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On the Spectrum of PP-Wave Matrix Theory

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Abstract: We study the spectrum of the recently proposed matrix model of DLCQ M-theory in a parallel plane (pp)-wave background. In contrast to matrix theory in a flat background this model contains mass terms, which lift the flat directions of the potential and renders its spectrum discrete. The supersymmetry algebra of the model groups the energy eigenstates into supermultiplets, whose members differ by fixed amounts of energy in great similarity to the representation of supersymmetry in AdS spaces. There is a unique and exact zero-energy groundstate along with a multitude of long and short multiplets of excited states. For large masses the quantum mechanical model may be treated perturbatively and we study the leading order energy shifts of the first excited states up to level two. Most interestingly we uncover a *protected* short multiplet at level two, whose energies do not receive perturbative corrections. Moreover, we conjecture the existence of an infinite series of similar protected multiplets in the pp-wave matrix model.

Contents

1. Introduction

The precise microscopic degrees of freedom of M-theory remain elusive even six years after its discovery [1]. The most promising candidate for such a description today is given by the large N limit of matrix theory [2], the maximally supersymmetric $U(N)$ gauge quantum mechanics [3, 4] which is intimately connected to the quantum supermembrane [5]. The study of this seemingly simple model has been plagued, however, by its nonlinearity and the existence of flat directions in the potential leading to a continuous spectrum [6].

Recently Berenstein, Maldacena and Nastase [7] realized that for eleven dimensional supergravity on the maximally supersymmetric parallel-plane (pp)-wave background [8]

$$\begin{aligned} ds^2 &= -2\,dx^+\,dx^- + \sum_{i=1}^{9}(dx^i)^2 - \Big(\sum_{a=1}^{3}\tfrac{\mu^2}{9}\,(x^a)^2 + \sum_{a'=4}^{9}\tfrac{\mu^2}{36}\,(x^{a'})^2\Big)(dx^+)^2 \\ F_{123+} &= \mu \end{aligned} \tag{1.1}$$

the corresponding matrix theory is - opposed to its $AdS_4\times S^7$ and $AdS_7\times S^4$ cousins [9] - rather simple. In this model the flat background matrix model is augmented by bosonic and fermionic mass terms with a scale set by μ along with a bosonic cubic interaction in the $SO(3)$ sector. As observed in [7] the mass terms remove the flat directions of the usual matrix theory potential and render its spectrum discrete.

In fact the introduction of the mass parameter μ into the model opens up a new perturbative window of pp-wave matrix theory for $\mu \gg 1$ which we shall study in this paper. First steps in this direction have been undertaken in [10]. Related work on the pp-wave matrix theory and supermembrane may be found in [11].

We begin our analysis after a careful statement of the model and its quantization with the supersymmetry algebra, which due to the non-rigidness of the associated supersymmetry variations displays some unusual features. The supersymmetry algebra groups the energy eigenstates into multiplets, whose members do not have degenerate energy eigenvalues any more, but differ by fixed amounts of energy in great similarity to the representation of supersymmetry in AdS spaces[1]. The Hamiltonian naturally splits into a free and an interacting piece in the limit $\mu \gg 1$, of which the free piece is given by a supersymmetric oscillator system with vanishing groundstate energy protected from perturbative corrections. We then go on to study the leading order energy shifts of the first excited states in perturbation theory and find some surprises. In particular we uncover a multiplet which does not receive any perturbative corrections to its energy eigenvalues in leading order perturbation theory. We argue that this result holds true to all orders. Motivated by additional perturbative evidence, we are led to conjecture the existence of an infinite series of protected states in the full pp-wave matrix model. Finally we end with some concluding remarks.

2. The Model and its Quantization

Our conventions are as follows: X^i_{rs} and θ^α_{rs} denote Hermitian $N \times N$ matrices, $i = 1, \ldots, 9$ are the transverse vector indices which split into $a = 1, 2, 3$ and $a' = 4, \ldots, 9$. Moreover for the SO(9) Majorana spinors we work with a charge conjugation matrix equaling unity, the Dirac matrices $\gamma^i_{\alpha\beta}, \gamma^{ijkl}_{\alpha\beta}$ are symmetric and $\gamma^{ij}_{\alpha\beta}, \gamma^{ijk}_{\alpha\beta}$ are antisymmetric running over the spinor indices $\alpha, \beta = 1, \ldots, 16$. It is useful to perform a rescaling of $t \to \tau/(2R)$ of the time variable of the matrix model proposed in [7] where R denotes the radius of the compactified direction in the DLCQ picture. With the help of this rescaling all parameters of the matrix quantum mechanics are cast into the single mass parameter $m = \mu/(2R)$. Then the pp-wave matrix model of [7] takes the simple form

$$S = S_{\text{flat}} + S_M \tag{2.1}$$

where

$$\begin{aligned} S_{\text{flat}} &= \int d\tau \, \text{Tr}\Big[\tfrac{1}{2}(DX^i)^2 - i\theta D\theta + \tfrac{1}{4}[X^i, X^j]^2 + \theta\gamma_i [X^i, \theta]\Big] \\ S_M &= \int d\tau \, \text{Tr}\Big[-\tfrac{1}{2}(\tfrac{m}{3})^2 (X^a)^2 - \tfrac{1}{2}(\tfrac{m}{6})^2 (X^{a'})^2 + \tfrac{m}{4}\, i\,\theta\gamma_{123}\theta + \tfrac{m}{3}\, i\epsilon_{abc}\, X^a X^b X^c\Big] \end{aligned}$$

[1]For a recent review see e.g. [12] and references therein.

and the covariant derivative is given by $D\mathcal{O} = \partial_\tau \mathcal{O} - i[\omega, \mathcal{O}]$. It is invariant under the 16+16 linearly and non-linearly realized supersymmetries

$$\begin{aligned}
\delta X^i &= 2\,\theta\gamma^i\epsilon(\tau)\\
\delta\theta &= \Big[iDX^i\gamma_i + \tfrac{1}{2}\,[X^i, X^j]\,\gamma_{ij} + \tfrac{m}{3}\,iX^a\gamma_a\gamma_{123} - \tfrac{m}{6}\,iX^{a'}\gamma_{a'}\gamma_{123}\Big]\,\epsilon(\tau) + \eta(\tau)\\
\delta\omega &= 2\,\theta\epsilon(\tau)
\end{aligned} \tag{2.2}$$

with

$$\epsilon(\tau) = e^{-\frac{m}{12}\gamma_{123}\,\tau}\,\epsilon_0 \qquad\qquad \eta(\tau) = e^{\frac{m}{4}\gamma_{123}\,\tau}\,\eta_0 \tag{2.3}$$

Note the non-rigid character of the supersymmetry transformations: The supersymmetry parameters depend explicitly on time. This is the reason why the supercharge will be shown to not commute with the Hamiltonian in the sequel. The cleanest way to derive this model is to start from the supermembrane action in $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds given in [9] and consider the pp-wave limit of the superspace geometry along with the standard κ gauge fixing condition ($\Gamma^+\theta = 0$) for the fermions. The resulting membrane model may then be discretized in the usual fashion [5] by approximating the group of area preserving diffeomorphisms by $U(N)$ in the limit $N \to \infty$. The outcome of this analysis is the model (2.1). This derivation is spelled out in detail in [10].

It is straightforward to go to a Hamiltonian description of the system. We choose the gauge $\omega = 0$ and find that the resulting Hamiltonian may be split into a free and an interacting piece

$$H = H_0 + H_{\text{INT}} \tag{2.4}$$

where

$$\begin{aligned}
H_0 &= \mathrm{Tr}\Big[\tfrac{1}{2}\,(P^i)^2 + \tfrac{1}{2}\,(\tfrac{m}{3}\,)^2\,(X^a)^2 + \tfrac{1}{2}\,(\tfrac{m}{6}\,)^2\,(X^{a'})^2 - \tfrac{m}{4}\,i\theta\gamma_{123}\theta\,\Big]\\
H_{\text{INT}} &= \mathrm{Tr}\Big[-\tfrac{m}{3}\,i\epsilon_{abc}\,X^a\,X^b\,X^c - \tfrac{1}{4}\,[X^i, X^j]^2 - \theta\gamma_i[X^i, \theta]\,\Big]
\end{aligned}$$

to be augmented by the gauge constraint

$$\mathcal{G} = [P^i, X^i] - i\{\theta, \theta\} = 0\,. \tag{2.5}$$

As we shall show for $m \gg 1$ the interacting piece of the Hamiltonian H_{INT} is suppressed and can be treated perturbatively.

Let us now turn to the quantization of the pp-wave matrix theory. The canonical (anti)-commutation relations for the matrix operators are given by

$$[P^i_{rs}, X^j_{tu}] = -i\,\delta^{ij}\,\delta_{st}\,\delta_{ru} \qquad \{\theta^\alpha_{rs}, \theta^\beta_{tu}\} = \tfrac{1}{2}\,\delta^{\alpha\beta}\,\delta_{st}\,\delta_{ru} \tag{2.6}$$

where the factor of 1/2 for the fermions arises from the Dirac procedure of treating the constraint $P^\alpha_\theta + i\theta^\alpha = 0$ properly. In view of H_0 in (2.4) it is natural to introduce the creation and annihilation operators in the bosonic sector as

$$a^a = \sqrt{\tfrac{3}{2m}}\,(P^a - i\tfrac{m}{3}\,X^a) \qquad b^{a'} = \sqrt{\tfrac{3}{m}}\,(P^{a'} - i\tfrac{m}{6}\,X^{a'}) \tag{2.7}$$

reflecting the 3+6 split of the masses. They obey the standard commutation relations

$$[a^a_{rs}, a^{\dagger\, b}_{tu}] = \delta^{ab}\,\delta_{st}\,\delta_{ru} \qquad [b^{a'}_{rs}, b^{\dagger\, b'}_{tu}] = \delta^{a'b'}\,\delta_{st}\,\delta_{ru}\,. \tag{2.8}$$

Using these relations the bosonic part of the free Hamiltonian H_0 takes the form

$$H_0^B = \mathrm{Tr}\Big[\,\tfrac{m}{3}\; a^{\dagger\, a}\, a^a + \tfrac{m}{6}\; b^{\dagger\, a'}\, b^{a'}\,\Big] + m\,N^2\,. \tag{2.9}$$

In the fermionic sector we complexify the real spinor matrices θ_{rs} via

$$\theta^{\pm}_{rs} = \Pi^{\pm}\,\theta_{rs} \quad \text{where} \quad \Pi^{\pm} = \tfrac{1}{2}\,(\mathbb{1} \pm i\gamma_{123}) \tag{2.10}$$

which yields the anticommutation relations

$$\{\theta^{+\,\alpha}_{rs}, \theta^{-\,\beta}_{tu}\} = \tfrac{1}{2}\,(\Pi^+)_{\alpha\beta}\,\delta_{st}\,\delta_{ru} \qquad \{\theta^{+\,\alpha}_{rs}, \theta^{+\,\beta}_{tu}\} = 0 = \{\theta^{-\,\alpha}_{rs}, \theta^{-\,\beta}_{tu}\}\,. \tag{2.11}$$

Note the chirality property of the complexified fermions $(i\gamma_{123})\,\theta^{\pm}_{rs} = \pm\theta^{\pm}_{rs}$ as well as $(\Pi^{\pm})^2 = \Pi^{\pm}$ and $\Pi^+\,\Pi^- = 0$. As $\theta = \theta^+ + \theta^-$ the fermionic term in H_0 is now given by

$$H_0^F = -i\tfrac{m}{4}\;\mathrm{Tr}\,[\,\theta\gamma_{123}\theta\,] = \tfrac{m}{2}\;\mathrm{Tr}[\,\theta^{+\,\alpha}\,\theta^{-\,\alpha}\,] - m\,N^2 \tag{2.12}$$

canceling precisely the zero point energy of the bosonic sector. The zero-energy groundstate of the resulting free Hamiltonian

$$H_0 = \mathrm{Tr}\Big[\,\tfrac{m}{3}\; a^{\dagger\, a}\, a^a + \tfrac{m}{6}\; b^{\dagger\, a'}\, b^{a'} + \tfrac{m}{2}\;\theta^{+\,\alpha}\,\theta^{-\,\alpha}\,\Big] \tag{2.13}$$

is denoted by $|0\rangle$ and is annihilated by

$$a^a_{rs}\,|0\rangle = 0 \qquad b^{a'}_{rs}\,|0\rangle = 0 \qquad \theta^{-\,\alpha}_{rs}\,|0\rangle = 0\,. \tag{2.14}$$

Physical states are required to be gauge invariant due to the gauge constraint (2.5). They are given by traces over words in the creation operators $a^\dagger, b^\dagger$ and θ^+, viz.

$$\mathrm{Tr}[\ldots a^{\dagger\, a}\ldots b^{\dagger\, a'}\ldots \theta^{+\,\alpha}\ldots]\ldots \mathrm{Tr}[\ldots a^{\dagger\, b}\ldots b^{\dagger\, b'}\ldots \theta^{+\,\beta}\ldots]\,|0\rangle\,. \tag{2.15}$$

In this paper we shall be interested in the spectrum of the full $U(N)$ Hamiltonian $H = H_0 + H_{\text{INT}}$. Clearly then the problem factorizes into the trivial free $U(1)$ sector spanned by the excitation operators of wordlength one, $\mathrm{Tr}[a^{\dagger a}]$, $\mathrm{Tr}[b^{\dagger a'}]$, $\mathrm{Tr}[\theta^{+\,\alpha}]$, and a complicated interacting $SU(N)$ sector spanned by excitation operators of wordlength two and larger.

3. Supersymmetry Algebra and Structure of the Spectrum

The derivation of the supersymmetry algebra is straightforward. One has two supercharges Q_α and q_α associated with the non-linearly and linearly realized supersymmetries of (2.2). Their form follows from the operator relations

$$\begin{aligned} \delta X^i &= 2i[Q\,\epsilon(\tau) + q\,\eta(\tau), X^i] \qquad & \epsilon(\tau) &= e^{-\frac{m}{12}\gamma_{123}\tau}\epsilon_0 \\ \delta\theta_\alpha &= 2i[Q\,\epsilon(\tau) + q\,\eta(\tau), \theta_\alpha] \qquad & \eta(\tau) &= e^{\frac{m}{4}\gamma_{123}\tau}\eta_0 \end{aligned}$$

One then deduces the two supercharges

$$Q_\alpha = \mathrm{Tr}\Big[\,[P^i\gamma_i - \tfrac{i}{2}\,[X^i,X^j]\,\gamma_{ij} + \tfrac{m}{3}\,X^a\,\gamma_a\gamma_{123} + \tfrac{m}{6}\,X^{a'}\gamma_{a'}\gamma_{123}\,]_\alpha{}^\gamma\,\theta_\gamma\Big] \tag{3.1}$$

$$q_\alpha = \mathrm{Tr}\,[\,\theta_\alpha\,]\,. \tag{3.2}$$

Note that q_α only acts in the free $U(1)$ sector of the model. We relegate the explicit evaluation of the supersymmetry algebra into appendix A. One finds in the interacting $SU(N)$ sector of the model

$$\begin{aligned}\{Q_\alpha,Q_\beta\} &= \delta_{\alpha\beta}H - \tfrac{m}{6}\,L^{ab}(\gamma_{ab}\,\gamma_{123})_{\alpha\beta} + \tfrac{m}{12}\,L^{a'b'}(\gamma_{a'b'}\,\gamma_{123})_{\alpha\beta} + i\mathrm{Tr}(X^i\mathcal{G})(\gamma_i)_{\alpha\beta}\\ [H,Q_\alpha] &= \frac{mi}{12}(Q\gamma_{123})_\alpha + \mathrm{Tr}(\theta_\alpha\mathcal{G})\end{aligned} \tag{3.3}$$

Compared to the Minkowski background superalgebra we thus see the emergence of additional terms coupling to the angular momentum operators L^{ab} and $L^{a'b'}$ in the anti-commutator of two supercharges, given by

$$L^{ij} = \mathrm{Tr}(\,P^i\,X^j - P^j\,X^i + \tfrac{i}{2}\,\theta\gamma^{ij}\theta) \tag{3.4}$$

and obeying $[L^{ij},L^{kl}] = -i\,(\,\delta^{jk}\,L^{il} + \delta^{il}\,L^{jk} - \delta^{ik}\,L^{jl} - \delta^{jl}\,L^{ik}\,)$. Despite the appearance of these angular momentum operators the old argument for the zero-energy groundstate still goes through: a maximally supersymmetric state (being annihilated by all the Q_α) will have zero energy and be a $SO(3)$ and $SO(6)$ singlet. Hence the vanishing energy of the groundstate $|0\rangle$ for $m\to\infty$ is protected from perturbative corrections and constitutes the unique groundstate of the interacting model. Moreover all excitations will have strictly positive energy.

We see in (3.3) that the supercharges do *not* commute with the Hamiltonian, which simply states that superpartners do not have the same mass in this model. This effect is induced by the time dependent supersymmetry parameter, and the coefficient $\frac{m}{12}$ can be easily inferred by computing the difference of bosonic and fermionic masses $\frac{m}{3}-\frac{m}{4}=\frac{m}{12}=\frac{m}{4}-\frac{m}{6}$. This phenomenon is analogous to the situation for representations of supersymmetry in AdS spaces.

The remaining (anti)-commutators read

$$\begin{aligned}\{q_\alpha,q_\beta\} &= \tfrac{N}{2}\,\delta_{\alpha\beta}\\ \{Q_\alpha,q_\beta\} &= \sqrt{\tfrac{2\,m}{3}}\,\mathrm{Tr}[(\not{a}\,\Pi^+)_{\alpha\beta} + (\not{a}^\dagger\,\Pi^-)_{\alpha\beta}] + \sqrt{\tfrac{m}{3}}\,\mathrm{Tr}[(\not{b}\,\Pi^+)_{\alpha\beta} + (\not{b}^\dagger\,\Pi^-)_{\alpha\beta}]\\ [H,q_\alpha] &= -\tfrac{m\,i}{4}\,(q\gamma_{123})_\alpha\end{aligned} \tag{3.5}$$

where the bosonic matrix ladder operators of (2.7) appear in the second line.

For the study of the spectrum of the pp-wave matrix model it is useful to perform a chirality split of the dynamical supercharges Q_α according to

$$Q^\pm = \Pi^\pm\,Q \qquad \text{where} \quad \Pi^\pm = \tfrac{1}{2}\,(\mathbb{1}\pm i\gamma_{123})\,. \tag{3.6}$$

The supersymmetry algebra then takes the compact form

$$\begin{aligned} \{Q^+_\alpha, Q^-_\beta\} &= (\Pi^+)_{\alpha\beta} H + i\tfrac{m}{6}\, L^{ab} (\Pi^+\, \gamma_{ab})_{\alpha\beta} - i\tfrac{m}{12}\, L^{a'b'} (\Pi^+\, \gamma_{a'b'})_{\alpha\beta} \\ \{Q^\pm_\alpha, Q^\pm_\beta\} &= 0 \\ [H, Q^\pm_\alpha] &= \mp \tfrac{m}{12}\, Q^\pm_\alpha \end{aligned} \tag{3.7}$$

where we have dropped the terms proportional to the gauge constraints $\mathcal{G}$ for transparency, as they do not affect gauge invariant states. It is instructive to spell out the form of $Q^\pm$ explicitly in terms of oscillators

$$\begin{aligned} Q^- &= \sqrt{\tfrac{m}{3}}\, \mathrm{Tr}\,[\, b^{a'}\, \gamma_{a'}\, \theta^+] + \sqrt{\tfrac{2m}{3}}\, \mathrm{Tr}\,[\, a^{\dagger\, a}\, \gamma_a\, \theta^-] - \tfrac{i}{2}\, \mathrm{Tr}\,(\, [X^i, X^j]\, \gamma_{ij}\, \theta^- \,) \\ Q^+ &= \sqrt{\tfrac{2m}{3}}\, \mathrm{Tr}\,[\, a^a\, \gamma_a\, \theta^+] + \sqrt{\tfrac{m}{3}}\, \mathrm{Tr}\,[\, b^{\dagger\, a'}\, \gamma_{a'}\, \theta^-] - \tfrac{i}{2}\, \mathrm{Tr}\,(\, [X^i, X^j]\, \gamma_{ij}\, \theta^+ \,) \end{aligned} \tag{3.8}$$

where X^i is given in terms of the oscillators $a^{(\dagger)\, a}$ and $b^{(\dagger)\, a'}$ through (2.7) and we have $(Q^+)^\dagger = Q^-$. Note that the free theory supersymmetry algebra generated by $Q_0^\pm$ takes the same form as (3.7), with H replaced by H_0 and $Q_0^\pm$ given by dropping the commutator terms in (3.8). Clearly now from (3.7) one observes that Q^+ lowers and Q^- raises the energy eigenvalue of a state by $m/12$. As there are 8 raising and 8 lowering operators a generic long multiplet will contain 256 states spread over 9 "floors" of equal energies and spanning an energy range from its smallest value E to $E + \frac{2m}{3}$.

The simplest long multiplet is built upon a $SO(3)$ and $SO(6)$ singlet on the "ground floor" and the entire multiplet has 256 states in total. It is straightforward to find out how the states of such a multiplet are grouped into irreducible representations of $SO(3) \times SO(6)$ on each floor. The result reads

Floor	$SO(3) \times SO(6)$ reps		
8	(1,1)		
7	$(2, \overline{4})$		
6	$(1, \overline{10})$	$(3, \overline{6})$	
5	$(2, \overline{20})$	(4,4)	
4	$(1, 20')$	(3,15)	(5,1)
3	(2,20)	$(4, \overline{4})$	
2	(1,10)	(3,6)	
1	(2,4)		
0	(1,1)		

(3.9)

The energy differences within one multiplet are fixed, however the lowest energy eigenvalue E of a generic multiplet may only be computed approximately in perturbation theory.

Let us now study the first excited states of the interacting $SU(N)$ sector, i.e. excitations of wordlength two, which are decomposed into irreducible representations as follows

State	$SO(3)\times SO(6)$ rep	Energy
$\|a'a'\rangle = \mathrm{Tr}[b^{\dagger\, a'}\, b^{\dagger\, a'}\,]\,\|0\rangle$	$(1,1)$	$\frac{m}{3}$
$\|a'b'\rangle = \mathrm{Tr}[b^{\dagger\,(a'}\, b^{\dagger\, b')}\,]\|0\rangle$	$(1,20')$	$\frac{m}{3}$
$\|aa'\rangle_B = \mathrm{Tr}[a^{\dagger\, a}\, b^{\dagger\, b'}\,]\,\|0\rangle$	$(3,6)$	$\frac{m}{2}$
$\|aa'\rangle_F = \mathrm{Tr}[\theta^+\gamma^{aa'}\theta^+]\,\|0\rangle$	$(3,6)$	$\frac{m}{2}$
$\|a'b'c'\rangle = \mathrm{Tr}[\theta^+\gamma^{a'b'c'}\theta^+]\,\|0\rangle$	$(1,\overline{10})$	$\frac{m}{2}$
$\|aa\rangle = \mathrm{Tr}[a^{\dagger\, a}\, a^{\dagger\, a}\,]\,\|0\rangle$	$(1,1)$	$\frac{2m}{3}$
$\|ab\rangle = \mathrm{Tr}[a^{\dagger\,(a}\, a^{\dagger\, b)}\,]\|0\rangle$	$(5,1)$	$\frac{2m}{3}$
$\|a'a';\alpha\rangle = \mathrm{Tr}[\not{b}^{\,\dagger}\theta^+_\alpha\,]\|0\rangle$	$(2,4)$	$\frac{5m}{12}$
$\|a'b';\alpha\rangle = \mathrm{Tr}[\gamma^{(a'}b^{\dagger\, b')}\theta^+_\alpha\,]\|0\rangle$	$(2,\overline{20})$	$\frac{5m}{12}$
$\|aa;\alpha\rangle = \mathrm{Tr}[\not{a}^{\,\dagger}\theta^+_\alpha\,]\|0\rangle$	$(2,\overline{4})$	$\frac{7m}{12}$
$\|ab;\alpha\rangle = \mathrm{Tr}[\gamma^{(a}a^{\dagger\, b)}\theta^+_\alpha\,]\|0\rangle$	$(4,\overline{4})$	$\frac{7m}{12}$

(3.10)

where the bifermion states are restricted to an odd number of $SO(6)$ vector indices, a consequence of the chirality property $\theta^+ = \Pi^+\theta^+$. In our notation (ij) refers to totally symmetrized indices without the trace part. Note that the states $|aa'\rangle_B$ and $|aa'\rangle_F$ are degenerate in mass and $SO(3)\times SO(6)$ representation and could potentially mix. Let us see how these states fit into multiplets of the free superalgebra generated by $Q_0^\pm$. As the free supercharges $Q_0^\pm$ preserve the wordlength it comes as no surprise, that the above states may be grouped into two multiplets. Also due to

$$\begin{aligned} Q_0^+|a'a'\rangle &= 0 \qquad & Q_0^+|a'b'\rangle &= 0 \\ Q_0^-|aa\rangle &= 0 \qquad & Q_0^-|ab\rangle &= 0 \end{aligned} \tag{3.11}$$

the multiplets begin with the states of energy $m/3$ and end with the states of energy $2m/3$ - they are short multiplets consisting of 5 floors. The relevant double step ladder operators connecting floors of bosonic states are

$$\begin{aligned} \text{2 floors up:} \qquad & Q_0^-\gamma^{aa'}Q_0^- \qquad Q_0^-\gamma^{a'b'c'}Q_0^- \\ \text{2 floors down:} \qquad & Q_0^+\gamma^{aa'}Q_0^+ \qquad Q_0^+\gamma^{a'b'c'}Q_0^+ \end{aligned} \tag{3.12}$$

which follow again from the chirality property of $Q_0^\pm$. Starting with the lightest state of $|a'a'\rangle$ one finds the multiplet "**A**"

$$\begin{aligned} |a'a'\rangle \ &\overset{Q_0^-\gamma^{aa'}Q_0^-}{\longrightarrow} \ (\,|aa'\rangle_F + \sqrt{2}\,|aa'\rangle_B\,) \ \overset{Q_0^+\gamma^{aa'}Q_0^+}{\longleftarrow} \ |ab\rangle \\ |a'a'\rangle \ &\overset{Q_0^-\gamma^{a'b'c'}Q_0^-}{\longrightarrow} \qquad\qquad 0 \qquad\qquad \overset{Q_0^+\gamma^{a'b'c'}Q_0^+}{\longleftarrow} \ |ab\rangle \end{aligned} \tag{3.13}$$

consisting of the 24 bosonic states $\{\,|a'a'\rangle, (\,|aa'\rangle_F + \sqrt{2}\,|aa'\rangle_B\,), |ab\rangle\,\}_{\mathbf{A}}$. Similarly starting with $|a'b'\rangle$ as the lightest state the multiplet "**B**" is obtained

$$|a'b'\rangle \ \overset{Q_0^-\gamma^{aa'}Q_0^-}{\longrightarrow} \ (\,|aa'\rangle_F - 2\sqrt{2}\,|aa'\rangle_B\,) \ \overset{Q_0^+\gamma^{aa'}Q_0^+}{\longleftarrow} \ |aa\rangle$$

$$|a'b'\rangle \quad \overset{Q_0^- \gamma^{a'b'c'} Q_0^-}{\longrightarrow} \qquad |a'b'c'\rangle \qquad \overset{Q_0^+ \gamma^{a'b'c'} Q_0^+}{\longleftarrow} \quad |aa\rangle \tag{3.14}$$

made out of the 49 bosonic states $\{\, |a'b'\rangle, (\, |aa'\rangle_F - 2\sqrt{2}\, |aa'\rangle_B\,), |a'b'c'\rangle, |aa\rangle \,\}_{\mathbf{B}}$. We indeed observe a mixing between the two $(3,6)$ states $|aa'\rangle_B$ and $|aa'\rangle_F$, which is orthogonal due to the norms

$${}_F\langle aa'|bb'\rangle_F = 4\,\delta_{ab}\,\delta_{a'b'}\,N^2 \qquad {}_B\langle aa'|bb'\rangle_B = \delta_{ab}\,\delta_{a'b'}\,N^2 \qquad {}_F\langle aa'|bb'\rangle_B = 0\,. \tag{3.15}$$

For the fermionic states it is obvious from $Q_0^-|a'a'\rangle = |a'a';\alpha\rangle$ and $Q_0^+|aa\rangle = |aa;\alpha\rangle$ that the fermionic states $|a'a';\alpha\rangle$ and $|ab;\alpha\rangle$ belong to the multiplet "**A**", whereas $|a'b';\alpha\rangle$ and $|aa;\alpha\rangle$ belong to "**B**". So the level two states make two irreducible supermultiplets,

$$\begin{aligned} &\mathbf{A}: \quad (1,1)+(2,4)+(3,6)+(4,\overline{4})+(5,1) \\ &\mathbf{B}: \quad (1,20')+(2,\overline{20})+[(1,\overline{10})+(3,6)]+(2,\overline{4})+(1,1)\,. \end{aligned} \tag{3.16}$$

We note that both of them can be part of the simplest long multiplet presented in (3.9). In the next section we shall study how the energies of these multiplets get corrected in perturbation theory.

4. The Perturbative Energy Spectrum and Protected States

The supersymmetry algebra derived in the last section implies that the energy of the maximally supersymmetric ground state, which is annihilated by all supercharges, must be exactly zero. Before we embark on the calculation of energy shifts for the excited states of (3.10) let us verify this in leading order perturbation theory [2]. The perturbative corrections to the spectrum are organized in an expansion in $1/m^2$. To consistently work out the leading correction of the groundstate energy it is then necessary to work up to second order in quantum mechanical perturbation theory and to evaluate the expression

$$\Delta E_0|_{\mathcal{O}(1/m^2)} = \langle 0|\, H_{\rm INT}|_{X^4}\, |0\rangle + \langle 0|\, H_{\rm INT}|_{X^3+X\theta^2}\, \frac{1}{E_0 - H_0}\, H_{\rm INT}|_{X^3+X\theta^2}\, |0\rangle \tag{4.1}$$

where the quartic interaction term contributes in first order perturbation theory whereas the cubic and the Yukawa term contribute at second order perturbation theory. The first term of the right-hand-side of (4.1) is given by the expectation value

$$\Delta E_0^1 = -\tfrac{1}{4}\,\langle 0|\,{\rm Tr}[X^i,X^j]^2\,|0\rangle = -\tfrac{1}{2}\,\langle 0|\,{\rm Tr}[X^i\,X^j\,X^i\,X^j - (X^i)^2\,(X^j)^2]\,|0\rangle \tag{4.2}$$

It is useful to maintain a unified language for the bosonic ladder operators by introducing the objects

$$X^i = -i\,(\,\tilde{a}^{\dagger\, i} - \tilde{a}^i) \tag{4.3}$$

[2] This check was also performed in [10].

which obey

$$[\tilde{a}^i_{rs},\tilde{a}^{\dagger j}_{tu}] = M^{ij}\,\delta_{st}\,\delta_{ru} \qquad M^{ij} = \tfrac{3}{m}\begin{pmatrix} \frac{1}{2}\,\delta^{ab} & 0 \\ 0 & \delta^{a'b'} \end{pmatrix} \tag{4.4}$$

With the help of these definitions one straightforwardly evaluates

$$\begin{aligned} \langle 0|\,\mathrm{Tr}[X^i\,X^j\,X^i\,X^j]\,|0\rangle &= 2N^3\,\mathrm{Tr}M^2 + N\,(\mathrm{Tr}M)^2 \\ \langle 0|\,\mathrm{Tr}[(X^i)^2\,(X^j)^2]\,|0\rangle &= N^3\,(\,\mathrm{Tr}M^2 + (\mathrm{Tr}M)^2\,) + N\,\mathrm{Tr}M^2 \end{aligned} \tag{4.5}$$

yielding the first contribution to the energy shift

$$\Delta E^1_0 = -\tfrac{1}{4}\,\langle 0|\,\mathrm{Tr}[X^i,X^j]^2\,|0\rangle = \frac{11\cdot 3^4}{4\,m^2}\,N\,(N^2-1) \tag{4.6}$$

The contributions to $\mathcal{O}(1/m^2)$ at second order perturbation theory come from the cubic and the Yukawa terms of $H_{\rm INT}$. For the first contribution one has

$$\Delta E^2_0 = \tfrac{m^2}{9}\,\langle 0|\mathrm{Tr}[e_{abc}\,\tilde{a}^a\,\tilde{a}^b\,\tilde{a}^c]\,\frac{1}{m}\,\mathrm{Tr}[e_{abc}\,\tilde{a}^{\dagger\,a}\,\tilde{a}^{\dagger\,b}\,\tilde{a}^{\dagger\,c}]\,|0\rangle \tag{4.7}$$

as here only a pure bosonic level 3 state is excited the free Hamiltonian in the denominator has been replaced by $3\cdot\frac{m}{3}$. Upon contracting this expression reduces to

$$\Delta E^2_0 = -\frac{3^3}{4\,m^2}\,N(N^2-1) \tag{4.8}$$

Turning to the final contribution from the Yukawa coupling we have

$$\begin{aligned} \Delta E^3_0 = &-\langle 0|\mathrm{Tr}(\theta^-\gamma_a\,[\tilde{a}^a,\theta^-]\,)\,\frac{1}{\frac{m}{3}+\frac{m}{2}}\,\mathrm{Tr}(\,\theta^+\gamma_a\,[\tilde{a}^{\dagger\,a},\theta^+\,]\,)\,|0\rangle \\ &-\langle 0|\mathrm{Tr}(\theta^-\gamma_{a'}\,[\tilde{a}^{a'},\theta^-])\,\frac{1}{\frac{m}{6}+\frac{m}{2}}\,\mathrm{Tr}(\,\theta^+\gamma_{a'}[\tilde{a}^{\dagger\,a'},\theta^+\,]\,)\,|0\rangle \end{aligned} \tag{4.9}$$

Note the two different mass channels appearing for the inverse free Hamiltonian in the above. Now one computes

$$\langle 0|\mathrm{Tr}(\theta^-\gamma_i\,[\tilde{a}^i,\theta^-]\,)\,\mathrm{Tr}(\,\theta^+\gamma_i\,[\tilde{a}^{\dagger\,i},\theta^+\,]\,)\,|0\rangle = N\,(N^2-1)\,\mathrm{tr}(\gamma^i\Pi^-\gamma^j\Pi^+)\,M_{ij} \tag{4.10}$$

where $\Pi^\pm = \frac{1}{2}\,(\mathbb{1}\pm i\gamma_{123})$ are the projectors appearing in (2.10). We note that $\mathrm{tr}(\gamma^a\Pi^-\gamma^a\Pi^+) = 0$ and $\mathrm{tr}(\gamma^{a'}\Pi^-\gamma^{a'}\Pi^+) = 6\,\mathrm{tr}\Pi^+$ which yields the final result

$$\Delta E^3_0 = -\frac{8\cdot 3^3}{m^2}\,N\,(N^2-1) \tag{4.11}$$

Summing up the three contributions (4.6),(4.8) and (4.11) we indeed find

$$\Delta E_0 = \Delta E^1_0 + \Delta E^2_0 + \Delta E^3_0 = 0 \tag{4.12}$$

the vanishing shift of the groundstate energy in leading order perturbation theory.

The computation of the energy shifts for the first excited states of (3.10) goes along the same lines, but is technically more involved. We report on the details of this computation in appendix B and simply state the complete result here. By virtue of the supersymmetry algebra (3.7) it is clear that states within one multiplet should receive the *same* perturbative correction to their energy eigenvalues. This is indeed what one finds. Taking care of the normalization of states the leading shift in energy for a generic state $|\phi\rangle$ is given by

$$\Delta E_{|\phi\rangle}\,|_{\mathcal{O}(1/m^2)} = \frac{1}{\langle\phi|\phi\rangle}\Big(\langle\phi|H_{\rm INT}|_{X^4} + H_{\rm INT}|_{X^3+X\theta^2}\,\frac{1}{E_0-H_0}\,H_{\rm INT}|_{X^3+X\theta^2}\,|\phi\rangle\Big)\,. \tag{4.13}$$

For the states of our multiplet "**A**" one obtains

$$\Delta E_{|\phi\rangle}\,|_{\mathcal{O}(1/m^2)} = \frac{108}{m^2}\,(N-\frac{1}{N})$$

$$\text{for}\qquad |\phi\rangle\in\Big\{\,h\,|a'a'\rangle,\;h_{aa'}\,(\,|aa'\rangle_F+\sqrt{2}\,|aa'\rangle_B\,),\;h_{ab}\,|ab\rangle\,\Big\} \tag{4.14}$$

introducing suitable polarization tensors $h, h_{aa'}$ and h_{ab}. Most interestingly, however, the energy shift for the members of our multiplet "**B**" precisely cancels!

$$\Delta E_{|\phi\rangle}\,|_{\mathcal{O}(1/m^2)} = 0$$

$$\text{for}\qquad |\phi\rangle\in\Big\{\,h_{a'b'}\,|a'b'\rangle,\;\tilde{h}_{aa'}\,(\,|aa'\rangle_F-2\,\sqrt{2}\,|aa'\rangle_B\,),\;g_{a'b'c'}\,|a'b'c'\rangle,\;h\,|aa\rangle\,\Big\} \tag{4.15}$$

We shall argue that this remains true to all orders in perturbation theory. The crucial input from the representation theory of Lie superalgebras here is that if a multiplet is short its energy is quantized by the symmetry algebra. One simple way of seeing this is to use the fact that the ground-floor state must be annihilated by a product of less than nine supercharges, since short multiplets do not span all of the 9 floors. One can calculate the norm of a state of the form $(\prod_i Q^-_{\alpha_i})|\Lambda\rangle$ using only the superalgebra and obtain a set of linear equations involving E and the Dynkin labels of $SO(3)\times SO(6)$ of the ground-floor state $|\Lambda\rangle$. In fact the representation theory of Lie superalgebras is known to some extent, original classifications and first important results are due to Kac [13]. A more detailed discussion of the representation theory of the M-theory pp-wave superalgebra will be presented in a separate publication [14].

Based on this insight, our calculation implies that the multiplet "**A**" should combine with other short multiplets of the free theory to make a long multiplet in the interacting theory for which there is no such quantization rule from the symmetry algebra. The multiplet "**B**" on the other hand should stay short. Because there are no lighter states in the free theory the ground-floor states $(1,1)$ and $(1,20')$ must remain as the ground floor of the two multiplets also with interactions. For the multiplet "**A**" we see from (3.9) that the first missing block is $(1,10)$ on the second floor with

$E = \frac{m}{2}$. In fact this state is provided from the level 3 spectrum: $\mathrm{Tr}[b^{\dagger\,[a'}b^{\dagger\,b'}b^{\dagger\,c']}]|0\rangle$, which also has $E = \frac{m}{2}$. One can show that the free theory spectrum can provide all the missing blocks of higher floors which are needed to complete "**A**" into a long multiplet. In order to turn multiplet "**B**" into a long multiplet starting from $(1, 20')$ we would need the states $(2, 4) \times (1, 20') = (2, \overline{20}) + (2, 60)$ on the first-floor, where in terms of Dynkin labels the (60) of $SO(6)$ is given by $[1, 2, 0]$. But here, unlike the situation for the multiplet "**A**", these additional $(2, 60)$ states at $E_0 = \frac{5m}{12}$ are simply not present in the free spectrum, even if we consider states of higher level. At $E_0 = \frac{5m}{12}$ there are no other states than the states listed in (3.9). So it turns out that the multiplet "**B**" is truly short even in the interacting theory, and its energy is free from corrections to all orders.

The fact that the multiplet "**B**" built upon the lightest state $|a'b'\rangle$ is protected is strongly reminiscent of the situation for the chiral primary operators in $\mathcal{N} = 4$ super Yang-Mills theory which do not receive any radiative corrections to their scaling dimensions. These operators are given by symmetric traceless combinations of the six scalar fields Φ^I with $I = 1, \ldots, 6$ of $\mathcal{N} = 4$ super Yang-Mills theory, i.e.

$$\mathcal{O}_n^A = C^A_{I_1 I_2 \ldots I_n} \,\mathrm{Tr}\,[\Phi^{I_1}\,\Phi^{I_2} \ldots \Phi^{I_n}] \tag{4.16}$$

with $C^A_{I_1 \ldots I_n}$ being totally symmetric in its lower indices and any contraction among the lower indices vanishing, $C^A_{I_1 \ldots I \ldots I \ldots I_n} = 0$. Let us therefore consider the multiplets built upon the lightest states

$$|C_{(n)}\rangle = C_{a'_1 a'_2 \ldots a'_n} \,\mathrm{Tr}\,[b^{\dagger a'_1}\, b^{\dagger a'_2} \ldots b^{\dagger a'_n}]\, |0\rangle \tag{4.17}$$

with $C_{a'_1 a'_2 \ldots a'_n}$ being totally symmetric and traceless, i.e. $C_{a'_1 \ldots a' \ldots a' \ldots a'_n} = 0$. These states have the mass $\frac{n \cdot m}{6}$ in the free theory. Clearly as

$$Q_0^+ |C_{(n)}\rangle = 0 \tag{4.18}$$

they constitute the lightest state in a multiplet of the free theory. It is tempting to speculate that the energy eigenvalue of these states is protected from perturbative corrections as well. We have computed the energy shifts for these states for $n = 3, 4$ and 5 in leading order perturbation theory and indeed find that they cancel! The explicit contributions are presented in appendix B. Based on this evidence we therefore conjecture that all the states contained in the multiplets built on (4.17) are protected and that their energy eigenvalues are exactly given by the free theory values. The proof of this conjecture should go along the same lines as the arguments presented in the above for the case $|C_{(2)}\rangle$, namely due to the absence of the required representations in the free theory at higher mass levels. We leave the detailed proof for future work.

5. Conclusions and Outlook

In this paper we have studied the spectrum of the recently found massive matrix quantum mechanics in a pp-wave background and performed a second order perturbation calculation. We uncovered a protected short multiplet of the theory whose energy eigenvalues are now known exactly in the full interacting model. Moreover we conjectured the existence of an infinite series of such protected states. Employing the matrix model conjecture this is a non-trivial statement about the light-cone Hamiltonian of M-theory in a pp-wave background. In the case of the maximally supersymmetric pp-wave solution of type IIB superstring, the precise energy spectrum in the light cone gauge was presented by Metsaev [15]. Our results can be thought of as the M-theory counterpart. Using the relation between the AdS space and the pp-wave, it was argued that the string spectrum in the pp-wave background must be related to the anomalous dimension of the dual CFT operator with large R-charge. It is very tempting to conjecture the same correspondence between the M-theory pp-wave solutions and the superconformal field theories of M2- and M5-branes. An intriguing fact is that the Penrose limit of both $AdS_4 \times S_7$ and $AdS_7 \times S_4$ lead to the same pp-wave solution with $SO(3) \times SO(6)$ symmetry, implying that they share essentially the same subsector. The M-brane field theories are still largely mysterious but it would be very interesting if we can compare the matrix theory calculation reported here with field theory calculations.

There are a number of further interesting open question emerging. For example the protected energy eigenvalues do not depend on N and should therefore survive the large N limiting procedure under which the matrix model approximates the pp-wave supermembrane. What is the picture of these states in the supermembrane theory? Furthermore, what can we learn from these considerations for the notorious flat matrix model in the limit $m \to 0$?

Finally, in our work we have exclusively studied the matrix model around the "trivial" vacuum $X^i = 0$. As discussed in [7, 10] there is a multitude of further maximally supersymmetric vacua in the bosonic $SO(3)$ sector corresponding to fuzzy sphere solutions of the equations of motion. As these vacua are subject to the same superalgebra we expect that similar protected multiplets exist in these sectors of the theory as well.

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A. The Supersymmetry Algebra

The nontrivial pieces of the supersymmetry algebra (3.3) lie in the (anti)-commmutators involving Q_α. In computing the anticommutator of two supercharges $\{Q_\alpha, Q_\beta\}$ we shall only focus on the terms proportional to m as we know from the work of [4] how the m independent terms work out. One then has

$$\{Q_\alpha, Q_\beta\} = [\ldots]_\alpha{}^\gamma{}_{rs}\,[\ldots]_\beta{}^\delta{}_{tu}\,\{\theta^{sr}_\gamma, \theta^{ut}_\delta\} - \Big[\,[\ldots]_\alpha{}^\gamma{}_{rs}, [\ldots]_\beta{}^\delta{}_{tu}\Big]\,\theta^{ut}_\delta\,\theta^{sr}_\gamma \tag{A.1}$$

where we have used the abbreviation

$$[\ldots]_\alpha{}^\gamma{}_{rs} = [P^i\,\gamma_i - \tfrac{i}{2}\,[X^i, X^j]\,\gamma_{ij} + m_i\,X^i\,\gamma_i\,\gamma_{123}\,]_\alpha{}^\gamma{}_{rs}$$

following from (3.1), r, s, t, u denote U(N) matrix indices. In the above we have moreover employed an extended summation convention for the index i where

$$m_i\,X^i\,\gamma_i \equiv \tfrac{m}{3}\;X^a\,\gamma_a + \tfrac{m}{6}\;X^{a'}\,\gamma_{a'}$$

capturing the two different SO(3) and SO(6) masses in m_i. The terms in (A.1) are then straightforwardly found to have the structure

$$\{Q_\alpha, Q_\beta\} = \tfrac{1}{2}\,(m_i\,X^i)^2 + +\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{F} + (m_i \text{ independent terms}) \tag{A.2}$$

where

$$\begin{aligned}
\mathcal{B}_1 &= -\tfrac{1}{2}\;m_i\,\Big(\mathrm{Tr}(P^k\,X^i)\;\gamma_k\gamma_{123}\gamma_i - \mathrm{Tr}(X^i\,P^k)\;\gamma_i\gamma_{123}\gamma_k\Big)\\
\mathcal{B}_2 &= \tfrac{i}{4}\;m_i\,\mathrm{Tr}(X^i\,[X^k, X^l]\,)\,\Big(\gamma_i\gamma_{123}\gamma_{kl} + \gamma_{kl}\gamma_{123}\gamma_i\,\Big)\\
\mathcal{F} &= i\,\Big(\gamma^i_{\alpha\gamma}\,(\gamma^i\gamma_{123})_{\beta\delta}\,m_i - (\gamma^i\gamma_{123})_{\alpha\gamma}\,\gamma^i_{\beta\delta}\,m_i\,\Big)\,\mathrm{Tr}(\theta_\delta\,\theta_\gamma)
\end{aligned} \tag{A.3}$$

Now the bosonic contributions $\mathcal{B}_1$ and $\mathcal{B}_2$ can be shown to be

$$\begin{aligned}
\mathcal{B}_1 &= -\tfrac{m}{3}\;\mathrm{Tr}(P^a\,X^b)\,(\gamma_{ab}\gamma_{123})_{\alpha\beta} + \tfrac{m}{6}\;\mathrm{Tr}(P^{a'}\,X^{b'})\,(\gamma_{a'b'}\gamma_{123})_{\alpha\beta}\\
\mathcal{B}_2 &= -i\,\tfrac{m}{3}\;\epsilon_{abc}\,\mathrm{Tr}(X^a\,X^b\,X^c)\,\delta_{\alpha\beta}
\end{aligned}$$

In order to reduce the fermionic contributions $\mathcal{F}$ in (A.2) we make use of the Fierz identity

$$\mathrm{Tr}(\theta_\delta\,\theta_\gamma) = \tfrac{N}{4}\;\delta_{\delta\gamma} + \tfrac{1}{32}\;\mathrm{Tr}(\theta\gamma^{jk}\theta)\,(\gamma_{jk})_{\delta\gamma} + \tfrac{1}{96}\;\mathrm{Tr}(\theta\gamma^{jkl}\theta)\,(\gamma_{jkl})_{\delta\gamma}$$

which after some algebra gives us the relation

$$\mathcal{F} = -i\,\tfrac{m}{12}\;\mathrm{Tr}(\theta\gamma^{ab}\theta)\,(\gamma_{ab}\gamma_{123})_{\alpha\beta} + i\,\tfrac{m}{24}\;\mathrm{Tr}(\theta\gamma^{a'b'}\theta)\,(\gamma_{a'b'}\gamma_{123})_{\alpha\beta} - i\,\tfrac{m}{4}\;\mathrm{Tr}(\theta\gamma_{123}\theta)\,\delta_{\alpha\beta} \tag{A.4}$$

We thus see the emergence of the angular momentum operators (3.4) in the algebra coupling to $(\gamma_{ab}\gamma_{123})_{\alpha\beta}$ and $(\gamma_{a'b'}\gamma_{123})_{\alpha\beta}$ respectively.

Summarizing we then find the following anticommutator relation

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta} H - \tfrac{m}{6}\, L^{ab}(\gamma_{ab}\,\gamma_{123})_{\alpha\beta} + \tfrac{m}{12}\, L^{a'b'}(\gamma_{a'b'}\,\gamma_{123})_{\alpha\beta} + i\mathrm{Tr}(X^i \mathcal{G})(\gamma_i)_{\alpha\beta} \quad \text{(A.5)}$$

putting back in the "old" m independent terms computed in [4].

Now we want to calculate $[H, Q_\alpha]$. First from the Jacobi identity one can easily show that

$$[H, Q_\alpha] = -\tfrac{1}{8}\,[\{Q_\alpha, Q_\beta\}, Q_\beta] \quad \text{(A.6)}$$

Now using the above result for $\{Q_\alpha, Q_\beta\}$ and the property

$$[L^{ij}, S_\alpha] = \tfrac{i}{2}\, S_\beta(\gamma^{ij})_{\beta\alpha} \quad \text{(A.7)}$$

for any spinor, we obtain

$$[H, Q_\alpha] = \frac{mi}{12}(Q\gamma_{123})_\alpha + \mathrm{Tr}(\theta_\alpha \mathcal{G}) \quad \text{(A.8)}$$

B. Details of the Perturbative Calculation

In this section we comment on the calculation of energy shifts for the excited states. Naturally these manipulations are algebraically more involved and we have performed them with the help of `Mathematica` and `FORM` [16] computer algebra systems.

The considered states of (3.10) which are excited by two raising operators are conveniently expressed with the help of the unified bosonic ladder operators of (4.3) and (4.4) through

$$\begin{aligned} |h\rangle &= h_{ij}\mathrm{Tr}(\tilde{a}^{\dagger\, i}\tilde{a}^{\dagger\, j})|0\rangle, \\ |f\rangle &= f_{aa'}\mathrm{Tr}(\theta^+\gamma^{aa'}\theta^+)|0\rangle, \\ |g\rangle &= g_{a'b'c'}\mathrm{Tr}(\theta^+\gamma^{a'b'c'}\theta^+)|0\rangle \end{aligned} \quad \text{(B.1)}$$

The interactions of H_{INT} respect the $SO(3)\times SO(6)$ split of the free Hamiltonian, therefore mixing can only appear for the degenerate states $|aa'\rangle_B$ and $|aa'\rangle_F$. We will deal with this problem at the end of this section and first study the diagonal elements of the interactions in quantum mechanical perturbation theory given by (4.13).

The states excited by bosonic oscillators will be dealt with first. For the calculations done in this paper it turns out that the three different interaction terms can be treated separately: the cross term of the Yukawa and the cubic bosonic term does not contribute. So just like the ground state we first consider the first order perturbation of the quartic Yang-Mills interaction, and then the Yukawa and the cubic bosonic term at second order. The results are summarized in a table at the end of this section.

First the Yang-Mills quartic interaction is calculated to give

$$
\begin{aligned}
&-\frac{1}{4}\langle h_1|\mathrm{Tr}[X^i,X^j]^2|h_2\rangle = \\
&(N^5-N^3)((\mathrm{Tr}M)^2-\mathrm{Tr}M^2)\mathrm{Tr}h_1Mh_2M \\
&+4(N^3-N)(\mathrm{Tr}h_1M^2\mathrm{Tr}h_2M^2-\mathrm{Tr}h_1M^2h_2M^2) \\
&+8(N^3-N)(\mathrm{Tr}M\mathrm{Tr}h_1M^2h_2M-\mathrm{Tr}h_1M^3h_2M)
\end{aligned} \tag{B.2}
$$

From this result one can easily check that there are no off-diagonal overlaps of the pure bosonic states as expected. Note that the normalization constants for the different sets of states have not been taken into account yet so we have to divide the above result by the norm

$$\langle h|h\rangle = 2N^2\mathrm{Tr}(hMhM) \tag{B.3}$$

What we have to do now is simply calculate the ratios of traces of the form $\mathrm{Tr}(hM^phM^q)$ which is straightforward.

Next we turn to the contributions from the cubic bosonic or Myers term. We write it in terms of raising and lowering operators and substitute appropriate values for E_0-H_0. Written more explicitly

$$
\begin{aligned}
\Delta E^2\langle h|h\rangle &= \frac{m^2}{9}\,\langle h|\mathrm{Tr}[e_{abc}\,X^a\,X^b\,X^c]\,\frac{1}{H_0-E_0}\,\mathrm{Tr}[e_{abc}\,X^a\,X^b\,X^c]\,|h\rangle \\
&= +\frac{m}{9}\,\langle h|\mathrm{Tr}[e_{abc}\,\tilde a^a\,\tilde a^b\,\tilde a^c]\ \mathrm{Tr}[e_{abc}\,\tilde a^{\dagger\,a}\,\tilde a^{\dagger\,b}\,\tilde a^{\dagger\,c}]\,|h\rangle \\
&\quad +3m\,\langle h|\mathrm{Tr}[e_{abc}\,\tilde a^a\,\tilde a^b\,\tilde a^{\dagger\,c}]\ \mathrm{Tr}[e_{abc}\,\tilde a^{\dagger\,a}\,\tilde a^{\dagger\,b}\,\tilde a^c]\,|h\rangle \\
&\quad -3m\,\langle h|\mathrm{Tr}[e_{abc}\,\tilde a^a\,\tilde a^{\dagger\,b}\,\tilde a^{\dagger\,c}]\ \mathrm{Tr}[e_{abc}\,\tilde a^{\dagger\,a}\,\tilde a^b\,\tilde a^c]\,|h\rangle \\
&\quad -\frac{m}{9}\,\langle h|\mathrm{Tr}[e_{abc}\,\tilde a^{\dagger\,a}\,\tilde a^{\dagger\,b}\,\tilde a^{\dagger\,c}]\ \mathrm{Tr}[e_{abc}\,\tilde a^{\dagger\,a}\,\tilde a^b\,\tilde a^c]\,|h\rangle
\end{aligned} \tag{B.4}
$$

When we evaluate this it turns out that for level 2 states we are interested in here only the first two channels have nonvanishing contributions. The result is summarized as

$$
\begin{aligned}
\Delta E^2 = &-\frac{36\mathrm{Tr}_3\,(MhMhM)+12m^2\epsilon^{abc}\epsilon^{def}M_{ad}(MhM)_{be}(MhM)_{cf}}{\mathrm{Tr}(hMhM)}\left(\frac{N^2-1}{Nm}\right) \\
&-\frac{27(N^3-N)}{4m^2}
\end{aligned} \tag{B.5}
$$

where Tr_3X means one should take the trace of the 3 dimensional part only, after calculating X as a 9-dimensional matrix. Again the result for different states can be found in table 1.

Now we can turn to the consideration of the Yukawa terms. For the states with bosonic oscillators only we can easily perform the Wick contraction of the fermionic oscillators in the interaction term. The result contains $\mathrm{tr}(\gamma^i\Pi^+\gamma^j\Pi^-)M_{ij}$,

States		Quartic	Myers	Yukawa	Total
$\vert 0\rangle$	(1,1)	$\frac{891(N^3-N)}{4m^2}$	$-\frac{27(N^3-N)}{4m^2}$	$-\frac{864(N^3-N)}{4m^2}$	0
$\vert aa\rangle$	(1,1)	$\frac{891N^4-351N^2-540}{4m^2N}$	$-\frac{27(N^4+19N^2-20)}{4m^2N}$	$-\frac{864(N^3-N)}{4m^2}$	0
$\vert ab\rangle$	(5,1)	$\frac{891N^4-405N^2-486}{4m^2N}$	$-\frac{27(N^4+N^2-2)}{4m^2N}$	$-\frac{864(N^3-N)}{4m^2}$	$\frac{108(N^2-1)}{m^2N}$
$\vert a'a'\rangle$	(1,1)	$\frac{891N^4+405N^2-1296}{4m^2N}$	$-\frac{27(N^3-N)}{4m^2}$	$-\frac{864(N^4-1)}{4m^2N}$	$\frac{108(N^2-1)}{m^2N}$
$\vert a'b'\rangle$	(1,20')	$\frac{891N^4-27N^2-864}{4m^2N}$	$-\frac{27(N^3-N)}{4m^2}$	$-\frac{864(N^4-1)}{4m^2N}$	0
$\vert aa'\rangle_B$	(3,6)	$\frac{891N^3-207N-684}{4m^2N}$	$-\frac{27(N^4+3N^2-4)}{4m^2N}$	$-\frac{432(2N^4-N^2-1)}{4m^2N}$	$\frac{36(N^2-1)}{m^2N}$
$\vert aa'\rangle_F$	(3,6)	$\frac{891(N^3-N)}{4m^2}$	$-\frac{27(N^3-N)}{4m^2}$	$-\frac{288(3N^4-4N^2+1)}{4m^2N}$	$\frac{72(N^2-1)}{m^2N}$
$\vert a'b'c'\rangle$	(1,10)	$\frac{891(N^3-N)}{4m^2}$	$-\frac{27(N^3-N)}{4m^2}$	$-\frac{864(N^3-N)}{4m^2}$	0

Table 1: The diagonal contributions of the second order perturbation calculation of the energy spectrum according to eq. (4.13). The states are defined in (3.10) and the numbers in the parenthesis represent the associated $SO(3)\times SO(6)$ representation. Only the states of represenation $(3,6)$ receive off-diagonal contributions, which are evaluated in (B.10).

and it is easy to see that only the $SO(6)$ part of the Yukawa term gives nontrivial contributions. So

$$\begin{aligned}\Delta E^3\langle h|h\rangle &= -\frac{3}{2m}\langle h|\mathrm{Tr}(\theta^-\gamma_{a'}[\tilde{a}^{a'},\theta^-])\mathrm{Tr}(\theta^+\gamma_{b'}[\tilde{a}^{\dagger\, b'},\theta^+])|h\rangle\\ &\quad-\frac{3}{m}\langle h|\mathrm{Tr}(\theta^+\gamma_{a'}[\tilde{a}^{\dagger\, a'},\theta^+])\mathrm{Tr}(\theta^-\gamma_{b'}[\tilde{a}^{b'},\theta^-])|h\rangle\\ &= -\frac{216}{m^2}(N^3-N)\langle h|h\rangle-\frac{36}{m}\langle h|N\mathrm{Tr}(\tilde{a}^{\dagger\, a'}\tilde{a}^{a'})-\mathrm{Tr}\tilde{a}^{\dagger\, a'}\mathrm{Tr}\tilde{a}^{a'}|h\rangle\\ &= -\frac{216}{m^2}(N^3-N)\langle h|h\rangle-\frac{144}{m}(N^3-N)\mathrm{Tr}_6\left(MhMhM\right)\end{aligned} \tag{B.6}$$

where Tr_6 means we take the trace of the six dimensional part only.

Now for the states excited by fermionic oscillators it is clear that the contribution of the quartic and the Myers term must be the same as the ground state. The Yukawa interaction can be considered as before, computing the different channels separately. For a state $\mathrm{Tr}(\theta^+_\alpha\Gamma_{\alpha\beta}\theta^+_\beta)|0\rangle$, we get

$$\Delta E^3 = -\frac{54(4N^4-5N^2+1)}{m^2N}+\frac{18(N^2-1)}{m^2N}\frac{\mathrm{tr}(\Gamma\gamma^a\Pi^+\Gamma\gamma^a)}{\mathrm{tr}(\Gamma^2\Pi^+)}$$

For $\Gamma=\gamma^{aa'}$, $\mathrm{tr}(\Gamma\gamma^a\Pi^+\Gamma\gamma^a)=\mathrm{tr}(\Gamma^2\Pi^+)$, so

$$\Delta E^3 = -\frac{72}{m^2}\left(3N^3-4N+1/N\right) \tag{B.7}$$

and for $\Gamma=\gamma^{a'b'c'}$ using $\mathrm{tr}(\Gamma\gamma^a\Pi^+\Gamma\gamma^a)=-3\mathrm{tr}(\Gamma^2\Pi^+)$,

$$\Delta E^3 = -\frac{216}{m^2}\left(N^3-N\right) \tag{B.8}$$

State	Quartic	Myers	Yukawa	Sum
$\vert C_{(3)}\rangle$	$\frac{297N^6+432N^4-729N^2}{4(m/3)^2}$	$-\frac{9(N^6-N^2)}{4(m/3)^2}$	$\frac{-288N^6-432N^4+720N^2}{4(m/3)^2}$	0
$\vert C_{(4)}\rangle$	$\frac{99N^7+588N^5-111N^3-576N}{(m/3)^2}$	$\frac{-3N^7-12N^5+15N^3}{(m/3)^2}$	$\frac{-96N^7-576N^5+96N^3+576N}{(m/3)^2}$	0
$\vert C_{(5)}\rangle$	$\frac{15N^2(33N^6+542N^4+649N^2-1224)}{4(m/3)^2}$	$\frac{-15N^2(N^6+14N^4-7N^2-8)}{4(m/3)^2}$	$\frac{-60N^2(2N^6+33N^4+41N^2-76)}{(m/3)^2}$	0

Table 2: The vanishing of the second order perturbation calculation of the energy shifts of the totally symmetrized $SO(6)$ higher level states defined in eq. (4.17).

Again this result is summarized in the table.

What remains to be done is the computation of the mixing of $|aa'\rangle_B$ and $|aa'\rangle_F$ under perturbation theory. Here it is essential to work with the properly normalized states

$$|aa'\rangle_{B_{\rm norm}} = \tfrac{1}{N}\ |aa'\rangle_B \qquad |aa'\rangle_{F_{\rm norm}} = \tfrac{1}{2\,N}\ |aa'\rangle_F \tag{B.9}$$

following from (3.15). For the cross term only the Yukawa interaction piece contributes and one finds

$${}_{B_{\rm norm}}\langle aa'|H_{\rm INT}|_{X\theta^2}\,\frac{1}{E_0-H_0}\,H_{\rm INT}|_{X\theta^2}|bb'\rangle_{F_{\rm norm}} = \sqrt{2}\,\frac{36\,(N^2-1)}{m^2\,N}\,\delta_{ab}\,\delta_{a'b'}\,. \tag{B.10}$$

Combining this with the result quoted in table 1 one thus obtains the mixing matrix

$$\frac{36\,(N^2-1)}{m^2\,N}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \tag{B.11}$$

for the normalized states $|aa'\rangle_{F_{\rm norm}}$ and $|aa'\rangle_{B_{\rm norm}}$. It indeed has the eigenvalues 0 and $\frac{108\,(N^2-1)}{m^2\,N}$ associated to the eigenvectors $(|aa'\rangle_{F_{\rm norm}} - \sqrt{2}|aa'\rangle_{B_{\rm norm}})$ and $(|aa'\rangle_{F_{\rm norm}} + \frac{1}{\sqrt{2}}|aa'\rangle_{B_{\rm norm}})$ respectively. These are precisely the combinations appearing in the $(3,6)$ sector of the free field multiplets "**A**" and "**B**" as stated in (4.14) and (4.15).

Finally we turn to the totally symmetrized $SO(6)$ higher level states $|C_{(n)}\rangle$ of (4.17). The explicit contributions from the three sectors of perturbation theory are stated in table 2 for $n=3,4,5$ which add up to zero.

During the calculation we have not distinguished connected and disconnected diagrams, so for each type of interaction the leading correction appears to be $\mathcal{O}(\frac{N^3}{m^2})$ for level-two states, however they always add up to zero due to the underlying supersymmetry. We can thus see that the physical coupling constant in a large N expansion is $\frac{N}{m^2}$.

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