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An algorithm for twisted fusion rules

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Abstract

We present an algorithm for an efficient calculation of the fusion rules of twisted representations of untwisted affine Lie algebras. These fusion rules appear in WZW orbifold theories and as annulus coefficients in boundary WZW theories; they provide NIM-reps of the WZW fusion rules.

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It is a well known fact that affine Lie algebras have twisted integrable highest weight representations, and also their fusion rules can be determined [1, 2]. The study of conformal field theories provides two interpretations for these algebraic objects: They appear as fusion rules in WZW orbifolds, and on surfaces with boundaries twisted representations label symmetry breaking boundary conditions; their fusion rules describe annulus coefficients [2], see also [3, 4, 5]. In this short note we present an algorithm to compute these fusion rules efficiently.

More precisely, we work in the following setting. Let $\hat{\mathfrak{g}}^{(1)}$ be an untwisted affine Lie algebra and ω an automorphism of order N of its horizontal subalgebra $\mathfrak{g}$. In the WZW theory based on $\hat{\mathfrak{g}}^{(1)}$ at level k with modular invariant given by charge conjugation, we consider boundary conditions for which left movers and right movers are related by the automorphism ω at the boundary. By T-duality, these boundary conditions correspond to symmetry preserving boundary conditions in a theory with modular invariant of automorphism type ω. This kind of boundary conditions was analysed in [2] and more recently again in [6, 7, 5, 8].

The spectrum of open strings living between two boundary conditions α, β is encoded in the boundary partition function

$$Z_{\beta\alpha}(q) = \sum_i N_{i\alpha}^{\beta}\, \chi_i(q)$$

where the sum over i runs over integrable highest weight representations of $\hat{\mathfrak{g}}^{(1)}$ at level k and $\chi_i(q)$ are the corresponding characters. The set of boundary conditions is given by twisted representations of $\hat{\mathfrak{g}}^{(1)}$ at level k and the annulus coefficients $N_{i\alpha}^{\beta}$ are the corresponding twisted fusion rules [2]. They form a representation of the fusion rules of the WZW theory at level k by matrices with non-negative integer entries, a so-called NIM-rep.

In order to describe the set of twisted representations we need to introduce some notation. We denote the weight lattice of the horizontal subalgebra $\mathfrak{g}$ by L.[1] A basis of this lattice are the fundamental weights $\Lambda_{(i)}$. The Killing form endows L with a bilinear form $(\cdot,\cdot)$, and on L we have the action of the Weyl group W which is generated by reflections $s_i(\lambda) = \lambda - 2\big(\lambda, \alpha_{(i)}\big)\alpha_{(i)}/\big(\alpha_{(i)}, \alpha_{(i)}\big)$ at the hyperplanes perpendicular to the simple roots $\alpha_{(i)}$. The lattice $L^\vee$ dual to L is the coroot lattice of $\mathfrak{g}$; a basis are the simple coroots $\alpha^\vee_{(i)}$. The lattices L and $L^\vee$ inherit an action of the automorphism ω, which can be decomposed into an outer automorphism ω_0 and an inner one ω_i, $\omega = \omega_i \circ \omega_0$. While the inner automorphism ω_i can be chosen

[1] Notice that L, in contrast to frequent use in the literature, refers to the weight lattice and *not* to the root lattice. This convention will be more economic later.

to be the adjoint action of an element of a Cartan subalgebra and therefore induces a trivial action on L and $L^\vee$, the outer part ω_0 can be chosen to be a diagram automorphism of the Dynkin diagram of $\mathfrak{g}$. It acts on the lattices L and $L^\vee$ by the permutations $\omega_0(\Lambda_{(i)}) = \Lambda_{(\omega_0 i)}$ and $\omega_0\big(\alpha^\vee_{(i)}\big) = \alpha^\vee_{(\omega_0 i)}$ of fundamental weights or simple coroots, respectively. Without loss of generality we can therefore assume ω to be a diagram automorphism. The length of the orbit $\big\{\Lambda_{(i)}, \omega(\Lambda_{(i)}), \omega^2(\Lambda_{(i)}), \ldots\big\}$ will be denoted by n_i. We also define the lattice of symmetric weights $L_\omega = \big\{\mu \in L \,\big|\, \omega(\mu) = \mu\big\}$ which inherits the scalar product from L.

An important ingredient in our algorithm is the subgroup [9, 10]

$$W_\omega = \big\{\, w \in W \,\big|\, w \circ \omega = \omega \circ w \,\big\}$$

of the Weyl group that commutes with the action of ω. It is a Coxeter group with the following generators $\tilde{s}_i$: for orbits of length 1, take $\tilde{s}_i = s_i$. If $i \neq \omega i$, take the product $\tilde{s}_i = s_i s_{\omega i} \ldots s_{\omega^{n_i-1} i}$. This prescription needs to be modified, if the element $A_{i,\omega i}$ of the Cartan matrix is non-vanishing, which in our situation only happens for the outer automorphism of A_{2n} and the orbit consisting of the two nodes in the middle of the Dynkin diagram. In this case, take $\tilde{s}_i = s_i s_{\omega i} s_i = s_{\omega i} s_i s_{\omega i}$.

We also need the orthogonal projection of weight space onto its symmetric subspace: $\mathcal{P}_\omega$ defined by $\mathcal{P}_\omega = \frac{1}{N}\big(1+\omega+\cdots+\omega^{N-1}\big)$, N being the order of ω. For the implementation on a computer, one uses directly the action of $\tilde{s}_i$ on symmetric weights:

$$\tilde{s}_i(\lambda) = \lambda - \frac{2(\lambda, \mathcal{P}_\omega \alpha_{(i)})}{(\mathcal{P}_\omega \alpha_{(i)}, \mathcal{P}_\omega \alpha_{(i)})} \mathcal{P}_\omega \alpha_{(i)} \,. \tag{1}$$

While the symmetric weight lattice L_ω is not invariant under the full Weyl group, it admits an action of W_ω.

We may also define a symmetric coroot lattice $(L^\vee)_\omega = \big\{\beta \in L^\vee \big| \omega(\beta) = \beta\big\}$. Note that L_ω and $(L^\vee)_\omega$ are not dual to each other. Instead one finds that the lattice $\big((L^\vee)_\omega\big)^\vee$ dual to $(L^\vee)_\omega$ involves fractional symmetric weights. $\mathcal{P}_\omega$ restricts to a surjective map from L to $\big((L^\vee)_\omega\big)^\vee$.

We summarise the expressions for the different lattices by comparing to the situation for inner automorphisms where just two lattices appear:

- Weight lattice: $L = \big\{\sum_i \lambda_i \Lambda_{(i)} \,\big|\, \lambda_i \in \mathbb{Z}\big\}$.
- Coroot lattice: $L^\vee = \big\{\sum_i \beta_i \alpha^\vee_{(i)} \,\big|\, \beta_i \in \mathbb{Z}\big\} \subset L$.

$\mathfrak{g}$	A_{2n}	A_3	A_{2n+1}	D_4 (triality)	D_n	E_6
θ_ω	$2\big(\Lambda_{(1)}+\Lambda_{(2n)}\big)$	$2\Lambda_{(2)}$	$\Lambda_{(2)}+\Lambda_{(2n)}$	$\Lambda_{(1)}+\Lambda_{(3)}+\Lambda_{(4)}$	$2\Lambda_{(1)}$	$\Lambda_{(1)}+\Lambda_{(5)}$

Table 1: The vector θ_ω in the labeling conventions of [12, p. 53].

In addition there are four lattices related to the automorphism ω.

- Symmetric weight lattice: $L_\omega = \big\{\sum_i \lambda_i \Lambda_{(i)} \,\big|\, \lambda_i \in \mathbb{Z},\ \lambda_{\omega i} = \lambda_i\big\} \subset L$.
- Symmetric coroot lattice:
  $(L^\vee)_\omega = \big\{\sum_i \beta_i \alpha^\vee_{(i)} \,\big|\, \beta_i \in \mathbb{Z},\ \beta_{\omega i} = \beta_i\big\} \subset L^\vee$.
- Fractional symmetric weight lattice:
  $\big((L^\vee)_\omega\big)^\vee = \big\{\sum_i \lambda_i \Lambda_{(i)} \,\big|\, n_i\lambda_i \in \mathbb{Z},\ \lambda_{\omega i} = \lambda_i\big\} \supset L_\omega$.
- Fractional symmetric coroot lattice:
  $(L_\omega)^\vee = \big\{\sum_i \beta_i \alpha^\vee_{(i)} \,\big|\, n_i\beta_i \in \mathbb{Z},\ \beta_{\omega i} = \beta_i\big\} \supset (L^\vee)_\omega$.

Recall that the n_i are the orbit lengths of fundamental weights.

The integrable highest weight modules of $\hat{\mathfrak{g}}^{(1)}$ at level k are in one-to-one correspondence with elements in $P_k^+ = L/(W \ltimes kL^\vee)$. The expression $W \ltimes kL^\vee$ is just the decomposition of the affine Weyl group into a semi-direct product of the finite Weyl group and the translations with respect to the scaled coroot lattice. Alternatively, the affine Weyl group is generated by finite Weyl reflections and one additional element, a shifted Weyl reflection. The latter is a combination of an elementary reflection at the highest root θ of $\mathfrak{g}$ and a translation. This amounts to an orthogonal reflection with respect to the hyperplane $(\theta,\cdot) = k$. An analogous construction can be performed with respect to the lattices L_ω and $\big((L^\vee)_\omega\big)^\vee$. This defines the sets $S_k^+ = L_\omega/\big(W_\omega \ltimes k(L^\vee)_\omega\big)$ and $B_k^+ = \big((L^\vee)_\omega\big)^\vee/\big(W_\omega \ltimes k(L_\omega)^\vee\big)$. While $W_\omega \ltimes k(L^\vee)_\omega$ is generated by W_ω and the shifted Weyl reflection at $(\theta,\cdot) = k$, the corresponding shifted Weyl reflection for $W_\omega \ltimes k(L_\omega)^\vee$ is at the hyperplane $(\theta_\omega,\cdot) = k$. The vector θ_ω in weight space is defined in Table 1. For each of the three subsets there is a natural choice of a fundamental domain.

- Integrable highest weights
  $P_k^+ = \big\{\lambda = \sum_i \lambda_i \Lambda_{(i)} \,\big|\, \lambda_i \in \mathbb{N}_0 \text{ and } (\theta,\lambda) \le k\big\}$.
- Symmetric integrable highest weights
  $S_k^+ = \big\{\lambda = \sum_i \lambda_i \Lambda_{(i)} \,\big|\, \lambda_i \in \mathbb{N}_0, (\theta,\lambda) \le k \text{ and } \lambda_{\omega i} = \lambda_i\big\}$.

- Boundary labels correspond to twisted highest weight representations [2] or, equivalently, to irreducible integrable highest weight representations of the corresponding twisted Lie algebra. They are labelled by $B_k^+ = \{\beta = \sum_i \beta_i \Lambda_{(i)} \mid n_i\beta_i \in \mathbb{N}_0, (\theta_\omega, \beta) \leq k \text{ and } \beta_i = \beta_{\omega i}\}$.

There is a distinguished vector $\rho_\omega = \sum_i n_i^{-1}\Lambda_{(i)}$ in the lattice $\big((L^\vee)_\omega\big)^\vee$ which is a fractional analogue of the Weyl vector $\rho = \sum_i \Lambda_{(i)}$. We denote by P_k^{++}, S_k^{++} and B_k^{++} the subsets obtained from P_k^+, S_k^+ or B_k^+ after dropping elements which belong to the boundary of the respective Weyl chamber, i.e. are left invariant by at least one nontrivial element of $W \ltimes kL^\vee$, $W_\omega \ltimes k(L^\vee)_\omega$ or $W_\omega \ltimes k(L_\omega)^\vee$, respectively. It is not difficult to see that there exist identifications of the form $P_k^+ + \rho = P_{k+g^\vee}^{++}$, $S_k^+ + \rho = S_{k+g^\vee}^{++}$ and $B_k^+ + \rho_\omega = B_{k+g^\vee}^{++}$ where $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$. These are a simple consequence of the fact that $(\theta, \rho) = (\theta_\omega, \rho_\omega) = g^\vee - 1$.

We are now prepared to state our result for the determination of twisted fusion rules. It is a generalisation of the Racah-Speiser algorithm for tensor product multiplicities (see e.g. [11]) and the Kac-Walton formula [12, 13] (see also [14, 15]) for ordinary fusion rules.

Theorem 1. *The decomposition of the fusion product*

$$i \star \alpha = \sum_{\beta \in B_k^+} N_{i\alpha}^{\beta}\, \beta$$

of an untwisted representation $i \in P_k^+$ of $\hat{\mathfrak{g}}$ and a twisted representation $\alpha \in B_k^+$ into twisted representations can be obtained by the following algorithm:

1. *Compute the weight system M_i, including multiplicities, of the finite dimensional irreducible highest weight representation i of the finite dimensional Lie algebra $\mathfrak{g}$.*
2. *Use $\mathcal{P}_\omega$ to project the set M_i to the lattice of fractional symmetric weights.*
3. *Add the weight α and the twisted Weyl vector ρ_ω to the resulting weights.*
4. *Use the reflections* (1) *in W_ω and the shifted reflection at the plane $(\theta_\omega, \cdot) = k + g^\vee$ to map the set $\mathcal{P}_\omega M_i + \alpha + \rho_\omega$ to the fundamental domain $B_{k+g^\vee}^+$.*
5. *Discard weights on the boundary $B_{k+g^\vee}^+ \backslash B_{k+g^\vee}^{++}$, i.e. those with at least one vanishing entry or scalar product with θ_ω equal to $k + g^\vee$. Supply each remaining contribution, counting multiplicities, with a sign depending on whether the number of reflections has been even or odd.*

6. *Subtract the twisted Weyl vector ρ_ω. Adding all contributions including the relevant multiplicities and signs gives the fusion product.*

We will split the proof into several steps. First, we summarise some earlier results which will be important in the sequel. It was shown in [2, (2.57)] that the twisted fusion coefficients for three weights $i \in P_k^+$ and $\alpha, \beta \in B_k^+$ are given by the formula

$$N_{i\alpha}^{\beta} = \sum_{\mu \in S_k^+} \frac{\bar{S}^{\omega}_{\beta\mu} S_{i\mu} S^{\omega}_{\alpha\mu}}{S_{0\mu}} \,. \tag{2}$$

where the matrix $S^{\omega}_{\alpha\mu}$ is given by [2, (4.6)]

$$S^{\omega}_{\alpha\mu} = (\text{phase}) \left| \, L_\omega \, / \, (k{+}g^\vee)(L^\vee)_\omega \right|^{-1/2} \sum_{w \in W_\omega} \epsilon_\omega(w) e^{-\frac{2\pi i}{k+g^\vee}\left(w(\alpha+\rho_\omega),\, \mu+\rho\right)} \tag{3}$$

(see also [12, Theorem 13.9]). Note that it carries two different labels $\alpha \in B_k^+$ and $\mu \in S_k^+$. The symbol ϵ_ω denotes the sign function of W_ω. As the generators of W_ω may be products of several generators of W, in general the sign function ϵ_ω of W_ω does not coincide with the restriction of the sign function ϵ of W to the subgroup W_ω. Using Weyl's character formula, the quotient of S matrices $S_{i\mu}/S_{0\mu}$ which appears in (2) may be expressed as

$$\frac{S_{i\mu}}{S_{0\mu}} = \chi_i\Big(-\frac{2\pi i}{k+g^\vee}(\mu+\rho)\Big) = \sum_{j \in M_i} e^{-\frac{2\pi i}{k+g^\vee}(j,\,\mu+\rho)} \tag{4}$$

where M_i denotes the weight system of the finite dimensional highest weight module i of $\mathfrak{g}$ including the multiplicities. If one inserts the expressions (3) and (4) into the definition (2) we may write

$$N_{i\alpha}^{\beta} = \sum_{\mu \in S_k^+} f(\mu+\rho) = \sum_{\nu \in S^{++}_{k+g^\vee}} f(\nu) \tag{5}$$

where we used the rule $S_k^+ + \rho = S_k^{++}$ and defined the function

$$f(\nu) = \left| L_\omega/(k+g^\vee)(L^\vee)_\omega \right|^{-1} \times \sum_{j \in M_i} \sum_{w_1, w_2 \in W_\omega} \epsilon_\omega(w_1)\epsilon_\omega(w_2) e^{-\frac{2\pi i}{k+g^\vee}\left(\mathcal{P}_\omega j + w_1(\alpha+\rho_\omega) - w_2(\beta+\rho_\omega),\, \nu\right)} \tag{6}$$

which takes symmetric weights $\nu \in L_\omega$ as arguments. Note that from the property $(\omega x, y) = (x, \omega^{-1} y)$ and the definition of $\mathcal{P}_\omega$ it follows $(\mathcal{P}_\omega j, \nu) = (j, \nu)$ for $\nu \in L_\omega$.

Lemma 1. *The function f is invariant under the action of $W_\omega \ltimes (k+g^\vee)(L^\vee)_\omega$ and vanishes for elements on the boundary of the Weyl chambers, in particular on $S^+_{k+g^\vee} \backslash S^{++}_{k+g^\vee}$.*

Proof. The property $f(w\nu) = f(\nu)$ for $w \in W_\omega$ is proved by using $(wx, y) = (x, w^{-1}y)$, invariance of the weight system M_i under Weyl transformations and redefinition of j, w_1, w_2. Due to $\epsilon_\omega(w)^2 = 1$ possible signs cancel. As $\mathcal{P}_\omega j + w_1(\alpha+\rho_\omega) - w_2(\beta+\rho_\omega) \in \big((L^\vee)_\omega\big)^\vee$, the property $f\big(\nu + (k+g^\vee)\beta\big) = f(\nu)$ for $\beta \in (L^\vee)_\omega$ is obvious. To prove the second statement let us define the auxiliary function $g(\nu) = S^\omega_{\alpha,\nu-\rho}$ which enters each summand of the function $f(\nu)$ as a factor. Similar as for $f(\nu)$ one shows that $g\big(w\nu + (k+g^\vee)\beta\big) = \epsilon_\omega(w) g(\nu)$ for all $\beta \in (L^\vee)_\omega$ and $w \in W_\omega$. Let ν be an element of the boundary of the fundamental Weyl chamber, i.e. $\nu \in S^+_{k+g^\vee} \backslash S^{++}_{k+g^\vee}$. Then it is either invariant under an elementary reflection or a combined action of a translation and an elementary reflection $w \in W_\omega$. The equation $g(\nu) = \epsilon_\omega(w) g(\nu)$ now implies that $g(\nu) = 0$ and thus $f(\nu) = 0$ for $\nu \in S^+_{k+g^\vee} \backslash S^{++}_{k+g^\vee}$. □

Corollary 1. *Eq.* (5) *can be rewritten as*

$$N^\beta_{i\alpha} = \frac{1}{|W_\omega|} \sum_{w \in W_\omega} \sum_{\nu \in S^+_{k+g^\vee}} f(w\nu) = \frac{1}{|W_\omega|} \sum_{\nu \in L_\omega/(k+g^\vee)(L^\vee)_\omega} f(\nu)\,. \tag{7}$$

Lemma 2. *Let Γ be a lattice and $\Gamma_s \subset \Gamma$ be a sublattice of the same rank as Γ. Let $\Gamma^\vee$ and $\Gamma_s^\vee$ be the dual lattices to Γ, Γ_s with respect to an inner product $(\cdot,\cdot)$. For any $h \in \mathbb{N}$ and $x \in \Gamma_s^\vee$ we have*

$$\sum_{y \in \Gamma/h\Gamma_s} e^{2\pi i (x,y)/h} = \big|\Gamma/h\Gamma_s\big| \cdot \delta_{x \in h\Gamma^\vee}\,.$$

Proof. We will use the fact that the characters χ of irreducible representations of a finite group G are orthogonal in the sense that $\sum_{g\in G} \chi(g)\overline{\chi'(g)} = |G| \cdot \delta_{\chi,\chi'}$. The quotient $\Gamma/h\Gamma_s$ is a finite abelian group. For $x \in \Gamma_s^\vee$ the function $\chi_x : \Gamma/h\Gamma_s \to \mathbb{C}$, $\chi_x(y) = e^{2\pi i (x,y)/h}$ is the character of an irreducible representation of $\Gamma/h\Gamma_s$ and the character χ_0 of the trivial representation is identical to one. The orthogonality relation reads, for $x \in \Gamma_s^\vee$,

$$\sum_{y \in \Gamma/h\Gamma_s} e^{2\pi i (x,y)/h} = \sum_{y \in \Gamma/h\Gamma_s} \chi_x(y)\overline{\chi_0(y)} = \big|\Gamma/h\Gamma_s\big| \cdot \delta_{\chi_x,\chi_0}\,.$$

But $\chi_x \equiv \chi_0$ is equivalent to $x \in h\Gamma^\vee$. □

Proof of Theorem 1. We insert expression (6) for $f(\nu)$ into (7) and apply Lemma 2 with $\Gamma = L_\omega$, $\Gamma_s = (L^\vee)_\omega$ and $h = k + g^\vee$. This results in

$$N_{i\alpha}^{\beta} = \frac{1}{|W_\omega|} \sum_{j \in M_i} \sum_{w_1, w_2 \in W_\omega} \epsilon_\omega(w_1 w_2)\, \delta_{\mathcal{P}_\omega j + w_1(\alpha+\rho_\omega) - w_2(\beta+\rho_\omega) \in (k+g^\vee)(L_\omega)^\vee} \,.$$

Using the invariance of all quantities under W_ω we are lead to the final result

$$N_{i\alpha}^{\beta} = \sum_{j \in M_i} \sum_{w \in W_\omega} \epsilon_\omega(w)\, \delta_{w(\mathcal{P}_\omega j + \alpha + \rho_\omega) - (\beta+\rho_\omega) \,\in\, (k+g^\vee)(L_\omega)^\vee} \,. \tag{8}$$

The interpretation of the last formula then amounts to the algorithm of the theorem. Step 5 follows since $\beta + \rho_\omega$ is always in $B_{k+g^\vee}^{++}$. □

Note that for inner automorphisms the sets P_k^+, S_k^+ and B_k^+ all coincide and we recover the Kac-Walton formula for ordinary fusion rules. Formula (8) directly shows that the twisted fusion rules are integer numbers but does not show that they are non-negative. (However, non-negativity follows from the general theory of symmetry breaking boundary conditions [16].) We have also implemented the algorithm on a computer and have verified that no negative integers appear for the first few levels in the cases listed in Table 1.

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