Some Comments on the Representation Theory of the Algebra Underlying Loop Quantum Gravity

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Abstract

Important characteristics of the loop approach to quantum gravity are a specific choice of the algebra $\mathfrak{A}$ of observables and of a representation of $\mathfrak{A}$ on a measure space over the space of generalized connections. This representation is singled out by its elegance and diffeomorphism covariance.

Recently, in the context of the quest for *semiclassical states*, states of the theory in which the quantum gravitational field is close to some classical geometry, it was realized that it might also be worthwhile to study different representations of the algebra $\mathfrak{A}$ of observables.

The content of the present note is the observation that under some mild assumptions, the mathematical structure of representations of $\mathfrak{A}$ can be analyzed rather effortlessly, to a certain extent: Each representation can be labeled by sets of functions and measures on the space of (generalized) connections that fulfill certain conditions.

These considerations are however mostly of mathematical nature. Their physical content remains to be clarified, and physically interesting examples are yet to be constructed.

1 Introduction

Loop quantum gravity (LQG for short) is a promising approach to the problem of finding a quantum theory of gravity, and has led to many interesting insights. It is based on the formulation of gravity as a constrained canonical system in terms of the Ashtekar variables [1], a canonical pair of an SU(2)-connection (in its real formulation) and a triad field.

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One of the interesting features of LQG (and perhaps one reason for its success) is its specific choice of basic observables: The configuration variables are holonomies along curves in the spacial slices of the spacetime, the basic momentum variables are integrals of a triad field over surfaces in the spacial slices of the spacetime. This is in contrast to ordinary quantum field theories, where both the configuration and the momentum observables are three dimensional integrals of the basic field and its conjugate momentum. The choice of basic variables in LQG is, however, well motivated, since in contrast to other possibilities, these variables can be defined without recurse to a fixed classical background geometry, and it furthermore leads to well defined operators for interesting geometric quantities such as area and volume.

A quantum theory for this type of basic variables was first given by Rovelli and Smolin in [2]. Since then, much work has gone into extracting the essence of this quantization and putting it onto firm mathematical ground. Key ideas in this context were the use of C^*-algebraic methods [3] and projective limit techniques [4, 5] resulting in what is now called the *connection representation*. This representation is based on a Hilbert space which is an L^2 space over the space of connections with respect to a certain measure, the Ashtekar-Lewandowski measure. The holonomies act as multiplication operators and the integrated triad fields as certain vector fields. Due to its diffeomorphism invariance and mathematical elegance, this representation is considered the fundamental representation of LQG.

That it might nevertheless be interesting to also consider representations other than the AL-representation was realized when attempts were made to construct states for LQG in which the quantum gravitational field behaves almost classical. The first proposal in this direction was contained in [6]. There, the goal was to find states for LQG that have semiclassical properties for spacetimes with non-compact spacial slices. Representations that are inequivalent to the AL-representation also seem to arise if one implements the ideas [7] about the use of statistical geometry for the construction of semiclassical states. Finally, in a series of recent works [8, 9, 10], measures on the space of generalized connections were constructed that derive from the Gaussian measure of ordinary (background dependent) free quantum field theory.

The representation theory for the holonomy algebra is well understood and many representations inequivalent to the AL-representation have been considered in the literature. Less attention has been paid to the question of what happens when one also takes the integrated triads into consideration. The main observation of the present note is that due to the structure of its commutation relations, representations of the combined algebra of holonomies and integrated triads can, without effort, be analyzed to a certain extent: Each representation can be labeled by sets of functions and measures on the space of (generalized) connections that fulfill certain conditions.

However, a cautionary remark is in order here. The considerations of the present note are mostly of mathematical nature. Truly interesting, albeit difficult, tasks would be to state physically motivated criteria for singling out interesting classes of representations, actually constructing such representations, and understanding the physical content of representations constructed by mathematical considerations. None of this will be addressed in this note. However, we hope that it can be used as a starting point when approaching those questions motivated by physics.

To finish this introduction, we should mention that the occurrence of inequivalent representations of the observable algebra is well known from quantum field theory and quantum statistical mechanics [11]: In that context, it was realized that the choice of representation for the observable algebra contains important physical information: Roughly speaking, whereas the algebraic structure of the theory encodes the physical system one is considering, the chosen representation carries the global information about the physical state, the system is in. It might for example decide whether the system is in a ground or in a thermal state or whether the state carries a global charge. Since the change of the global properties of a state of the system is not always physically realizable (it might necessitate an infinite amount of energy or the creation of charges) the emergence of inequivalent representations is quite natural. The considerations of representations different from the AL-representation in the quest for the semiclassical regime of LQG fits quite nicely into this general picture.

2 CQG briefing

It is sometimes helpful to quantize a given classical system in two steps. The first consists in associating to each member of a chosen set of classical observables, an operator in some (abstract) *-algebra $\mathfrak{A}$, such that

- The Poisson structure of the classical observables is mirrored as closely as possible by the commutators within the algebra ("Poisson brackets go to commutators").
- Complex conjugate on classical observables are mapped to conjugates under the $*$-operation on $\mathfrak{A}$.

The importance of the second condition lies in the fact that it ensures that real classical quantities will be associated with symmetric operators, which in turn have spectrum on the real line and real expectation values. If this would not be the case, the interpretation of the resulting quantum theory would be completely obscure.

The second step consists in choosing a $*$-representation of the algebra $\mathfrak{A}$, thus enabling one to compute expectation values and hence make physical predictions.

The purpose of this section is to look at the first of these two steps in the context of LQG. It has been extensively studied there and the choice of the set of classical observables as well as the corresponding $*$-algebra which is made can be regarded as the very essence of LQG. In this section, we will briefly review these developments to make the paper self contained as well as fix the notation. Thus the pedagogical value of this section will be very low. For an extensive recent review see [12].

As a first step recall that the canonical pair in LQG is a SU(2) connection one-form A and a frame field E_I with a nontrivial density weight. Both of these take values on a spacial slice Σ of the four-manifold M. Being a one-form, A can be integrated naturally (that is, without recurse to background structure) along curves e in Σ, to form *holonomies*

$$h_e[A] = \mathcal{P}\exp\left[i\int_e A_a ds^a\right].$$

It turns out to be convenient to consider functions of A which are slightly more general.

Definition 2.1. *A graph in Σ is a collection of analytic, oriented curves in Σ which intersect each other at most in their endpoints.*

A function c depending on connections A on Σ just in terms of their holonomies along the edges of a graph, i.e.

$$c[A] \equiv c(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_n}[A]), \qquad e_1, e_2, \ldots, e_n \quad \textit{edges of some } \gamma,$$

where $c(g_1, \ldots, e_n)$ viewed as a function on $\mathrm{SU}(2)^n$ is continuous, will be called cylindrical.

Analyticity of the edges is required to exclude certain pathological intersection structures of the edges with surfaces which would render the Poisson brakets which will be introduced below ill-defined.

It turns out that the set of cylindrical functions can be equipped with a norm (essentially the sup-norm for functions on $\mathrm{SU}(2)^n$) such that its closure Cyl with respect to that norm is a commutative C*-algebra. We will not spell out the details of this construction but refer the reader to the beautiful presentations [4, 5]. We note furthermore that by changing the word "continuous" in the above definition to "n times differentiable", we can define subsets Cyl^n of Cyl and, most importantly for us,

$$\mathrm{Cyl}^\infty \doteq \bigcap_n \mathrm{Cyl}^n,$$

the space of smooth cylindrical functions.

The density weight of E on the other hand is such that, using an additional real (co-)vector field f^i, it can be naturally integrated over oriented surfaces S to form a quantity

$$E_{S,f} = \int_S E^a_i f^i \epsilon_{abc}\, dx^b\, dx^c$$

analogous to the electric flux through S.

One of the defining choices of LQG is to base the quantization precisely on the elements of Cyl and the fluxes $E_{S,f}$ as classical observables. From the Poisson brackets of A and E on can compute the Poisson brackets for the c, $E_{S,f}$. Call γ adapted to S if all p in $S\cap\gamma$ are vertices of γ. Let c be a function cylindrical on γ and S some analytical surface. Without restriction of generality we assume that γ is adapted to S[1]. Then

$$\{E_{S,f}, c\} = \frac{\kappa}{2} \sum_{p\in S\cap\gamma} \sum_{e_p} \kappa(e_p) f_i X^i_{e_p}[c],$$

where the second sum is over the edges of γ adjacent to p,

$$\kappa(e_p) = \begin{cases} 1 & \text{if } e \text{ lies above } S \\ 0 & \text{if } e \text{ is tangential to } S \\ -1 & \text{if } e \text{ lies below } S \end{cases},$$

and $X^I_{e_p}$ is the Ith left-invariant (right-invariant) vector field on SU(2) acting on the argument of c corresponding to the holonomy h_{e_p} if e_p is pointing away from (towards) S. κ is the coupling constant of gravity.

Surprisingly, the Poisson brackets of the $E_{S,f}$ among themselves do not vanish as one would expect for the momentum observables. This poses two questions: Can one nevertheless give some well defined "Poisson bracket goes to commutator"-prescription to associate algebra elements to classical observables? And: Can one understand where this non-commutativity of the momentum observables come from? As shown in [14], both questions can be answered affirmatively. We do not want to repeat the discussion of [14] here but just give its result, condensed in a definition of the algebra $\mathfrak{A}$ on which the quantum theory will be based, as well as the association of classical observables to algebra elements. Let

$$X_{S,f}[c] := \frac{il_P^2}{2} \sum_{p\in S\cap\gamma} \sum_{e_p} \kappa(e_p) f_i X^i_{e_p}[c],$$

where we have used the notation introduced above.

Definition 2.2. *Let $\mathfrak{A}$ be the algebra generated by the cylindrical functions* Cyl*, together with the derivations $X_{S,f}$ on* Cyl*. On $\mathfrak{A}$ a $*$-operation is given by the usual complex conjugation on* Cyl *and the trivial $(X_{S,f})^* := X_{S,f}$ on X.*

[1] There is always γ' that contains γ such that γ' is adapted to S. A c cylindrical on γ is clearly also cylindrical on γ'. For details see [13].

The association of the classical functionals Cyl, E *with elements of* $\mathfrak{A}$ *is then given by*

$$\begin{aligned} c &\mapsto c, \\ E_{S,f} &\mapsto X_{S,f}, \\ \left\{E_{S,f}, E_{S',f'}\right\} &\mapsto i\hbar^{-1}\left[X_{S,f}\,,\,X_{S',f'}\right], \end{aligned}$$

and higher order Poisson brackets of elements of E *are mapped to the higher order commutators of the corresponding derivations.*

Note that since $\mathfrak{A}$ is generated by the elements of Cyl and X, a representation π of $\mathfrak{A}$ is completely determined once the representors $\pi\left(\mathrm{Cyl}\right)$ and $\pi\left(X\right)$ are known.

3 Remarks on the representation theory of $\mathfrak{A}$

In the present section we will make some simple observations on the structure of representations of $\mathfrak{A}$. Before we proceed to the details, let us give a brief outline of what we are going to do.

We assume that a representation of $\mathfrak{A}$ is given. As a first step, we appeal to the powerful machinery available for representation of C*-algebras, to decompose the representation space into subspaces on which Cyl acts cyclic. Then we look at the action of the representors of the $E_{S,f}$ with respect to this decomposition. Since the respective operators will be unbounded, we have to make an assumption that gets possible domain problems out of the way: We assume that the $\pi\left(E_{S,f}\right)$ share a certain dense set in their domains. Then we make the central observation that under this assumption, the action of the $\pi\left(E_{S,f}\right)$ is rather simple: Roughly speaking it is the sum of a derivative defined by $X_{S,f}$ and a multiplication operator. As a consequence, we can show that each representation of $\mathfrak{A}$ is uniquely determined by a set of measures and functions on the space of (generalized) connections fulfilling certain compatibility conditions. Since despite our assumption the considerations might appear exceedingly general, we will finish by giving a useful corollary of our results in a rather simple case.

Before we start our analysis of the representations of $\mathfrak{A}$ we want to recall some basic facts about the representation theory of Cyl. As was realized in [3, 4], many powerful results are at hand because Cyl is an unital Abelian C*-algebra. Firstly we recall that, due to a theorem of Gelfand (see for example [15]), since Cyl is Abelian, it is isomorphic, via some isomorphism ι, to the algebra of continuous functions on the *spectrum* $\overline{\mathcal{A}}$, a compact Hausdorff space, of Cyl. From this and the Riesz-Markov Theorem (see for example [16]) it follows that every positive linear functional on Cyl is given by a positive Baire measure on $\overline{\mathcal{A}}$. The converse trivially holds true: Every positive Baire measure on $\overline{\mathcal{A}}$ gives a positive functional on Cyl.

Now let $(\pi, \mathcal{H})$ be a *cyclic* representation of Cyl. Since the representation is cyclic, it defines a positive linear functional ω on Cyl and certainly is unitarily equivalent to the GNS representation coming from ω. Moreover as concluded above ω must be given by a positive Baire measure on $\overline{\mathcal{A}}$. Vice versa, every cyclic representation of Cyl is given by a positive Baire measure on $\overline{\mathcal{A}}$. Thus, we conclude that the cyclic representations of Cyl are all of the form

$$\mathcal{H}_\nu = L^2(\overline{\mathcal{A}}, d\mu_\nu), \qquad \pi_\nu(c) = \iota(c), \tag{1}$$

where μ_ν is some positive Baire measure on $\overline{\mathcal{A}}$. Note that we did not have to assume continuity of the cyclic representation. Rather, continuity follows automatically from cyclicity.

It is important for the rest of this work, that because of their structure (1), for cyclic representations the $\pi\,(c)$ play a double role: On the one hand they are operators on the representation space, on the other hand they are L^2 functions. Let us note the following Lemma which will be useful, later on:

Lemma 3.1. *Let $(\pi, \mathcal{H})$ be some cyclic representation of* Cyl*. Then $\pi\,(\mathrm{Cyl}^\infty)$ is dense in $L^2(\overline{\mathcal{A}}, d\mu)$.*

Sketch of the proof. Since we do not want to introduce the projective limit machinery that is used to define the closure Cyl of the set of cylindrical functions, we will only sketch the proof. The details can however be easily fixed using the methods of [4, 5].

The idea for the proof is that functions in Cyl^∞ can essentially be viewed as subset of the continuous functions on a compact space. They are separating points and the constant functions are among them, so the Stone-Weierstrass Theorem (see for example [17]) applies, showing that they are dense in Cyl (wrt. its C* norm). Now cyclicity of the representation just means that $\pi\,(\mathrm{Cyl})$ is dense in $\mathcal{H}$, whence $\pi\,(\mathrm{Cyl}^\infty)$ is dense in $\mathcal{H}$ as well. $\square$

Let now a representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ be given. It is well known that every representation of a C*-algebra is a direct sum of cyclic representations (see for example [18]). Applying this to the representation $\pi|_{\mathrm{Cyl}}$ of Cyl yields

$$\mathcal{H} \cong \bigoplus_\nu \mathcal{H}_\nu, \qquad \pi \cong \oplus_\nu \pi_\nu.$$

where the $(\pi_\nu, \mathcal{H}_\nu)$ are cyclic and therefore

$$\mathcal{H}_\nu \cong L^2(\overline{\mathcal{A}}, d\mu_\nu), \qquad \pi_\nu(c) \cong \iota(c).$$

To simplify notation in what follows, we will take all isometries as identities. Furthermore, we denote by I_ν the canonical inclusion

$$I_\nu : \mathcal{H}_\nu \hookrightarrow \mathcal{H}$$

and by P_ν the canonical projection followed by the inverse of I_ν

$$P_\nu : \mathcal{H} \to \mathcal{H}_\nu.$$

Now we have to analyze the action of the operators $\pi\left(E_{S,f}\right)$ on $\mathcal{H}$. This gets complicated by the fact that they represent vector fields and will therefore be unbounded operators. To get these complications out of the way, we will make an assumption on π. To this end, let us define the following subspace of $\mathcal{H}$:

$$\mathfrak{h} := \operatorname{span}\left[\bigcup_\nu I_\nu\left(\mathrm{Cyl}^\infty\right)\right].$$

Note that $\mathfrak{h}$ is dense in $\mathcal{H}$ because Cyl^∞ is dense in $\mathcal{H}_\nu$. With this definition at hand, we can state our assumption:[2]

Assumption 3.2. *The representation π should be such that $\mathfrak{h} \subset \operatorname{dom}\left(\pi\left(E_{S,f}\right)\right)$ for all surfaces S and co-vector fields f on S.*

Under this assumption, the action of the $\pi\left(E_{S,f}\right)$ can be computed rather explicitely: Let c be a cylindrical function. Then

$$\begin{aligned}
\pi\left(E_{S,f}\right)\left[I_\nu(c)\right] &= \pi\left(E_{S,f}\right)\pi\left(c\right)\left[I_\nu(\mathbf{1})\right]\\
&= \left[\pi\left(E_{S,f}\right),\, \pi\left(c\right)\right]\left[I_\nu(\mathbf{1})\right] + \pi\left(c\right)\pi\left(E_{S,f}\right)\left[I_\nu(\mathbf{1})\right]\\
&= I_\nu\left(X_{S,f}[c]\right) + \sum_\iota I_\iota\left(cF^{\iota\nu}_{S,f}\right)
\end{aligned}$$

where we have made the definition

$$F^{\iota\nu}_{S,f} := P_\iota\left(\pi\left(E_{S,f}\right)\left[I_\nu(\mathbf{1})\right]\right). \in \mathcal{H}_\iota$$

Thus the action of the fluxes on $\mathfrak{h}$ and hence on $\mathcal{H}$ is completely determined by the $F^{\iota\nu}_{S,f}$. Let us exhibit some further properties of this family:

Because of Assumption 3.2, we have that[3]

$$\sum_\iota I_\iota\left(F^{\iota\nu}_{S,f}\right) \in \mathcal{H}. \qquad \text{(dom)}$$

[2]Note that this assumption *does not* automatically follow from the perhaps more natural one that $\{\pi\left(c\right) \mid c \in \mathrm{Cyl}\}$ should be contained in the domains of the $E_{S,f}$.

[3]Note that from Assumption 3.2 we directly get $c\sum_\iota I_\iota\left(F^{\iota\nu}_{S,f}\right) \in \mathcal{H}$ for cylindrical functions c. But since Cyl also contains the constant functions, (dom) follows.

More properties come from the fact that the $\pi(E_{S,f})$ represent the $E_{S,f}$. First of all, for co-vector fields f, f'' on a surface S and f' on S'

$$\begin{aligned} F^{\iota\nu}_{S,f+f''} &= F^{\iota\nu}_{S,f} + F^{\iota\nu}_{S,f''}, \\ F^{\iota\nu}_{S,f} + F^{\iota\nu}_{S',f'} &= F^{\iota\nu}_{S\setminus S',f} + F^{\iota\nu}_{S'\setminus S,f'} + F^{\iota\nu}_{S\cap S',f\pm f'}, \end{aligned} \tag{rep}$$

where in the second line $S \cap S'$ is given the orientation of S and the sign depends on the relative orientation of S and S' on their intersection.

Further relations come from the fact that π is a *-representation: For $c, c' \in \mathrm{Cyl}$ let

$$\Delta^{(\iota)}_{S,f}(c,c') := \left\langle X_{S,f}[c]\,,\, c'\right\rangle_{\mathcal{H}_\iota} - \left\langle c\,,\, X_{S,f}[c']\right\rangle_{\mathcal{H}_\iota}$$

denote the divergence of the vector field $X_{S,f}$ with respect to the measure μ_ι. As the $\pi(E_{S,f})$ have to be symmetric, we have

$$\begin{aligned} &\left\langle \pi(E_{S,f})\, I_\nu(c)\,,\, I_\iota(c')\right\rangle = \left\langle I_\nu(c)\,,\, \pi(E_{S,f})\, I_\iota(c')\right\rangle \\ &\Leftrightarrow \delta_{\nu\iota}\Delta^{\iota}_{S,f}(c,c') = \left\langle c\,,\, F^{\nu\iota}_{S,f} c'\right\rangle_{\mathcal{H}_\nu} - \left\langle F^{\iota\nu}_{S,f} c\,,\, c'\right\rangle_{\mathcal{H}_\iota} \end{aligned}$$

which can easily seen to be equivalent to

$$\begin{aligned} \Delta^{(\iota)}_{S,f}(c,c') &= 2i\left\langle c\,,\, \mathrm{Im}\left(F^{\iota\iota}_{S,f}\right) c'\right\rangle, \\ F^{\nu\iota}\, d\mu_\nu &= \overline{F^{\iota\nu}}\, d\mu_\iota \qquad \text{for } \iota \neq \nu. \end{aligned} \tag{div}$$

Let us summarize our findings

Proposition 3.3. *Any representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$, fulfilling our Assumption 3.2, determines*

- *A family of positive measures $\{\mu_\nu\}$ on $\overline{\mathcal{A}}$,*
- *A family of functions $\{F^{\iota\nu}_{S,f}\}$, where $F^{\iota\nu}_{S,f} \in L^2(\overline{\mathcal{A}}, d\mu_\iota)$,*

such that (dom), (rep), (div) *are fulfilled.*

It is probably more interesting to note that also the converse holds true:

Proposition 3.4. *Let a family of measures $\{\mu_\nu\}$ on $\overline{\mathcal{A}}$ and a family of functions $\{F^{\iota\nu}_{S,f}\}$, where $F^{\iota\nu}_{S,f} \in L^2(\overline{\mathcal{A}}, d\mu_\nu)$ that fulfill* (dom), (rep), (div). *From this data, one can construct a representation π of $\mathfrak{A}$ that fulfills the Assumption 3.2.*

Proof. The proof is quite obvious: Let $\{\mu_\nu\}$, $\{F^{\iota\nu}_{S,f}\}$ fulfilling (dom), (rep), (div) be given. The representation space is defined as

$$\mathcal{H} \doteq \bigoplus_\nu L^2(\overline{\mathcal{A}}, d\mu_\nu)$$

whence

$$\pi(c) \oplus_\nu f_\nu \doteq \oplus_\nu c f_\nu, \qquad f_\nu \in L^2(\overline{\mathcal{A}}, d\mu_\nu).$$

Now

$$\pi(E_{S,f})\left[\oplus_\nu c_\nu\right] \doteq \oplus_\nu \left(X_{S,f}[c_\nu] + c_\nu \sum_\iota F_{S,f}^{\nu\iota} \right), \qquad c_\nu \in \mathrm{Cyl}^\infty \tag{2}$$

defines operators that are well defined on $\mathfrak{h}$ because of (dom), hence Assumption 3.2 is fulfilled. Moreover they are symmetric because of (div) and give a representation of the $E_{S,f}$ since the commutator with the representors of cylindrical functions is right and (rep) holds. Finally, (2) completely determines the action of the $\pi(E_{S,f})$ on $\mathfrak{h}$ and hence on $\mathcal{H}$. □

The above results may appear exceedingly general. Let us therefore reduce consideration to representations in which Cyl acts cyclic and state the following corollary which is perhaps closer to applications than the above general results:

Proposition 3.5. *Let a cyclic representation $(\pi, \mathcal{H})$ of* Cyl *be given. Then a necessary and sufficient condition for π to be extendable to a representation, fulfilling Assumption 3.2, of the whole $\mathfrak{A}$ is that for each surface S and co-vector field f on S there exists a constant $C_{S,f}$ such that*

$$|\Delta_{S,f}(c, \mathbf{1})| \leq C_{S,f} \, \|c\|_{\mathcal{H}} \qquad \textit{for all } c \in \mathrm{Cyl}^\infty \tag{3}$$

where the sesquilinear form $\Delta_{S,f}$ is given by

$$\Delta_{S,f}(c, c') \doteq \left\langle \pi\left(X_{S,f}[c]\right), \pi\left(c'\right) \right\rangle_{\mathcal{H}} - \left\langle \pi(c), \pi\left(X_{S,f}[c']\right) \right\rangle_{\mathcal{H}}, \qquad c, c' \in \mathrm{Cyl}.$$

Proof. Let us first prove necessity: Let a representation $(\tilde{\pi}, \mathcal{H})$ of $\mathfrak{A}$ be given such that $\tilde{\pi}|_{\mathrm{Cyl}} = \pi$. Application of Proposition 3.3 then yields a measure μ and a family of functions $\{F_{S,f}\}$ satisfying (dom), (rep), (div). Thus we can finish by noting that (div), (dom) imply (3).

Sufficiency can be proved by straightforward construction: Let a cyclic representation $(\pi, \mathcal{H})$ of Cyl, fulfilling the condition (3), be given. Because of cyclicity, $\mathcal{H}$ is isomorphic to $L^2(\overline{\mathcal{A}}, d\mu)$ for some positive regular measure μ. Moreover, $\pi(\mathrm{Cyl}^\infty)$ is dense in $\mathcal{H}$. Therefore the Riesz Representation Theorem (see for example [16]) shows that (3) implies that $\Delta_{S,f}(c, \mathbf{1})$ is given by an element $\widetilde{F}_{S,f}$ of $\mathcal{H}$, i.e. $\Delta_{S,f}(c, \mathbf{1}) = \langle \pi(c), \widetilde{F}_{S,f} \rangle$. Using the fact that $\mathcal{H}$ is an L^2 space, one can easily see that $\Delta_{S,f}(c, c')$ is determined by $\widetilde{F}_{S,f}$ as well and that $\mathrm{Re}(\widetilde{F}_{S,f}) = 0$. Set $F_{S,f} \doteq \widetilde{F}_{S,f}/2$. The $F_{S,f}$ fulfill (dom), (rep) because the $\widetilde{F}_{S,f}$ do. Moreover, the $F_{S,f}$ satisfy (div). So one can construct a representation of $\mathfrak{A}$ from the data $\mu, \{F_{S,f}\}$ by Proposition 3.4. □

4 Discussion

Let us start the discussion of the above results by describing the simplest case, the AL-representation. In that representation, Cyl acts cyclic, the corresponding measure on $\overline{\mathcal{A}}$ is the Ashtekar-Lewandowski measure μ_{AL} constructed in [4] and the $F_{S,f}$ are all equal to zero.

Next, we remark that Assumption 3.2 precludes the possibility that the generalized divergences $\Delta_{S,f}$ *are* given by functions which however *are not* L^2. This case does not seem unnatural, so it might appear too restrictive to exclude it. Note however that admitting that case as well would mean that not all of Cyl (especially not the constant functions) would be contained in the domains of the $\pi(E_{S,f})$. On the other hand the cylindrical functions are the only ones we have direct control on, and removing some of them would most likely leave us with a set that is not dense anymore. Thus it would be extremely difficult to work with such more general representations.

Also, we would like to make some remarks concerning Proposition 3.5: As we saw, it is simple to derive that result. It turns out to be much more difficult to actually come up with an example for a measure on $\overline{\mathcal{A}}$ fulfilling the condition, other then the AL-measure. All measures constructed so far, with the remarkable exception of the AL-measure, violate (3). The interested reader is referred to [19] for a closer investigation of this subject. A class of representations that avoids this problem is the one obtained by using the AL-measure but having the $F_{S,f}$ real and not equal to zero. In such representations, the $\pi(E_{S,f})$ have non vanishing expectation values.

As the last remark showed, this note merely provides a starting point for the analysis of the representations of $\mathfrak{A}$, and much more difficult and interesting problems remain to be tackled. Nevertheless, we hope that this note is a useful preparation for that task, and we would like to come back to some of the questions related to it in future publications.

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