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Some stationary points of gauged N=16 D=3 supergravity

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Abstract

Five nontrivial stationary points are found for maximal gauged N=16 supergravity in three dimensions with gauge group $SO(8) \times SO(8)$ by restricting the potential to a submanifold of the space of $SU(3) \subset (SO(8) \times SO(8))_{\mathrm{diag}}$ singlets. The construction presented here uses the embedding of $E_{7(+7)} \subset E_{8(+8)}$ to lift the analysis of $N = 8, D = 4$ supergravity performed by N. Warner to $N = 16, D = 3$, and hence, these stationary points correspond to some of the known extrema of gauged $N = 8, D = 4$ supergravity.

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1 Introduction

In $N > 1$ extended supergravity theories in dimensions $D \geq 4$, it is possible to gauge the $SO(N)$ symmetry that rotates the supersymmetry generators into each other. In three dimensions, the case of gauged $N = 16$ supergravity is of particular interest, since it exhibits a very rich structure; here, scalar fields are on-shell equivalent to vector fields, and due to the freedom in the choice of the number of vector fields defined as nonlocal functions of the scalar fields, a large number of gauge groups become possible [1, 2]. Furthermore, in contrast to $D \geq 4$ supergravity, maximal three-dimensional gauged supergravities are not derivable by any technique known so far from any of the known higher-dimensional maximal gauged supergravity theories, since the vector fields show up in the Lagrangian via a Chern-Simons term and not via a kinetic term as in a Lagrangian obtained by Kaluza-Klein compactification.

Gauging of any extended supergravity introduces into the Lagrangian, among other terms, at second order in the gauge coupling constant g a potential for the scalar fields. For $D = 4$, $N = 2, 3$, the scalar potential is just a cosmological constant $-6\,g^2$, while for $N \geq 5$, it features a rich extremal structure which defies an exhaustive analysis for $N = 8$. Some nontrivial extrema of $D = 4, N = 8$ supergravity have been determined in [3, 4] by employing group-theoretical arguments. In order to shed light on the question how the extremal structure of maximal gauged $D = 3, N = 16$ supergravity constructed in [1, 2] is related to that of $D = 4, N = 8$ when one chooses $SO(8) \times SO(8)$ as gauge group, it is interesting to try to generalize the construction given in [4] to this case.

The potential of gauged maximal $N = 16$ supergravity with maximal compact $SO(8) \times SO(8)$ gauge group which we investigate here is considerably more complicated than any supergravity potential previously considered and may well be the most complicated analytic potential ever studied. The significance of the deeply involved structure of the exceptional Lie group E_8 showing up in this case still remains to be elucidated.

2 The scalar potential

The space of the 128 scalars of $N = 16, D = 3$ supergravity can be identified with the symmetric space $E_{8(+8)}/SO(16)$, with $SO(16)$ being the maximal compact subgroup of $E_{8(+8)}$ and as $E_{8(+8)}$ is obtained by fusing the adjoint representation of $SO(16)$ with the Majorana-Weyl spinor representation of $SO(16)$, we split E_8 indices $\mathcal{A}, \mathcal{B}, \ldots$ via $\mathcal{A} = (A, [IJ])$, where indices $A, B, \ldots$ denote $SO(16)$ spinors and indices $I, J, \ldots$ belong to the fundamental representation of $SO(16)$. The structure constants of $E_{8(+8)}$ are

$$f_{IJ\,KL}{}^{MN} = -8\,\delta_{[I[K}\delta^{MN}_{L]J]}, \qquad f_{IJ\,A}{}^{B} = \tfrac{1}{2}\,\Gamma^{IJ}_{AB}. \tag{2.1}$$

The rationale of the common convention to introduce an extra factor $1/2$ for every antisymmetric index pair $[IJ]$ that is summed over is explained in the appendix.

From the $E_{8(+8)}$ matrix generators $t_{\mathcal{A}}{}^{\mathcal{C}}{}_{\mathcal{B}} = f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}$, one forms the Cartan-Killing metric

$$\eta_{\mathcal{A}\mathcal{B}} = \tfrac{1}{60}\mathrm{tr}\, t_{\mathcal{A}} t_{\mathcal{B}}; \qquad \eta_{AB} = \delta_{AB}, \quad \eta_{IJ\,KL} = -2\delta^{IJ}_{KL}. \tag{2.2}$$

In order to obtain the potential, we first introduce the zweihundert-achtundvierzigbein $\mathcal{V}$ in an unitary gauge via

$$\mathcal{V} = \exp\left(\psi^A t_A\right) \tag{2.3}$$

where ψ^A is a $SO(16)$ Majorana-Weyl-Spinor and t_A are the corresponding generators of $E_{8(+8)}$.

Due to the on-shell equivalence of scalars and vectors in three dimensions, the structure of gauged maximal $D = 3, N = 16$ supergravity is much richer than in higher dimensions; here, besides the maximal compact gauge group $SO(8) \times SO(8)$ and its noncompact forms, it is possible to also have a variety of noncompact exceptional gauge groups. The choice of gauge group G_0 is parametrized by the Cartan-Killing metric of G_0 embedded in $E_{8(+8)}$. The requirement of maximal supersymmetry reduces to a single algebraic condition for this symmetric tensor Θ which states that it must not have a component in the $\mathbf{27000}$ of the $E_{8(+8)}$ tensor product decomposition $(\mathbf{248} \times \mathbf{248})_{\mathrm{sym}} = \mathbf{1} + \mathbf{3875} + \mathbf{27000}$. Obviously, one extremal case is $G_0 = E_{8(+8)}$. In this case, the scalar potential again reduces to just a cosmological constant, but the smaller we choose the gauge group, the richer the extremal structure of the corresponding potential becomes.

From this embedding tensor $\Theta_{\mathcal{M}N}$, the T-tensor now is formed by

$$T_{\mathcal{A}\mathcal{B}} = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{V}^{\mathcal{N}}{}_{\mathcal{B}}\Theta_{\mathcal{M}\mathcal{N}}. \tag{2.4}$$

With $\theta = \frac{1}{248}\eta^{\mathcal{K}\mathcal{L}}\Theta_{\mathcal{K}\mathcal{L}}$, we form the tensors

$$\begin{aligned} A_1^{IJ} &= \tfrac{8}{7}\,\theta\delta_{IJ} + \tfrac{1}{7}T_{IK\,JK} \\ A_2^{I\dot{A}} &= -\tfrac{1}{7}\Gamma^J_{A\dot{A}}T_{IJ\,A} \\ A_3^{\dot{A}\dot{B}} &= 2\theta\delta_{\dot{A}\dot{B}} + \tfrac{1}{48}\Gamma^{IJKL}_{\dot{A}\dot{B}}T_{IJ\,KL}. \end{aligned} \tag{2.5}$$

The potential is given by

$$V\left(\psi^A\right) = -\frac{1}{8}\,g^2\left(A_1^{IJ}A_1^{IJ} - \frac{1}{2}A_2^{I\dot{A}}A_2^{I\dot{A}}\right). \tag{2.6}$$

If we split the $SO(16)$ vector index I into $SO(8)$ Indices $I = (i, \bar{j})$, the nonzero components of the embedding tensor for the gauge group $SO(8) \times SO(8)$ considered here are given by

$$\Theta_{ij\,kl} = 8\,\delta^{ij}_{kl}, \qquad \Theta_{\overline{ij}\,\overline{kl}} = -8\,\delta^{\overline{ij}}_{\overline{kl}} \tag{2.7}$$

where we use the same normalization as in [2].

The most fruitful technique for a study of the extremal structure of these potentials known so far appears to be that introduced in [4]: first, choose a subgroup H of the gauge group G ($SO(8)$ for $N = 8, D = 4$, $SO(8) \times SO(8)$ in for the case considered here); then, determine a parametrization of the submanifold M of H-singlets of the manifold of physical scalars P. Every point of this submanifold for which all derivatives within M vanish must also have vanishing derivatives within P. The reason is that, with the potential V being invariant under G and hence also under H, the power series expansion of a variation δz of V around a stationary point z_0 in M where δz points out of the submanifold M of H-singlets can not have a $\mathcal{O}(\delta z)$ term, since each term of this expansion must be invariant under H and it is not possible to form a H-singlet from just one H-nonsinglet. All the stationary points found that way will break the gauge group down to a symmetry group that contains H.

The general tendency is that, with H getting smaller, the number of H-singlets among the supergravity scalars will increase. For H-singlet spaces of low dimension, it easily happens that the scalar potential does not feature any nontrivial stationary points at all, while for higher-dimensional singlet spaces, the potential soon becomes intractably complicated. Using the embedding of $SU(3) \subset SO(8)$ under which the scalars, vectors and co-vectors of $SO(8)$ decompose into $\mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1}$ for $N = 8, D = 4$ gives a case of manageable complexity with six-dimensional scalar manifold for which five nontrivial extrema were given in a complete analysis in [4]. (It seems reasonable to expect further yet undiscovered extrema breaking $SO(8)$ down to groups smaller than $SU(3)$.)

Since it is interesting to see how the extremal structure of $N = 8, D = 4$ gauged $SO(8)$ supergravity is related to $N = 16, D = 3$ $SO(8) \times SO(8)$ gauged supergravity, it is reasonable to try to lift the construction given in [4] to this case via the embedding of $E_{7(+7)}$ in $E_{8(+8)}$ described in the appendix. As explained there in detail, the 128 spinor components ψ^A decompose into

2×1 $SO(8)$ scalars which we call $\psi^{\pm}$, 2×28 2-forms $\psi^{\pm}_{\mathtt{i}_1\mathtt{i}_2}$, and 2×35 4-forms $\psi^{\pm}_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4}$.[1]

Re-identifying the $E_{8(+8)}$ generators corresponding to the $SU(3)$ singlets, resp. the $SU(8)$ rotations used to parametrize the singlet manifold given in [4] is straightforward; exponentiating them, however, is not. Looking closely at explicit 248×248 matrix representations of these generators reveals that, after suitable re-ordering of coordinates, they decompose into blocks of maximal size 8×8 and are (by using a computer) sufficiently easy to diagonalize. Considerable simplification of the task of computing explicit analytic expressions for the scalar potential by making use of as much group theoretical structure as possible is expected, but nowadays computers are powerful enough to allow a head-on approach using explicit 248-dimensional component notation and symbolic algebra on sparsely occupied tensors.[2] The original motivation to invest time into the design of aggressively optimized explicit symbolic tensor algebra code comes from the wealth of different cases due to the large number of possible gauge groups of $D=3, N=16$ supergravity.

One important complication arises from the fact that the $(\mathbf{56},\mathbf{2})$ and $(\mathbf{1},\mathbf{3})$ representations give additional $SU(3)$ singlets: from the $(\mathbf{1},\mathbf{3})$, these are the generators corresponding to $\psi^{\pm}$, while each of the 2-forms $F^{\pm}$ which are defined as intermediate quantities in [4] appears twice (once for $\psi^{+}_{\mathtt{i}_1\mathtt{i}_2}$, once for $\psi^{-}_{\mathtt{i}_1\mathtt{i}_2}$), giving a total of six extra scalars, so our scalar manifold M now is 12-dimensional. While explicit analytic calculation of the potential on a submanifold of M reveals that the full 12-dimensional potential definitely is way out of reach of a complete analysis using standard techniques, it is nevertheless possible to make progress by making educated guesses at the possible locations of extrema; for example, one notes that for four of the five stationary points given in [4], the angular parameters are just such that the sines and cosines appearing in the potential all assume values $\{-1;0;+1\}$. Hence it seems reasonable to try to search for stationary points by letting these compact coordinates run through a discrete set of special values only, thereby reducing the number of coordinates.

The immediate problem with the consideration of only submanifolds M' of the full manifold M of singlets is that, aside from not being able to prove the nonexistence of further stationary points on M, the vanishing of derivatives within M' does not guarantee to have a stationary point of the full potential.

[1] Here, $\mathtt{i},\mathtt{k},\ldots$ denote $SO(8)$ indices of the diagonal $SO(8)$ of $SO(8)\times SO(8)$.

[2] Maple as well as Mathematica do not perform well enough here, while FORM only has comparatively poor support for tensors in explicit component representation; hence, all the symbolic algebra was implemented from scratch using the CMUCL Common LISP compiler.

A sieve for true solutions is given by the stationarity condition (4.12) in [2]:

$$3\,A_1^{IM}A_2^{M\dot{A}} = A_2^{I\dot{B}}A_3^{\dot{A}\dot{B}}. \tag{2.8}$$

3 The SU(3) singlets

Explicitly, the 12 singlets are

- The 6 selfdual complex 4-form singlets in the $(\mathbf{133},\mathbf{1})$ (using the same terminology as in [4]):

$$\begin{array}{rcl}
G_1^+ &=& \left(\psi^-_{1234}+\psi^-_{1256}+\psi^-_{1278}\right)^A\ t_A\\
G_1^- &=& \left(\psi^+_{1234}+\psi^+_{1256}-\psi^+_{1278}\right)^A\ t_A\\
G_2^+ &=& \left(-\psi^-_{1357}+\psi^-_{1368}+\psi^-_{1458}+\psi^-_{1467}\right)^A\ t_A\\
G_2^- &=& \left(\psi^+_{1357}-\psi^+_{1467}+\psi^+_{1458}+\psi^+_{1368}\right)^A\ t_A\\
G_3^+ &=& \left(-\psi^-_{1468}+\psi^-_{1367}+\psi^-_{1358}+\psi^-_{1457}\right)^A\ t_A\\
G_3^- &=& \left(\psi^+_{1468}-\psi^+_{1358}+\psi^+_{1367}+\psi^+_{1457}\right)^A\ t_A
\end{array} \tag{3.1}$$

- The two scalars from $(\mathbf{1},\mathbf{3})$:

$$\begin{array}{rcl}
S_1 &=& \psi^{-A}\,t_A\\
S_2 &=& \psi^{+A}\,t_A
\end{array} \tag{3.2}$$

- The spinors corresponding to the two-forms $F^\pm$ of [4], once built from the $\psi^+_{\mathtt{i}_1\mathtt{i}_2}$, and once built from the $\psi^-_{\mathtt{i}_1\mathtt{i}_2}$:

$$\begin{array}{rcl}
F^{++} &=& \left(\psi^+_{12}+\psi^+_{34}+\psi^+_{56}+\psi^+_{78}\right)^A\ t_A\\
F^{+-} &=& \left(\psi^+_{12}+\psi^+_{34}+\psi^+_{56}-\psi^+_{78}\right)^A\ t_A\\
F^{-+} &=& \left(\psi^-_{12}+\psi^-_{34}+\psi^-_{56}+\psi^-_{78}\right)^A\ t_A\\
F^{--} &=& \left(\psi^-_{12}+\psi^-_{34}+\psi^-_{56}-\psi^-_{78}\right)^A\ t_A
\end{array} \tag{3.3}$$

Using the same parametrization of the six singlets from $(\mathbf{133},\mathbf{1})$ as in [4], that is, writing them as

$$\psi = S\left(\lambda_1\,G_1^+ + \lambda_2\,G_2^+\right) \tag{3.4}$$

with

$$
\begin{aligned}
S &= \operatorname{diag}(\omega,\omega,\omega,\omega,\omega,\omega,\omega^{-3}P)\,, \qquad \omega = e^{ia/4}, \\
P &= \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix},
\end{aligned} \tag{3.5}
$$

and using the translation of $SU(8)$ generators to $E_{8(+8)}$ generators of the appendix, the formula corresponding to (2.10) in [4] reads

$$
\begin{aligned}
-8\,g^{-2}V \;=\;& \tfrac{189}{2} - \tfrac{3}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
&+\tfrac{3}{2}\,\mathtt{K}(3\lambda_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) + 12\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
&-12\,\mathtt{K}(2\lambda_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) + \tfrac{3}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
&-\tfrac{3}{2}\,\mathtt{K}(\lambda_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) - 12\,\mathtt{K}(2\lambda_2)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
&+12\,\mathtt{c}(2a)\mathtt{c}(2\varphi) + \tfrac{23}{8}\,\mathtt{K}(3\lambda_1) - \tfrac{19}{8}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2) \\
&-\tfrac{1}{8}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{c}(4\varphi) - \tfrac{1}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(2\lambda_2) \\
&+\tfrac{1}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{c}(4\varphi) - \tfrac{3}{8}\,\mathtt{K}(3\lambda_1)\mathtt{c}(4\varphi) \\
&+12\,\mathtt{K}(2\lambda_1) + 36\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) + \tfrac{405}{8}\,\mathtt{K}(\lambda_1) \\
&-\tfrac{9}{8}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2) - \tfrac{3}{8}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{c}(4\varphi) \\
&+\tfrac{93}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2) + \tfrac{3}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{c}(4\varphi) \\
&-\tfrac{9}{8}\,\mathtt{K}(\lambda_1)\mathtt{c}(4\varphi) + \tfrac{7}{2}\,\mathtt{K}(4\lambda_2) + \tfrac{1}{2}\,\mathtt{K}(4\lambda_2)\mathtt{c}(4\varphi) \\
&+14\,\mathtt{K}(2\lambda_2) - 2\,\mathtt{K}(2\lambda_2)\mathtt{c}(4\varphi) + \tfrac{3}{2}\,\mathtt{c}(4\varphi).
\end{aligned} \tag{3.6}
$$

Here and in what follows, we use the abbreviations

$$
\mathtt{c}(\alpha) = \cos(\alpha), \qquad \mathtt{K}(\sigma) = \cosh(\sigma), \qquad \mathtt{S}(\tau) = \sinh(\tau). \tag{3.7}
$$

Note that just as in [4], we have ϑ-independence due to $SO(8)$ invariance *as well as* an additional independence of Ψ.

A detailed analysis of this restricted potential shows that there are seven candidates for nontrivial stationary points, but none of these is a true stationary point of the full potential. However, it is observed numerically that the derivative at many of these points lies in the $\psi^{\pm}$ plane, hence we extend (3.4) to

$$
\psi = S\left(\lambda_1\,G_1^+ + \lambda_2\,G_2^+\right) + \sigma_1\,S_1 \tag{3.8}
$$

and obtain the potential

$$
\begin{array}{rcl}
-8g^{-2}V & = & \frac{189}{2} + \frac{1}{8}\,\mathtt{S}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a)\mathtt{c}(4\varphi) \\
& & +\frac{1}{8}\,\mathtt{S}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a) - \frac{1}{8}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1)\mathtt{c}(4\varphi) \\
& & -\frac{1}{2}\,\mathtt{S}(3\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a)\mathtt{c}(4\varphi) + \frac{1}{2}\,\mathtt{S}(3\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a) \\
& & +\frac{3}{8}\,\mathtt{S}(3\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(3a)\mathtt{c}(4\varphi) - \frac{5}{8}\,\mathtt{S}(3\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(3a) \\
& & -\frac{3}{8}\,\mathtt{S}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a)\mathtt{c}(4\varphi) - \frac{3}{8}\,\mathtt{S}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a) \\
& & -\frac{3}{8}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1)\mathtt{c}(4\varphi) + \frac{3}{2}\,\mathtt{S}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a)\mathtt{c}(4\varphi) \\
& & -\frac{3}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
& & +\frac{3}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
& & +\frac{3}{2}\,\mathtt{S}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(a)\mathtt{c}(2\varphi) \\
& & +\frac{3}{2}\,\mathtt{S}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(a)\mathtt{c}(2\varphi) \\
& & -\frac{3}{4}\,\mathtt{S}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(a) - \frac{3}{8}\,\mathtt{K}(3\lambda_1)\mathtt{K}(\sigma_1)\mathtt{c}(4\varphi) \\
& & -\frac{3}{2}\,\mathtt{S}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(3a) + \frac{3}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(\sigma_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
& & -\frac{3}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(\sigma_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) - \frac{3}{2}\,\mathtt{S}(3\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(a)\mathtt{c}(2\varphi) \\
& & +\frac{3}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1)\mathtt{c}(4\varphi) - \frac{9}{8}\,\mathtt{S}(\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(3a)\mathtt{c}(4\varphi) \\
& & +\frac{15}{8}\,\mathtt{S}(\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(3a) + 12\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
& & -24\,\mathtt{S}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(a)\mathtt{c}(2\varphi) + \frac{3}{2}\,\mathtt{c}(4\varphi) \\
& & -12\,\mathtt{K}(2\lambda_1)\mathtt{c}(2a)\mathtt{c}(2\varphi) - 12\,\mathtt{K}(2\lambda_2)\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
& & -24\,\mathtt{S}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(a) + 12\,\mathtt{c}(2a)\mathtt{c}(2\varphi) \\
& & +\frac{9}{4}\,\mathtt{S}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{S}(\sigma_1)\mathtt{c}(a) - \frac{9}{4}\,\mathtt{S}(3\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(a) \\
& & +\frac{45}{2}\,\mathtt{S}(\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(a)\mathtt{c}(2\varphi) + \frac{99}{4}\,\mathtt{S}(\lambda_1)\mathtt{S}(\sigma_1)\mathtt{c}(a) \\
& & -\frac{19}{8}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) + \frac{1}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1)\mathtt{c}(4\varphi) \\
& & -\frac{1}{2}\,\mathtt{K}(3\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) + \frac{23}{8}\,\mathtt{K}(3\lambda_1)\mathtt{K}(\sigma_1) + 12\,\mathtt{K}(2\lambda_1) \\
& & +36\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) - \frac{9}{8}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) \\
& & -\frac{9}{8}\,\mathtt{K}(\lambda_1)\mathtt{K}(\sigma_1)\mathtt{c}(4\varphi) + \frac{93}{2}\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) \\
& & +\frac{405}{8}\,\mathtt{K}(\lambda_1)\mathtt{K}(\sigma_1) + \frac{7}{2}\,\mathtt{K}(4\lambda_2) + \frac{1}{2}\,\mathtt{K}(4\lambda_2)\mathtt{c}(4\varphi) \\
& & +14\,\mathtt{K}(2\lambda_2) - 2\,\mathtt{K}(2\lambda_2)\mathtt{c}(4\varphi)
\end{array}
\tag{3.9}
$$

where we again find independence of ϑ and Ψ.[3] It must be emphasized that despite its complexity this is still not the potential on the *complete* subspace of $SU(3)$ singlets, since the F and S_2 singlets have not been included yet.

This potential defies a complete analysis on the symbolic level using technology available today. Nevertheless, it is possible to extract further candidates for stationary points by either employing numerics or making educated guesses at the values of some coordinates. Here, we take (as explained above)

[3] Even if one uses ϑ, Ψ-independence from start, the head-on calculation in explicit component notation produces as an intermediate quantity an (admittedly not maximally reduced) T-tensor containing 83192 summands which in turn contain 550148 trigonometric functions, not counting powers; today, with some careful programming, this is quite manageable for a desktop machine, but it clearly shows the futility of this approach if one were to do such calculations by hand.

$(a,\phi) \in \left\{0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi\right\} \times \left\{0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi\right\}$ and thus obtain the following six cases:

1. $a = \pi, \phi = \frac{1}{2}\pi$:

$$
\begin{aligned}
-8g^{-2}V \;=\;& 84 - \mathtt{K}(3\lambda_1+\sigma_1)\mathtt{K}(4\lambda_2) + \mathtt{K}(3\lambda_1+\sigma_1) \\
& +24\,\mathtt{K}(2\lambda_1) + 24\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) \\
& -3\,\mathtt{K}(\lambda_1-\sigma_1)\mathtt{K}(4\lambda_2) + 48\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) \\
& +24\,\mathtt{K}(\lambda_1+\sigma_1) + 27\,\mathtt{K}(\lambda_1-\sigma_1) \\
& +4\,\mathtt{K}(4\lambda_2) + 24\,\mathtt{K}(2\lambda_2)
\end{aligned} \tag{3.10}
$$

2. $a = \pi, \phi \in \left\{\frac{1}{4}\pi, \frac{3}{4}\pi\right\}$:

$$
\begin{aligned}
-8g^{-2}V \;=\;& 93 - \tfrac{9}{4}\,\mathtt{K}(3\lambda_1+\sigma_1)\mathtt{K}(4\lambda_2) - \mathtt{K}(3\lambda_1+\sigma_1)\mathtt{K}(2\lambda_2) \\
& +\tfrac{13}{4}\,\mathtt{K}(3\lambda_1+\sigma_1) + 12\,\mathtt{K}(2\lambda_1) + 36\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) \\
& -\tfrac{3}{4}\,\mathtt{K}(\lambda_1-\sigma_1)\mathtt{K}(4\lambda_2) + 36\,\mathtt{K}(\lambda_1+\sigma_1)\mathtt{K}(2\lambda_2) \\
& +9\,\mathtt{K}(\lambda_1-\sigma_1)\mathtt{K}(2\lambda_2) + 12\,\mathtt{K}(\lambda_1+\sigma_1) \\
& +\tfrac{159}{4}\,\mathtt{K}(\lambda_1-\sigma_1) + 3\,\mathtt{K}(4\lambda_2) + 16\,\mathtt{K}(2\lambda_2)
\end{aligned} \tag{3.11}
$$

3. $a = \pi, \phi = 0$:

$$
\begin{aligned}
-8g^{-2}V \;=\;& 108 - 4\,\mathtt{K}(3\lambda_1+\sigma_1)\mathtt{K}(4\lambda_2) + 4\,\mathtt{K}(3\lambda_1+\sigma_1) \\
& +48\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) + 48\,\mathtt{K}(\lambda_1+\sigma_1)\mathtt{K}(2\lambda_2) \\
& +48\,\mathtt{K}(\lambda_1-\sigma_1) + 4\,\mathtt{K}(4\lambda_2)
\end{aligned} \tag{3.12}
$$

4. $a \in \left\{\frac{1}{2}\pi, \frac{3}{2}\pi\right\}, \phi = \frac{1}{2}\pi$:

$$
\begin{aligned}
-8g^{-2}V \;=\;& 108 - 4\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) + 4\,\mathtt{K}(3\lambda_1)\mathtt{K}(\sigma_1) \\
& +48\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) + 48\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) \\
& +48\,\mathtt{K}(\lambda_1)\mathtt{K}(\sigma_1) + 4\,\mathtt{K}(4\lambda_2)
\end{aligned} \tag{3.13}
$$

5. $a \in \left\{\frac{1}{2}\pi, \frac{3}{2}\pi\right\}, \phi \in \left\{\frac{1}{4}\pi, \frac{3}{4}\pi\right\}$:

$$\begin{aligned} -8g^{-2}V &= 93 - \tfrac{9}{4}\,\mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) - \mathtt{K}(3\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) \\ &\quad + \tfrac{13}{4}\,\mathtt{K}(3\lambda_1)\mathtt{K}(\sigma_1) + 12\,\mathtt{K}(2\lambda_1) + 36\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) \\ &\quad - \tfrac{3}{4}\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) + 45\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) \\ &\quad + \tfrac{207}{4}\,\mathtt{K}(\lambda_1)\mathtt{K}(\sigma_1) + 3\,\mathtt{K}(4\lambda_2) + 16\,\mathtt{K}(2\lambda_2) \end{aligned} \tag{3.14}$$

6. $a \in \left\{\frac{1}{2}\pi, \frac{3}{2}\pi\right\}, \phi = 0$:

$$\begin{aligned} -8g^{-2}V &= 84 - \mathtt{K}(3\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) + \mathtt{K}(3\lambda_1)\mathtt{K}(\sigma_1) \\ &\quad + 24\,\mathtt{K}(2\lambda_1) + 24\,\mathtt{K}(2\lambda_1)\mathtt{K}(2\lambda_2) - 3\,\mathtt{K}(\lambda_1)\mathtt{K}(4\lambda_2)\mathtt{K}(\sigma_1) \\ &\quad + 48\,\mathtt{K}(\lambda_1)\mathtt{K}(2\lambda_2)\mathtt{K}(\sigma_1) + 51\,\mathtt{K}(\lambda_1)\mathtt{K}(\sigma_1) \\ &\quad + 4\,\mathtt{K}(4\lambda_2) + 24\,\mathtt{K}(2\lambda_2) \end{aligned} \tag{3.15}$$

The case $a = 0, \pi = \frac{1}{2}\pi$ gives just the same potential as $a = \pi, \phi = \frac{1}{2}\pi$, but with $\sigma_1 \leftrightarrow -\sigma_1$. Furthermore, $a = 0, \phi \in \left\{\frac{1}{4}\pi, \frac{3}{4}\pi\right\}$ and $a = 0, \phi = 0$ correspond to $a = \pi, \phi \in \left\{\frac{1}{4}\pi, \frac{3}{4}\pi\right\}$, respectively $a = \pi, \phi = 0$, both with $\sigma_1 \leftrightarrow -\sigma_1$. In each of the cases $(2), (3), (6)$, a detailed analysis produces a subcase of unmanageable complexity; aside from these, cases $(1), (3), (4)$ feature nontrivial stationary points that turn out to be true solutions of eq. (2.8).

4 Five Extrema

Many of the extrema of the potentials listed in the last section re-appear multiple times; of every set of coordinates connected by various sign flips or coordinate degeneracies, we only list one representative.

Extremum	Location $(\sigma_1, \lambda_1, \lambda_2, a, \phi)$	Form of the scalar field	Cosmological constant $\Lambda = 4V$	Remaining group symmetry	Remaining super-symmetry
X_1	$(-K_1, K_1, K_1, 0, 0)$	$\exp\left(K_1\ \left(G_1^+ + G_2^+ - S_1\right)\right)$	$-200\, g^2$	$SO(7)^+ \times SO(7)^+$	None
X_2	$(-K_2, K_2, -2\,K_1, \pi, \frac{\pi}{2})$	$\exp\left(-K_2\ \left(G_1^+ + S_1\right) - 2K_1\, G_2^-\right)$	$-416\, g^2$	$SU(4)$	None
X_3	$(-K_3, -K_3, -K_3, \pi, \frac{\pi}{2})$	$\exp\left(K_3\ \left(G_1^+ - S_1 - G_2^-\right)\right)$	$-288\, g^2$	$SU(3) \times SU(3) \times$ $\times U(1) \times U(1)$	$(n_L, n_R) = (2, 2)$
X_4	$(K_6, K_7, K_9, \pi, \frac{\pi}{2})$	$\exp\left(-K_7\, G_1^+ + K_9\, G_2^- + K_6\, S_1\right)$	$K_{10}\, g^2$	$SU(3) \times U(1) \times U(1)$	None
X_5	$(-2\,K_2, 0, -2\,K_1, \frac{\pi}{2}, \frac{\pi}{2})$	$\exp\left(-2K_2\, S_1 - 2K_1\, G_2^-\right)$	$-416\, g^2$	$SU(4)^-$	None

where

$$K_1 = \tfrac{1}{4}\,\ln\left(7 + 4\sqrt{3}\right) \approx 0.6584789$$

$$K_2 = \tfrac{1}{2}\,\ln\left(\tfrac{5}{2} + \tfrac{1}{2}\sqrt{21}\right) \approx 0.7833996$$

$$K_3 = \ln(1 + \sqrt{2}) \approx 0.8813736$$

$$K_4 = \sqrt{6 + 6\sqrt{33}} \approx 6.3613973$$

$$K_5 = 18 + 6\sqrt{33} + 6\,K_4 \approx 90.6357598$$

$$K_6 = \ln\left(\tfrac{1}{18}\sqrt{K_5}\left(19 + \tfrac{5}{3}\sqrt{33} + \tfrac{5}{3}\,K_4 + \tfrac{1}{144}\,K_5^2 - \tfrac{1}{7776}\,K_5^3\right)\right) \approx -1.3849948$$

$$K_7 = \ln\left(\tfrac{1}{6}\sqrt{K_5}\right) \approx 0.4616649$$

$$K_8 = \sqrt{78 + 14\sqrt{33}} \approx 12.5866547$$

$$K_9 = \tfrac{1}{2}\,\ln\left(\tfrac{7}{2} + \tfrac{1}{2}\sqrt{33} - \tfrac{1}{2}\,K_8\right) \approx -1.2694452$$

$$K_{10} = -28512\left(1453 + 253\sqrt{33} + 116\,K_8 + 20\sqrt{33}\,K_8\right)\left(3 + \sqrt{33} + K_4\right)^{-2}\left(7 + \sqrt{33} + K_8\right)^{-2} \left(24 + 6\sqrt{33} - 3\,K_4 - \sqrt{33}\,K_4\right)^{-1} \approx -398.5705673$$

Figure 1: The Extrema

The extremum X_1 breaks the diagonal $SO(8)$ of $SO(8) \times SO(8)$ down to a $SO(7)$ under which the $SO(8)$ spinor decomposes as $\mathbf{8} \rightarrow \mathbf{7} + \mathbf{1}$. In the notation of [4] this is called $SO(7)^+$; it is easily checked that both $SO(8)$ of $SO(8) \times SO(8)$ are broken in the same way. Hence this extremum corresponds to E_2 in [4]. The extrema X_1, X_2, X_3 all break the diagonal $SO(8)$ down to $SU(3) \times U(1)$ and hence loosely correspond to E_5 of [4]. It is easy to check that the remaining symmetry of X_4 is 10-dimensional and hence has to be $SU(3) \times U(1) \times U(1)$. Likewise, the remaining symmetry of X_3 is 18-dimensional, and since its derivative is 16-dimensional, it has to be $SU(3) \times SU(3) \times U(1) \times U(1)$. X_5 breaks the diagonal $SO(8)$ down to $SU(4)^-$ (again using the nomenclature of [4]), and since the remaining symmetry is only 15-dimensional, this is all that remains. This extremum corresponds to E_4 of [4]. There is strong evidence that the extrema X_2 and X_5 indeed are equivalent, despite breaking the *diagonal* $SO(8)$ to different subgroups; a detailed examination of these stationary points will have to show whether this is really the case and what this might tell us about the $D = 4$ vacua E_4 and E_5. Furthermore, it is a bit unexpected to see all cosmological constants except one have rational values.

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Appendix A: $E_{8(+8)}$ and $E_{7(+7)}$ conventions

For quick reference, we assemble all used conventions in this appendix.

Structure constants of $E_{8(+8)}$ are explicitly given as follows: using the conventions of [5], we define

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma_e &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{A.1}$$

to construct $SO(8)$ Γ-matrices $\Gamma^a_{\alpha\dot\beta}$ via

$$\begin{aligned} \Gamma^1 &= \sigma_e \times \sigma_e \times \sigma_e & \Gamma^2 &= \sigma_1 \times \sigma_z \times \sigma_e \\ \Gamma^3 &= \sigma_e \times \sigma_1 \times \sigma_z & \Gamma^4 &= \sigma_z \times \sigma_e \times \sigma_1 \\ \Gamma^5 &= \sigma_1 \times \sigma_x \times \sigma_e & \Gamma^6 &= \sigma_e \times \sigma_1 \times \sigma_x \\ \Gamma^7 &= \sigma_x \times \sigma_e \times \sigma_1 & \Gamma^8 &= \sigma_1 \times \sigma_1 \times \sigma_1 . \end{aligned} \tag{A.2}$$

With the decomposition $I = (\alpha, \dot\beta)$ for $SO(16)$ vector indices in terms of $SO(8)$ indices as well as $A = (\alpha\dot\beta, ab)$ for spinor and $\dot A = (\alpha a, b\dot\beta)$ for co-spinor indices, we define $SO(16)$ Γ-matrices $\Gamma^I_{A\dot A}$ following the conventions of [6]:

$$\begin{aligned} \Gamma^{\alpha}_{\beta\dot\gamma\ \delta b} &= \delta_{\beta\delta}\Gamma^b_{\alpha\dot\gamma} & \Gamma^{\alpha}_{ab\ c\dot\delta} &= \delta_{ac}\Gamma^b_{a\dot\delta} \\ \Gamma^{\dot\alpha}_{ab\ \beta c} &= \delta_{bc}\Gamma^a_{\beta\dot\alpha} & \Gamma^{\dot\alpha}_{\beta\dot\gamma\ b\dot\delta} &= -\delta_{\dot\gamma\dot\delta}\Gamma^b_{\beta\dot\alpha} . \end{aligned} \tag{A.3}$$

Splitting E_8 indices $\mathcal{A}, \mathcal{B}, \ldots$ via $\mathcal{A} = (A, [IJ])$, the structure constants of $E_{8(+8)}$ are given by

$$f_{IJ\,KL}{}^{MN} = -8\,\delta_{{}_{[}I[K}\delta^{MN}_{L]J_{]}}, \qquad f_{IJ\,A}{}^{B} = \tfrac{1}{2}\,\Gamma^{IJ}_{AB} . \tag{A.4}$$

following the conventions of [1, 2].

The common convention that for every antisymmetric index pair $[IJ]$ that is summed over, an extra factor 1/2 has to be introduced more explicitly corresponds to splitting $E_{8(+8)}$ indices not like $\mathcal{A} \to (A, [IJ])$, but instead like $\mathcal{A} \to (A, \underline{[IJ]}) \to (A, [IJ])$, where $\underline{[IJ]}$ is treated as a single index in the range $1 \ldots 120$ (and hence only summed over once) and the split $\underline{[IJ]} \to [IJ]$ is performed using the map $M^{\underline{[IJ]}}{}_{KL} = 2\,\delta^{IJ}_{KL}$. If we sum over $[IJ]$ after this split, we have to include a factor 1/2.

From the generators $t_{\mathcal{A}}{}^{\mathcal{C}}{}_{\mathcal{B}} = f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}$, we form the Cartan-Killing metric

$$\eta_{\mathcal{A}\mathcal{B}} = \tfrac{1}{60}\mathrm{tr}\, t_{\mathcal{A}} t_{\mathcal{B}}; \qquad \eta_{AB} = \delta_{AB}, \quad \eta_{IJ\,KL} = -2\delta^{IJ}_{KL} . \tag{A.5}$$

If we further use lexicographical order for $[IJ]$ index pairs, taking only $I < J$ and use as the canonical $SO(16)$ Cartan subalgebra

$$t_{[1\,2]} = T_{129}, t_{[3\,4]} = T_{158}, \ldots, t_{[15\,16]} = T_{248},$$

then the generators corresponding to the simple roots of E_8 are explicitly

$$\begin{array}{lcl} T_{+-------+} & = & T_{35} + i\,T_{36} + i\,T_{43} - T_{44} \\ T_{+2-3} & = & T_{159} - i\,T_{160} + i\,T_{171} + T_{172} \\ T_{+3-4} & = & T_{184} - i\,T_{185} + i\,T_{194} + T_{195} \\ T_{+4-5} & = & T_{205} - i\,T_{206} + i\,T_{213} + T_{214} \\ T_{+5-6} & = & T_{222} - i\,T_{223} + i\,T_{228} + T_{229} \\ T_{+6-7} & = & T_{235} - i\,T_{236} + i\,T_{239} + T_{240} \\ T_{+7-8} & = & T_{244} - i\,T_{245} + i\,T_{246} + T_{247} \\ T_{+7+8} & = & T_{244} + i\,T_{245} + i\,T_{246} - T_{247}. \end{array} \tag{A.6}$$

The fundamental 56-dimensional matrix representation of the $E_{7(+7)}$ Lie algebra decomposes into 28×28 submatrices under its maximal compact subgroup $SU(8)$

$$\begin{pmatrix} 2A_{[\mathtt{i}}{}^{[\mathtt{I}}\delta_{\mathtt{j}]}{}^{\mathtt{J}]} & \Sigma_{\mathtt{ijKL}} \\ \Sigma^{\mathtt{klIJ}} & 2A_{[\mathtt{k}}{}^{[\mathtt{K}}\delta_{\mathtt{l}]}{}^{\mathtt{L}]} \end{pmatrix} \tag{A.7}$$

where $A_{\mathtt{i}}{}^{\mathtt{I}}$ is an anti-hermitian traceless complex 8×8 matrix generator of SU(8) and $\Sigma_{\mathtt{ijKL}} = \overline{\Sigma^{\mathtt{ijKL}}}$ is complex, self-dual and totally antisymmetric.

We obtain this subalgebra from the $E_{8(+8)}$ algebra as follows: we form $U(8)$ indices from $SO(16)$ indices via

$$x^{\mathtt{j}} + i\,x^{(\mathtt{j}+8)} = z^{\mathtt{j}}, \qquad \mathtt{j} = 1\ldots 8 \tag{A.8}$$

and thus identify the corresponding $SU(8)$ subalgebra within $SO(16)$. Under this embedding, $SU(8)$ generators are lifted to $SO(16)$ via

$$\begin{array}{lcl} G^{\{SO(16)\}\mathtt{i}}{}_{\mathtt{j}} & = & \Re\left(G^{\{SU(8)\}}{}_{\mathtt{i}}{}^{\mathtt{j}}\right) \\ G^{\{SO(16)\}\mathtt{i}+8}{}_{\mathtt{j}+8} & = & \Re\left(G^{\{SU(8)\}}{}_{\mathtt{i}}{}^{\mathtt{j}}\right) \\ G^{\{SO(16)\}\mathtt{i}+8}{}_{\mathtt{j}} & = & \Im\left(G^{\{SU(8)\}}{}_{\mathtt{i}}{}^{\mathtt{j}}\right) \\ G^{\{SO(16)\}\mathtt{i}}{}_{\mathtt{j}+8} & = & -\Im\left(G^{\{SU(8)\}}{}_{\mathtt{i}}{}^{\mathtt{j}}\right) \end{array} \tag{A.9}$$

and then to $E_{8(+8)}$ by

$$t^{\mathcal{M}}{}_{\mathcal{N}} = f_{\underline{[IJ]}\mathcal{N}}{}^{\mathcal{M}} M^{\underline{[IJ]}}{}_I{}^J G^{\{SO(16)\}I}{}_J. \tag{A.10}$$

Furthermore, we form raising and lowering operators from $SO(16)$ Γ-matrices:

$$\Gamma^{\mathtt{j}\pm} = \frac{1}{2}\left(\Gamma^{\mathtt{j}} \pm i\,\Gamma^{\mathtt{j}+8}\right), \qquad \mathtt{j} = 1\ldots 8 \tag{A.11}$$

We call the $SO(16)$ Weyl-Spinor that is annihilated by all $\Gamma^{\mathtt{j}-}$ ψ_0. With our conventions, the complex E_8 generator corresponding to this spinor is

$$\begin{aligned} T \;=\; & \tfrac{1}{4}\,(T_1+T_{10}+T_{19}+T_{28}+T_{37}+T_{46}+T_{55}+T_{64} \\ & +i\,(T_{65}+T_{74}+T_{83}+T_{92}+T_{101}+T_{110}+T_{119}-T_{128}))\,. \end{aligned} \tag{A.12}$$

Since the charge-conjugation matrix is just the identity in our basis, the real as well as the imaginary part of every spinor obtained by acting with an even number of $\Gamma^{\mathtt{j}+}$ on ψ_0 is a Majorana-Weyl-spinor. We use the terminology

$$\begin{aligned} \psi^+_{\mathtt{i}_1\mathtt{i}_2\ldots\mathtt{i}_{2n}} &= \mathrm{Re}\,\Gamma^{\mathtt{i}_1+}\Gamma^{\mathtt{i}_2+}\cdots\Gamma^{\mathtt{i}_{2n}+}\psi_0 \\ \psi^-_{\mathtt{i}_1\mathtt{i}_2\ldots\mathtt{i}_{2n}} &= -\mathrm{Im}\,\Gamma^{\mathtt{i}_1+}\Gamma^{\mathtt{i}_2+}\cdots\Gamma^{\mathtt{i}_{2n}+}\psi_0. \end{aligned} \tag{A.13}$$

Note that

$$\begin{aligned} \psi^+ &= \tfrac{1}{2}\,(\psi_0-\psi_{1\ldots8}) \\ \psi^- &= \tfrac{i}{2}\,(\psi_0+\psi_{1\ldots8}) \\ \psi^+_{\mathtt{i}_1\mathtt{i}_2} &= -\tfrac{1}{2}\,\left(\psi_{\mathtt{i}_1\mathtt{i}_2}+\delta^{\mathtt{i}_1\ldots\mathtt{i}_8}_{1\ldots8}\,\psi_{\mathtt{i}_3\ldots\mathtt{i}_8}\right) \\ \psi^-_{\mathtt{i}_1\mathtt{i}_2} &= -\tfrac{i}{2}\,\left(\psi_{\mathtt{i}_1\mathtt{i}_2}-\delta^{\mathtt{i}_1\ldots\mathtt{i}_8}_{1\ldots8}\psi_{\mathtt{i}_3\ldots\mathtt{i}_8}\right) \\ \psi^+_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4} &= \tfrac{1}{2}\,\left(\psi_{\mathtt{i}_1\ldots\mathtt{i}_4}-\delta^{\mathtt{i}_1\ldots\mathtt{i}_8}_{1\ldots8}\,\psi_{\mathtt{i}_5\ldots\mathtt{i}_8}\right) \\ \psi^-_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4} &= \tfrac{i}{2}\,\left(\psi_{\mathtt{i}_1\ldots\mathtt{i}_4}+\delta^{\mathtt{i}_1\ldots\mathtt{i}_8}_{1\ldots8}\,\psi_{\mathtt{i}_5\ldots\mathtt{i}_8}\right). \end{aligned} \tag{A.14}$$

This way, the 35 $E_{8(+8)}$ generators $t_A{}^{\mathcal{C}}{}_{\mathcal{B}}\psi^A$ corresponding to the $\psi^-_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4}$ are the $E_{7(+7)}$ generators for the real parts of the $\Sigma_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4}$ while the generators corresponding to $\psi^+_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4}$ carry the imaginary parts.

All in all, under the $E_{7(+7)}\times SL(2)\subset E_{8(+8)}$ embedding considered here the $\mathbf{248}=(\mathbf{133},\mathbf{1})+(\mathbf{56},\mathbf{2})+(\mathbf{1},\mathbf{3})$ generators of $E_{8(+8)}$ which decompose into 120 compact generators corresponding to the adjoint representation of $SO(16)$ and 128 noncompact Majorana-Weyl-Spinors of $SO(16)$ further decompose as follows: the compact 63 $SU(8)$ generators form the maximal compact subgroup of the $(\mathbf{133},\mathbf{1})$ adjoint representation of $E_{7(+7)}$ while the extra $U(1)$ generator provides the compact generator of the adjoint $SL(2)$ representation $(\mathbf{1},\mathbf{3})$. The 2×35 real spinors $\psi^\pm_{\mathtt{i}_1\mathtt{i}_2\mathtt{i}_3\mathtt{i}_4}$ provide the 70 noncompact generators in $(\mathbf{133},\mathbf{1})$ while the spinors $\psi^\pm$ give the two noncompact generators in $(\mathbf{1},\mathbf{3})$. The remaining 56 compact generators group together with the $\psi^\pm_{\mathtt{i}_1\mathtt{i}_2}$ spinors and form the $(\mathbf{56},\mathbf{2})$ fundamental representation of $E_{7(+7)}\times SL(2)$.

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