}$$

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If we trust the steps of our reasoning, the two formulas (4.6) and (4.7) provide a semi-classical description of the spherical branes in SU(2) which replaces the simpler formula (2.25) in the first lecture.

Quantization and matrices. Now let us recall that the Moyal-Weyl product shows up for brane geometry in flat space with constant B-field is obtained from the constant Poisson bracket (2.25) on $\mathbb{R}^d$ through quantization. By analogy, our semi-classical analysis for branes on SU(2) $\cong S^3$ suggests that the quantization of 2-spheres in $\mathbb{R}^3$ with Poisson bracket (4.6) becomes relevant for branes on SU(2) in the limit where $\mathsf{k} \to \infty$. This is sufficient motivation for us to try quantizing the geometry (4.6), (4.7).

Quantization requires to find some operators $\hat{y}^\nu = Q(y^\nu)$ acting on a state space V such that

$$[\hat{y}^\mu, \hat{y}^\nu] = i f^{\mu\nu}{}_\rho \, \hat{y}^\rho \tag{4.8}$$

$$\mathrm{C} := \sum (\hat{y}^\mu)^2 = \mathsf{c} \, \mathbf{1} \tag{4.9}$$

where $\mathbf{1}$ denotes the identity operator on the state space V. These two requirements are the quantum analogues of the classical relations (4.6), (4.7) and the quantization problem they pose is easy to solve. By the commutation relation (4.8), the operators $\hat{y}^\mu$ have to form a representation of su(2). Condition (4.9) states that in this representation, the quadratic Casimir element C must be proportional to the identity $\mathbf{1}$. This is true if the representation on V is irreducible. Hence, any irreducible representation of su(2) can be used to quantize our Poisson geometry.

Irreducible representations of the Lie algebra su(2) are labeled by one discrete parameter $J = 0, 1/2, 1, \ldots$. This implies that only a discrete set of 2-spheres in $\mathbb{R}^3$ is quantizable and their radii increase with the value of the quadratic Casimir in the corresponding irreducible representation. For each quantizable 2-sphere $S_J^2 \subset \mathbb{R}^3$, we obtain a state space V_J of dimension $\dim V_J = 2J + 1$ equipped with an action of the quantized coordinate functions $\hat{y}^\mu$ on V_J. The latter generate the matrix algebra $\mathrm{Mat}(2J + 1)$. Note that our quantized 2-spheres have all the features we anticipated in the introduction above. But this should not be considered a derivation since our arguments relied on extending formula (2.11) to the present context. While such a step is certainly suggestive, we gave no evidence for it to be correct.

Matrix (fuzzy) geometry. It is useful to go a bit deeper into exploring these quantized 2-spheres. Let us start by recalling that the space $\text{Fun}(S^2)$ of functions on a 2-sphere is spanned by spherical harmonics $Y_a^j \in \text{Fun}(S^2)$ where j runs through all integer isospins. A product of any two spherical harmonics is again a function on the 2-sphere and hence it can be written as a linear combination of spherical harmonics,

$$Y_a^i \, Y_b^j \;=\; \sum_{k,c} c_{ijk} \left[\begin{smallmatrix} i & j & k \\ a & b & c \end{smallmatrix} \right] Y_c^k \tag{4.10}$$

with $[:::]$ denoting the Clebsch-Gordan coefficients of su(2). The structure constants c_{ijk} can be found at many places in the literature. We also note that elements of the vector multiplet Y_ν^1 may be identified with the restriction of the three coordinate functions y^ν to the 2-sphere in $\mathbb{R}^3$.

The algebra $\text{Fun}(S^2)$ admits an action of su(2) which is generated by infinitesimal rotations in $\mathbb{R}^3$,

$$L_\mu \;:=\; f_{\mu\rho}{}^\nu \, y^\rho \partial_\nu \;\;.$$

Even though the differential operators L_μ are initially defined for arbitrary functions on $\mathbb{R}$, they obviously descend down to $\text{Fun}(S^2)$. Under the action of L_μ, spherical harmonics Y_a^j transform in the representation j of su(2).

This classical symmetry survives quantization, i.e. there exists an analogous action of su(2) on $\text{Mat}(2J+1)$. It is given by

$$\text{L}_\mu \text{A} \;:=\; [\, t_\mu^J, \, \text{A}\,] \quad \text{for all} \quad \text{A} \in \text{Mat}(2J+1) \;. \tag{4.11}$$

Here t_μ^J denote the generators of su(2) evaluated in the $(2J+1)$-dimensional irreducible representation. One can easily decompose this reducible action of su(2) on $\text{Mat}(2J+1)$ into its irreducible sub-representations to find the following equivalence

$$\text{Mat}(2J+1) \;\cong\; \bigoplus_{j=0}^{2J} V_j \;\;, \tag{4.12}$$

where the sum on the right hand side runs over integer labels j. Each irreducible component V_j in this expansion is spanned by $(2j+1)$ matrices which we denote by $Y_a^j, j \leq 2J$.

In the case $J = 1/2$, only the scalar and the vector multiplet appear and explicit expressions for the corresponding 2×2 matrices are of course well-known: they are given by the identity and the Pauli-matrices, respectively.

Since Y_a^j span $\mathrm{Mat}(2J + 1)$, the product of any two such matrices may be expressed as a linear combination of matrices Y_c^k,

$$\mathrm{Y}_a^i \, \mathrm{Y}_b^j \;\; = \;\; \sum_{k \leq 2J, c} \; \{ {\textstyle i \; j \; k \atop \textstyle J \; J \; J} \} [{\textstyle i \; j \; k \atop \textstyle a \; b \; c}] \, \mathrm{Y}_c^k \;\; . \tag{4.13}$$

Here $\{::\}$ denote the re-coupling coefficients (or 6J-symbols) of su(2). This relation should be considered as a quantization of the expansions (4.10) and the classical expression is recovered from eq. (4.13) upon taking the limit $J \to \infty$ [79]. Hence, the matrices Y_a^j in the quantized theories are a proper replacement for spherical harmonics. Note, however, that the angular momentum $j \leq 2J$ is bounded from above. This may be interpreted as 'fuzziness' of the quantized 2-spheres on which short distances cannot be resolved [80]. We shall eventually refer to Y_a^j as 'fuzzy spherical harmonics'.

4.2 The exact CFT solution

Having gained some intuition into the main features of branes on S^3, at least in the limit of weak curvature, it is now time to let the microscopic machinery work for us. Since the exact solutions we wrote down in the previous lecture used a lot of data from the representation theory of the symmetry algebra, we will first list some of these data for affine Lie algebras, which are the chiral algebras relevant for strings on group manifolds. Then we present the exact formulas for the one-point functions of bulk fields, the open string partition function and the boundary OPE, and we discuss them in the light of our geometric insights.

Affine Lie algebras. In the following we collect a few basic facts on WZW models and affine Lie algebras. Many more details and references to the original literature can be found e.g. in [81, 7]. The fundamental SU(2) valued field g of the WZW model is known to satisfy the following classical equation of motion in the bulk

$$\partial \left(g^{-1}(z \, \bar{z}) \bar{\partial} g(z, \bar{z}) \right) \;\; = \;\; 0 \;\; . \tag{4.14}$$

This should be considered as a non-linear version of the Laplace equation which governs the string motion in flat backgrounds. It follows immediately that the fields

$$J(z) := -\mathsf{k}\,\partial g(z,\bar{z})\,g^{-1}(z,\bar{z}) \quad , \quad \bar{J}(\bar{z}) := \mathsf{k}\,g^{-1}(z,\bar{z})\bar{\partial}g(z,\bar{z}) \tag{4.15}$$

are chiral. Since they take values in the Lie algebra su(2), we can expand each of these two currents in terms of three component fields, i.e. $J(z) =: J^\mu(z)t_\mu$ and similarly for the anti-holomorphic partner $\bar{J}$.

The chiral fields J^μ of the SU(2) WZW model form an affine Lie algebra denoted by $\widehat{\mathrm{SU}}(2)_\mathsf{k}$. It is generated by Laurent modes J_n^μ which obey the following commutation relations

$$[\,J_n^\mu,\,J_m^\nu\,] = if^{\mu\nu}{}_\rho\,J_{n+m}^\rho + \mathsf{k}\,n\,\delta^{\mu,\nu}\,\delta_{n,-m} \tag{4.16}$$

along with the usual reality property $(J_n^\mu)^* = J_{-n}^\mu$. The commutators (4.16) differ from the corresponding relations (2.5), (2.9) in the flat space theory by the first term on the right hand side. This signals the presence of a non-vanishing background curvature.

Zero-modes J_0^μ of $\widehat{\mathrm{SU}}(2)_\mathsf{k}$ satisfy the usual relations for generators of the finite dimensional Lie algebra su(2). Hence, the sectors $\mathcal{V}_j$ of the theory are created out of the $(2j+1)$-dimensional representation spaces $V_j^0 = V_j$ of su(2). But only if $j \leq \mathsf{k}/2$, these sectors are free of negative norm states and hence unitarity leaves us with just a finite number of physical representations which we can label through $j \in \mathcal{J} = \{0, 1/2, \ldots, \mathsf{k}/2\}$. Their conformal weights are given by $h_j = j(j+1)/(\mathsf{k}+2)$ and for the modular S-matrix one finds

$$S_{ij} = \sqrt{\frac{2}{\mathsf{k}+2}}\,\sin\frac{\pi(2i+1)(2j+1)}{\mathsf{k}+2} \quad . \tag{4.17}$$

With the help of the Verlinde formula (3.10) it is not difficult to compute the following fusion rules,

$$N_{ij}{}^k = \begin{cases} 1 & \text{for } k = |i-j|, \ldots, \min(i+j, \mathsf{k}-i-j) \\ 0 & \text{otherwise} \end{cases} \quad . \tag{4.18}$$

They are similar to the Clebsch-Gordan multiplicities of su(2), apart from the truncation which appears whenever $i + j > \mathsf{k}/2$. As in the case of su(2), the trivial representation $k = 0$ occurs only in the fusion of j with itself, i.e. the conjugate j^+ of j is given by $j^+ = j$.

Formulas for the fusing matrix also exist and they can be found in the literature. Since they are rather complicated we will not spell them out. Let us only mention one property concerning their limiting behavior as we send $k \to \infty$,

$$\lim_{k \to \infty} F_{Jk} \left[\begin{smallmatrix} i & j \\ I & K \end{smallmatrix} \right] = \left\{ \begin{smallmatrix} i & j & k \\ I & J & K \end{smallmatrix} \right\} \ . \tag{4.19}$$

This concludes our list of representation theoretic data for the affine Lie algebra. We will now use these quantities to solve our boundary problem for the SU(2) WZW model.

The closed string sector. Our first task is to check whether our two basic requirements, namely the gluing condition (3.16) and the assumptions (3.36), (3.37) in Cardy's solution, are fulfilled by the boundary condition (4.2) we would like to impose. In terms of the chiral currents (4.15), we can rewrite eq. (4.2) as follows

$$J^\mu(z) = \bar{J}^\mu(\bar{z}) \quad \text{for} \quad z = \bar{z}$$

and $\mu = 1, 2, 3$. This indeed has the form of the gluing condition (3.16) with $\Omega = $ id and, moreover, it also implies the gluing property (3.15) for the usual Sugawara-Virasoro field

$$T(z) := \frac{1}{2} \frac{\delta_{\mu,\nu}}{k+2} \, {}_\circ^\circ J^\mu(z) J^\nu(z) {}_\circ^\circ \ .$$

The same formula with $\bar{J}$ instead of J is used for $\overline{T}$. Hence, spherical branes on S^3 preserve the full chiral algebra of the WZW model, including its conformal symmetry. This puts us into an excellent position to succeed with the exact solution.

Let us now turn to Cardy's assumptions (3.36), (3.37). The space of bulk fields for our WZW theory is given by the charge conjugate modular invariant (recall that $j^+ = j$),

$$\mathcal{H}^{(P)} = \bigoplus_j \mathcal{V}_j \otimes \bar{\mathcal{V}}_j \ ,$$

which has the necessary form (3.36). Since our gluing map Ω is trivial, it induces the identity map $\omega(j) = j$ on the labels $j \in \mathcal{J}$. Using our earlier observation that sectors of the affine su(2) are self-conjugate, $j^+ = j$, we conclude $j^\omega = \omega(j^+) = j = \bar{j}$. This guarantees that the WZW model with our choice of boundary conditions (4.2) can be solved by the formulas we provided at the end of the previous lecture !

To begin with, we learn from Cardy's solution that there are $\mathsf{k}+1$ possible boundary theories which we label by $J = 0, 1/2, \ldots, \mathsf{k}/2$, just as we enumerate the sectors of the corresponding affine Lie algebra. In the different boundary theories, the bulk fields possess different one-point functions. These are given by (see eq. (3.38))

$$\langle \phi_{j,j}^{ab}(z, \bar{z}) \rangle_J \;=\; \left(\frac{2}{\mathsf{k}+2} \right)^{\frac{1}{4}} \frac{\sin \frac{\pi(2j+1)(2J+1)}{\mathsf{k}+2}}{\left(\sin \frac{\pi(2j+1)}{\mathsf{k}+2} \right)^{1/2}} \frac{\delta^{a,b}}{|z - \bar{z}|^{2h_j}} \tag{4.20}$$

where $h_j = j(j+1)/(\mathsf{k}+2)$. The superscripts a, b placed at the symbol ϕ label different components within the tensor multiplet $\phi_{j,j}$ (see eq. (3.5)). Each of them runs through a basis in the representation space V_j^0 of su(2). Since Ω is trivial, the intertwiner $\mathcal{U}_{jj}$ between the left and right representation of su(2) is trivial as well, i.e. $\mathcal{U}_{jj}^{ab} = \delta^{a,b}$.

When we discussed flat space models, we have described how to read off a brane's location from its set of one-point functions. Let us now try to repeat the same procedure in the case of branes on SU(2) (see [66, 53]). According to the Peter-Weyl theory for compact groups, the space of functions on SU(2) is spanned by the matrix elements $D_{ab}^j(g)$ of finite dimensional unitary representations D^j. More precisely, the functions

$$\phi_j^{ab}(g) \;:=\; \sqrt{2j+1}\, D_{ab}^j(g) \tag{4.21}$$

form a complete orthonormal basis of $\mathrm{Fun}(S^3)$ just in the same way as the exponentials $\exp(ikx)$ do for $\mathrm{Fun}(\mathbb{R}^D)$. Writing $\phi_j^{ab}(g)$ in terms of three Euler angles on S^3, we find in particular

$$\sum_{c=-j}^{j} \phi_j^{cc}(g) \;=\; \sqrt{2j+1}\, \frac{\sin \vartheta(2j+1)}{\sin \vartheta}$$

where $\vartheta \in [0, \pi]$ parametrizes the azimuthal angle that is transverse to conjugacy classes on SU(2). From the completeness of $\sin(n\vartheta)$ on the interval $[0, \pi]$ one then concludes

$$\frac{1}{\sin \vartheta_0} \delta(\vartheta - \vartheta_0) \;=\; \frac{4}{\pi} \sum_{j} \sum_{c=-j}^{j} \frac{\sin(\vartheta_0(2j+1))}{\sqrt{2j+1}} \phi_j^{cc}(g) \;\; . \tag{4.22}$$

Except from a numerical factor, the coefficients in this expansion agree with the limit of our one-point functions (4.20) as k tends to infinity. We can also phrase this observation in a form similar to eq. (2.23),

$$\langle \phi_{j,j}^{ab} \rangle_{J(\mathsf{k})} \stackrel{\mathsf{k} \to \infty}{\sim} \int_{\mathrm{SU}(2)} d\mu(g) \, \rho_0 \, \delta(\vartheta(g) - \vartheta_0) \, \phi_j^{ab}(g) \tag{4.23}$$

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where ρ_0 is some constant and $d\mu(g)$ denotes the Haar measure on SU(2). In taking the limit, we allowed the boundary label J to depend on the level k and we defined $\vartheta_0 := 2\pi \lim J(k)/k$. Since $0 \leq J \leq k/2$, the angle ϑ_0 lies in the interval $[0, \pi]$. If we want to obtain a 2-sphere with non-zero angle ϑ_0 (and hence with non-zero radius), the boundary label J has to be scaled up at the same rate as the level k. The necessity for this rescaling is simple to understand. As we have seen, boundary WZW theory at level k allows us to fit $k + 1$ different branes on the 3-sphere. Therefore, the difference of the azimuthal angles between two neighboring branes is roughly $\Delta\vartheta_0 = \pi/k$. We numbered these branes with $J = 0, 1/2, 1, \ldots$ according to the angle at which they appear and starting with the smallest one at the group unit. When we keep J fixed but increase the level k, then the Jth brane moves closer and closer to the unit and in the limit it shrinks to a point. This is precisely the behavior that forces us to set $J(k) \sim \vartheta_0 k/2\pi$. Later on we will introduce another large k limit in which open string data are kept fixed rather than the angle ϑ_0.

The open string sector. As in the first lecture, we shall restrict our discussion to open strings on a single brane. Strings which have their two ends on different branes present no additional difficulty. The state space of the Jth boundary theory is determined by equation (3.39) and it has the form

$$\mathcal{H}_J = \mathcal{H}_{JJ} = \bigoplus_j N_{JJ}{}^j \mathcal{V}_j \qquad (4.24)$$

where $\mathcal{V}_j$, $j = 0, 1/2, \ldots, k/2$, denote irreducible highest weight representations of the affine Lie algebra $\widehat{SU}(2)_k$, and where $N_{JJ}{}^j$ are the associated fusion rules (see eq. (4.18)). Note that only integer spins j appear on the right hand side of (4.24) and that the sum is truncated at $j_{max} = \min(2J, k - 2J)$. In the limit $k \to \infty$, we obtain $j_{max} = 2J$ so that the decomposition of $\mathcal{H}_J$ is as close as it can be to the decomposition (4.12) of $\mathrm{Mat}(2J + 1)$ into su(2) multiplets. More precisely, there is a unique correspondence between fuzzy spherical harmonics $Y_a^j \in \mathrm{Mat}(2J + 1)$ and ground states in $\mathcal{V}_j \subset \mathcal{H}_J$. This generalizes the relation between the Weyl operators $\exp(ik\hat{x})$ and the boundary fields ${}^\circ_\circ \exp(ikX(u)){}^\circ_\circ$ that we found for flat branes in the first lecture.

By the state field correspondence Ψ, each state $|\psi\rangle \in \mathcal{H}_J$ gives rise to a boundary field $\psi(u) = \Psi(|\psi\rangle; u)$. Since we deal with a single boundary condition J, we will omit

the superscripts J that we used in rel. (3.32). Ground states in $\mathcal{H}_J$ furnish a muliplet of boundary primary fields

$$\psi_j^a(u) \; := \; \Psi(|e_j^a\rangle; u) \quad \text{where} \quad |a| \leq j$$

and $j = 0, 1, \ldots, j_{max}$. The operator product expansion (3.40) of these open string vertex operators reads [82]

$$\psi_i^a(u_1) \, \psi_j^b(u_2) \; = \; \sum_{k,c} u_{12}^{h_k - h_i - h_j} \left[\begin{smallmatrix} i & j & k \\ a & b & c \end{smallmatrix} \right] F_{Jk} \left[\begin{smallmatrix} j & i \\ J & J \end{smallmatrix} \right] \psi_k^c(u_2) + \ldots \tag{4.25}$$

where $h_j = j(j+1)/(\mathsf{k}+2)$ is the conformal dimension of ψ_j^a and the symbol in square brackets stands for the Clebsch-Gordan coefficients of su(2). The latter provide the intertwiners $\mathcal{U}_{ij;k}$ which appear in the general formula. As we send the level k to infinity (while keeping the boundary label J fixed), the conformal dimensions h_j tend to zero so that the OPE (4.25) of boundary fields becomes regular as in a topological model,

$$\left(\psi_i^a(u_1) \, \psi_j^b(u_2) \right)^{\mathsf{k} \to \infty} \; = \; \sum_{k,c} \left\{ \begin{smallmatrix} j & i & k \\ J & J & J \end{smallmatrix} \right\} \left[\begin{smallmatrix} i & j & k \\ a & b & c \end{smallmatrix} \right] \psi_k^c(u_2) + \ldots \tag{4.26}$$

Here we have also used the property (4.19) of the fusing matrix. Comparing (4.26) with eq. (4.13) we make a striking observation [82]: In fact, the large k limit of the operator product expansion exactly reproduces the multiplication of matrices ! We can use this fact to evaluate arbitrary n-point functions of the boundary fields ψ_j^a in the limiting regime. Before we spell out the result, let us introduce the notation

$$\psi[\mathrm{A}](u) \; := \; \sum \mathrm{a}_{jb} \, \psi_j^b(u) \quad \text{for all} \quad \mathrm{A} \; = \; \sum \mathrm{a}_{jb} \mathrm{Y}_b^j \; \in \; \mathrm{Mat}(2J+1) \quad .$$

For an arbitrary set of matrices $\mathrm{A}_r \in \mathrm{Mat}(2j+1)$, the operator product expansion formula (4.26) then implies

$$\langle \psi[\mathrm{A}_1](u_1) \, \psi[\mathrm{A}_2](u_2) \, \cdots \, \psi[\mathrm{A}_n](u_n) \rangle^{\mathsf{k} \to \infty} \; = \; \mathrm{tr} \, (\mathrm{A}_1 \, \mathrm{A}_2 \, \cdots \, \mathrm{A}_n) \quad . \tag{4.27}$$

The trace appears because the vacuum expectation value is SU(2) invariant and the trace maps matrices to their SU(2) invariant component.

Our final expression (4.27) for the correlators of open string vertex operators shares many features with the formula (2.32) that we encountered through our investigation of branes in flat space. In both cases, the vertex operators are in one-to-one correspondence

with elements of the non-commutative algebra of 'functions' on the world-volume of the brane. Furthermore, in a limiting regime, the correlators are independent of the insertion points u_r and they can be evaluated using the multiplication and integration (trace) on the non-commutative world-volume algebra. There remains, however, one important difference between the two cases: For branes on SU(2), the world-volume algebra is cut off at some angular momentum $2J$ so that there are only finitely many linearly independent 'functions' (resp. boundary primary fields).

Before we conclude our discussion of the solution for the boundary problem on SU(2), we would like to make one more remark. The attentive reader may have wondered already, why all the spherical branes in this section were centered around the group unit $e \in SU(2)$. Since no point on a group manifold is distinguished from any other, there should exist spherical branes with arbitrary locations. It is not hard to spot the place in our analysis at which we broke the SU(2) translation symmetry. In fact, it is the gluing condition (4.2) that forced all the branes to have their center at e. This lack of democracy, however, is easy to overcome if we admit gluing automorphisms Ω taken from the group of inner automorphisms on SU(2). By definition, an inner automorphism is of the form $\Omega = \text{Ad}_g$ with some element $g \in SU(2)$. Branes surrounding the point g are obtained with the boundary condition

$$ J(z) \;=\; \text{Ad}_g \, \bar{J}(\bar{z}) \quad \text{for} \quad z \;=\; \bar{z} \;\;. \tag{4.28} $$

Needless to say that the WZW boundary problem can be solved for any of these gluing conditions and the solution is essentially the same as above. Only the $\delta^{a,b}$ in the formula (4.20) for the bulk one-point functions must be replaced through the matrix $D_{ab}^j(g)$.

4.3   Fuzzy gauge theory and dynamics

Now that we have an exact CFT-solution for spherical branes on S^3, we also want to mention some applications. As in the case of branes in flat space, a non-commutative gauge theory can be associated with stacks of branes on group manifolds. But since the underlying world-volume geometry is described by matrix algebras, the gauge theories turn out to be matrix models. We sketch their derivation in the first subsection and then study some classical solutions. The latter possess an interpretation in terms of bound

state formation. We finish this lecture with some brief remarks on brane dynamics in the stringy regime.

Fuzzy gauge theory.   In the first lecture we have seen how information about some limiting behavior of certain boundary correlators can be stored in a non-commutative Yang-Mills theory on the world-volume of the brane. Our aim here is to repeat this analysis for correlators of the fields

$$\substack{\circ\\\circ} J^\mu(u)\psi_j^a(u)\substack{\circ\\\circ} \;=\; J_<^\mu(u)\psi_j^a(u) + \psi_j^a(u)J_<^\mu(u) \;\;.$$

Here we have used the same split into raising and lowering modes as before. The computation of three- and four-point functions of these fields follows exactly the same strategy as in the first lecture. In particular, it requires expressions for the operator product expansions of the currents among each other and with the primary boundary fields (see eqs. (2.34), (2.35) for comparison). From the commutation relations (4.16) in the affine Lie algebra we obtain

$$\left(J^\mu(u_1)\,J^\nu(u_2)\right)_{sing} \;:=\; \left[\,J_>^\mu(u_1)\,,\,J^\nu(u_2)\,\right] \;=\; \mathsf{k}\,\frac{\delta^{\mu,\nu}}{(u_1-u_2)^2} + \frac{if^{\mu\nu}{}_\rho}{(u_1-u_2)}\,J^\rho(u_2) \;\;. \quad (4.29)$$

The operator product expansion between currents and primary fields is a special case of eq. (3.35),

$$\left(J^\mu(u_1)\,\psi_j^a(u_2)\right)_{sing} \;:=\; \left[\,J_>^\mu(u_1)\,,\,\psi_j^a(u_2)\,\right] \;=\; \frac{(t_\mu^j)_{ab}}{(u_1-u_2)}\,\psi_j^b(u_2) \;\;. \quad (4.30)$$

Equipped with all these relations, we are able to calculate the low-energy effective action for massless open string modes. With respect to the flat space case, there occur three important changes during the computation. First of all, it follows from eq. (4.27) that all Moyal-Weyl products get replaced by matrix multiplication. Second, there appears a new term $f^{\mu\nu}{}_\rho J^\rho$ in the operator product expansion of currents (4.29). This term leads to an extra contribution of the form $f_{\mu\nu\rho}A^\mu A^\nu A^\rho$ in the scattering amplitude of three massless open string modes. Consequently, the resulting effective action is not given by Yang-Mills theory on a fuzzy 2-sphere alone but involves also a Chern-Simons like term. Finally, all momenta k_μ that arise from contractions of currents with primaries in the flat space computation, must be substituted by the representation matrices t_μ^j (see eq. (4.30)). After

Fourier transformation, the momenta turned into derivatives, and similarly our matrices t_μ^j give rise to the generators (4.11) of infinitesimal rotations.

After these sketchy remarks on the derivation, let us now display the final answer (many more details can be found in [75]). For M branes of type J on top of each other, the results of a complete computation can be summarized in the following formula (see [83] for normalizations),

$$\mathcal{S}_{(M,J)} \;=\; \mathcal{S}_{\mathrm{YM}} + \mathcal{S}_{\mathrm{CS}} \;=\; \frac{\pi^2}{\mathsf{k}^2 \, d_J \, M} \; \left(\frac{1}{4} \; \mathrm{tr}\; (\mathrm{F}_{\mu\nu} \; \mathrm{F}^{\mu\nu}) - \frac{i}{2} \; \mathrm{tr}\; (f^{\mu\nu\rho} \; \mathrm{CS}_{\mu\nu\rho}) \right) \qquad (4.31)$$

where $d_J = 2J + 1$. We defined the 'curvature form' $\mathrm{F}_{\mu\nu}$ by the expression

$$\mathrm{F}_{\mu\nu}(\mathrm{A}) \;=\; i \, \mathrm{L}_\mu \mathrm{A}_\nu - i \, \mathrm{L}_\nu \mathrm{A}_\mu + i \, [\mathrm{A}_\mu \,,\, \mathrm{A}_\nu] + f_{\mu\nu\rho} \mathrm{A}^\rho \qquad (4.32)$$

and a non-commutative analogue of the Chern-Simons form through

$$\mathrm{CS}_{\mu\nu\rho}(\mathrm{A}) \;=\; \mathrm{L}_\mu \mathrm{A}_\nu \, \mathrm{A}_\rho + \frac{1}{3} \, \mathrm{A}_\mu \, [\mathrm{A}_\nu \,,\, \mathrm{A}_\rho] - \frac{i}{2} \, f_{\mu\nu\sigma} \, \mathrm{A}^\sigma \, \mathrm{A}_\rho \quad . \qquad (4.33)$$

The three fields $\mathrm{A}^\mu = \sum \mathrm{a}_{ja}^\mu \mathrm{Y}_a^j$ on the fuzzy 2-sphere $S_J^2 \in \mathbb{R}^3$ take values in $\mathrm{Mat}(M)$, i.e. all their Chan-Paton coefficients a_{ja}^μ should be considered as $M \times M$ matrices. Hence, the fields A^μ are elements of $\mathrm{Mat}(M) \otimes \mathrm{Mat}(2J + 1)$. Infinitesimal rotations L_μ act exclusively on the fuzzy spherical harmonics Y_a^j and they commute with the Chan-Paton coefficients a_{ja}^μ.

It follows from a straightforward computation that the action (4.31) is invariant under the gauge transformations

$$\mathrm{A}_\mu \;\to\; \mathrm{A}_\mu + i \, \mathrm{L}_\mu \lambda \;+\; i \, [\mathrm{A}_\mu \,,\, \lambda] \quad \text{for} \quad \lambda \in \mathrm{Mat}(M) \otimes \mathrm{Mat}(2J + 1) \quad .$$

Note that the 'mass term' in the Chern-Simons form (4.33) guarantees the gauge invariance of $\mathcal{S}_{\mathrm{CS}}$. On the other hand, the effective action (4.31) is the unique linear combination of $\mathcal{S}_{\mathrm{YM}}$ and $\mathcal{S}_{\mathrm{CS}}$ from which mass terms cancel. As we shall see below, it is this special feature of our action that allows solutions describing translations of the branes on the group manifold. The action $\mathcal{S}_{\mathrm{YM}}$ was already considered in the non-commutative geometry literature [84, 85, 86, 87, 88], where it was derived from a Connes spectral triple and viewed as describing Maxwell theory on the fuzzy sphere. Arbitrary linear combinations of non-commutative Yang-Mills and Chern-Simons terms were considered in [89].

Classical solutions and brane dynamics. Stationary points of the action (4.31) describe condensation processes on a brane configuration $Q = (M, J)$ which drive the whole system into another configuration Q'. To identify the latter, we have two different types of information at our disposal. On the one hand, we can compare the tension of D-branes in the final configuration Q' with the value of the action $\mathcal{S}_Q(\Lambda)$ at the classical solution Λ. On the other hand, we can look at fluctuations around the chosen stationary point and compare their dynamics with the low-energy effective theory $\mathcal{S}_{Q'}$ of the brane configuration Q'. In formulas, this means that

$$\mathcal{S}_Q(\Lambda + \delta A) \overset{!}{=} \mathcal{S}_Q(\Lambda) + \mathcal{S}_{Q'}(\delta A) \quad \text{with} \quad \mathcal{S}_Q(\Lambda) \overset{!}{=} \ln \frac{g_{Q'}}{g_Q} \quad . \tag{4.34}$$

The second requirement expresses the comparison of tensions in terms of the g-factors [90] of the involved conformal field theories

$$g_{(M,J)} := M \, g_J := M \, \langle \phi_{0,0}(z, \bar{z}) \rangle_J = M \, \frac{S_{J0}}{\sqrt{S_{00}}} \quad . \tag{4.35}$$

All equalities must hold to the same order in $(1/k)$ that we used when we constructed the effective actions. We say that the brane configuration Q decays into Q' if Q' has lower mass, i.e. whenever $g_{Q'} < g_Q$.

In terms of the world-sheet description, each classical solution of the effective action is linked to a conformal boundary perturbation in the CFT of the brane configuration Q (cf. results in [90]). Adding the corresponding boundary terms to the original theory causes the boundary condition to change so that we end up with the boundary conformal field theory of another brane configuration Q'. Recall, however, that all these statements only apply to a limiting regime in which the level k is sent to infinity.

Let us now become more specific. From eq. (4.31) we obtain the following equations of motion for the elements $A^\mu \in \mathrm{Mat}(M) \otimes \mathrm{Mat}(2J + 1)$

$$\mathrm{L}_\mu \, \mathrm{F}^{\mu\nu} + [\,\mathrm{A}_\mu \,, \mathrm{F}^{\mu\nu}\,] \;=\; 0 \quad . \tag{4.36}$$

It is easy to find two very different types of solutions. For the first one, the gauge fields A_μ are of the special form $A^\mu = \mathsf{a}_{00}^\mu Y_0^0$ with three pairwise commuting Chan-Paton matrices $\mathsf{a}_{00}^\mu \in \mathrm{Mat}(M)$. These solutions come as a $3M$ parameter family corresponding to

the number of eigenvalues appearing in $\{a_{00}^\mu\}$. The same kind of solutions appears for branes in flat backgrounds. They describe rigid translations of the M branes on the group manifold. Since each brane's position is specified by 3 coordinates, the number of parameters matches nicely with the interpretation. Moving branes around in the background is a rather trivial symmetry and the corresponding conformal field theories are easy to construct, either directly (see remarks after eq. (4.28)) or through conformal perturbation theory (see [58, 91]). As we have mentioned before, the existence of such continuous families of solutions is guaranteed by the absence of the mass term in the full effective action.

There exists a second type of solutions to eqs. (4.36) which is a lot more interesting. In fact, any $M(2J+1)$-dimensional representation of the Lie algebra su(2) can be used to solve the equations of motion. Their interpretation was found in [75]. Here, we describe the answer for a stack of M branes of type $J = 0$, i.e. of M point-like branes at the origin of SU(2). In this case, $A_\mu \in \mathrm{Mat}(M) \otimes \mathrm{Mat}(1) \cong \mathrm{Mat}(M)$ so that we need an M-dimensional representation of su(2) to solve the equations of motion. Let us choose the M-dimensional irreducible representation of isospin $J_M = (M-1)/2$. Our claim then is that this drives the initial stack of M point-like branes at the origin into a final configuration containing only a single brane wrapping the sphere of type J_M, i.e.

$$(M,\ J = 0) \longrightarrow (1,\ J_M = (M-1)/2) \ \ .$$

Support for this statement comes from both the open string sector and the coupling to closed strings. In the open string sector one can study small fluctuations δA_μ of the fields $A_\mu = \Lambda_\mu + \delta A_\mu \in \mathrm{Mat}(M)$ around the stationary point $\Lambda_\mu \in \mathrm{Mat}(M)$. If Λ_μ form an irreducible representation of su(2), we find

$$\mathcal{S}_{(M,0)}(\Lambda_\mu + \delta A_\mu) \ = \ \mathcal{S}_{(1,J_M)}(\delta A_\mu) \ + \ \mathrm{const} \ \ .$$

In the closed string channel, the leading term (in the $1/\mathsf{k}$-expansion) from the exact 'mass' formula (4.35) gives [75]

$$\ln \frac{g_{(1,J_M)}}{g_{(M,0)}} \ = \ -\frac{\pi^2}{6}\frac{M^2 - 1}{\mathsf{k}^2} \ = \ \mathcal{S}_{(M,0)}(\Lambda_\mu) \ \ .$$

Note that the mass of the final state is lower than the mass of the initial configuration. This means that a stack of M point-like branes on a 3-sphere is unstable against decay

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into a single spherical brane. Stationary points of the action (4.31) and the formation of spherical branes on S^3 were also discussed in [92, 93, 94]. Similar effects have been described for branes in RR background fields [95]. The advantage of our scenario with NSNS background fields is that it can be treated in perturbative string theory so that string effects may be taken into account (see [96, 97, 98] and below).
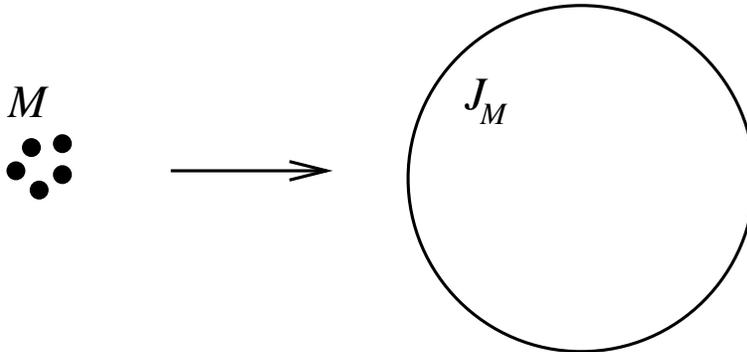
Figure 7: M point-like branes stacked at the origin of a weakly curved $S^3 \sim \mathbb{R}$ are unstable against decay into a single spherical brane with label $J_M = (M-1)/2$.

Dynamics in stringy regime and K-theory. Now we would like to understand the dynamics of branes in the stringy regime where k is finite. Proceeding along the lines of the previous discussion would force us to include higher order corrections to the effective action. Unfortunately, such a complete control of the brane dynamics in the stringy regime is out of reach.

But we could be somewhat less ambitious and ask whether at least some of the solutions we found in the large volume limit possess a deformation into the small volume theory and if so, which boundary conformal field theories they correspond to. It turns out that this is possible for all the processes that are obtained from constant gauge fields on the brane. In this way we may overlook new stationary points of the stringy effective action that have no well behaved large k limit. On the other hand, the reduced program has a positive and very beautiful solution that is known from the work on the Kondo effect.

The Kondo model is designed to understand the effect of magnetic impurities on the

low-temperature conductance properties of a 3D conductor. The latter can have electrons in a number k of conduction bands. If the impurities are far apart, their effect may be understood within an s-wave approximation of scattering events between a conduction electron and the impurity. This allows to formulate the whole problem on a 2-dimensional world-sheet for which the coordinates (u, v) are associated with the time and the radial distance from the impurity, respectively. One can build several currents out of the basic fermionic fields. Among them is a *spin current* $\vec{J}(u, v)$. It satisfies the relations (4.29) of a $\widehat{\mathrm{SU}}(2)_{\mathsf{k}}$ current algebra. This spin current is the one that couples to the magnetic impurity of spin J_M which is sitting at the boundary $v = 0$,

$$S_{\mathrm{pert}} \ \sim \ \lambda \int_{-\infty}^{\infty} du\, \Lambda_\mu J^\mu(u, 0) \quad . \tag{4.37}$$

Here, $\vec{\Lambda} = (\Lambda_\mu, \mu = 1, 2, 3)$ is a $(2J_M + 1)$-dimensional irreducible representation of $su(2)$ and the parameter λ controls the strength of the coupling. The term (4.37) is identical to the coupling of open string ends to a background gauge field $\mathsf{A}_\mu = \Lambda_\mu \in \mathrm{Mat}(2J_M + 1)$. Hence, Λ_μ may be interpreted as a constant gauge field on a Chan-Paton bundle of rank $M = 2J_M + 1$.

Fortunately, a lot of techniques have been developed to deal with perturbations of the form (4.37) going back even to the work of Wilson [99]. In fact, this problem is what Wilson's renormalization group techniques were originally designed for. From the old analysis we know that there are two different cases to be distinguished. When $2J_M > \mathsf{k}$ ('under-screening') the low-temperature fixed point of the Kondo model appears only at infinite values of λ. On the other hand, the fixed point is reached at a finite value $\lambda = \lambda^*$ of the renormalized coupling constant λ if $2J_M \leq \mathsf{k}$ (exact- or over-screening resp.). In the latter case, the fixed points are described by non-trivial (interacting) conformal field theories. One can summarize the results on the spectrum of the fixed points through the so-called 'absorption of the boundary spin'-principle [100, 101]

$$\mathrm{tr}\ _{V_{J_M} \otimes \mathcal{V}_j}\left(q^{H_0 + H_{\mathrm{pert}}}\right)^{\mathrm{ren}}_{\lambda = \lambda^*} := \sum_l N_{J_M j}{}^l \chi_l(q) \quad . \tag{4.38}$$

Here, $H_0 = L_0 + c/24$ is the unperturbed Hamiltonian, the superscript $^{\mathrm{ren}}$ stands for 'renormalized' and V_{J_M} denotes the representation space of the representation J_M of $su(2)$.

In the rule (4.38), $\mathcal{V}_j$ can be any of the sectors in the state space $\mathcal{H}$ of the boundary theory. Formula (4.38) means that our perturbation with some irreducible representation with spin J_M interpolates continuously between a building block $M\chi_j(q)$ of the partition function of the UV-fixed point (i.e. $\lambda = 0$) and the sum of characters on the right hand side of the previous formula,

$$M\,\chi_j(q) \; \longrightarrow \; \sum_l N_{J_M j}{}^l \, \chi_l(q) \quad, \tag{4.39}$$

where $M = 2J_M + 1$. In particular, we can use this formula to determine the decay product of a stack of M point-like branes. Since each of the two string ends can be attached to any of the M branes, the partition function of the whole stack is M^2 times the partition function for a single brane. For this system we find

$$
\begin{aligned}
Z_{(M,0)}(q) \;&=\; M^2\, Z_{(1,0)}(q) \;=\; M^2\, \chi_0(q) \\
&\longrightarrow\; M\,\chi_{J_M}(q) \;\longrightarrow\; \sum_j N_{J_M J_M}{}^j \chi_j(q) \;=\; Z_{(1,J_M)}(q) \quad.
\end{aligned}
$$

We applied the rule (4.39) twice because both endpoints of an open string couple to the background field. The result can be summarized in the following process

$$(M,\,0) \;\longrightarrow\; (1,\,J_M) \quad. \tag{4.40}$$

This is formally identical to the decay process we found in the large k regime, except that this time $2J_M$ is bounded from above by the level k. Our final answer may not seem very surprising, but it is still remarkable that there exists such a solid derivation even deep in the stringy regime.

Charges and twisted K-theory. The analysis of brane dynamics on S^3 brings us to the last subject of this lecture, namely the issue of brane charges. It is a traditional conception to measure brane charges through their coupling to closed string (RR) modes. When applied to branes on S^3, however, this naive idea of charge seems to fail. In particular, the couplings of branes to closed strings are not quantized [73], at least as long as we are not at the limit point of infinite level. For this reason, an alternative definition of brane charges was proposed in [96]. There it was suggested to define charges

as quantities which are invariant under conformal perturbations in the world-sheet theory. More precisely, two brane configurations Q and Q' are called *dynamically equivalent* if there exists a conformal boundary perturbation that relates the two boundary theories. The space of all (anti-)brane configurations modulo this dynamical equivalence is the group C of brane charges. The latter is a property of the background.

To determine the brane charges on S^3 along with the group they generate, let us apply the rule (4.40) to a supersymmetric theory on a 3-sphere with $K = \mathsf{k} + 2$ units of NSNS flux passing through. In this case one can have (anti-)branes wrapping $\mathsf{k} + 1$ different conjugacy classes labeled by $J = 0, 1/2, \ldots, \mathsf{k}/2$ (see e.g. [96]). If we stack more and more point-like branes at the origin, the radius of the sphere that is wrapped by the resulting object will first grow, then decrease, and finally a stack of $\mathsf{k} + 1$ point-like branes at e will decay to a single point-like object at $-e$ (see Figure 8). By taking orientations into account, one can see that the final point-like object is the translate of an anti-brane at e. Hence, we conclude that the stack of $\mathsf{k} + 1$ point-like branes at e has decayed into a single point-like anti-brane at $-e$.
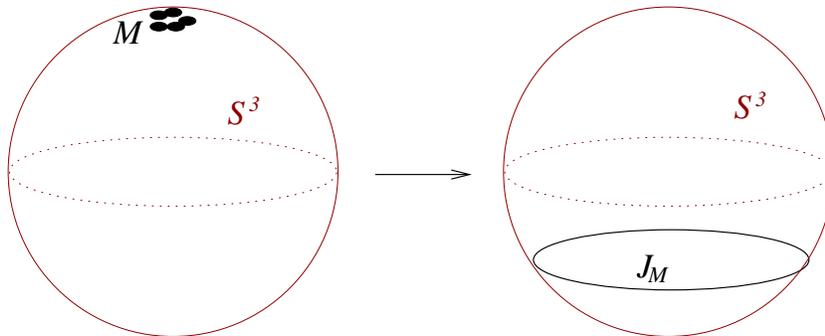
Figure 8: Brane dynamics on S^3: A stack of point-like branes at e can decay into a single spherical object. The distance of the latter increases with the number of branes in the stack until one obtains a single point-like object at $-e$.

In this concrete example, we may assign charge 1 to the point-like branes at e and if we want the charge to be conserved, the decay product of $\mathsf{k} + 1$ such point-like branes must have charge $\mathsf{k}+1$. On the other hand we identified the latter with a single anti-brane

which has charge -1. Thus we have to identify $\mathsf{k}+1$ and -1 which means that charge is only well-defined modulo $K = \mathsf{k} + 2$ [96], i.e.

$$C(\mathrm{SU}(2), K) \;=\; \mathbb{Z}_K \quad .$$

According to a proposal of Bouwknegt and Mathai [102], the brane charges on a background X with non-vanishing NSNS 3-form field H take values in some twisted K-groups $K_H^*(X)$ which feel the presence of $H \in H^3(X, \mathbb{Z})$ (see also [71, 103] for related proposals when H is torsion). For S^3 this twisted K-group is known to be $K_H^*(\mathrm{SU}(2)) = \mathbb{Z}_K$ and hence

$$C(\mathrm{SU}(2), K) \;=\; K_H^*(\mathrm{SU}(2))$$

as predicted in [102]. Many more details about this twisted K-theory, its computation through spectral sequences and the relation with string theory can be found in [104]. Geometric construction of brane charges on S^3 and other group manifolds have also been discussed in [105, 106, 96, 107].

5 Some further results and directions

During the last years, all ingredients of the technology we have used so far were generalized in a variety of different directions. Our aim in this final part is to touch upon some of these extensions, trying to provide some ideas of the current status in the field along with a very incomplete guide to the existing literature.

We shall begin with several remarks on the various approaches that more recent research has followed to generalize the construction of boundary conformal field theories beyond the Cardy-case. These include orbifold and simple current techniques and the use of conformal embeddings. Once more, group manifolds serve as a stage on which we can nicely present some of the progress that has been made in obtaining exact solutions. These lectures finally end with a short summary of some first steps in extending the whole program to non-compact backgrounds, focusing on some of the main new ingredients and difficulties.

5.1 Solutions beyond the Cardy case

For the exact solutions we have outlined and applied above, we had to make several assumptions. Not only were we restricted to a very particular class (3.36) of bulk modular invariants, but also it was necessary to preserve the maximal chiral symmetry $\mathcal{W}$ of the model through our eqs. (3.16). The classification of all conformal boundary theories, on the other hand, would require to impose the gluing condition (3.15) for the Virasoro field only. Since chiral symmetries of solvable theories with $c \geq 1$ are much larger than the Virasoro algebra, there remains a lot of room for new symmetry breaking conformal boundary conditions. In fact, constructing boundary theories with the minimal Virasoro symmetry tends to lead into non-rational models which are notoriously difficult to control. Nevertheless, some progress has been made in this direction. Boundary conditions with the minimal Virasoro symmetry were systematically investigated for 1-dimensional flat targets [51, 52, 108, 109]. In spite of this remarkable progress, such a complete control over conformal boundary conditions should be considered exceptional and it is probably very difficult to achieve for more complicated backgrounds. Less ambitious programs therefore focus on intermediate symmetries which are carefully selected so as to render

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the boundary theory rational. We shall sketch the two main approaches in this direction and then end this subsection with a few remarks on other solution generating techniques.

Orbifold constructions. In this part we discuss some elements of branes in so-called simple current orbifolds. It is advantageous at first, to think of the following constructions as providing boundary theories for a new class of bulk modular invariants, more general than eq. (3.36). Later we shall then argue that applications of the general formalism also include the construction of new boundary theories for backgrounds with a bulk invariant of the form (3.36) by analyzing the model with respect to a certain orbifold chiral algebra.

Investigations of branes in orbifolds have a long history and it is not possible to give a complete account here of all the existing results. Much of the work was devoted to orbifold constructions in flat space (see e.g. [110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120]). The basis for most of these developments were laid in [110] which uses earlier ideas originating from [121, 122]. Open string theory in more general conformal field theory orbifolds was pioneered by Sagnotti and collaborators starting from [121] (see also e.g. [16, 17, 123]). Important contributions were made later by Behrend et al. [56, 65] and by Fuchs et al. [124, 125, 126]. The latter extends the simple current techniques that were developed for closed strings in [127, 128, 129, 130] to the case of open strings (see also [131, 132, 153, 196]).

Geometrically, branes in an orbifold background are understood through branes on its covering space. More specifically, we represent an orbifold brane by several pre-images on the covering space which are mapped onto each other by the action of the orbifold group. If the latter has fixed points, the corresponding branes can be resolved so that several different branes are associated with the same pre-images on the covering space. These basic ideas are common to all orbifolds, and they apply in particular to exactly solvable models in which the orbifold action is generated by simple currents. Before we state some of the main results that have been obtained in this context, we need a few new notions from conformal field theory.

Within the set $\mathcal{J}$ of $\mathcal{W}$ sectors one often finds non-trivial elements $\gamma \in \mathcal{J}$ such that the fusion product of γ with any other $j \in \mathcal{J}$ gives again a single primary $\gamma \circ j = \gamma j \in \mathcal{J}$. Such elements γ are called *simple currents* and the set of all these simple currents forms

an abelian group whose product is inherited from the fusion product. From now on, let Γ denote the group of simple currents or some subgroup thereof. Through the fusion of representations, the group Γ acts on the index set $\mathcal{J}$. This action can be diagonalized by the S-matrix in the sense that

$$S_{\gamma i j} = e^{2\pi i Q_\gamma(j)} S_{ij} \quad . \tag{5.1}$$

The quantity $Q_\gamma(j)$ is known as the *monodromy charge* and it may be computed from the conformal weights by means of the formula $Q_\gamma(j) = h_j + h_\gamma - h_{\gamma j} \mod 1$. As one can see from definition (5.1), the monodromy charge gives rise to a character of the group Γ.

Simple current techniques allow to solve the boundary problems for bulk partition functions of the following form (see e.g. [130])

$$Z^{\mathrm{orb}}(q, \bar{q}) = \sum_{[j], Q_\Gamma(j) = 0} |\mathcal{S}_{[j]}| \left(\sum_{j' \in [j]} \chi_{j'}(q) \right) \left(\sum_{\bar{j}' \in [\bar{j}]} \chi_{\bar{j}'}(\bar{q}) \right) \quad . \tag{5.2}$$

Here, we use the symbol $[j]$ to denote the orbit of j under the action of Γ and we define the stabilizer subgroup $\mathcal{S}_{[j]} \subset \Gamma$ by

$$\mathcal{S}_{[j]} = \{ \gamma \in \Gamma \mid \gamma \circ j = j \} \quad . \tag{5.3}$$

Note that the partition function (5.2) does not have the simple form (3.36) so that Cardy's theory for the classification and construction of branes does not apply directly.

But if we choose the gluing map Ω such that the assumption (3.37) is satisfied, then the solution to the corresponding boundary problem is inherited from Cardy's solution of the theory with bulk invariant (3.36) through a simple construction that follows very closely the geometric procedure we sketched above (our presentation follows [133, 134, 135]). In fact, let us recall that the boundary states in Cardy's theory are given by

$$|J\rangle_\Omega = \sum_i \frac{S_{Jj^+}}{\sqrt{S_{0j^+}}} |j\rangle\!\rangle_\Omega \quad . \tag{5.4}$$

Here, J runs through $J \in \mathcal{J}$ and we have seen that in some sense it encodes the brane's transverse position. The geometric ideas suggest to introduce

$$|[J]\rangle_\Omega^{\mathrm{orb}} = \frac{1}{\sqrt{|\Gamma|}} \sum_{\gamma \in \Gamma} |\gamma J\rangle_\Omega \quad . \tag{5.5}$$

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On the right hand side, we sum over the orbit of 'pre-images' of the brane $[J]$. To see that the sum (5.5) is a boundary state of the orbifold theory (5.2), we insert the expression (5.4) into (5.5). Using the relation (5.1) we find

$$|[J]\rangle_\Omega^{\mathrm{orb}} = \frac{1}{\sqrt{|\Gamma|}} \sum_j \Big(\sum_{\gamma \in \Gamma} e^{2\pi i Q_\gamma(j)} \Big) \frac{S_{Jj^+}}{\sqrt{S_{0j^+}}} |j\rangle\!\rangle_\Omega \ . \tag{5.6}$$

The term in brackets is non-zero, if and only if $Q_\Gamma(j) = 0$. Hence, all the generalized coherent states that appear on the right hand side of the previous formula are indeed associated with sectors in the bulk theory (5.2).

From the boundary state (5.5) it is easy to work out the corresponding boundary partition functions of the orbifold model. They are given by

$$Z_{[I][J]}^{\mathrm{orb}}(q) = \sum_{\gamma,k} N_{I\ k}^{\ \gamma J} \chi_k(q) \ . \tag{5.7}$$

This agrees precisely with the prediction from the geometric picture of branes on orbifolds. In fact, the I, J can be considered as geometric labels specifying the position of the brane on the covering space. To compute spectrum of two branes $[I]$ and $[J]$ of the orbifold theory, we lift $[I]$ to one of its pre-images I on the covering space and include all the open strings that stretch between this fixed brane I on the cover and an arbitrary pre-image γJ of the second brane $[J]$.

It is important to notice that in many cases the boundary conditions $[I]$ are not elementary and can be further resolved, i.e. there exists a larger set of boundary theories such that $[I]$ can be written as a superposition of boundary theories with integer coefficients. This happens whenever the stabilizer subgroup $\mathcal{S}_{[I]}$ is non-trivial. In the absence of discrete torsion, the elementary branes resolving the boundary condition $[I]$ are labeled by irreducible representations of $\mathcal{S}_{[I]}$.[1] Geometrically, this corresponds to the fact that the Chan-Paton factors of branes at orbifold fixed points can carry different representations of the stabilizer subgroup. General formulas for the resolved boundary states and the corresponding boundary partition functions can be found e.g. in [56, 124, 65].

Under some additional technical assumptions, operator product expansions of boundary fields in simple current orbifolds have been studied in [134, 135]. The results in [135]

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[1]More general possibilities including discrete torsion have been discussed in [113, 114, 116, 115, 118, 119]. The extension to conformal field theory backgrounds can be found in [136]

also cover part of the results on boundary operator product expansions for the D-type modular invariant of minimal model [64]. There exist several classes of important models to which all these findings on branes in simple current orbifolds apply. Among them are the Gepner models [137] which are used to study aspects of string theory on complete intersection Calabi-Yau spaces. The projection method that is encoded in formula (5.6) was used in [18] to obtain boundary states of Gepner models. Some of these states had to be resolved and at least for so-called A-type branes this was done explicitly in [133] (see also [138, 139, 140]). First geometric interpretations for boundary states in Gepner models were found in [141]. There has been a lot of substantial, more recent work on branes in Gepner models (see e.g. [142, 136, 143, 144] and references therein).

As we have briefly mentioned above, there is one rather interesting application of the constructions we have presented in this subsection. Namely, they can be used to build new branes in backgrounds with an invariant of the form (3.36). For branes in compact group manifolds, this will be addressed below. Here we shall only take a look at the simplest example, namely we show how to recover Neumann boundary conditions for a single free bosonic field from Cardy's solution. Let us recall that our application of Cardy's solution to the flat space theory in section 3.4 only gave theories with Dirichlet boundary conditions. Neumann gluing conditions for a 1-dimensional flat space, i.e. the relations $J(z) = \bar{J}(\bar{z})$, preserve the same amount of symmetry and they should not be much harder to construct. If we compare the two gluing conditions we find that they differ only by a reflection $\lambda(J)(z) = -J(z)$. This defines an action of the group $\mathbb{Z}_2$ on the chiral algebra which obviously comes from the geometric reflection symmetry of the underlying background. Operators which contain an even number of oscillators α_n are invariant under the reflection and we denote the corresponding chiral algebra by $\mathcal{W} = \widehat{U}(1)^{\mathbb{Z}_2}$. The idea is now to start with the Cardy theory for trivial gluing conditions on $\mathcal{W}$. The above constructions then turn out to furnish all maximally symmetric branes for the theory whose target is real line.

The chiral algebra $\widehat{U}(1)^{\mathbb{Z}_2}$ is known to possess the following sectors (see e.g. [145]). To begin with, there is a continuous family of sectors which are labeled by positive momenta $k > 0$. The vacuum sector of the U(1) theory splits into two sectors of $\widehat{U}(1)^{\mathbb{Z}_2}$ which we denote by $0^\pm$. Finally, there are two twisted sectors $\tau^\pm$. Even though not all the

data from the representation theory of $\widehat{\mathrm{U}}(1)^{\mathbb{Z}_2}$ have been worked out explicitly, we can in principle apply Cardy's theory to construct boundary theories for the charge conjugate bulk modular invariant. The latter is known to appear when we describe closed strings moving on the quotient $\mathbb{R}/\mathbb{Z}_2$. Cardy's theory provides us with a continuum of boundary states $|x_0\rangle$ which are labeled by some positive real number $x_0 > 0$. These can be identified with point-like branes at regular points of the quotient space. In addition, there are four discrete boundary states $|0^{\pm}\rangle$ and $|N^{\pm}\rangle$. Among them are two point-like branes sitting at the singularity and a pair of 1-dimensional branes which extend throughout the whole background.

Now we would like to descend from this theory on $\mathbb{R}/\mathbb{Z}_2$ to some simple current orbifold. Note that $\widehat{\mathrm{U}}(1)^{\mathbb{Z}_2}$ possesses a simple current $\gamma = 0^-$ which acts on $\mathcal{J}$ according to $\gamma \circ 0^{\pm} = 0^{\mp}, \gamma \circ \tau^{\pm} = \tau^{\mp}$ and leaves all sectors $k > 0$ from the continuous series invariant. If we use this simple current in our formulas above, the bulk modular invariant (5.2) coincides with the diagonal modular invariant (3.7) of an uncompactified free bosonic field. According to the general rules we stated above, the simple current orbifold possesses two boundary states which are associated with the two orbits $[0^+]$ and $[\tau^+]$ of discrete sectors. These boundary states describe a point-like brane at $x_0 = 0$ and the 1-dimensional brane on $\mathbb{R}$ that we were after. In addition, each sector $k > 0$ gives rise to two boundary states labeled by the irreducible representations of $\mathcal{S}_{[k]} = \mathbb{Z}_2$. These belong to point-like branes at points $\pm x_0$, respectively. In this way, all maximally symmetric branes on $\mathbb{R}$ can be constructed in terms of representation theoretic data of the chiral algebra $\widehat{\mathrm{U}}(1)^{\mathbb{Z}_2}$.

Twisted branes in WZW models. The non-abelian generalization of the ideas that allowed to obtain Neumann boundary conditions from Cardy's solution provides us with new maximally symmetric branes on group manifolds. Recall that maximally symmetric boundary conditions in the WZW model require to choose some gluing automorphism Ω of the Lie algebra $\mathfrak{g}$ so that we can glue holomorphic and anti-holomorphic currents along the boundary. Even though most of our discussion was restricted to $\Omega = \mathrm{id}$, we commented briefly on more general situations in which $\Omega = \mathrm{Ad}_g$ is an inner automorphism (see remarks around eq. (4.28)). But this does not exhaust all possibilities. In fact, the

most general form of the gluing condition is

$$\Omega \;=\; \lambda \circ \mathrm{Ad}_g \quad \text{for some} \quad g \in G \tag{5.8}$$

with some outer automorphism λ of $\mathfrak{g}$, i.e. an automorphism that does not come from conjugation with some element g. For simple compact Lie algebras $\mathfrak{g}$, such outer automorphisms come with symmetries σ of the Dynkin diagram of $\mathfrak{g}$ and it suffices to let λ run through such diagram automorphisms λ_σ (see e.g. [146]). The Dynkin diagrams for $A_{n>1}, D_{n>4}, E_6, E_8$ possess only one non-trivial symmetry so that in these cases we shall find one new family of so-called *twisted* branes. D_4 possesses 2 symmetries, while there are no non-trivial actions on all the other Dynkin diagrams. The absence of such symmetries for A_1 implies that we will not find any more maximally symmetric branes on SU(2). If the group G consists of several simple factors, one can have further maximally symmetric branes whenever some of the factors coincide. In such cases, holomorphic currents from the different isomorphic factors of the same level may be permuted before gluing them to their anti-holomorphic partners. The corresponding 'permutation branes' were studied in [144] and many of our statements below hold for such boundary theories as well.

Before we describe the results from boundary conformal field theory, let us briefly look at the geometric scenario these boundary conditions are associated with. We have seen in the third lecture that branes constructed with $\lambda = \mathrm{id}$ are localized along conjugacy classes or translates thereof (if $\mathrm{Ad}_g \neq \mathrm{id}$). It turns out that the general case has an equally simple and elegant interpretation [66]. Note that after exponentiation, any automorphism Ω of the Lie algebra $\mathfrak{g}$ furnishes an automorphism Ω^G of the group G. Following the same steps as in the third lecture, one can then show that the gluing map Ω forces the string ends to stay on one of the following Ω-twisted conjugacy classes

$$\mathcal{C}_u^\Omega \;:=\; \{\, g\,u\,\Omega^G(g^{-1}) \mid g \in G \,\} \;\; .$$

The subsets $\mathcal{C}_u^\Omega \subset G$ are parametrized through equivalence classes of group elements u where the equivalence relation between two elements $u, v \in G$ is given by: $u \sim_\Omega v$ iff $v \in C_u^\Omega$. This parameter space U^Ω of equivalence classes is not a manifold, i.e. it contains singular subspaces at which the geometry of the associated twisted conjugacy classes changes. In the example of SU(2), the parameter space for conjugacy classes is an

interval and generic conjugacy classes are spherical, but they degenerate to a single point at the two end-points of the interval. Similar issues for twisted and untwisted branes on SU(3) have been analyzed in great detail in [147] (see also [148, 149]). For us all this rich structure is of little concern. It suffices to know that generic (twisted) conjugacy classes have a transverse space of dimension $r_\sigma(G)$ given by the number of orbits that the vertices of the Dynkin diagram form under the action of the diagram symmetry σ. This means in particular that $r_{\mathrm{id}}(G) = \mathrm{rank}(G)$. Let us also note that (twisted) conjugacy classes come equipped with a B-field.

As in the SU(2) example, only a discrete set of these (twisted) conjugacy classes on G can be wrapped by a brane. This certainly comes out of the exact constructions [126], but it can also be understood as a quantization condition within a semi-classical approach (see [150, 76] for a related analysis) or from the brane's stability [152]. All these arguments show that there is only a finite set of allowed branes whose number depends on the level k. They are labeled by points on a finite $r_\sigma(G)$-dimensional lattice which can be considered as a discrete version of the transverse space for a generic (twisted) conjugacy class. The precise mathematical nature of the labels for non-trivially twisted branes has been investigated by several groups (see [126, 153, 154]). Formulas for their boundary states and partition functions were originally provided in [126] though their expressions are not fully explicit. More efficient constructions, at least for a large number of cases, have been spelled out in [153, 154, 151]. The boundary operator product expansions are known for branes wrapping ordinary conjugacy classes. Since these theories are of Cardy-type, the coefficients in the operator products are obtained from the fusion matrix, as usual (see subsection 3.4). For twisted conjugacy classes, similar results are only known in the limit $\mathsf{k} \to \infty$ (see [83] and below).

Going into more details of the exact solutions would require too much of new terminology. But to sketch the general picture, we can once more look at the limit of infinite level k [155, 83] where many results may be stated in more classical terms. In this regime, the number of branes becomes infinite and they are labeled by the representations of another group $G_\sigma \subset G$. It consists of all elements $g \in G$ which are invariant under the diagram automorphism λ_σ^G of the group G.

We can see this group G_σ emerging on a semiclassical level already. In fact, it is not

difficult to show that a generic twisted conjugacy class "close to" the twisted conjugacy class of the group unit can be represented in the form $\mathcal{C}^{\lambda_\sigma} = G \times_{G_\sigma} \mathcal{C}'$. Here, $\mathcal{C}'$ is an ordinary conjugacy class of G_σ and G_σ acts on G by multiplication from the right. In other words, $\mathcal{C}^{\lambda_\sigma}$ may be considered as a bundle over G/G_σ with fiber $\mathcal{C}'$. In the $\mathsf{k} \to \infty$ limit, we keep $\mathcal{C}'$ small by rescaling its radius. Arguments similar to the ones discussed in subsection 4.1 (see also [76]) show that $\mathcal{C}'$ then turns into a co-adjoint orbit with its usual linear Poisson structure. At the same time, the volume of G/G_σ grows with k so that the corresponding Poisson structure scales down and vanishes in the limit. After quantization, we obtain a bundle with non-commutative fibers. It possesses a classical base G/G_σ which is the same for all branes, but the fibers depend on $\mathcal{C}'$ and they are labeled by irreducible representations of the group G_σ.

This semi-classical picture for the twisted conjugacy classes can also be used to motivate the following proposal for the non-commutative world-volume algebra of twisted branes [83] which generalizes the matrix geometries we found for spherical branes on S^3. 'Functions' on the quantized co-adjoint orbits $\mathcal{C}'$ of G_σ are represented by matrices $\mathrm{Mat}(d_J)$ where d_J is the dimension of an irreducible representation of G_σ. The space of such matrices carries an action of G_σ or its Lie algebra $\mathfrak{g}_\sigma$ which is defined as in eq. (4.11). To built the algebra $\mathcal{F}$ of 'functions' on the entire brane, we combine the matrices with the commutative algebra of functions on G and restrict to G_σ invariants,

$$\mathcal{F}_J \;\cong\; \mathrm{Inv}_{\mathfrak{g}_\sigma}\Big(\mathrm{Fun}(G) \otimes \mathrm{Mat}(d_J) \Big) \;. \tag{5.9}$$

Here, the Lie algebra $\mathfrak{g}_\sigma \subset \mathfrak{g}$ acts on $\mathrm{Fun}(G)$ through right derivatives. If we specialize to $SU(2)$, we have $G_\sigma = G$ and hence we recover $\mathcal{F}_J \cong \mathrm{Mat}(2J+1)$. By construction, $\mathcal{F}_J$ is an associative non-commutative algebra and it comes equipped with an action of $\mathfrak{g}$ through left derivatives on $\mathrm{Fun}(G)$.

Under the action of $\mathfrak{g}$, the algebras $\mathcal{F}$ decompose into irreducible multiplets. We met such a decomposition in the case of the fuzzy 2-spheres in eq. (4.12) and also saw by comparison with eq. (4.24) that they mimic the decomposition of boundary partition functions into sectors of the chiral algebra, at least up to certain truncations. Similar statements can be made in the more general case of Ω-twisted branes. The coefficients in the decomposition of the algebra (5.9) are easily worked out and they are combinations

of the so-called branching coefficients for the embedding $G_\sigma \to G$ and the familiar fusion rules of G (see [155, 83] for concrete formulas). Related expressions for finite level were found in [153, 154].

With the existing control over the boundary conformal field theory of maximally symmetric branes on group manifolds it is possible to study their dynamics along the lines of subsection 4.3. The effective non-commutative field theories were found in [83] along with a large number of interesting solutions. It turns out that all branes which are localized along ordinary conjugacy classes, i.e. for which λ is trivial, are obtained from condensates on a stack of point-like branes and hence they carry only a single charge (see [97]). One new charge appears with each diagram automorphism. Once more, all branes associated with the same symmetry σ of the Dynkin diagram appear as bound states of stacks built from a single 'generating' twisted brane. Among the many processes that have been studied in [83] the ones that are generated by constant gauge fields admit a deformation into the stringy regime [97]. As in the case of SU(2), one can employ the absorption of the boundary spin principle (cf. the rule (4.39)) to determine the final brane configurations after condensation. The set of processes one obtains in this way suffice to determine the order of the charge carried by the generating branes (see [97]) and the results agree nicely with the findings from twisted K-theory (see [104] and references therein).

Conformal embeddings and cosets. The construction of all maximally symmetric branes on group manifolds is a remarkable achievement of orbifold and simple current methods. We shall now see that symmetry breaking branes on group manifolds (and in other backgrounds) can be obtained with another classical technique, namely through the systematic use of so-called conformal embeddings. A chiral algebra $\mathcal{W}'$ is said to be conformally embedded into $\mathcal{W}$ if the respective Virasoro elements are mapped onto each other. Such an embedding is called rational if all $\mathcal{W}$-sectors decompose into a *finite* sum of sectors for the conformally embedded algebra $\mathcal{W}'$. Hence, if a rational model with maximal chiral algebra $\mathcal{W}$ is analyzed with respect to $\mathcal{W}'$, it stays rational.

The first application of rational conformal embeddings to the construction of boundary theories can be found in [156] where they were used to break the symmetry of a $c = 2$ torus compactification at some particular radius. A related idea then appeared later in

the work [53] to build symmetry breaking branes on the group manifold SU(2). Actually, Maldacena et al. employ the parafermionic coset SU(2)/U(1), a free bosonic U(1) model and some orbifold ideas in their construction of (unstable) 1- and 3- dimensional branes on SU(2). A generalization to SU(N) which uses the abelian cosets SU(N)/U(1)$^{N-1}$ was proposed in [104]. Our presentation here will follow the approach of [54]. The latter allows to incorporate non-abelian cosets and leads to a very large set of new symmetry breaking branes on group manifolds.

In [53, 104, 54], coset chiral algebras are a central ingredient of the whole procedure. Let us therefore briefly outline a few basics from the coset or GKO construction [158]. Suppose we are given some affine Lie algebra $\widehat{\mathcal{G}}$ or its associated chiral algebra $\mathcal{W}(G)$ along with some affine subalgebra $\mathcal{W}(U) \subset \mathcal{W}(G)$. Then we can look for the maximal chiral subalgebra $\mathcal{W}(G/U)$ within $\mathcal{W}(G)$ which commutes with $\mathcal{W}(U)$. As one can easily check, the algebra $\mathcal{W}(G/U)$ contains a Virasoro field $T^{G/H} = T^G - T^H$ of central charge $c^{G/U} = c^G - c^U$. By construction, the chiral algebra $\mathcal{W}' = \mathcal{W}(G/U) \otimes \mathcal{W}(U)$ is conformally embedded into $\mathcal{W}(G)$. In fact, the Virasoro field $T^{G/U} + T^H$ of $\mathcal{W}'$ coincides with the Virasoro field T^G. Moreover, each representation of $\mathcal{W}^G$ can be shown to decompose into a finite number of representations for $\mathcal{W}' \subset \mathcal{W}(G)$ (see e.g. [7]). The construction of $\mathcal{W}'$ that we have just outlined can be applied repeatedly if we set $U = U_1$ and select a chiral algebra $\mathcal{W}(U_2) \subset \mathcal{W}(U_1)$ etc. In this way, we obtain a large number of conformally embedded chiral algebras

$$\mathcal{W}' = \mathcal{W}(G/U_1) \otimes \mathcal{W}(U_1/U_2) \otimes \cdots \otimes \mathcal{W}(U_{N-1}/U_N) \otimes \mathcal{W}(U_N) \subset \mathcal{W}(G) \ .$$

Except from some subtleties that may arise from the centers of the groups U_s (see e.g. [159, 160]), the formulas in [54] provide a set of boundary theories which preserve such chiral algebras $\mathcal{W}'$.

Here we shall content ourselves with a description of their geometry [161]. As in our construction of $\mathcal{W}'$, we must choose a chain of groups $U_s, s = 1, \ldots, N$, along with homomorphism $\epsilon_s : U_s \to U_{s-1}$ (we set $U_0 = G$). The latter are assumed to induce embeddings of the corresponding Lie algebras. Furthermore, we select an automorpism Ω_s on each group U_s. Given these data, it is possible to construct a set of branes which

preserve an U_N group symmetry. These are localized along the following sets

$$\mathcal{C}^{\underline{\Omega}}_{\underline{\epsilon};\underline{u}} \;\; = \;\; \mathcal{C}^0_{u_0} \cdot \mathcal{C}^1_{u_1} \cdot \ldots \cdot \mathcal{C}^N_{u_N} \;\; \subset \;\; G \quad \text{where} \tag{5.10}$$

$$\mathcal{C}^s_{u_s} \;\; = \;\; \Omega_0 \circ \epsilon_1 \circ \cdots \circ \Omega_{s-1} \circ \epsilon_s (\mathcal{C}^{\Omega_s}_{u_s}) \;\; \subset \;\; G \quad \text{for} \quad u_s \; \in \; U_s$$

and $\mathcal{C}^0_{u_0} = \mathcal{C}^{\Omega_0}_{u_0}$ for $u_0 \in G$. The $\cdot$ indicates that we consider the set of all points in G which can be written as products (with group multiplication) of elements from the various subsets. One should stress that branes may be folded onto the subsets (5.10) such that a given point is covered several times. This phenomenon has been observed for a special case in [53]. In some examples, depending on the choice of $\underline{u}$, several different branes can wrap the same set (5.10). Such a situation occurs e.g. for the volume filling brane on S^3 (which requires an even level k) but is it not understood in general.

The new symmetry breaking boundary theories have various applications, in particular when dealing with groups $G = G_1 \times \cdots \times G_n$ which factorize into several simple factors. Let us note that many interesting solvable string backgrounds are factorizable or orbifolds of factorizable backgrounds. Some boundary states for such theories can be factored accordingly so that they are simply products of boundary states for each of the individual factors. But this does certainly not exhaust all possibilities. Generically, branes preserving the maximal chiral symmetry are factorizing. Only the permutation branes that exist for backgrounds G with several identical simple factors are non-factorizable and maximally symmetric at the same time. Many more interesting examples of non-factorizable branes are obtained from constructions of symmetry breaking branes (see [54, 161]).

There exists another - superficially very different - setup which leads exactly to the same type of problems. It arises by considering a one-dimensional quantum system with a defect (see e.g. [162, 112, 163, 164, 25, 165] and [166, 167] for higher dimensional analogues), or, more generally, two different systems on the half-lines $v < 0$ and $v > 0$ which are in contact at the origin. The defect or contact at $v = 0$ could be totally reflecting, or more interestingly it could be partially (or fully) transmitting. To fit such a system into our general discussion, we apply the usual folding trick (see Figure 9). After such a folding, the defect or contact is located at the boundary of a new system on the half-line. In the bulk, the new theory is simply a product of the two models that were initially placed to both sides of the contact at $v = 0$.
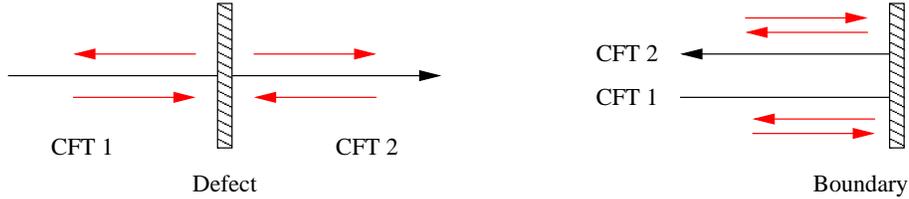
Figure 9: The folding trick relates a system on the real line with a defect to a tensor product theory on the half line.

Factorizing boundary states for the new product theory on the half-line correspond to totally reflecting defects or contacts. With our new boundary states we can go further and couple the two systems in a non-trivial way. Since we always start with conformal field theories with chiral algebras $\mathcal{W}_1 = \mathcal{W}_<$ and $\mathcal{W}_2 = \mathcal{W}_>$ on either side of $v = 0$, it is natural to look for contacts that preserve conformal invariance. This requires to preserve the sum of the two Virasoro algebras of the individual theories. After folding the system, the preserved Virasoro algebra is diagonally embedded into the product theory $\mathcal{W} = \mathcal{W}_1 \otimes \mathcal{W}_2$. Of course, one can often embed a larger chiral algebra $\mathcal{W}'$ and then look for defects that preserve the extended symmetry. This is exactly the setup to which our general ideas apply. Note that they are capable of constructing defect lines which join two conformal field theories with different central charge. Such defects are known to appear on the boundary of an AdS-space when there is a brane in the bulk which extends all the way to the boundary [168, 169, 170, 171]. Simpler examples without jumps in the central charge have also been analyzed in [172, 173].

Finally, let us briefly mention that the branes we have discussed here and in the pervious sections descend to (asymmetric) coset models and therefore many of our insights directly apply to this very large class of backgrounds. Coset theories possess a maximal chiral algebra of the form $\mathcal{W}(G/H)$ and they describe strings moving on the space G/H with non-constant background fields (see e.g. [174, 175, 176, 177, 178] for some early work on the geometry of the bulk theories). The geometry of Cardy-type branes in such cosets was exhibited in [179] (see also [97, 53, 180, 181, 182]). Non-commutative gauge theories for branes in coset models were found and studied in [182] (see also [183, 184] for related

work in a more traditional conformal field theory framework). These effective actions have interesting implications on brane bound state formation in some limiting regime. The resulting structure of possible condensates is much richer than for group manifolds since there are more conserved charges [182]. Many of the processes in [182] admit a deformation into the stringy regime. In fact, an extension of the 'absorption or the boundary spin'-principle (4.38), (4.39) to coset models was formulated in [185] and it was tested against known results on the boundary flows in minimal models [186, 187, 188, 189, 190] (see also [191] for additional examples). More recent work on the construction of branes in coset models and related issues includes [192, 193, 194, 195, 196, 197].

Other approaches to exact solutions. In these notes we presented a conventional approach to the construction of exact solutions in which we start from the bulk and then work our way through to the boundary by first solving the factorization constraints (3.21) for the one-point functions, then computing the boundary partition function through eq. (3.30) and finally solving the relations (3.34) for the structure constants of the boundary operator product expansions. It is interesting, however, to turn the whole procedure upside down and to start from the boundary. This may seem almost hopeless at first, partly because solving the complicated non-linear equations (3.34) requires us to guess some appropriate multiplicities $n_{\alpha\beta}{}^j$ for the sectors of the boundary theory. A closer look reveals, however, that much of this problem can be linearized. Once eqs. (3.34) have been solved for a certain choice of the multiplicities $n_{\alpha\beta}{}^j$, one may go back and determine the coefficients A_j^α from eqs. (3.30). Finally, even the structure constants of the bulk operator products can be calculated through formula (3.22). Such an approach has been suggested by Petkova and Zuber [198]. It was then rephrased and extended systematically in [199, 173]. This whole program is quite elegant and it is somewhat similar in spirit to recent attempts in string field theory to reconstruct the closed from open strings (see e.g. [200]).

There is another very interesting solution generating technique that has been applied very successfully in the past, namely the use of boundary deformation theory (see [58, 91] for some general results). Unlike the ideas we presented above, boundary deformation theory is capable of constructing non-rational boundary theories from rational ones. The

idea is to start from some rational boundary model and to look for exactly marginal operators among its boundary fields. If they exist, they generate a continuous family of new boundary theories. The latter correspond to translates of the original branes if the marginal field is taken from the chiral algebra itself, but they typically break much of the symmetry otherwise. At least in some examples [201, 202, 58] even such symmetry breaking deformations can be constructed perturbatively, to all orders in perturbation theory. The corresponding boundary theories play an important role for our understanding of open string tachyon condensation and time dependent open string backgrounds (see e.g. [203, 204]).

5.2   Towards non-compact backgrounds

Our presentation above was mainly tailored towards compact curved backgrounds, or, in world-sheet terms, rational (boundary) conformal field theories. Technically, our assumptions implied that there were only finitely many primary fields and hence the various constraining equations (see (3.21), (3.30), (3.34)) on the structure constants A_i^α and C had to be solved for a finite number of unknowns. All this changes drastically when we deal with non-compact backgrounds and even though some of the general ideas do carry over, at least after appropriate modifications, there is no generic exact construction to replace Cardy's solution for rational models. This is mainly due to the fact that analogues of the Cardy condition (3.30) are less restrictive and therefore one cannot get away without deriving and solving some factorization constraints (see below).

The same problems are certainly present in the bulk theories already so that there is only a small number of exactly solved models to begin with. Most attention in the past has been devoted to Liouville theory. The exact solution of this model (in a certain regime) was proposed in [205, 206] (the proposal was partly based on [207]) and then more thoroughly analyzed in [208, 209, 210, 211]. The boundary problem for Liouville theory was treated by several authors [212, 213, 214, 215, 216].

Except from the supersymmetric versions of the Liouville model (see e.g. [217, 218] and [219, 220] for the boundary problem), there exists only one other non-rational model that has been solved in the bulk. It describes strings moving on a Euclidean analogue of AdS_3. Since this background is a very close relative of the 3-sphere that we studied

extensively in the third lecture, we shall use it here to explain some of the similarities and differences between solving rational and non-rational boundary conformal field theories. We begin with a brief introduction to the bulk theory and then turn to the possible branes in this background. After a short review of their classical geometry, it is explained how to master the various subtleties that arise when we try to obtain factorization constraints for the one-point functions. Then we analyze the open string sector. In particular, we shall motivate and explain the concept of a *relative partition function* and its relation to the *reflection amplitude.*

The bulk of the H_3^+ WZW model. The model we are interested in describes strings moving on the space H_3^+ of hermitian unimodular 2×2 matrices with positive trace,

$$H_3^+ = \{h \in \mathrm{SL}(2, \mathbb{C}) \,|\, h^* = h\,, \ \mathrm{tr}\, h > 0\} \ .$$

It is easy to see that H_3^+ is a non-compact coset $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$. Following the standard rules, one can write down the classical WZW model for this geometry. The associated quantum theory has been solved by Teschner in a series of papers [221, 222, 223, 224] after Gawedzki computed its bulk partition function in [225].

There are several good reasons to study the H_3^+ model. As we mentioned before, H_3^+ is a Euclidean version of AdS_3 (with NSNS 3-form) and much of the recent progress towards the construction of perturbative closed string theory for AdS_3, see [226, 227, 228] and references therein, has been based on the Euclidean background. Furthermore, one can descend from H_3^+ to a coset describing the 2D Euclidean black hole [229] which is part of many interesting string backgrounds (see e.g. [230, 231, 232]). We shall not return to this coset below, but we want to mention that its partition function was recently computed in [233] and the results confirm expectations which go back to the work of Dijkgraaf et al. [234]. The theory was conjectured by Fateev, Zamolodchikov and Zamolodchikov to be T-dual to sine-Liouville theory (see [235] for a more precise description of the conjecture). The bulk operator product expansions of sine-Liouville were studied [236]. A supersymmetric version of the T-duality [237] which involves the $N = 2$ $\mathrm{SL}(2, \mathbb{R})/U(1)$ Kazama-Suzuki quotient on one side and $N = 2$ Liouville theory on the other was proven in [238].

To proceed with our outline of the H_3^+ model, it is convenient to parametrize this space through coordinates $(\phi, \gamma, \bar{\gamma})$ such that

$$h = \begin{pmatrix} e^{\phi} & e^{\phi}\bar{\gamma} \\ e^{\phi}\gamma & e^{\phi}\gamma\bar{\gamma} + e^{-\phi} \end{pmatrix} \quad . \tag{5.11}$$

Here, ϕ runs through the real numbers and γ is a complex coordinate with conjugate $\bar{\gamma}$. We can visualize the geometric content of these coordinates most easily by expressing them in terms of the more familiar coordinates (ρ, τ, θ) (see Figure 10),

$$\gamma = e^{\tau + i\theta} \tanh \rho \quad \text{and} \quad e^{\phi} = e^{-\tau} \cosh \rho \quad .$$

At fixed $\gamma, \bar{\gamma}$, the boundary of H_3^+ is reached in the limit of infinite ϕ. The boundary is now represented as the complex plane with coordinates $\gamma, \bar{\gamma}$.
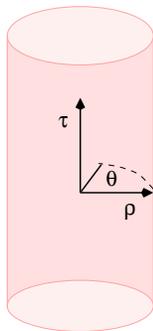
Figure 10: The coordinates (ρ, τ, θ) parametrize H_3^+ as shown. The boundary of H_3^+ appears at $\rho = \infty$.
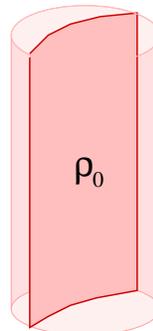
Figure 11: AdS_2-branes in AdS_3 are parametrized by a parameter ϱ_0 measuring the distance from $\rho = 0$.

Any wave function on H_3^+ can be expanded in terms of eigen-functions of the Laplace operator on H_3^+. Since there exists an action of $\mathrm{SL}(2,\mathbb{C})$ on H_3^+ which commutes with the Laplace operator, each eigen-space must carry some representation of $\mathrm{SL}(2,\mathbb{C})$. It is not difficult to show that the possible eigenvalues of the Laplacian are given by $j(j+1), j = -\frac{1}{2} + iP$, where P is a non-negative real number and that the associated eigen-spaces carry the irreducible representation D^j from the principal continuous series. Explicitly,

the eigen-functions are given by the following formula

$$\varphi_j(w, \bar{w}|\phi, \gamma, \bar{\gamma}) \;=\; -\frac{2j+1}{\pi}\,(v_w\, h\, v_w^*)^{2j} \tag{5.12}$$

$$\;=\; -\frac{2j+1}{\pi}\left(|w-\gamma|^2 e^\phi + e^{-\phi}\right)^{2j}.$$

Here, $v_w = (-w, 1)$ depends on a complex coordinate w. The latter is a continuous analogue of the discrete labels $a, b, \dots$ we used for functions Φ_j^{ab} on S^3 (see eq. (4.21)). In fact, $\mathrm{SL}(2, \mathbb{C})$ acts on w by the usual rational transformations and thereby it generalizes the role played by the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of left and right translations on S^3. One may consider the functions $\varphi_j(w, \bar{w}|h)$ as wave-function of some particle that was created with 'radial momentum' j at the boundary point with coordinates $w, \bar{w}$ [239]. They form a basis in the space of square integrable functions on H_3^+ and are in one-to-one correspondence with the ground states of the bulk conformal field theory on H_3^+ (see [225]). The state-field correspondence then provides us with the following set of bulk fields

$$\phi_j(w, \bar{w}; z, \bar{z}) \;=\; \Phi^{(P)}\big(\,|\varphi_j(w, \bar{w})\rangle\,;\, z, \bar{z}\,\big)\;.$$

We will not spell out the coefficients of their operator product expansions, but they can be found in [221, 223].

Introduction to branes in H_3^+. The microscopic study of brane geometries in the Lorentzian model began with the work of Stanciu [240] who used the relation between AdS_3 and the group $\mathrm{SL}(2, \mathbb{R})$ to apply the known results about branes in group manifolds. It was later shown by Bachas and Petropoulos [241] that the most interesting branes on AdS_3 are associated with twisted conjugacy classes in the sense of [66] (cf. our discussion in the previous subsection). These are localized along $AdS_2 \subset AdS_3$ (see Figure 11) and they are parametrized by a single real parameter ϱ_0. In addition one can have branes localized along H_2, dS_2, the light cone, as well as point-like branes. Not all of these geometric possibilities correspond to physical brane configurations, though: The branes localized along dS_2, for example, were found to have a supercritical electric field on their world-volume [241].

Most of the branes we have just listed possess a Euclidean counterpart. Here we will be concerned mainly with the Euclidean AdS_2 branes in H_3^+ which are localized along the

surfaces

$$\text{tr} \, (\, \varpi \, h \,) \; = \; 2 \sinh(\varrho_0) \quad \text{where} \quad \varpi \; = \; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.13}$$

and parametrized by $\varrho_0 \in \mathbb{R}$. After rotation with a particular $\text{SL}(2,\mathbb{C})$ symmetry transformation of H_3^+ these branes are localized along one of the connected component $H_2^{\pm}$ of the Euclidean H_2 brane, so that we do not have to treat these two types of branes seperately. While the non-compact hyper-surfaces (5.13) preserve an $\text{SL}(2,\mathbb{R}) \subset \text{SL}(2,\mathbb{C})$ symmetry, a family of compact $\text{SU}(2)$ symmetric 2-spheres is obtained through the equations

$$\text{tr} \, (\, h \,) \; = \; 2 \cosh(\varsigma_0) \tag{5.14}$$

with $\varsigma_0 \geq 0$. These spheres degenerate to a single point for $\varsigma_0 = 0$. The boundary conformal field theory analysis shows that there exist related (unstable) branes with a spherical symmetry, but they seem to have an imaginary radius ς_0. Otherwise, these boundaries behave very much in the way one would expect from spherical branes. In particular, they possess a finite dimensional space of boundary primary fields. We will make a few more remarks about these spherical boundary conditions as we proceed, but will focus mainly on the non-compact branes.

The space of ground states in the boundary theory for a single AdS_2 brane consists of eigen-functions for the Laplacian on AdS_2. As in the case of AdS_3 (see eq. 5.12), it is easy to find an exact expression for the eigen-functions with eigen-value $j = j(j+1), j \in -\frac{1}{2} + i\mathbb{R}_0^+$ by restricting

$$\psi_j(w|h) \; := \; (v'_w \, h \, v'^*_w)^j \tag{5.15}$$

to the 2-dimensional surface (5.13). These functions are parametrized through some real coordinate w which appears in $v'_w = (iw, 1)$. The eigen-functions transform according to the (infinite dimensional) irreducible representations from the principal continuous series of the symmetry group $\text{SL}(2,\mathbb{R})$. Once more, we obtain a continuous spectrum of momenta j which run through the same values as for the bulk fields, but this time j labels representations of $\text{SL}(2,\mathbb{R})$ rather than $\text{SL}(2,\mathbb{C})$. The state field correspondence Ψ associates a boundary field to each function (5.15),

$$\psi_j(w; u) \; = \; \Psi(\, |\psi_j(w)\rangle \, ; u \,) \quad .$$

The closed string sector. In constructing the exact boundary CFTs that describe branes in H_3^+ we follow the same strategy as before, i.e. we try to find the one-point functions of the bulk fields $\phi_j(w, \bar{w}; z, \bar{z})$ by solving appropriate factorization constraints. Once again the simplest factorization constraint arises from the two-point functions of the theory and even the main idea behind its derivation is similar to the compact situation (see Figure 3). If we imagine the two bulk fields close to each other it is most natural to use the bulk operator product expansion to get a factorization in the closed string channel, leading to a representation of the two-point function as an integral over one-point functions. This is very much the same as in the rational models, apart from the fact that the bulk operator product expansion contains a continuum of primary fields.

The second regime, however, in which the fields are far apart from each other differs more drastically from what we have seen in the second lecture. In fact, for compact backgrounds we projected this regime onto the channel in which the identity field with label 0 propagates along the boundary. But now such a projection would vanish for a very simple reason: our bulk fields ϕ_j correspond to normalizable states of the model. The boundary identity field, on the other hand, can certainly not be associated with a normalizable state simply because the constant function on a non-compact brane is not normalizable. Since it is impossible to create a non-normalizable excitation on the boundary with a normalizable excitation in the bulk, the strategy of our derivation of the constraint (3.21) breaks down for non-compact branes.

Fortunately, there exists a way out [221, 213]. In fact, the fields ϕ_j we have considered so far are not the only ones in the theory. They are the fields that are in one-to-one correspondence with the normalizable states of the model. By analytic continuation in j, however, we obtain additional fields which are still perfectly well defined even though they do not correspond to any normalizable state (see [223, 224] for a rigorous justification). For certain discrete values of j, the new fields are associated with degenerate representations of the current algebra. This implies that the operator product of these degenerate fields with any other field of the theory contains only finitely many primary fields and, more importantly, that the factorization in the open string channel includes a contribution from the identity boundary field.

The factorization constraints that arise from the degenerate field $\phi_{1/2}$ have been worked

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out and solved in [242, 243]. For the one-point functions one finds

$$\langle \phi_j(w, \bar{w}; z, \bar{z}) \rangle_{\varrho_0} \;\; = \;\; \pi \, 2^{\frac{3}{4}} \, \sqrt{b} \, \nu_b^{j+\frac{1}{2}} \, \Gamma(1 + b^2(2j + 1)) \; \frac{|w + \bar{w}|^{2j} \, e^{-\varrho_0(2j+1)\mathrm{sgn}(w+\bar{w})}}{|z - \bar{z}|^{2h_j}} \tag{5.16}$$

$$\text{where} \qquad \nu_b \;\; = \;\; \frac{\Gamma(1 - b^2)}{\Gamma(1 + b^2)} \quad , \quad h_j \;\; = \;\; -b^2 j(j + 1) \tag{5.17}$$

and b is related to the level k by $b^2 = (\mathsf{k} - 2)^{-1}$. A short analysis shows that the w-dependent terms $\mathrm{sgn}(w + \bar{w})^\epsilon |w + \bar{w}|^{2j}$ for $\epsilon = 0, 1$ are analogues of the intertwiner $\mathcal{U}$ in the general formula (3.19) for one-point functions. We should stress here that the factorization constraints obtained with the field $\phi_{1/2}$ do not fix the solution uniquely. Therefore, it would be quite interesting to investigate further conditions that arise e.g. from the degenerate field with $j = 1/2b^2$. These relations have not been worked out yet, but the one-point function was shown to pass further consistency conditions that come with the open string sector (see [242] and below).

The open string sector. If we could follow the same strategy as in the compact case, the next step would be to compute the boundary partition function from the coefficients of the one-point function. Once more, things are not that simple for non-compact branes. In fact, if we would naivly copy the old computation we would end up with a divergent result. The reasons for this are very general and we will explain them in a simple quantum mechanical setup first before returning to our AdS_2 branes.

In systems with a continuous energy spectrum, the spectral set itself does not contain much information about the dynamics. Consider, for example, a 1-dimensional quantum system with a positive potential $V(x)$ which vanishes at $x \to \infty$ and diverges as we approach $x = -\infty$. Such a system has a continuous spectrum which is bounded from below by $E = 0$ and under some mild assumptions, the spectrum does not depend at all on details of the potential V (see e.g. [244]). There is much more dynamical information stored in the so-called reflection amplitude of the system. Recall that for each value $E > 0$ of the energy, our system admits a unique (up to normalization) wave function. It has the form

$$\psi_E(x) \;\; = \;\; e^{-ipx} \; + \; R(p) \, e^{+ipx} \qquad \text{where } p \;\; = \;\; \sqrt{E} \; . \tag{5.18}$$

The phase in front of the second term is called the *reflection amplitude*. It is a functional

of the potential which is very sensitive to small changes of V. In fact, it even encodes enough data to reconstruct the whole potential.

From the reflection amplitude $R(p)$ we can extract some spectral density function ρ. To this end, let us regularize the system by placing a reflecting wall at $x = L$, with large positive L. Later we will remove the cutoff L, i.e. send it to infinity. But as long as L is finite, our system has a discrete spectrum so that we can count the number of energy or momentum levels in each interval of some fixed size and thereby we define a density of the spectrum. Its expansion around $L = \infty$ starts with the following two terms

$$\rho^L(p) \;\; = \;\; \frac{L}{\pi} + \frac{1}{2\pi i}\frac{\partial}{\partial p}\ln R(p) \;\; + \;\; \ldots \tag{5.19}$$

where the first one diverges for $L \to \infty$. This divergence is associated with the infinite region of large x in which the whole system approximates a free theory and consequently it is universal, i.e. under only mild assumptions it is independent of the potential $V(x)$. The sub-leading term, however, is much more interesting and we can extract it from the regularized theory if we compute relative spectral densities before taking the limit $L \to \infty$. This can be done by fixing one reference potential V_* whose regularized spectral density we denote by ρ_*^L. The relative spectral density for a potential V is then given by

$$\rho_{\rm rel}(p) \;\; := \;\; \lim_{L\to\infty}\left(\rho^L(p) - \rho_*^L(p)\right) \;\; = \;\; \frac{1}{2\pi i}\frac{\partial}{\partial p}\ln\frac{R(p)}{R_*(p)} \;\; .$$

It is not difficult to transfer these observations from quantum mechanics to the investigation of non-compact branes. As in the toy model, the naive partition functions of our boundary theories for AdS_2-branes diverge. But it is possible to introduce a cutoff L and to construct partition functions relative to a fixed reference brane with parameter ϱ_* [242],

$$Z_{\rm rel}(q|\varrho_0;\varrho_*) \;\; = \;\; \lim_{L\to\infty}\left(Tr_{\mathcal{H}^{(H)}_{\varrho_0}}\left(q^{\mathsf{H}^{(H)}_{\varrho_0}}\right)^L - Tr_{\mathcal{H}^{(H)}_{\varrho_0}}\left(q^{\mathsf{H}^{(H)}_{\varrho_*}}\right)^L\right)$$

$$\sim \;\; \int\limits_0^\infty dP\,\frac{1}{2\pi i}\frac{\partial}{\partial P}\log\frac{R\left(-\frac{1}{2}+iP|\varrho_0\right)}{R\left(-\frac{1}{2}+iP|\varrho_*\right)}\,\chi^j(q) \tag{5.20}$$

Here we have used the regularized characters $\chi^j(q) \;\; = \;\; q^{b^2 P^2}\eta^{-3}(q)$. Formulas for the reflection amplitude $R(j|\varrho_0) \;\; = \;\; R(-1/2 + iP)$ can be obtained in two different ways.

One possibility is to employ world-sheet duality to derive the partition function from the one-point functions (5.16), just as it was done in rational theories. This leads to the expressions [242]

$$R(j|\varrho_0) = \nu_b^{-iP} \frac{\Gamma_k^2(b^{-2} - iP + \frac{1}{2})}{\Gamma_k^2(b^{-2} + iP + \frac{1}{2})} \frac{\Gamma_k(b^{-2} + 2iP)}{\Gamma_k(b^{-2} - 2iP)} \frac{S_k(2R + P)}{S_k(2R - P)} \quad , \qquad (5.21)$$

where $R \equiv \varrho_0/2\pi b^2$ and the two special functions $S_k(x)$ and $\Gamma_k(x)$ are defined through

$$\log S_k(x) = i \int_0^\infty \frac{dt}{t} \left(\frac{\sin 2tb^2 x}{2 \sinh b^2 t \sinh t} - \frac{x}{t} \right) \quad , \qquad (5.22)$$

$$\Gamma_k(x) = b^{b^2 x(x - b^{-2})} (2\pi)^{\frac{x}{2}} \Gamma_2^{-1}(x|1, b^{-2}) \quad . \qquad (5.23)$$

Here, $\Gamma_2(x|\omega_1, \omega_2)$ denotes Barnes Double Gamma function. If we follow this route to compute the partition function from the boundary states, then world-sheet duality does not give rise to a constraint on one-point functions simply because for systems with a continuous spectrum there are no a priori integrality conditions.

But there is another way of obtaining the relative partition function through a direct construction of the stringy reflection amplitude $R(j|\varrho_0)$. Just as in rational models, open string states may be created by boundary operators $\psi_j(w|u)$ where u is the usual coordinate for the boundary of the world-sheet. One can then study the scattering amplitude for an open string that is sent in with momentum j_1 from the boundary of AdS_3 into an outgoing open string with momentum j_2. The reflection amplitude is obtained from the two-point function of boundary operators through

$$\langle \psi_{j_1}(w_1|u_1) \, \psi_{j_2}(w_2|u_2) \rangle_{\varrho_0} \sim \delta(j_1 - j_2) \, R(j_1|\varrho_0) \frac{1}{|u_1 - u_2|^{h_j}} \quad . \qquad (5.24)$$

Here, $j_i \in -1/2 + i\mathbb{R}^+$ and we omitted some (j_i, w_i) dependent factor that is determined by the symmetry. This leaves us with the problem to find an expression for the boundary two-point functions. The latter are subject to factorization constraints for the boundary three point functions which have been worked out and solved in [242]. The resulting expression for the reflection amplitude is the one we have spelled out in eq. (5.21). Let us stress once more that world-sheet duality only provides us with a consistency condition for the one-point functions of bulk fields after some boundary factorization problem has been solved.

Now we are only missing one more piece of data, namely the boundary operator product expansions. Except from some very special subset which is encoded in the boundary 2-point function, these operator products of boundary fields are not yet known. But since the few coefficients that have been found in [242] are still related to elements of the fusing matrix (though with an interesting shift between boundary and representation labels), it is very likely that the solution comes once again from the representation theory of chiral algebras. Let us briefly note that in the case of spherical branes the situation is better. Their couplings to closed string modes were obtained in [242] (correcting earlier formulas in [245, 246]) and they were shown to possess a discrete set of boundary primary fields [245]. Operator products for the latter have been spelled out in [247].

This brings us to the end of our discussion of non-compact branes. We have tried to sketch how some of the fundamental ideas we developed during the investigation of compact backgrounds need to be modified. But as we have seen, there appear many new problems that so far have only been solved in a few models. Finding model independent exact solutions as in the case of compact backgrounds remains one of the many challenging problems for future research.

References

[1] M. R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, `hep-th/0011017`.

[2] M. R. Douglas, *D-branes and $N = 1$ supersymmetry*, `hep-th/0105014`.

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity, Phys. Rept.* **323** (2000) 183–386, `hep-th/9905111`.

[4] J. Polchinski, *TASI lectures on D-branes*, `hep-th/9611050`.

[5] P. Ginsparg, *Applied conformal field theory*, Lectures given at Les Houches Summer School in Theoretical Physics, Les Houches, France, Jun 28 - Aug 5, 1988.

[6] J. L. Cardy, *Conformal invariance and statistical mechanics*, Lectures given at Les Houches Summer School in Theoretical Physics, Les Houches, France, Jun 28 - Aug 5.

[7] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer, New York, 1999.

[8] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984) 333–380.

[9] J. L. Cardy, *Conformal invariance and surface critical behaviour*, Nucl. Phys. **B240** (1984) 514–532.

[10] J. L. Cardy, *Effect of boundary conditions on the operator content of two-dimensional conformally invariant theories*, Nucl. Phys. **B275** (1986) 200–218.

[11] J. L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. **B324** (1989) 581.

[12] D. C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, Nucl. Phys. **B372** (1992) 654–682.

[13] M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, Phys. Lett. **B247** (1990) 517–524.

[14] M. Bianchi, G. Pradisi, and A. Sagnotti, *Planar duality in the discrete series*, Phys. Lett. **B273** (1991) 389–398.

[15] M. Bianchi and A. Sagnotti, *Twist symmetry and open string Wilson lines*, Nucl. Phys. **B361** (1991) 519–538.

[16] G. Pradisi, A. Sagnotti, and Y. S. Stanev, *Planar duality in SU(2) WZW models*, Phys. Lett. **B354** (1995) 279–286, `hep-th/9503207`.

[17] G. Pradisi, A. Sagnotti, and Y. S. Stanev, *The open descendants of nondiagonal SU(2) WZW models*, Phys. Lett. **B356** (1995) 230–238, `hep-th/9506014`.

[18] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, Nucl. Phys. **B531** (1998) 185–225, `hep-th/9712186`.

[19] J. Fuchs and C. Schweigert, *Branes: From free fields to general backgrounds*, Nucl. Phys. **B530** (1998) 99–136, `hep-th/9712257`.

[20] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory I*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1987.

[21] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory II*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1987.

[22] J. G. Polchinski, *String Theory : An Introduction to the Bosonic String.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1998.

[23] J. G. Polchinski, *String Theory : Superstring Theory and Beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1998.

[24] V. B. Petkova and J.-B. Zuber, *Conformal boundary conditions and what they teach us*, `hep-th/0103007`.

[25] H. Saleur, *Lectures on non perturbative field theory and quantum impurity problems*, `cond-mat/9812110`.

[26] M. R. Gaberdiel, *Lectures on non-BPS Dirichlet branes*, *Class. Quant. Grav.* **17** (2000) 3483–3520, `hep-th/0005029`.

[27] C. Angelantonj and A. Sagnotti, *Open strings*, `hep-th/0204089`.

[28] H. Weyl, *Quantum mechanics and group theory*, *Z. Phys.* **46** (1927) 1.

[29] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry, Part II*, *Phys. Rept.* **360** (2002) 353–465, `hep-th/0107251`.

[30] J. E. Moyal, *Quantum mechanics as a statistical theory*, *Proc. Cambridge Phil. Soc.* **45** (1949) 99–124.

[31] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032, `hep-th/9908142`.

[32] M. R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *JHEP* **02** (1998) 008, `hep-th/9711165`.

[33] Y.-K. E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B528** (1998) 185–196, `hep-th/9803031`.

[34] F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, *JHEP* **02** (1999) 016, `hep-th/9810072`.

[35] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B550** (1999) 151, `hep-th/9812219`.

[36] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030, `hep-th/9903205`.

[37] C. G. Callan and L. Thorlacius, *Open string theory as dissipative quantum mechanics*, *Nucl. Phys.* **B329** (1990) 117.

[38] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689–703, `hep-th/9802068`.

[39] R. Gopakumar, S. Minwalla, and A. Strominger, *Noncommutative solitons*, *JHEP* **05** (2000) 020, `hep-th/0003160`.

[40] D. J. Gross and N. A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *JHEP* **07** (2000) 034, `hep-th/0005204`.

[41] A. P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B495** (2000) 407–412, `hep-th/0007043`.

[42] M. Aganagic, R. Gopakumar, S. Minwalla, and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *JHEP* **04** (2001) 001, `hep-th/0009142`.

[43] J. A. Harvey, P. Kraus, and F. Larsen, *Exact noncommutative solitons*, *JHEP* **12** (2000) 024, `hep-th/0010060`.

[44] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, *Phys. Rept.* **360** (2002) 353–465, `hep-th/0012145`.

[45] J. A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, `hep-th/0102076`.

[46] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2002) 977–1029, `hep-th/0106048`.

[47] P. Bouwknegt and K. Schoutens, *W - symmetry: reprints*. World Scientific, 1995.

[48] R. Blumenhagen *et. al.*, *W algebras with two and three generators*, *Nucl. Phys.* **B361** (1991) 255–289.

[49] E. Verlinde, *Fusion rules and modular transformations in 2-D conformal field theory*, *Nucl. Phys.* **B300** (1988) 360.

[50] G. W. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Commun. Math. Phys.* **123** (1989) 177.

[51] M. R. Gaberdiel, A. Recknagel, and G. M. T. Watts, *The conformal boundary states for SU(2) at level 1*, *Nucl. Phys.* **B626** (2002) 344–362, `hep-th/0108102`.

[52] M. R. Gaberdiel and A. Recknagel, *Conformal boundary states for free bosons and fermions*, *JHEP* **11** (2001) 016, `hep-th/0108238`.

[53] J. Maldacena, G. W. Moore, and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *JHEP* **07** (2001) 046, `hep-th/0105038`.

[54] T. Quella and V. Schomerus, *Symmetry breaking boundary states and defect lines*, *JHEP* **06** (2002) 028, `hep-th/0203161`.

[55] J. L. Cardy and D. C. Lewellen, *Bulk and boundary operators in conformal field theory*, *Phys. Lett.* **B259** (1991) 274–278.

[56] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber, *On the classification of bulk and boundary conformal field theories*, *Phys. Lett.* **B444** (1998) 163, `hep-th/9809097`.

[57] J. Fuchs and C. Schweigert, *A classifying algebra for boundary conditions*, *Phys. Lett.* **B414** (1997) 251–259, `hep-th/9708141`.

[58] A. Recknagel and V. Schomerus, *Boundary deformation theory and moduli spaces of D-branes*, *Nucl. Phys.* **B545** (1999) 233, `hep-th/9811237`.

[59] V. Pasquier, *Operator content of the ADE lattice models*, *J. Phys.* **A20** (1987) 5707.

[60] V. B. Petkova and J. B. Zuber, *On structure constants of sl(2) theories*, *Nucl. Phys.* **B438** (1995) 347–372, `hep-th/9410209`.

[61] V. B. Petkova and J. B. Zuber, *From CFT's to graphs*, *Nucl. Phys.* **B463** (1996) 161–193, `hep-th/9510175`.

[62] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A4** (1989) 251.

[63] I. Runkel, *Boundary structure constants for the A-series Virasoro minimal models*, *Nucl. Phys.* **B549** (1999) 563, `hep-th/9811178`.

[64] I. Runkel, *Structure constants for the D-series Virasoro minimal models*, *Nucl. Phys.* **B579** (2000) 561, `hep-th/9908046`.

[65] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B570** (2000) 525–589, `hep-th/9908036`.

[66] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162–190, `hep-th/9909030`.

[67] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Conformal boundary conditions and three-dimensional topological field theory*, *Phys. Rev. Lett.* **84** (2000) 1659, `hep-th/9909140`.

[68] J.-B. Zuber, *CFT, BCFT, ADE and all that*, `hep-th/0006151`.

[69] C. G. Callan, J. A. Harvey, and A. Strominger, *World sheet approach to heterotic instantons and solitons*, *Nucl. Phys.* **B359** (1991) 611–634.

[70] C. G. Callan, J. A. Harvey, and A. Strominger, *Supersymmetric string solitons*, `hep-th/9112030`.

[71] E. Witten, *D-branes and K-theory*, *JHEP* **12** (1998) 019, `hep-th/9810188`.

[72] D. S. Freed and E. Witten, *Anomalies in string theory with D-branes*, `hep-th/9907189`.

[73] C. Bachas, M. R. Douglas, and C. Schweigert, *Flux stabilization of D-branes*, *JHEP* **05** (2000) 048, `hep-th/0003037`.

[74] J. Pawelczyk, *SU(2) WZW D-branes and their noncommutative geometry from DBI action*, *JHEP* **08** (2000) 006, `hep-th/0003057`.

[75] A. Yu. Alekseev, A. Recknagel, and V. Schomerus, *Brane dynamics in background fluxes and non-commutative geometry*, *JHEP* **05** (2000) 010, `hep-th/0003187`.

[76] A. Yu. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D60** (1999) 061901, `hep-th/9812193`.

[77] K. Gawedzki, *Conformal field theory: A case study*, `hep-th/9904145`.

[78] S. Stanciu, *D-branes in group manifolds*, *JHEP* **01** (2000) 025, `hep-th/9909163`.

[79] J. Hoppe, *Diffeomorphism groups, quantization and $SU(\infty)$*, *Int. J. Mod. Phys.* **A4** (1989) 5235.

[80] J. Madore, *The fuzzy sphere*, *Class. Quant. Grav.* **9** (1992) 69–88.

[81] P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, *Int. J. Mod. Phys.* **A1** (1986) 303.

[82] A. Yu. Alekseev, A. Recknagel, and V. Schomerus, *Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres*, *JHEP* **09** (1999) 023, `hep-th/9908040`.

[83] A. Y. Alekseev, S. Fredenhagen, T. Quella, and V. Schomerus, *Non-commutative gauge theory of twisted D-branes*, `hep-th/0205123`.

[84] H. Grosse, C. Klimcik, and P. Presnajder, *Towards finite quantum field theory in noncommutative geometry*, *Int. J. Theor. Phys.* **35** (1996) 231–244, `hep-th/9505175`.

[85] H. Grosse, C. Klimcik, and P. Presnajder, *Field theory on a supersymmetric lattice*, *Commun. Math. Phys.* **185** (1997) 155, `hep-th/9507074`.

[86] H. Grosse, C. Klimcik, and P. Presnajder, *Simple field theoretical models on noncommutative manifolds*, `hep-th/9510177`.

[87] J. Madore, *An introduction to noncommutative differential geometry and its physical applications*. Cambridge University Press, 1999.

[88] U. Carow-Watamura and S. Watamura, *Noncommutative geometry and gauge theory on fuzzy sphere*, *Commun. Math. Phys.* **212** (2000) 395, `hep-th/9801195`.

[89] C. Klimcik, *A nonperturbative regularization of the supersymmetric Schwinger model*, *Commun. Math. Phys.* **206** (1999) 567, `hep-th/9903112`.

[90] I. Affleck and A. W. W. Ludwig, *Universal noninteger 'ground state degeneracy' in critical quantum systems*, *Phys. Rev. Lett.* **67** (1991) 161–164.

[91] A. Recknagel and V. Schomerus, *Moduli spaces of D-branes in CFT-backgrounds*, *Fortsch. Phys.* **48** (2000) 195, `hep-th/9903139`.

[92] K. Hashimoto and K. Krasnov, *D-brane solutions in non-commutative gauge theory on fuzzy sphere*, *Phys. Rev.* **D64** (2001) 046007, `hep-th/0101145`.

[93] Y. Hikida, M. Nozaki, and Y. Sugawara, *Formation of spherical D2-brane from multiple D0-branes*, *Nucl. Phys.* **B617** (2001) 117–150, `hep-th/0101211`.

[94] D. P. Jatkar, G. Mandal, S. R. Wadia, and K. P. Yogendran, *Matrix dynamics of fuzzy spheres*, *JHEP* **01** (2002) 039, `hep-th/0110172`.

[95] R. C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022, `hep-th/9910053`.

[96] A. Yu. Alekseev and V. Schomerus, *RR charges of D2-branes in the WZW model*, `hep-th/0007096`.

[97] S. Fredenhagen and V. Schomerus, *Branes on group manifolds, gluon condensates, and twisted K-theory*, *JHEP* **04** (2001) 007, `hep-th/0012164`.

[98] S. Fredenhagen and V. Schomerus, *Brane dynamics in CFT backgrounds*, `hep-th/0104043`.

[99] K. G. Wilson, *The renormalization group: Critical phenomena and the Kondo problem*, *Rev. Mod. Phys.* **47** (1975) 773.

[100] I. Affleck and A. W. W. Ludwig, *The Kondo effect, conformal field theory and fusion rules*, *Nucl. Phys.* **B352** (1991) 849–862.

[101] I. Affleck and A. W. W. Ludwig, *Critical theory of overscreened Kondo fixed points*, *Nucl. Phys.* **B360** (1991) 641–696.

[102] P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted K-theory*, *JHEP* **03** (2000) 007, `hep-th/0002023`.

[103] A. Kapustin, *D-branes in a topologically nontrivial B-field*, *Adv. Theor. Math. Phys.* **4** (2000) 127–154, `hep-th/9909089`.

[104] J. Maldacena, G. W. Moore, and N. Seiberg, *D-brane instantons and K-theory charges*, *JHEP* **11** (2001) 062, `hep-th/0108100`.

[105] W. Taylor, *D2-branes in B-fields*, *JHEP* **07** (2000) 039, `hep-th/0004141`.

[106] S. Stanciu, *A note on D-branes in group manifolds: Flux quantization and D0-charge*, *JHEP* **10** (2000) 015, `hep-th/0006145`.

[107] J. M. Figueroa-O'Farrill and S. Stanciu, *D-brane charge, flux quantization and relative (co)homology*, *JHEP* **01** (2001) 006, `hep-th/0008038`.

[108] R. A. Janik, *Exceptional boundary states at c = 1*, *Nucl. Phys.* **B618** (2001) 675–688, `hep-th/0109021`.

[109] A. Cappelli and G. D'Appollonio, *Boundary states of c = 1 and 3/2 rational conformal field theories*, *JHEP* **02** (2002) 039, `hep-th/0201173`.

[110] M. R. Douglas and G. W. Moore, *D-branes, Quivers, and ALE Instantons*, `hep-th/9603167`.

[111] M. Oshikawa and I. Affleck, *Defect lines in the Ising model and boundary states on orbifolds*, *Phys. Rev. Lett.* **77** (1996) 2604–2607, `hep-th/9606177`.

[112] M. Oshikawa and I. Affleck, *Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line*, *Nucl. Phys.* **B495** (1997) 533–582, `cond-mat/9612187`.

[113] M. R. Douglas, *D-branes and discrete torsion*, `hep-th/9807235`.

[114] M. R. Douglas and B. Fiol, *D-branes and discrete torsion. II*, `hep-th/9903031`.

[115] D.-E. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *JHEP* **10** (2000) 001, `hep-th/9906242`.

[116] J. Gomis, *D-branes on orbifolds with discrete torsion and topological obstruction*, *JHEP* **05** (2000) 006, `hep-th/0001200`.

[117] M. R. Gaberdiel and J. Stefanski, Bogdan, *Dirichlet branes on orbifolds*, *Nucl. Phys.* **B578** (2000) 58–84, `hep-th/9910109`.

[118] M. R. Gaberdiel, *Discrete torsion orbifolds and D-branes*, *JHEP* **11** (2000) 026, `hep-th/0008230`.

[119] B. Craps and M. R. Gaberdiel, *Discrete torsion orbifolds and D-branes. II*, *JHEP* **04** (2001) 013, `hep-th/0101143`.

[120] M. Billo, B. Craps, and F. Roose, *Orbifold boundary states from Cardy's condition*, *JHEP* **01** (2001) 038, `hep-th/0011060`.

[121] G. Pradisi and A. Sagnotti, *Open string orbifolds*, *Phys. Lett.* **B216** (1989) 59.

[122] E. G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, *Phys. Rev.* **D54** (1996) 1667–1676, `hep-th/9601038`.

[123] E. Dudas, J. Mourad, and A. Sagnotti, *Charged and uncharged D-branes in various string theories*, *Nucl. Phys.* **B620** (2002) 109–151, `hep-th/0107081`.

[124] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. I: General theory*, *Nucl. Phys.* **B558** (1999) 419, `hep-th/9902132`.

[125] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. II: More structures, examples*, *Nucl. Phys.* **B568** (2000) 543, `hep-th/9908025`.

[126] L. Birke, J. Fuchs, and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, *Adv. Theor. Math. Phys.* **3** (1999) 671–726, `hep-th/9905038`.

[127] M. B. Halpern, *Direct approach to operator conformal constructions: From fermions to primary fields*, *Ann. Phys.* **194** (1989) 247. Erratum-ibid. **200** (1989) 422.

[128] K. A. Intriligator, *Bonus symmetry in conformal field theory*, Nucl. Phys. **B332** (1990) 541.

[129] A. N. Schellekens and S. Yankielowicz, *Extended chiral algebras and modular invariant partition functions*, Nucl. Phys. **B327** (1989) 673.

[130] A. N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants and fixed points*, Int. J. Mod. Phys. **A5** (1990) 2903–2952.

[131] L. R. Huiszoon, A. N. Schellekens, and N. Sousa, *Open descendants of non-diagonal invariants*, Nucl. Phys. **B575** (2000) 401–415, `hep-th/9911229`.

[132] A. N. Schellekens, *Open strings, simple currents and fixed points*, `hep-th/0001198`.

[133] I. Brunner and V. Schomerus, *D-branes at singular curves of Calabi-Yau compactifications*, JHEP **04** (2000) 020, `hep-th/0001132`.

[134] I. Brunner and V. Schomerus, *On superpotentials for D-branes in Gepner models*, JHEP **10** (2000) 016, `hep-th/0008194`.

[135] K. Matsubara, V. Schomerus, and M. Smedback, *Open strings in simple current orbifolds*, Nucl. Phys. **B626** (2002) 53–72, `hep-th/0108126`.

[136] J. Fuchs, L. R. Huiszoon, A. N. Schellekens, C. Schweigert, and J. Walcher, *Boundaries, crosscaps and simple currents*, Phys. Lett. **B495** (2000) 427–434, `hep-th/0007174`.

[137] D. Gepner, *Space-time supersymmetry in compactified string theory and superconformal models*, Nucl. Phys. **B296** (1988) 757.

[138] M. Naka and M. Nozaki, *Boundary states in Gepner models*, JHEP **05** (2000) 027, `hep-th/0001037`.

[139] S. Govindarajan, T. Jayaraman, and T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, Nucl. Phys. **B580** (2000) 519–547, `hep-th/9907131`.

[140] J. Fuchs, C. Schweigert, and J. Walcher, *Projections in string theory and boundary states for Gepner models*, Nucl. Phys. **B588** (2000) 110–148, `hep-th/0003298`.

[141] I. Brunner, M. R. Douglas, A. E. Lawrence, and C. Romelsberger, *D-branes on the quintic*, JHEP **08** (2000) 015, `hep-th/9906200`.

[142] J. Fuchs *et. al.*, *Boundary fixed points, enhanced gauge symmetry and singular bundles on K3*, Nucl. Phys. **B598** (2001) 57–72, `hep-th/0007145`.

[143] I. Brunner and J. Distler, *Torsion D-branes in nongeometrical phases*, `hep-th/0102018`.

[144] A. Recknagel, *Permutation branes*, `hep-th/0208119`.

[145] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *The operator algebra of orbifold models*, *Commun. Math. Phys.* **123** (1989) 485.

[146] V. G. Kac, *Infinite Dimensional Lie Algebras*. Cambridge University Press, Cambridge, 1990.

[147] S. Stanciu, *An illustrated guide to D-branes in SU(3)*, `hep-th/0111221`.

[148] T. Itoh and S.-J. Sin, *A note on singular D-branes in group manifolds*, `hep-th/0206238`.

[149] T. Itoh and S.-J. Sin, *Classification and quantum moduli space of D-branes in group manifolds*, `hep-th/0207077`.

[150] A. Y. Alekseev, P. Schaller, and T. Strobl, *The topological G/G WZW model in the generalized momentum representation*, *Phys. Rev.* **D52** (1995) 7146–7160, `hep-th/9505012`.

[151] T. Quella, I. Runkel and C. Schweigert, *An algorithm for twisted fusion rules*, `math.qa/0203133`.

[152] P. Bordalo, S. Ribault, and C. Schweigert, *Flux stabilization in compact groups*, *JHEP* **10** (2001) 036, `hep-th/0108201`.

[153] M. R. Gaberdiel and T. Gannon, *Boundary states for WZW models*, `hep-th/0202067`.

[154] V. B. Petkova and J. B. Zuber, *Boundary conditions in charge conjugate sl(N) WZW theories*, `hep-th/0201239`.

[155] T. Quella, *Branching rules of semi-simple Lie algebras using affine extensions*, `math-ph/0111020`.

[156] I. Affleck, M. Oshikawa, and H. Saleur, *Quantum Brownian motion on a triangular lattice and c = 2 boundary conformal field theory*, *Nucl. Phys.* **B594** (2001) 535–606, `cond-mat/0009084`.

[157] J. Maldacena, G. W. Moore, and N. Seiberg, *D-brane charges in five-brane backgrounds*, *JHEP* **10** (2001) 005, `hep-th/0108152`.

[158] P. Goddard, A. Kent, and D. I. Olive, *Virasoro algebras and coset space models*, *Phys. Lett.* **B152** (1985) 88.

[159] D. Gepner, *Field identification in coset conformal field theories*, *Phys. Lett.* **B222** (1989) 207.

[160] A. N. Schellekens and S. Yankielowicz, *Field identification fixed points in the coset construction*, *Nucl. Phys.* **B334** (1990) 67.

[161] T. Quella, *Hierarchy of symmetry breaking branes on group manifolds*, `hep-th/0209157`.

[162] E. Wong and I. Affleck, *Tunneling In Quantum Wires: A Boundary Conformal Field Theory Approach*, *Nucl. Phys.* **B417** (1994) 403.

[163] A. LeClair and A. W. W. Ludwig, *Minimal models with integrable local defects*, *Nucl. Phys.* **B549** (1999) 546–562, `hep-th/9708135`.

[164] C. Nayak, M. Fisher, A. W. Ludwig, and H. H. Lin, *Resonant multilead point-contact tunneling*, *Phys. Rev.* **B59** (1999) 15694.

[165] H. Saleur, *Lectures on non perturbative field theory and quantum impurity problems. II*, `cond-mat/0007309`.

[166] D. M. McAvity and H. Osborn, *Conformal field theories near a boundary in general dimensions*, *Nucl. Phys.* **B455** (1995) 522–576, `cond-mat/9505127`.

[167] J. Erdmenger, Z. Guralnik, and I. Kirsch, *Four-dimensional superconformal theories with interacting boundaries or defects*, `hep-th/0203020`.

[168] A. Karch and L. Randall, *Locally localized gravity*, *JHEP* **05** (2001) 008, `hep-th/0011156`.

[169] A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, *JHEP* **06** (2001) 063, `hep-th/0105132`.

[170] O. DeWolfe, D. Z. Freedman, and H. Ooguri, *Holography and defect conformal field theories*, `hep-th/0111135`.

[171] C. Bachas, J. de Boer, R. Dijkgraaf, and H. Ooguri, *Permeable conformal walls and holography*, `hep-th/0111210`.

[172] V. B. Petkova and J. B. Zuber, *Generalised twisted partition functions*, *Phys. Lett.* **B504** (2001) 157–164, `hep-th/0011021`.

[173] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators. I: Parition functions*, `hep-th/0204148`.

[174] W. Nahm, *Gauging symmetries of two-dimensional conformally invariant models,*. UCD-88-02.

[175] K. Bardakci, E. Rabinovici, and B. Saering, *String models with $c < 1$ components*, *Nucl. Phys.* **B299** (1988) 151.

[176] K. Gawedzki and A. Kupiainen, *G/H conformal field theory from gauged WZW model*, *Phys. Lett.* **B215** (1988) 119–123.

[177] K. Gawedzki and A. Kupiainen, *Coset construction from functional integrals*, *Nucl. Phys.* **B320** (1989) 625.

[178] A. A. Tseytlin, *Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories*, *Nucl. Phys.* **B411** (1994) 509–558, `hep-th/9302083`.

[179] K. Gawedzki, *Boundary WZW, G/H, G/G and CS theories*, `hep-th/0108044`.

[180] S. Elitzur and G. Sarkissian, *D-branes on a gauged WZW model*, *Nucl. Phys.* **B625** (2002) 166–178, `hep-th/0108142`.

[181] F. Falceto and K. Gawedzki, *Boundary G/G theory and topological Poisson-Lie sigma model*, `hep-th/0108206`.

[182] S. Fredenhagen and V. Schomerus, *D-branes in coset models*, *JHEP* **02** (2002) 005, `hep-th/0111189`.

[183] A. Recknagel, D. Roggenkamp, and V. Schomerus, *On relevant boundary perturbations of unitary minimal models*, *Nucl. Phys.* **B588** (2000) 552–564, `hep-th/0003110`.

[184] K. Graham, *On perturbations of unitary minimal models by boundary condition changing operators*, *JHEP* **03** (2002) 028, `hep-th/0111205`.

[185] S. Fredenhagen and V. Schomerus, *On boundary RG-flows in coset conformal field theories*, `hep-th/0205011`.

[186] L. Chim, *Boundary S-matrix for the tricritical Ising model*, *Int. J. Mod. Phys.* **A11** (1996) 4491–4512, `hep-th/9510008`.

[187] F. Lesage, H. Saleur, and P. Simonetti, *Boundary flows in minimal models*, *Phys. Lett.* **B427** (1998) 85–92, `hep-th/9802061`.

[188] C. Ahn and C. Rim, *Boundary flows in the general coset theories*, *J. Phys.* **A32** (1999) 2509–2525, `hep-th/9805101`.

[189] P. Dorey, I. Runkel, R. Tateo, and G. Watts, *g-function flow in perturbed boundary conformal field theories*, *Nucl. Phys.* **B578** (2000) 85–122, `hep-th/9909216`.

[190] K. Graham, I. Runkel, and G. M. T. Watts, *Minimal model boundary flows and c = 1 CFT*, *Nucl. Phys.* **B608** (2001) 527–556, `hep-th/0101187`.

[191] S. Fredenhagen, *Dynamics of D-branes in curved backgrounds,*. PhD thesis, unpublished.

[192] H. Ishikawa, *Boundary states in coset conformal field theories*, `hep-th/0111230`.

[193] T. Kubota, J. Rasmussen, M. A. Walton, and J.-G. Zhou, *Maximally symmetric D-branes in gauged WZW models*, `hep-th/0112078`.

[194] M. A. Walton and J.-G. Zhou, *D-branes in asymmetrically gauged WZW models and axial- vector duality*, `hep-th/0205161`.

[195] G. Sarkissian, *Non-maximally symmetric D-branes on group manifold in the Lagrangian approach*, *JHEP* **07** (2002) 033, `hep-th/0205097`.

[196] H. Ishikawa and T. Tani, *Novel construction of boundary states in coset conformal field theories*, `hep-th/0207177`.

[197] T. Quella and V. Schomerus, *Asymmetric cosets*, `hep-th/`.

[198] V. B. Petkova and J. B. Zuber, *BCFT: From the boundary to the bulk*, `hep-th/0009219`.

[199] J. Fuchs, I. Runkel, and C. Schweigert, *Conformal correlation functions, Frobenius algebras and triangulations*, *Nucl. Phys.* **B624** (2002) 452–468, `hep-th/0110133`.

[200] S. L. Shatashvili, *On field theory of open strings, tachyon condensation and closed strings*, `hep-th/0105076`.

[201] C. G. Callan, I. R. Klebanov, A. W. W. Ludwig, and J. M. Maldacena, *Exact solution of a boundary conformal field theory*, *Nucl. Phys.* **B422** (1994) 417–448, `hep-th/9402113`.

[202] J. Polchinski and L. Thorlacius, *Free fermion representation of a boundary conformal field theory*, *Phys. Rev.* **D50** (1994) 622–626, `hep-th/9404008`.

[203] A. Sen, *Descent relations among bosonic D-branes, Int. J. Mod. Phys.* **A14** (1999) 4061–4078, `hep-th/9902105`.

[204] A. Sen, *Time evolution in open string theory*, `hep-th/0207105`.

[205] H. Dorn and H. J. Otto, *Two and three point functions in Liouville theory, Nucl. Phys.* **B429** (1994) 375–388, `hep-th/9403141`.

[206] A. B. Zamolodchikov and A. B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory, Nucl. Phys.* **B477** (1996) 577–605, `hep-th/9506136`.

[207] M. Goulian and M. Li, *Correlation functions in Liouville theory, Phys. Rev. Lett.* **66** (1991) 2051–2055.

[208] J. Teschner, *On the Liouville three point function, Phys. Lett.* **B363** (1995) 65–70, `hep-th/9507109`.

[209] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, `hep-th/9911110`.

[210] B. Ponsot and J. Teschner, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U_q(\mathrm{sl}(2,\mathrm{R}))$, Commun. Math. Phys.* **224** (2001) 613–655, `math.qa/0007097`.

[211] J. Teschner, *Liouville theory revisited, Class. Quant. Grav.* **18** (2001) R153–R222, `hep-th/0104158`.

[212] E. Cremmer and J.-L. Gervais, *The quantum strip: Liouville theory for open strings, Commun. Math. Phys.* **144** (1992) 279–302.

[213] V. Fateev, A. B. Zamolodchikov, and A. B. Zamolodchikov, *Boundary Liouville field theory. I: Boundary state and boundary two-point function*, `hep-th/0001012`.

[214] J. Teschner, *Remarks on Liouville theory with boundary*, `hep-th/0009138`.

[215] K. Hosomichi, *Bulk boundary propagator in Liouville theory on a disc, JHEP* **11** (2001) 044, `hep-th/0108093`.

[216] B. Ponsot and J. Teschner, *Boundary Liouville field theory: Boundary three point function, Nucl. Phys.* **B622** (2002) 309–327, `hep-th/0110244`.

[217] R. C. Rashkov and M. Stanishkov, *Three-point correlation functions in $N = 1$ super-Liouville theory, Phys. Lett.* **B380** (1996) 49–58, `hep-th/9602148`.

[218] R. H. Poghosian, *Structure constants in the $N = 1$ super-Liouville field theory,* Nucl. Phys. **B496** (1997) 451–464, `hep-th/9607120`.

[219] T. Fukuda and K. Hosomichi, *Super Liouville theory with boundary,* Nucl. Phys. **B635** (2002) 215–254, `hep-th/0202032`.

[220] C. Ahn, C. Rim, and M. Stanishkov, *Exact one-point function of $N = 1$ super-Liouville theory with boundary,* Nucl. Phys. **B636** (2002) 497–513, `hep-th/0202043`.

[221] J. Teschner, *On structure constants and fusion rules in the SL(2,C)/SU(2) WZNW model,* Nucl. Phys. **B546** (1999) 390–422, `hep-th/9712256`.

[222] J. Teschner, *The mini-superspace limit of the SL(2,C)/SU(2) WZNW model,* Nucl. Phys. **B546** (1999) 369–389, `hep-th/9712258`.

[223] J. Teschner, *Operator product expansion and factorization in the H_3^+ WZNW model,* Nucl. Phys. **B571** (2000) 555–582, `hep-th/9906215`.

[224] J. Teschner, *Crossing symmetry in the H_3^+ WZNW model,* Phys. Lett. **B521** (2001) 127–132, `hep-th/0108121`.

[225] K. Gawedzki, *Noncompact WZW conformal field theories,* `hep-th/9110076`.

[226] J. M. Maldacena and H. Ooguri, *Strings in AdS_3 and SL(2,R) WZW model. I,* J. Math. Phys. **42** (2001) 2929–2960, `hep-th/0001053`.

[227] J. M. Maldacena, H. Ooguri, and J. Son, *Strings in AdS_3 and the SL(2,R) WZW model. II: Euclidean black hole,* J. Math. Phys. **42** (2001) 2961–2977, `hep-th/0005183`.

[228] J. M. Maldacena and H. Ooguri, *Strings in AdS_3 and the SL(2,R) WZW model. III: Correlation functions,* Phys. Rev. **D65** (2002) 106006, `hep-th/0111180`.

[229] E. Witten, *On string theory and black holes,* Phys. Rev. **D44** (1991) 314–324.

[230] G. T. Horowitz and A. Strominger, *Black strings and p-branes,* Nucl. Phys. **B360** (1991) 197–209.

[231] J. M. Maldacena and A. Strominger, *Semiclassical decay of near-extremal fivebranes,* JHEP **12** (1997) 008, `hep-th/9710014`.

[232] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds,* Nucl. Phys. **B463** (1996) 55–72, `hep-th/9511164`.

[233] A. Hanany, N. Prezas, and J. Troost, *The partition function of the two-dimensional black hole conformal field theory*, *JHEP* **04** (2002) 014, `hep-th/0202129`.

[234] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B371** (1992) 269–314.

[235] V. Kazakov, I. K. Kostov, and D. Kutasov, *A matrix model for the two-dimensional black hole*, *Nucl. Phys.* **B622** (2002) 141–188, `hep-th/0101011`.

[236] T. Fukuda and K. Hosomichi, *Three-point functions in sine-Liouville theory*, *JHEP* **09** (2001) 003, `hep-th/0105217`.

[237] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034, `hep-th/9909110`.

[238] K. Hori and A. Kapustin, *Duality of the fermionic 2d black hole and $N = 2$ Liouville theory as mirror symmetry*, *JHEP* **08** (2001) 045, `hep-th/0104202`.

[239] J. de Boer, H. Ooguri, H. Robins, and J. Tannenhauser, *String theory on AdS_3*, *JHEP* **12** (1998) 026, `hep-th/9812046`.

[240] S. Stanciu, *D-branes in an AdS_3 background*, *JHEP* **09** (1999) 028, `hep-th/9901122`.

[241] C. Bachas and M. Petropoulos, *Anti-de-Sitter D-branes*, *JHEP* **02** (2001) 025, `hep-th/0012234`.

[242] B. Ponsot, V. Schomerus, and J. Teschner, *Branes in the Euclidean AdS_3*, *JHEP* **02** (2002) 016, `hep-th/0112198`.

[243] P. Lee, H. Ooguri, and J. Park, *Boundary states for AdS_2 branes in AdS_3*, *Nucl. Phys.* **B632** (2002) 283–302, `hep-th/0112188`.

[244] A. Messiah, *Quantum mechanics, Vol. 1/2*. North-Holland, 1975.

[245] A. Giveon, D. Kutasov, and A. Schwimmer, *Comments on D-branes in AdS_3*, *Nucl. Phys.* **B615** (2001) 133–168, `hep-th/0106005`.

[246] A. Parnachev and D. A. Sahakyan, *Some remarks on D-branes in AdS_3*, *JHEP* **10** (2001) 022, `hep-th/0109150`.

[247] B. Ponsot, *Monodromy of solutions of the Knizhnik-Zamolodchikov equation: SL(2,C)/SU(2) WZNW model*, `hep-th/0204085`.