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Fractional Calculus via Functional Calculus: Theory and Applications

SIEGMAR KEMPFLE and INGO SCHÄFER
Fachbereich Maschinenbau, Universität der Bundeswehr, Holstenhofweg 85, D-22043 Hamburg, Germany

HORST BEYER
MPI für Gravitationsphysik, Am Mühlenberg 5, D-14476 Golm, Germany

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Abstract. This paper demonstrates the power of the functional-calculus definition of linear fractional (pseudo-) differential operators via generalised Fourier transforms.

Firstly, we describe in detail how to get global causal solutions of linear fractional differential equations via this calculus. The solutions are represented as convolutions of the input functions with the related impulse responses. The suggested method via residue calculus separates an impulse response automatically into an exponentially damped (possibly oscillatory) part and a 'slow' relaxation. If an impulse response is stable it becomes automatically causal, otherwise one has to add a homogeneous solution to get causality.

Secondly, we present examples and, moreover, verify the approach along experiments on viscolelastic rods. The quality of the method as an effective few-parameter model is impressively demonstrated: the chosen reference example PTFE (Teflon) shows that in contrast to standard classical models our model describes the behaviour in a wide frequency range within the accuracy of the measurement. Even dispersion effects are included.

Thirdly, we conclude the paper with a survey of the required theory. There the attention is directed to the extension from the $\mathbf{L}_2$-approach on the space of distributions $\mathcal{D}'$.

Keywords: Fractional calculus, functional calculus, residue calculus, viscoelasticity, mechanics of rods.

1. Introduction

Fractional derivatives have become an often used tool in hereditary physics. In particular the damping behaviour of viscoelastic media can be modelled with only few parameters by replacing integer differential orders of damping terms by fractional orders. Since fractional operators are global ones, they are predestinated to describe moreover memory effects.

The nearly exclusively used technique is to start from an *a priori* definition of fractional derivatives like Riemann–Liouville or Caputo integrals [1–4]. But the mainly used fractional calculus via Laplace transforms together with initial conditions runs into difficulties concerning the physical interpretation of such conditions. Additionally, it can only take into account memory effects within the scope of the Laplace transforms.

We have shown in [5] that any change in the past of the input function of a fractional order system changes the future of the solution. Hence the past of such systems cannot be represented by a finite set of local initial conditions.

From an analytical point of view, such an approach has moreover many disadvantages, e.g., the loss of the important semigroup property of integer ordered derivatives, the loss of the translation invariance in time (see [6]) and last but not least it does not specify an admissible function space for the solutions.

Thus there were reasons enough for introducing another, functional analytic based approach. We did this in [7] with the intension to get criteria for the existence and the behaviour of solutions and particularly, to get *a priori* criteria for causality. In that paper we developed a $\mathbf{L}_2$-theory together with representations and properties of the solutions along a fractional model of a single-mass-oscillator:

$$(D^2 + aD^\nu + b)\,x(t) = f(t), \quad 0 < \nu < 2. \tag{1}$$

A more general theory was given in [8]. Further investigations were done on 2- and 3-term operators in [9] as well as comparisons with other approaches in [5, 6]. Apart from the mathematical stringency we attain the main justification for our method from accompanying experiments (see [5, 10–12]). There the comparison of measurement with the calculated solutions of impulse and frequency responses in viscoelastic rods have shown good agreement.

Due to the fact that in contrast to the theory its realisation is very simple, we firstly describe the handling of our approach. Then we proceed with applications. The quality of the method becomes evident by comparing the numerical results with measurement.

Finally, the handling and applications are backed up with a description of the required theory.

2. Handling

2.1. Tools

2.1.1. *The Fourier Transformation*

We remind of the formula for the Fourier transform $\widehat{f}(\omega)$ of some function $f(t)$

$$\widehat{f}(\omega) := \mathcal{F}\{f(t)\} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t, \tag{2}$$

which is valid for all f with $\int_{\mathbb{R}} |f(t)|\,\mathrm{d}t < \infty$. If we demand further $(\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t)^{1/2} < \infty$ and name this quantity the $\mathbf{L}_2$-norm $\|f\|_2$, then the Parseval formula holds:

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{3}$$

We remark that there is already a wide field of applications for the deduced $\mathbf{L}_2$-approach. For example, stable damped causal systems are completely covered and we have concentrated our former investigations on the related theory. Truly, the scope of the approach is much larger. We will see later on that by functional analytic embeddings the scope of Fourier transforms can be extremely extended to functions of exponential increasement, distributions, etc. Thus we have not to take care about restrictions for the rest of this section.

2.1.2. *Residue Calculus*

For the purpose of this paper it is sufficient to give the needed formulas without any proof. For more details see any book about complex analysis, e.g. [13, 14].

Let $z_0 \in \mathbb{C}$ be a pole of n-th order of $f : \mathbb{C} \to \mathbb{C}$. Put $g(z) = (z - z_0)^n f(z)$. Then the residuum of f in z_0 is

$$\mathrm{Res}_{z_0}(f(z)) = \frac{1}{(n-1)!}\, g^{(n-1)}(z_0). \tag{4}$$

Mostly, there are single poles and (4) can be reduced to:

Let $f(z) = \varphi(z)/\psi(z)$ and z_0 be a single zero of ψ, but $\varphi(z_0) \neq 0$. Then

$$\mathrm{Res}_{z_0}(f(z)) = \frac{\varphi(z_0)}{\psi'(z_0)}. \tag{5}$$

Now the following version of the main theorem holds:

RESIDUE THEOREM. *Let $\mathcal{G} \subset \mathbb{C}$ be a simply connected region and $\Gamma \subset \mathcal{G}$ a simply closed (positive orientated) path. Let $\{z_k \mid k = 1, \ldots, n\}$ be the set of isolated singularities inside Γ such that f is holomorphic on $\mathcal{G}\backslash\{z_k\}$. Then*

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z_k}(f(z)).$$

2.2. Starting Point

We consider the formal linear differential expression

$$\begin{aligned} \mathcal{A} \;:=\; & D^{\nu_n} + a_{n-1}D^{\nu_{n-1}} + \cdots + a_1 D^{\nu_1} + a_0, \\ & (a_k \in \mathbb{R}, \quad 0 < \nu_1 < \ldots < \nu_n =: \deg \mathcal{A}). \end{aligned} \tag{6}$$

It is called 'fractional', if at least one ν_k is non-integer, otherwise it is called 'integer'. We look for solutions of an associated differential equation

$$\mathcal{A}\,x(t) = f(t), \quad t \in \mathbb{R}. \tag{7}$$

As a reference example we may take model (1). Generally, $\mathcal{A}$ may be interpreted as model of some physical system, e.g., a composition of one-dimensional viscoelastic oscillators. Then $f(t)$ acts as noise, external force etc. The fractionality of the model is appropriate, if the system shows memory. Thus f is defined on the whole time-axis and initial conditions would moreover contradict the physics of the system. Consequently, we expect a unique solution of Equation (7).

We introduce now two versions of the so called symbol which is associated to $\mathcal{A}$. The first one is the Fourier type representation:

$$a(\omega) := (i\omega)^{\nu_n} + a_{n-1}(i\omega)^{\nu_{n-1}} + \cdots + a_1(i\omega)^{\nu_1} + a_0. \tag{8}$$

For applications the Laplace type is more convenient:

$$A(s) := s^{\nu_n} + a_{n-1}s^{\nu_{n-1}} + \cdots + a_1 s^{\nu_1} + a_0. \tag{9}$$

2.3. Strategy

To characterise the behaviour of the system $\mathcal{A}$ and to describe its interplay with the input $f(t)$ we rewrite f as (Fourier-) convolution $(\delta * f)(t)$ with the Dirac δ-impact. If $K(t)$ denotes the impulse response of the system $\mathcal{A}$, i.e., the solution of

$$\mathcal{A}\,K(t) = \delta(t), \tag{10}$$

then the solution of Equation (7) is given by

$$x(t) = (K * f)(t) = \int_{-\infty}^{\infty} K(\tau) f(t-\tau)\,\mathrm{d}\tau. \tag{11}$$

If we look for physically consistent solutions to Equation (7), then we think of *causal* solutions, i.e.,

$$(f(t) = 0 \quad \text{for } t < t_0) \Rightarrow (x(t) = 0 \quad \text{for } t < t_0). \tag{12}$$

We firstly conclude from (11) that causality of the solutions to Equation (7) is equivalent to the causality of the impulse response, i.e., $K(t) = 0$ for $t < 0$. Secondly, if (12) holds, the range of the integral is actually the intervall $[t_0, t]$ and (11) reads as

$$x(t) = \int_{t_0}^{t} K(\tau - t_0) f(t + t_0 - \tau)\,\mathrm{d}\tau. \tag{13}$$

We see moreover from this representation that we have not really to take care on global function spaces for f. Only the existence of the integral in the relevant interval is required.

Thus we can concentrate our attention on impulse responses.

We will clarify in the theoretical part that the following handling is not only admissible but is based on a mathematically strong definition of the operator (6). The reader may verify further that the method includes the cases with only integer derivatives.

2.4. The Impulse Response

For hurried readers we give the gross formula for the causal impulse response.

2.4.1. *The Causal Impulse Response*

Let the s_n denote the different zeros of $A(s)$. Then

$$K_{\text{causal}}(t) = K_{\text{s}}(t) + K_{\text{u}}(t). \tag{14}$$

The 'stable' component K_{s} is calculated for $t \geq 0$ via

$$K_{\text{s}}(t) = K_{\text{rel}}(t) + \sum_{\Re(s_n)<0} Res_{s_n}\left(\frac{\mathrm{e}^{st}}{A(s)}\right). \tag{15}$$

For K_{rel} take the below formula (20).

The 'unstable' component K_{u} results for $t \geq 0$ as

$$K_{\text{u}}(t) = -\sum_{\Re(s_n)\geq 0} \mathrm{Res}_{s_n}\left(\frac{\mathrm{e}^{st}}{A(s)}\right). \tag{16}$$

In detail we proceed in the following steps:

2.4.2. *The Integral Representation*

Fourier transforms of Equation (10) yields formally

$$a(\omega)\,\widehat{K}(\omega) = 1,$$

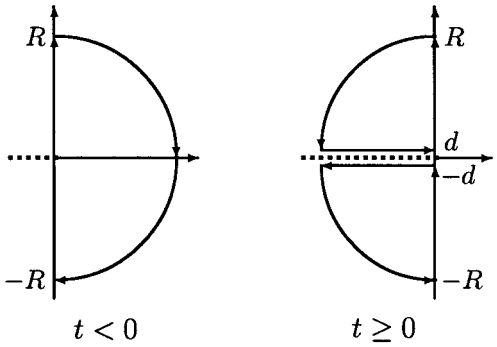

Figure 1. Integration paths.

where $a(\omega)$ is the symbol (8).

Clearly, $a(\omega)$ has to be well defined. We have shown in [7] that physically consistent solutions require all non-integer powers of $i\omega$ to be defined as their principal branches. But this is what computer programs like *Mathematica*, *Maple*, etc., do automatically. For the further handling it is advisible to locate the associated branch cut on the positive imaginary axis.

Now, a solution of (10) is obtained by inverse Fourier transforms:

$$K(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \frac{\mathrm{e}^{i\omega t}}{a(\omega)}\,\mathrm{d}\omega.$$

It is convenient to substitute further $\omega \mapsto -is$ such that the branch cut turns to the negative real axis and the symbol becomes $A(s)$ (Equation (9)). We arrive at

$$K(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-iR}^{iR} \frac{\mathrm{e}^{st}}{A(s)}\,\mathrm{d}s. \tag{17}$$

We emphasise that this formula may be used as black box and starting point for the explicit evaluation.

To evaluate (17) we suggest the residue theorem method sketched in Section 2.1.2. Thus we look for the principal-branch-roots s_k of $A(s)$ (i.e., $-\pi < \arg(s_k) \leq \pi$). This can easily be done by computer-algebra systems. It has been shown in [8, prop. 3.1], that there is always only a finite number of such zeros. We choose the integration paths of Figure 1, due to the vanishing integrals along the arcs for $R \to \infty$.

Respectively, let $K(t) = K^-(t) + K^+(t)$, where $K^-(t) = 0$ for $t \geq 0$ and vice versa $K^+(t) = 0$ for $t < 0$. Thus the left-hand side of Figure 1 provides K^-, the right-hand side provides K^+.

If and only if there is no s_k with $\Re(s_k) \geq 0$ then we get $K^- \equiv 0$ and $K(t)$ becomes causal, otherwise it becomes non-causal. We describe in Section 2.4.4 how to transfer such a K^- to the unstable component K_u of the causal impulse response K_causal. $K^+(t)$ is ex. def. always causal and turns now out to be the stable component K_s. To get this result let us for the moment 'forget' roots in the right halfplane and consider the stable case.

2.4.3. *The Stable Case*

Assume no roots s_k with $\Re(s_k) \geq 0$. Then $K(t) = 0$ for $t < 0$ (causality). For $t \geq 0$ the residue method separates automatically the impulse response into an exponentially damped part $K_{\exp}$ and a slow relaxation K_{rel} of power-law order, i.e.,

$$K(t) = K_{\mathrm{s}}(t) = \begin{cases} 0, & t < 0, \\ K_{\exp}(t) + K_{\mathrm{rel}}(t), & t \geq 0. \end{cases} \tag{18}$$

If there is no zero then $K_{\exp}(t) \equiv 0$. Otherwise, in case of some non-integer derivatives, there are always pairs of conjugate complex zeros $s_k, \bar{s}_k = -\rho_k \pm i\sigma_k (\rho_k, \sigma_k > 0)$ and possibly some on the branch cut ($s_k = -\rho_k$, $(\rho_k > 0)$, i.e. $\arg(s_k) = \pi$). We get from (4) for $t \geq 0$

$$K_{\exp}(t) = \sum_{\Re(s_k<0)} \mathrm{Res}_{s_k}\left(\frac{\mathrm{e}^{st}}{A(s)}\right) = \sum_{\Re(s_k<0)} p_k(t)\,\mathrm{e}^{-\rho_k t}\sin(\sigma_k t + \theta_k), \tag{19}$$

where p_k is a polynomial factor if the zero s_k is multiple. The θ_k are resulting phaseshifts (see, e.g., Equation (26) below).

The fractional parts of the symbol (9) cause that the integrals along the branch cut (negative real axis) do not cancel. If we split the symbol along the branch cut into $A(r) =: C(r) + iS(r)$ $(-\infty < r \leq 0)$, $C(r),\ S(r) \in \mathbb{R}$ then we get for $t \geq 0$

$$K_{\mathrm{rel}}(t) = \frac{1}{\pi}\int_0^\infty \frac{S(r)\,\mathrm{e}^{-rt}\,\mathrm{d}r}{C^2(r) + S^2(r)}. \tag{20}$$

Apart from possible representations of K_{rel} via Mittag–Leffler functions we suggest a numerical treatment of this integral. We will characterise K_{rel} later (Equation (23)) more precisely. Obviously, $K_{\mathrm{rel}}(t) \to 0$ for $t \to \infty$. Thus the heading of this subsection is correct: Equation (18) becomes asymptotically stable.

Remark 1. Truly, the 'integer' case, i.e. only integer derivatives, is included. Firstly, $K_{\exp}$ keeps his form. In case of only negative zeros all σ_k vanish such that no oscillations occur. Secondly, $K_{\mathrm{rel}}(t) \equiv 0$, because $S(r) \equiv 0$ along the real negative axis. This is of course consistent to the trivial fact that no branch cut is needed in this case.

2.4.4. *The Unstable Case*

Assume that there are roots s_m with $\Re(s_m) \geq 0$. Firstly, $K_{\mathrm{s}} := K^+$ is calculated via Section 2.4.3 (from the roots with negative real part, if there are some. The relaxational part exists always, if there is at least one fractional derivative). For $t < 0$ we proceed analogously: we have simply to summarise all residues of all zeros $s_m = \rho_m \pm i\sigma_m$ $(rho_m, \sigma_m \geq 0)$. Thus $K^-(t) = 0$ for $t \geq 0$ and analogous to (19) for $t < 0$

$$K^-(t) = \sum_{\Re(s_m)\geq 0} \mathrm{Res}_{s_m}\left(\frac{\mathrm{e}^{st}}{A(s)}\right) = \sum_{\Re(s_m)\geq 0} q_m(t)\,\mathrm{e}^{\rho_m t}\sin(\sigma_m t + \theta_m). \tag{21}$$

Thus we get a stable but non-causal 'impulse response'. Apart from the mathematical justification in the theoretical part we conclude from the causality and the instability of the physical system that K^- is the superposition of a homogeneous solution of equation (7) and

the unstable component K_{u} of the causal impulse response. Obviously, because of $K_{\mathrm{u}}(t) = 0$ for $t < 0$, this homogeneous solution must be $K^-|_{t<0}$ extended to $\mathbf{C}^\infty$. Substracting it from K^- we get

$$K_{\mathrm{u}}(t) = \begin{cases} 0, & t < 0, \\ -\sum\limits_{\Re(s_m)\geq 0} q_m(t)\, \mathrm{e}^{\rho_m t} \sin(\sigma_m t + \theta_m), & t \geq 0. \end{cases} \tag{22}$$

We have now arrived at the above given result (14) for the causal impulse response. We emphasise once more that it includes the well-known results from the 'integer' theory. But we have essentially one more detail because of the decomposition of the stable component (see equation (18)).

Remark 2. It must be emphasised that other approaches, e.g. Riemann–Liouville, yield the same integral (17), if the initial conditions are all chosen zero. Thus we could have followed now the commonly used representation via special functions, mostly of Mittag–Leffler-type [2, 4, 15–17]. From a mathematical point of view this is satisfactory. But from those formulas nobody can see any significant properties of the impulse responses (as pointed out in the following) nor is it (until now) easy to calculate them. Thus we advise emphatically [7, 8] the described method.

3. Illustration

In consequence to the last section, the 'fractional' causal system shows all 'integer' effects as there are oscillations in case of non-real zeros of $A(s)$, exponential behaviour in case of zeros with non-zero real parts, etc. But there is an additional relaxation in the stable component which needs some more attention. Thus let us firstly consider the stable component.

3.1. The Stable Component

In contrast to the 'integer' case there is always a stable component in case of a fractional operator, even if there are no zeros with negative real parts. Due to the necessary branch cut we get at least a relaxation K_{rel}. From its representation (20) we are able to quantify its asymptotic behaviour. It is easily seen that K_{rel} decays for large t totally monotonic and becomes totally dominant over K_{exp}: all derivatives of K_{rel} decrease with powers of t whereas the derivatives of K_{exp} decay exponentially (if $K_{\mathrm{exp}} \not\equiv 0$). More precisely, Watson's lemma [8] yields for $t \to \infty$

$$K_{\mathrm{rel}}^{(n)} \asymp (-1)^n \frac{a_q \sin(\nu_q \pi)}{a_0^2 \pi} \Gamma(\nu_q + n + 1)\, t^{-(\nu_q + n + 1)}\,, \tag{23}$$

where ν_q denotes the lowest fractional (i.e. non-integer) derivative in the operator (6). We emphasise that K_{rel} cannot be modelled by 'integer' linear differential operators with constant coefficients. If $K_{\mathrm{exp}} \not\equiv 0$, we get an exponentially damped oscillation round K_{rel}. Consequently the stable impulse response has only a finite number of zeros and decays in the end monotonically to the t-axis.

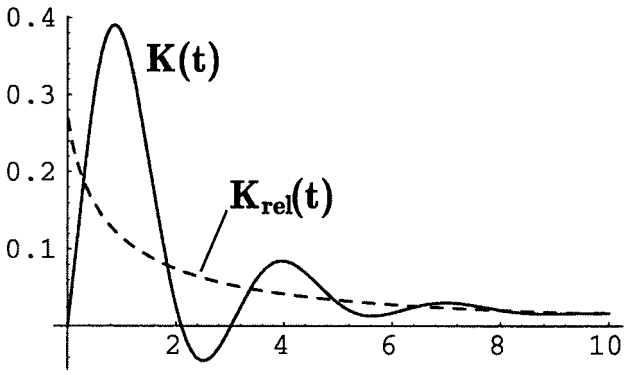

Figure 2. Impulse response and relaxation.

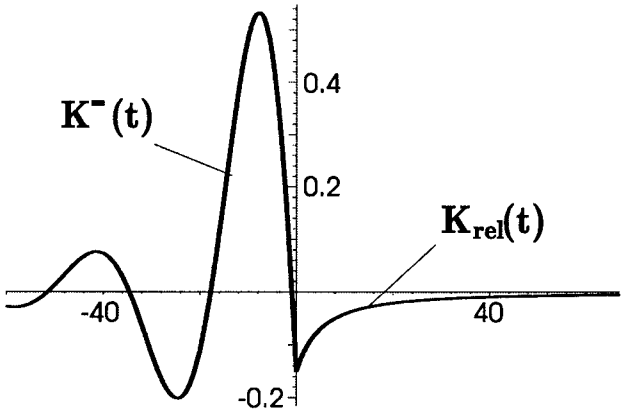

Figure 3. Non-causal stable impulse response.

To display this typical behaviour we consider the 3-term-operator $D^2 + 3\,D^{0.6} + 1$, which belongs to a stable system (the zeros of $A(s)$ are $-1.19794 \pm 1.97672\,i$). In Figure 2 we depict its impulse response $K(t)$ and its relaxation K_{rel} (dashed line).

3.2. The Unstable Case

Consider the operator $5D^2 - D^{0.1} + 1$. Since the symbol $A(s)$ has exactly one pair of zeros $0.057687 \pm 0.18663\,i$ in the right halfplane, we get from Section 2.4.3 that the stable component consists of only K_{rel}, whereas $K^-(t)$ is for $t < 0$ an exponentially increasing oscillation. Figure 3 shows the resulting non-causal but stable 'impulse response'. In agreement with the integral representation it is continuous.

Following the in Section 2.4.4 described further procedure, i.e. the superposition with a homogeneous solution, we get the causal unstable impulse response which is depicted in Figure 4.

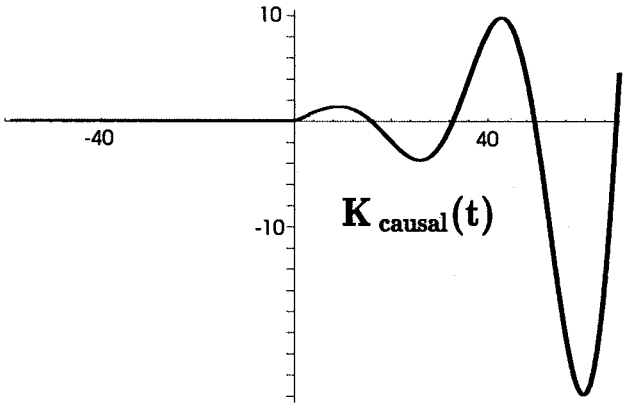

Figure 4. Causal unstable impulse response.

4. Applications

4.1. Viscoelastic Rods

Firstly, the frequency responses of viscoelastic rods of materials like PTFE (Teflon = Poly-TetraFluorEthylen), PA (PolyAmid), PU (PolyUrethan), PVC (PolyVinylChlorid), PE (Poly-Ethylen), etc., were measured. To demonstrate the power of the just described method the results are compared to the calculated frequency responses from the fractional model as well as from two classical damping models.

4.1.1. *The Models*

We start from the well-known Kelvin–Voigt model for the longitudinal displacement $u(x, t)$ of a one-dimensional rod

$$\rho u_{,tt}(x, t) = E u_{,xx}(x, t) + \eta u_{,xxt}(x, t). \tag{KV}$$

The fractional model is now concluded by substituting the order of the time derivative in the damping term by a fractional one

$$\rho u_{,tt}(x, t) = E u_{,xx}(x, t) + \eta u_{,xxt^\alpha}(x, t). \tag{FR}$$

The third considered model is the one with constant complex Young modulus

$$\rho u_{,tt}(x, t) = (E_\Re + i E_\Im) u_{,xx}(x, t). \tag{CY}$$

In these equations we have used the following notations:

ρ: mass density, E: real Young modulus,

η: viscosity, α: order of fractional derivative,

$E_\Re + i E_\Im$: constant complex Young modulus.

The boundary conditions BC are

$x = 0$

$\Rightarrow F(0, t) = 0 = E A u_{,x}(0, t) + \eta A u_{,xt^\alpha}(0, t)$

$x = l$ (cosinus exciting force)

$\Rightarrow F(l, t) = \tilde{F} \cos(\Omega t) = E A u_{,x}(l, t) + A \eta u_{,xt^\alpha}(l, t).$ (BC)

Figure 5. Experimental setup.

A more detailed and illustrative description of the above damping models is given in [11].

4.1.2. *Calculated Solutions*

The models KV and CY can be interpreted as special cases of model FR ($a = 1$ or $\eta = 0$, $E \in \mathbb{C}$, resp.). To calculate the solution of FR (with BC) we shall take the Fourier transform with respect to time t. The solution of the resulting ordinary differential equation for x is, in the frequency domain

$$\hat{u}(x,\omega) = \frac{\tilde{F}\pi(\delta(\omega-\Omega)+\delta(\omega+\Omega))\cosh\left(x\sqrt{\frac{-\rho\omega^2}{E+\eta(i\omega)^\alpha}}\right)}{Ai\omega\sqrt{\rho(E+\eta(i\omega)^\alpha)}\sinh\left(l\sqrt{\frac{-\rho\omega^2}{E+\eta(i\omega)^\alpha}}\right)}.$$

In the time domain this is a phase shifted stationary oscillation of the form

$$u(x,t) = U(x,\Omega)\cos(\Omega t + \varphi(x,\Omega)). \tag{24}$$

4.1.3. *Measurement*

We took rods of length 1 m. A force transducer, linked with a shaker was placed at one end. At the other (free) end an acceleration transducer was placed (Figure 5).

The measured frequency response of our 'reference' material PTFE (Teflon) with bronce ($\rho = 3850$ kg/m^3, $E = 0.12 \cdot 10^{10}$ N/m^2) is shown in Figure 6.

4.1.4. *Comparison*

To compare the measured data with the calculated solutions of the models KV, CY and FR we fitted the parameters $\alpha = 0.086$, $\eta_{\mathrm{FR}} = 0.44 \cdot 10^9$ Ns$^\alpha$/m^2, $\eta_{\mathrm{KV}} = 2 \cdot 10^3$ Ns/m^2, $E_\infty = 0.8 \cdot 10^8$ N/m^2. The upper hull-curves of the corresponding frequency responses are depicted in Figure 7. The very good agreement of the fractional model is obvious. Whereas CY may be used with some restrictions the model KV appears as not admissible.

Another important effect is the so called dispersion of the eigenfrequencies which is caused by the non-constant phase-velocity

$$c_l = \left(\Re\left(\sqrt{\frac{\rho}{E+\eta(i\Omega)^\alpha}}\right)\right)^{-1}.$$

Figure 8 shows the resonance frequencies divided by their order. Whereas the models KV, CY yield a constant for all values and thus do not show any dispersion, FR models the increasing quotient in surprisingly good agreement with the measurement.

We emphasise that the results for other materials (including elastic ones) are as good as in this example. In [12] the reader may found more details. Finally we have to remark with

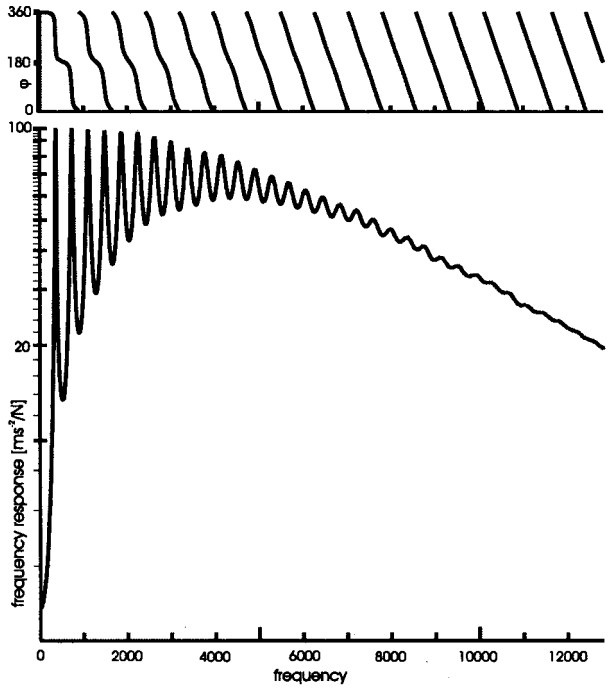

Figure 6. Frequency response of PTFE.

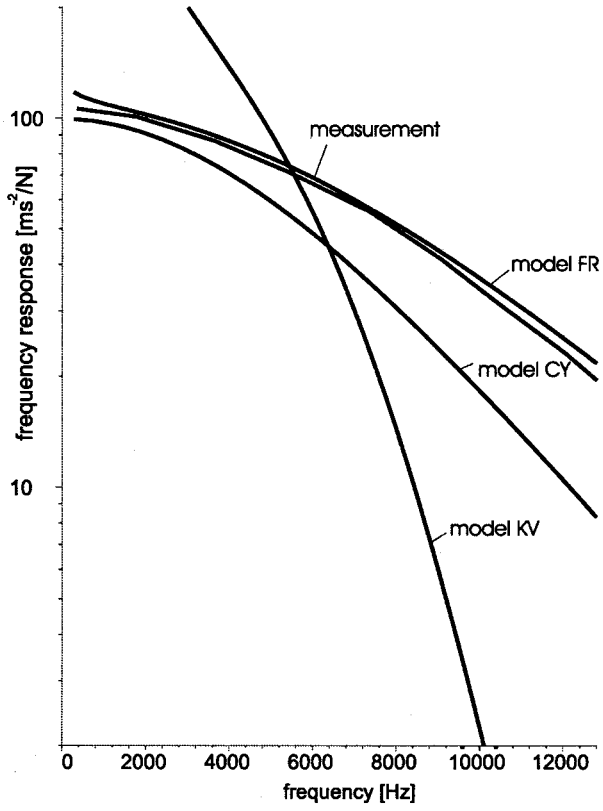

Figure 7. Comparison of the considered models.

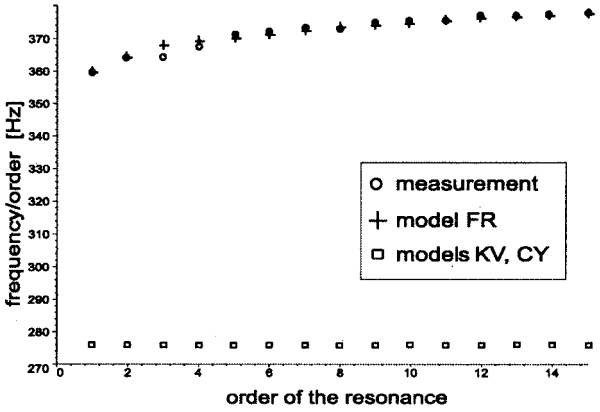

Figure 8. Dispersion.

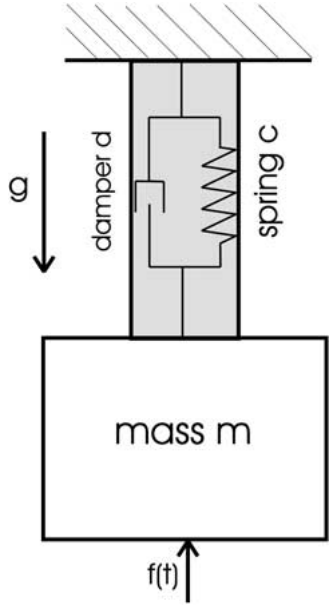

Figure 9. Experimental setup.

respect to other fractional calculi that due to the stationarity of the solutions all approaches with one-sided Laplace transforms are not admissible (apart from the problem of setting initial conditions).

4.2. The One-Mass Oscillator

The aim is to check further the model of a viscoelastic one-mass oscillator (see [7])

$$m\,D^2 y(t) + d\,D^{\nu} y(t) + c\,y(t) = -f(t)\,. \tag{25}$$

For the realisation we use as spring/damper a piece of the just described rods. It is fixed at the top whereas the oscillator mass m is attached at the bottom and moved vertically by the force $f(t)$ (Figure 9).

The main problem is to get *a priori* the values of $d,\ c,\ \nu$.

We start from the related model (FR) where all constants are known. Since this model has shown its validity, it is advisable to extract model (25) from it. This will be sketched now briefly.

Firstly, we neglect the mass of the rod, i.e. $\rho = 0$. We can do this, since the mass m of the oscillator is big compared with the mass of the damper. The appropriate boundary conditions are now

$$u(0,t) = 0,$$

$$m\,u_{,tt}(l,t) = -EA\,u_{,x}(l,t) - \eta A\,u_{,xt^\alpha}(l,t) - f(t),$$

with damper-length $l = 0.12$ m and cross-section $A = 7.85 \cdot 10^{-5}$ m^2.

Integration of (FR) along the rod yields finally

$$m\,u_{,tt}(l,t) + \frac{A\eta}{l}\,u_{,t^\alpha}(l,t) + \frac{AE}{l}\,u(l,t) = -f(t).$$

Thus we have arrived at (25) and can identify the constants as

$$d = \frac{A\eta}{l}, \quad c = \frac{AE}{l}, \quad \nu = \alpha.$$

Beating the oscillator from below simply with a hammer (i.e. $f(t) = c_0\,\delta(t)$) we have realised impulse responses for several masses m. In [7] the impulse response of (25) has been solved (as also sketched in Section 2.4.3). We cite the result for the normalised equation (i.e. $m = c_0 = 1$).

Due to $\alpha \neq 1$ and $d, c > 0$ the symbol $A(s) = s^2 + d\,s^\alpha + c$ has exactly one pair of conjugate complex zeros $s_{1,2} = -\sigma \pm i\omega$ $(\sigma, \omega > 0)$ if $A(s)$ is defined by the principal branch of s^α. Thus

$$K_{\text{causal}}(t) = c_1\,\mathrm{e}^{-\sigma t}\sin(\omega t + \varphi_1) + K_{\text{rel}}(t), \tag{26}$$

with $c_1 = 2/\sqrt{u^2+v^2}$, $\varphi_1 = \arctan(u/v)$, where $u \pm iv = a'(s_{1,2})$.

The relaxation function (20) is given by

$$K_{\text{rel}}(t) = \frac{1}{\pi}\int_0^\infty \frac{d\sin(\alpha\pi)\,r^\alpha\,\mathrm{e}^{-rt}\,\mathrm{d}r}{(r^2 + r^\alpha d\cos(\alpha\pi) + c)^2 + (r^\alpha d\sin(\alpha\pi))^2}.$$

It turns out that due to the parameters of our materials the relaxation $K_{\text{rel}}(t)$ is such small that – although it is domininant later on – it can be neglected together with the phaseshift φ_1. Taking into account that the equation has been normalised the solution of (25) becomes

$$y(t) \approx c_2\,\mathrm{e}^{-\sigma t}\sin(\omega t). \tag{27}$$

We will show later on that measurement and calculation of σ and ω match very good for a large range of oscillator masses. To get an idea of the quality of the fractional approach we compare the results with those from the classical damped oscillator:

$$m\,\ddot{x}(t) + \widetilde{d}\,\dot{x}(t) + \widetilde{c}\,x(t) = -c_0\,\delta(t). \tag{28}$$

Here we have $\widetilde{c} = c$ whereas $\widetilde{d}$ can be fitted and optimised during the measurement. The zeros of the accompanying characteristic equation are: $\widetilde{s}_{1,2} := -\widetilde{\sigma} \pm i\widetilde{\omega}$ with $\widetilde{\sigma} = \widetilde{d}/2m$ and $\widetilde{\omega} = \sqrt{\widetilde{\sigma}^2 - \widetilde{c}/m}$. Hence Equation (28) has the solution

Table 1. Comparison with measurement.

	Measured		Calculated			
m [kg]	σ	ω	σ	ω	$\widetilde{\sigma}$	$\widetilde{\omega}$
0.975	30	880	29.3	845	92.0	660
2.475	18	527	17.6	525	36.3	414
5.475	13	351	11.4	350	16.4	278
8.475	10	289	9.0	280	10.6	224
9.975	9	263	8.2	258	9.0	206

$$x(t) = c_3\, \mathrm{e}^{-\widetilde{\sigma} t}\, \sin(\widetilde{\omega} t).$$

To compare both models the measured and calculated damping and frequency values are listed in Table 1.

We have to make some comments. The constant $\widetilde{d}$ has been optimised for the mass 9.975 kg to $\widetilde{d} = 179.22$ Ns/m. Hence the last row shows the exact damping constant $\widetilde{\sigma} = 9.0$ for the classical model. Truly, this is the only admissible value of all $\widetilde{\sigma}$, $\widetilde{\omega}$. An optimation for another mass would only move but not change the bad coincidence of the classical model with the measurement. The fractional model however covers the whole shown range of masses very good (σ and ω) with one constant d. In contrast the classical model would at last need for every mass a new parameter $\widetilde{d}$ to match the correct damping behaviour. But it fails totally matching the frequency. We emphasise further that the constants of the fractional model are only dependent on the material and have not been optimised for this experiment. Consistent to the above results for the viscoelastic rod 4.1 the model (25) of the one-mass oscillator has shown similar good results for other materials too (see [18]).

5. Theory

We remind on the starting point in Section 2.2. There we considered differential equations

$$\mathcal{A}\, x(t) = f(t), \quad t \in \mathbb{R}, \tag{29}$$

where $\mathcal{A}$ denotes a differential expression

$$\begin{aligned} \mathcal{A} \;:=\; & D^{\nu_n} + a_{n-1} D^{\nu_{n-1}} + \cdots + a_1 D^{\nu_1} + a_0, \\ & a_k \in \mathbb{R}, \quad 0 < \nu_1 < \ldots < \nu_n =: \deg \mathcal{A}. \end{aligned} \tag{30}$$

We remark that the now following theory can be easily extended to n dimensions (x maps from $\mathbb{R}^n$ to $\mathbb{C}$), if the ν_k are interpreted as multiindices. Only some normalising factors have to be generalised [19, 20].

5.1. $\mathbf{L}_2$-Theory

5.1.1. *The Functional Calculus Definition*

In order to make the formal expression (30) a well-defined operator $\mathcal{A}$ and to get global and global causal solutions of the associated differential equation (29) we start from a $\mathbf{L}_2$-theory.

We consider $\mathcal{A}$ as a pseudo-differential operator in the sense of the usual functional calculus (see, e.g., [21, 22]). If $a(\omega)$ denotes the associated symbol (Equation (8)) and $\mathcal{F}$ denotes the extension of the Fourier transformation (2) to a unitary isomorphism on $\mathbf{L}_2$ (see, e.g., [20, 22]) we get

$$\mathcal{A} := \mathcal{F}^{-1} a(\omega) \mathcal{F}. \tag{31}$$

$\mathcal{A}$ is well defined, if the symbol $a(\omega)$ or $A(s)$ (equation (9)), respectively, is measurable. Thus there are as many well-defined operators as there are well-defined symbols. To be precise: the domain of the analytic continuation of $A(s)$ is a Riemann (logarithmic) surface. If one changes its sheets along $s \in \mathbb{R}$ countably often, A remains measurable and $\mathcal{A}$ becomes densely defined, linear and closed. Thus there are infinitely many ways to fix $\mathcal{A}$. This is the purely mathematical aspect. We have shown in [7] that physically consistent solutions of (29) require that all non-integer powers of the symbol must be defined by their principal branches. 'Physically consistent' means at least that in case of a continuous input f the solution should be continuous too.

5.1.2. $\mathbf{L}_2$*-solutions*

To solve Equation (29) it is necessary that $\mathcal{A}$ is injective. This is true if $1/A(s) \in \mathbf{L}_2$. (For continuous $A(s)$: if and only if $\deg \mathcal{A} > 1/2$ and $A(s)$ has no zeros along the imaginary axis.)

If additionally $1/A(s) \in \mathbf{L}_1$ (i.e. $\deg \mathcal{A} > 1$) and $f \in \mathbf{L}_\infty$ then $x(t)$ results to a continuous $\mathbf{L}_2$-function (see [8]).

Since the product at the right-hand side of (31) is associative $\mathcal{A}^{-1}$ is given by

$$\mathcal{A}^{-1} := \mathcal{F}^{-1} \left(\frac{1}{a(\omega)}\right) \mathcal{F}. \tag{32}$$

5.1.3. *Causal* $\mathbf{L}_2$*-Impulse Responses*

Since the Dirac impact $\delta(t)$ is no $\mathbf{L}_2$-function we cannot define the impulse response $K(t)$ from $\mathcal{A}K = \delta$. But the functional calculus approach provides from $\mathcal{A}x = f$ via Fourier transform

$$a(\omega)\widehat{x}(\omega) = \widehat{f}(\omega). \tag{33}$$

Now we rewrite $\widehat{f}(\omega) = 1 \cdot \widehat{f}(\omega)$ such that the Fourier transform $\widehat{K}(\omega)$ of the impulse response $K(t)$ appears as solution of

$$a(\omega)\widehat{K}(\omega) = 1$$

and the convolution theorem yields again

$$x(t) = (K * f)(t).$$

Conditions for causal $\mathbf{L}_2$-impulse responses $K(t)$ we obtained from [7, 8] from a well-known Paley–Wiener theorem (see, e.g., [23, VI.4, theorem 2]). The result is the following theorem:

THEOREM 1. *Let* $\deg \mathcal{A} > 1/2$ *and let* $A(s)$ *be defined by the principal branches of all non-integer powers. If* $A(s)$ *has no zeros* s_k *with* $\Re(s_k) \geq 0$, *then the impulse response is a causal* $\mathbf{L}_2$*-function given by*

$$K(t) := \mathcal{A}^{-1}1 = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-iR}^{iR} \frac{e^{st}}{A(s)}\, ds \quad (t \in \mathbb{R}). \tag{34}$$

K is continuous on $\mathbb{R}$ *(i.e.,* $K(0) = 0$*), if* $\deg \mathcal{A} > 1$. *For* $1/2 < \deg \mathcal{A} \leq 1$ *it is continuous on* $\mathbb{R}^+ \cup \{0\}$ *(i.e., a finite jump at* $t = 0$*).*

5.1.4. *A First Extension*

From the integral representation we conclude that the integral (34) still exists (and vanishes for $t < 0$), if in Theorem 1 the condition $\deg \mathcal{A} > 1/2$ is weakened to $\deg \mathcal{A} > 0$. Consequently we get still causal and stable impulse responses which are moreover continuous for $t > 0$ (see, e.g., [8]). Truly, for $0 < \deg \mathcal{A} < 1/2$ the global $\mathbf{L}_2$-property is weakened to $\mathbf{L}_2(\mathbb{R}^+)$, because we run into a right-sided pole at $t = 0$.

5.1.5. *A Priori Criteria for Causality*

First we state that in case of $\mathbf{L}_2$-solutions causality and (asymptotic) stability are exactly the same. For both it is sufficient and necessary that the symbol $A(s)$ must not have any zeros s_k with $\Re(s_k) \geq 0$.

Since the fractional symbol $A(s)$ looks similar to a polynomial, it might be expected that (at least necessary) conditions on the coefficients are similar. It turns out that this is true only for the operator $D^2 + aD^\nu + b$ treated in [7].

But, for instance, the symbol $s^{2/3} - s^{1/2} + 1/2$ has the principal-branch-roots $s_{1,2} \approx -1.0798 \pm 0.4087\,i$. This shows that neither the well-known necessary condition of positive coefficients is valid nor that the number of zeros is simply determined by the degree of the operator.

This problem was attacked in [9] by geometrical methods and totally solved for the 2- and 3-term operators $D^\mu + aD^\nu + b$, $b \neq 0$. The results are also given in [10]. Since our operators can be composed and decomposed multiplicatively (see next subsection), we cover in fact a broad range of applications.

5.1.6. *Properties*

In contrast to all other fractional calculi our operators inherit the algebraic properties from the class of symbols. In fact our operators form a (commutative) $\mathbb{R}$-algebra. Particularly we get from the associativity of the operator product

COROLLARY 2. *Let* a_1, a_2 *denote the symbols of* $\mathcal{A}_1, \mathcal{A}_2$. *Let further* $a_{12} = a_1 a_2$ *such that* $\mathcal{A}_{12}$ *is the associated differential operator. Then*

$$\mathcal{A}_{12} = \mathcal{A}_1 \mathcal{A}_2 = \mathcal{A}_2 \mathcal{A}_1.$$

Thus we have neither problems to include operators with only integer derivatives nor do we have problems to compose or decompose such operators. Moreover, the inversion formula (32) is consistent to this corollary.

A great disadvantage of approaches via one-sided Laplace transform is, from a physical point of view, their missing translation invariance in time (see [6]). But obviously our operators have this property, namely

LEMMA 3. *Let* $x_\tau(t) := x(t - \tau)$ *denote a shifted function (fixed* τ*). Then*

$$\mathcal{A}x_\tau(t) = \mathcal{A}x(t - \tau) \quad \textit{for all } t \in \mathbb{R}.$$

5.1.7. *Fractional Derivatives*

We may now briefly answer the always asked question about explicit formulas for a fractional derivative. From (31) we get two slightly different possibilities to get $D^q f, q > 0$. Firstly, if the relevant Fourier transforms are easily obtainable,

$$D^q f(t) = \mathcal{F}^{-1}(i\omega)^q \widehat{f}(\omega). \tag{35}$$

Secondly for $f \in \mathbf{L}_2$, the integral representation

$$D^q f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (i\omega)^q \, \mathrm{e}^{i\omega(t-\tau)} \, f(\tau) \, \mathrm{d}\tau \, \mathrm{d}\omega. \tag{36}$$

Some elementary manipulations lead to the well-known Riemann–Liouville integral

$$D^q f(t) = {}_{-\infty}D_t^q \, f(t) = \frac{1}{\Gamma(k-q)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \int_{-\infty}^{t} \frac{f(\tau) \, \mathrm{d}\tau}{(t-\tau)^{q+1-k}}, \tag{37}$$

where $k - 1 \leq q < k, k \in \{1, 2, 3, \ldots\}$.

But it must be emphasised that our approach remains a qualitative different one than starting from this formula. This will become evident by the now following extensions to a much larger class for f than the possible scope of (37).

5.2. Extension to Tempered Distributions

It is not the intent of this paper to give a detailed theory on this topic. But we try to emphasise the main points and refer the interested reader (in particular for proofs) to the rich literature, e.g., [19, 20, 23, 24].

5.2.1. *The Idea*

We look for a larger function space such that the Fourier transformation acts as an isomorphism on it. The idea is, to look firstly for a dense subspace ($\mathcal{S}$) of $\mathbf{L}_2 \cap \mathbf{C}^\infty$ with this property. This $\mathcal{S}$ is provided with a metric which makes it a topological vector space. The Fourier transform can now be easily embedded in the dual space ($\mathcal{S}'$) which is provided with the weak*-topology from $\mathcal{S}$. This way $\mathcal{F}$ will be established as an isomorphism on $\mathcal{S}'$. We sketch the steps.

5.2.2. *The Testing Space $\mathcal{S}$*

DEFINITION 1 (Schwartz space). (1) The testing space (Schwartz space) $\mathcal{S}$ of 'rapidly decreasing functions' consists of all complex valued $\varphi(x) \in \mathbf{C}^\infty$, such that

$$\sup_{x \in \mathbb{R}} |x^a \varphi^{(b)}(x)| < \infty \quad \text{for all } a, b \in \{0, 1, 2, 3, \ldots\}.$$

(2) $\mathcal{S}$ becomes a topological vector space, if we introduce a metric d as follows

(a) Define a countable family of seminorms

$$\rho_{ab}(\varphi) := \sup_{x \in \mathbb{R}} |x^a \varphi^{(b)}(x)|, \varphi \in \mathcal{S}.$$

The induced metrics $d'_{ab}(\varphi, \psi) := \rho_{ab}(\varphi - \psi)$ may be ordered to a sequence $(d'_n)_{n \in \mathbb{N}}$.

(b) As this sequence is generally not bounded, we define the equivalent metrics

$$d_n := \frac{d'_n}{1 + d'_n} < 1,$$

(c) which, with the weight factor 2^{-n}, sum up to the metric d on $\mathcal{S}$:

$$d := \sum_{n=0}^{\infty} 2^{-n} d_n.$$

(3) From now we always consider $\mathcal{S}$ under the topology induced by d.

Remark 3. (1) This topology can briefly be characterised by characterising convergence in $\mathcal{S}$:

$\varphi_k \to 0 \to |\varphi_k^{(b)}(x)| \to 0$ uniformly in x for all $b \in \mathbb{N}$.

(2) $\mathcal{S}$ is separable and a dense subset of all $\mathbf{L}_p$-spaces:

$\mathcal{S} \Subset \mathbf{L}_p, \quad 1 \le p \le \infty.$

Remark 4. We use '$\Subset$' to denote embeddings of topological spaces. Thus $A \Subset B$ means that A is a dense subset of B with respect to the B-topology.

The basis for the further theory is (see, e.g., [20, 3.2])

THEOREM 4. *The Fourier transformation $\mathcal{F}$ acts as a unitary isomorphism on $\mathcal{S}$. Briefly:*

$\mathcal{F}(\mathcal{S}) = \mathcal{F}^{-1}(\mathcal{S}) = \mathcal{S}$ *and* $\|\widehat{\varphi}\|_2 = \|\varphi\|_2$ *for all* $\varphi \in \mathcal{S}$.

5.2.3. *The Space $\mathcal{S}'$ of Tempered Distributions*

DEFINITION 2. The dual space $\mathcal{S}'$ of $\mathcal{S}$ is defined as the space of all continuous linear functionals $\mathcal{S} \to \mathbb{C}$:

$\mathcal{S}' = \{\langle f \mid \varphi \rangle \in \mathbb{C} \mid \varphi \in \mathcal{S}\}.$

We name it the space of 'tempered distributions'.

We remind of the scope of $\mathcal{S}'$, which consists of

1. *Regular tempered distributions*, given by

$$\langle f \mid \varphi \rangle := \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x, \tag{38}$$

which can simply be identified with the 'normal' functions $f(x)$. Particularly, all functions g are included which fulfil

$$\frac{g(x)}{(1 + |x|^2)^a} \in \mathbf{L}_p \quad \text{for some} \quad \begin{cases} a \in \{0, 1, 2, \ldots\}, \\ 1 \le p \le \infty. \end{cases} \tag{39}$$

They are called 'tempered' or 'slowly increasing' functions.

2. *Singular tempered distributions*, which cannot be represented via (38). The most famous one is the Dirac-impact δ:

$$\langle \delta \mid \varphi \rangle := \varphi(0). \tag{40}$$

A big class of singular distributions are defined via

$$\langle \mu \mid \varphi \rangle = \int_{\mathbb{R}} \varphi(x)\, \mathrm{d}\mu(x), \tag{41}$$

with some finite Borel-measure μ. These distributions are simply called 'measures'. Thus (40) can be interpreted as point measure with total mass 1 in the origin.

Remark 5. We end this subsection with the remark that $\mathcal{S}'$ contains all $\mathbf{L}_p$-spaces as dense subsets, i.e.,

$$\mathbf{L}_p \Subset \mathcal{S}', \quad 1 \leq p \leq \infty.$$

5.2.4. *Topological Remarks*
To get well-defined calculation rules as well as to extend the Fourier transformation on $\mathcal{S}'$ we may establish the weak*-topology, induced by the topology of $\mathcal{S}$. This way, convergence in $\mathcal{S}'$ is simply given via

DEFINITION 3. Let $f_k \in \mathcal{S}'$ for all $k \in \mathbb{N}$.

$$f_k \to f \Leftrightarrow \langle f_k \mid \varphi \rangle \to \langle f \mid \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Very important for applications and numerics is

LEMMA 5. *Every singular distribution g can be approximated by a sequence $\{g_k\}$ of regular distributions*

$$\langle g_k \mid \varphi \rangle \to \langle g \mid \varphi \rangle \quad \textit{for all } \varphi \in \mathcal{S}.$$

The continuity of linear functionals causes a characterisation of $\mathcal{S}'$:

LEMMA 6. *A linear functional $\langle f \mid \varphi \rangle$ belongs to $\mathcal{S}'$ if and only if*

$$\varphi_k \to 0 \textit{ in } \mathcal{S} \Rightarrow \langle f \mid \varphi_k \rangle \to 0 (\textit{ in } \mathbb{C}).$$

It becomes immediately clear how to get now proper calculation rules on $\mathcal{S}'$ from the ones on $\mathcal{S}$.

5.2.5. *Calculation Rules*

1. *Multiplication.* gf is defined for all $f \in \mathcal{S}'$ via

$$\langle gf \mid \varphi \rangle := \langle f \mid g\varphi \rangle,$$

if $g\varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$ and $g\varphi_k \to 0$ if $\varphi_k \to 0$.
Functions that will do are tempered functions, see (39).

2. *Differentiation*

$$\langle D^k f \mid \varphi \rangle := (-1)^k \langle f \mid \varphi^{(k)} \rangle, \quad k \in \mathbb{N}.$$

3. *Convolution (1)*. Let $f \in \mathcal{S}'$, $\psi \in \mathcal{S}$. Then

$$\langle f * \psi \mid \varphi \rangle := \langle f \mid -\psi * \varphi \rangle.$$

This convolution has smoothing properties. The following lemma holds:

LEMMA 7. *The just defined convolution $f * \psi$ is regular and a slowly increasing $\mathbf{C}^\infty$-function.*

4. *Convolution (2)*. Let $f,\ g \in \mathcal{S}'$. Then

$$\langle f * g \mid \varphi \rangle := \langle f \mid -g * \varphi \rangle$$

is well defined, if either g has bounded support, or the supports of f and g are one-sided bounded to the same side.

5. *Differentiation of convolutions*. Let $f,\ g$ such that $f * g \in \mathcal{S}'$. Then

$$D^k(f * g) = (D^k f) * g = f * (D^k g), \quad k \in \{1, 2, 3, \ldots\}.$$

5.2.6. *Fourier Transforms on $\mathcal{S}'$*

DEFINITION 4. Let $f \in \mathcal{S}'$. Then $\widehat{f}$ and $\mathcal{F}^{-1} f$ are defined via

$$\langle \widehat{f} \mid \varphi \rangle := \langle f \mid \widehat{\varphi} \rangle, \quad \text{for all } \varphi \in \mathcal{S},$$

$$\langle \mathcal{F}^{-1} f \mid \varphi \rangle := \langle f \mid \mathcal{F}^{-1} \varphi \rangle, \quad \text{for all } \varphi \in \mathcal{S}.$$

This way $\mathcal{F}$ is an isomorphism on $\mathcal{S}'$. Truly, it is not unitary, because we have no inner product. But, since we can approximate all $\mathcal{S}'$-elements by $\mathbf{L}_2$-functions (or better $\mathcal{S}$-functions provided with $\mathbf{L}_2$-topology), the Parseval formula (3) is useful also in the generalised sense.

5.2.7. *The Functional Calculus Definition on $\mathcal{S}'$*
From $\mathcal{F}(\mathcal{S}') = \mathcal{S}'$ we can now extend the scope of the functional calculus from $\mathbf{L}_2$ on $\mathcal{S}'$.

LEMMA 8. *Let $\mathcal{A}$ be the formal differential expression* (30) *with symbol $a(\omega)$ as above. Then*

$$\mathcal{A} := \mathcal{F}^{-1} a(\omega) \mathcal{F}$$

is a well-defined (pseudo-)differential operator on $\mathcal{S}'$. It fulfills moreover all algebraic properties of Section 5.1.6.

Before we start the final extension we give a short illustration of the achieved state.

5.2.8. *Examples*
The Heaviside step function is denoted as $\theta(t)$.

1. $\widehat{\delta} = (2\pi)^{-1/2}$ and $\mathcal{F}^{-1}1 = \sqrt{2\pi}\,\delta$,
2. $D^q\theta(t) = \dfrac{\theta(t)}{\Gamma(1-q)}\,t^{-q} \quad (0 \le q < 1)$,
3. $D^q\delta(t) = \dfrac{\delta(t)}{\Gamma(1-q)}\,t^{-q} + \dfrac{\theta(t)}{\Gamma(-q)}\,t^{-q-1} \quad (0 \le q < 1)$,
4. $D^q(\theta(t)\,t^p) = \dfrac{\theta(t)\Gamma(p+1)}{\Gamma(p+1-q)}\,t^{p-q} \quad (0 \le q \le p)$,
5. $\langle D^k\delta \mid \varphi\rangle = (-1)^k\varphi^{(k)}(0)$ for all $k \in \mathbb{N}$,
6. $D^q 1 = 0$ for all $q > 0$,
7. $D^q \sin(at+b) = a^q \sin\left(at + b + \dfrac{a\pi}{2}\right)$ for $a > 0$,
8. $D^q(f * g) = (D^q f) * g = f * (D^q g)$ for all $q > 0$.

5.3. Extension to $\mathcal{D}'$

Although a wide field of applications (including all stable systems) is already covered by the $\mathcal{S}'$-approach, there are reasons to look for a further extension. The first one is the apparently non-causal impulse response from the integral representation (17) in the unstable case (Section 2.4.4). We can only get rid of this problem by adding homogeneous solutions of obviously exponential increasement (which are not in $\mathcal{S}'$). A second reason is of theoretical nature: our approach includes inside $\mathcal{S}'$ linear operators with solely integer derivatives. But the integer theory provides global solutions of exponential behaviour, too. Again, the extension to a function space is required which includes at least all homogeneous solutions of 'integer' operators. Such a space can be constructed by some modifications of the procedure that established $\mathcal{S}'$.

5.3.1. *The Testing Space $\mathcal{D}$*

DEFINITION 5. The testing space $\mathcal{D}$ consists of all $\mathbf{C}^\infty$-functions with compact support.

Obviously, we cannot simply use the topology of $\mathcal{S}$or an equivalent one (e.g. starting from $\rho_n(\phi) := \sup_{\mathbb{R}}\{|D^n\phi(x)|\}$, $n \in \mathbb{N}$), because it is not complete. (One can easily construct Cauchy sequences which converge to a $\mathcal{S}$-element with unbounded support, i.e. $\notin \mathcal{D}$.) We despense with a detailed description of the little bit sophisticated procedure how to get a complete topology (which is truly not metrisable) (see, e.g., [19, 6.3]). Anyway, such a topology can be reconstructed from the following definition of convergence in $\mathcal{D}$ (see, e.g., [24, 2.vi]). A small addition to the $\mathcal{S}$-convergence in Remark 3(1) is sufficient.

DEFINITION 6. Let $\phi_k \in \mathcal{D}$ with supports R_k.

$\phi_k \to 0$ in $\mathcal{D}$ if and only if

1. $|D^b\phi_k(x)| \to 0$ uniformly for all $b \in \{0, 1, 2, \ldots\}$.
2. There is a bounded region $R \subset \mathbb{R}$ such that $R_k \subset R$ for all k.

We get immediately the important property

$$\mathcal{D} \Subset \mathcal{S}. \tag{42}$$

Remark 6. Convergence in $\mathcal{D}$ causes convergence in $\mathcal{S}$. Thus the $\mathcal{D}$-topology can be embedded in the above $\mathcal{S}$-topology.

We proceed now totally analogous (Sections 5.2.3–5.2.5).

5.3.2. *The Space of Distributions $\mathcal{D}'$*

DEFINITION 7. The dual space $\mathcal{D}'$ of $\mathcal{D}$ is defined as space of all continuous linear functionals on $\mathcal{D}$:

$$\mathcal{D}' = \{\langle f \mid \phi\rangle \in \mathbb{C} \mid \phi \in \mathcal{D}\}.$$

We name it the space of distributions. It becomes a topological vector space via the weak*-topology induced by the $\mathcal{D}$-topology.

Consistent with this is the following definition of convergence:

DEFINITION 8. Let $f_k \in \mathcal{D}'$ for all $k \in \{1, 2, 3, \ldots\}$.

$$f_k \to f \text{ in } \mathcal{D}' \Leftrightarrow \langle f_k \mid \phi\rangle \to \langle f \mid \phi\rangle \quad \text{for all } \phi \in \mathcal{D}.$$

Since $\mathcal{D}$ is a dense subspace of $\mathcal{S}$, it becomes evident that

$$\mathcal{S}' \Subset \mathcal{D}'.$$

But different from the bounded increasement of regular $\mathcal{S}'$-elements ('functions') regular $\mathcal{D}'$-elements only need to be locally integrable. Thus we get from Lemma 5

LEMMA 9. *Let* $\mathbf{L}_{\text{loc}}$ *denote the space of locally integrable functions. Then*

$$\mathbf{L}_{\text{loc}} \Subset \mathcal{D}'.$$

Thus $\mathcal{D}'$ is sufficiently large for our purposes. In particular exponentially increasing functions are included as well as, e.g., point measures.

5.3.3. *Properties*

We can now keep it very short: all statements about $\mathcal{S}'$ in Sections 5.2.3–5.2.5. remain valid, if we simply replace $\mathcal{S}$ by $\mathcal{D}$.

But to establish now a functional calculus on $\mathcal{D}'$, we can not use the same procedure as in case of $\mathcal{S}'$. It is easily seen that Fourier transforms do not preserve the compact-support property. Consequently, $\mathcal{F}(\mathcal{D}) \neq \mathcal{D}$. To overcome this problem we introduce now the Fourier–Laplace transformation.

5.3.4. *The Fourier–Laplace Transformation*

DEFINITION 9.

$$\widetilde{f}(\zeta) := \mathcal{F}_{\mathbb{C}}[f](\zeta) := (2\pi)^{-1/2} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-i\,\zeta\, t}\, \mathrm{d}t, \quad \zeta \in \mathbb{C}.$$

Remark 7. Firstly, we see that $\widehat{f} = \widetilde{f}|_{\mathbb{R}}$. Secondly and *vice versa*, we conclude from the Riemann–Lebesgue lemma for $\mathbf{L}_1$-functions f that $\widehat{f}$ has an analytic continuation in $\mathbb{C}$. Thus $\widetilde{f}$ is well defined at least in some neighbourhood of $\mathbb{R}$.

There are several Paley–Wiener theorems which describe exactly scope and properties of $\mathcal{F}_{\mathbb{C}}$-transforms from scope and properties of f. One we have already used to establish Theorem 1 [7, 8]. The second one we need is (see, e.g., [19, 7.22])

THEOREM 10 (Paley–Wiener). *Let*

$$\mathbf{B}_r := \{|t| \leq r \mid r > 0\}.$$

For every $\phi \in \mathcal{D}$ with support in $\mathbf{B}_r$ it holds

1. $\widetilde{\phi}(\zeta), \zeta = \xi + i\eta$ $(\xi, \eta \in \mathbb{R})$*is entire.*
2. *There are constants $0 < c_k < \infty$ such that*

$$|\widetilde{\phi}(\zeta)\ | \leq c_k(1 + |\zeta|)^{-k}\, \mathrm{e}^{r|\eta|}, \quad k \in \{0, 1, 2, \ldots\}.$$

3. *Conversely, if an entire function $\psi(\zeta)$ satisfies item* (2)*, then there exists $\phi \in \mathcal{D}$, with support in $\mathbf{B}_r$, such that item* (1) *holds, i.e.,*

$$\psi(\zeta) = \widetilde{\phi}(\zeta), \quad \zeta \in \mathbb{C}.$$

DEFINITION 10. As commonly done, we denote by $\mathcal{Z}$ the space of all entire functions $\psi : \mathbb{C} \to \mathbb{C}$ which satisfy, for some $r > 0$

$$|\psi(\zeta)| \leq c_k(1 + |\zeta|)^{-k}\, \mathrm{e}^{r|\Im(\zeta)|}, \quad k \in \{0, 1, 2, \ldots\}$$

with constants $0 < c_k < \infty$.

5.3.5. *Properties of $\mathcal{Z}$*

1. The $\mathcal{F}_{\mathbb{C}}$-transformation provides a 1-1-mapping between $\mathcal{D}$ and $\mathcal{Z}$.
2. $\mathcal{Z} = \mathcal{F}_{\mathbb{C}}(\mathcal{D})$ is the unique analytic continuation of $\mathcal{F}(\mathcal{D})$ on $\mathbb{C}$, i.e.,

$$\psi(t) \in \mathcal{F}(\mathcal{D}), \quad t \in \mathbb{R} \Rightarrow \psi(\zeta) \in \mathcal{Z}, \quad \zeta \in \mathbb{C}.$$

3. Vice versa, briefly

$$\mathcal{F}(\mathcal{D}) = \mathcal{Z}|_{\mathbb{R}}.$$

4. Wheras inverse Fourier transforms are Fourier transforms themselves, this not true for Fourier–Laplace transforms. We get

$$\mathcal{F}_{\mathbb{C}}^{-1}[\psi] := \mathcal{F}^{-1}[\psi|_{\mathbb{R}}], \quad \text{for all } \psi \in \mathcal{Z}.$$

5. Because of Remark 6 and the continuity of $\mathcal{F}$, the topology of D induces a complete topology on $\mathcal{F}(\mathcal{D})$ which can be extended directly on $\mathcal{Z}$ via item (2): a subset Ψ in $\mathcal{Z}$ is open, if and only if $\Psi|_{\mathbb{R}}$ is open. This is consistent with the following definition of $\mathcal{Z}$-convergence.

DEFINITION 11.

$$\psi_k \to 0 \text{ in } \mathcal{Z} \Leftrightarrow \psi_k|_{\mathbb{R}} \to 0 \text{ in } \mathcal{Z}|_{\mathbb{R}}.$$

Thus, in connection with the $\mathcal{D}$-topology

$$\phi_k \to 0 \text{ in } \mathcal{D} \Leftrightarrow \widehat{\phi}_k \to 0 \text{ in } \mathcal{F}(\mathcal{D}) \Leftrightarrow \widetilde{\phi}_k \to 0 \text{ in } \mathcal{Z}.$$

6. $\mathcal{D} \Subset \mathcal{F}(\mathcal{D}) \Subset \mathcal{S}$.
7. For all $\psi(\zeta) \in \mathcal{Z}$ the well-known power series expansion for entire functions leads to

$$\psi(\zeta + \zeta_0) = \sum_{n=0}^{\infty} \frac{\zeta_0^n}{n!} D^n \psi(\zeta) \quad \text{for all } \zeta_0 \in \mathbb{C}.$$

   Trivially, this sum converges also in the $\mathcal{Z}$-topology.
8. Finally, the convolution can be extended on $\mathcal{Z}$:

$$(\psi * \phi)(\zeta) := \int_{\mathbb{R}} \psi(\tau)\phi(\zeta - \tau)\,\mathrm{d}\tau = (\phi * \psi)(\zeta).$$

5.3.6. *The Space $\mathcal{Z}'$*
We define the dual space $\mathcal{Z}'$ again as the space of continuous linear functionals on $\mathcal{Z}$

$$\mathcal{Z}' = \{\langle f \mid \psi \rangle \in \mathbb{C} \mid \psi \in \mathcal{Z}\}, \tag{43}$$

provided with the weak*-topology induced by the $\mathcal{Z}$-topology from Section 5.3.5 (5). Thus convergence in $\mathcal{Z}'$ means

$$f_k \to 0 \text{ in } \mathcal{Z}' \Leftrightarrow |\langle f_k \mid \psi \rangle| \to 0 \quad \text{for all } \psi \in \mathcal{Z}. \tag{44}$$

We omit the proof that all rules in Section 5.2.5 hold in $\mathcal{Z}'$ but emphasise as an important feature of $\mathcal{Z}'$.

LEMMA 11. *All elements of $\mathcal{Z}'$ are entire. Particularly,*

$$f(\zeta + \zeta_0) = \sum_{n=0}^{\infty} \frac{\zeta_0^n}{n!} D^n f(\zeta) \quad \textit{for all } f \in \mathcal{Z}', \zeta_0 \in \mathbb{C}.$$

Proof.

$$\langle f(\zeta + \zeta_0) \mid \psi \rangle = \left\langle \sum_{n=0}^{\infty} \frac{\zeta_0^n}{n!} D^n f \,\middle|\, \psi \right\rangle$$

is equivalent to

$$\langle f \mid \psi(\zeta - \zeta_0) \rangle = \left\langle f \,\middle|\, \sum_{n=0}^{\infty} \frac{(-\zeta_0)^n}{n!} \psi^{(n)}(\zeta) \right\rangle.$$

The last functional is well defined in Section 5.3.5. Thus $f \in \mathcal{Z}'$ inherits the 'entire'-property from $\mathcal{Z}$. □

5.3.7. *Examples*

1. $\langle \delta(\zeta - \zeta_0) \mid \psi \rangle = \sum_{n=0}^{\infty} \frac{(-\zeta_0)^n}{n!} D^n \psi(0) = \psi(\zeta_0).$
2. From Section 5.2.5, items (1) and (2), it follows:

PROPOSITION 12. *$f\, D^k\delta$, $k \in \mathbb{N}$ is well defined in $\mathcal{Z}$:*

$$\langle f\, D^k\delta \mid \psi \rangle = (-1)^k \sum_{m=0}^{k} \binom{k}{m} f^{(m)}(0)\psi^{(k-m)}(0).$$

Remark 8. We stress the fact that the sometimes introduced 'analytic functionals' $\langle f \mid \psi \rangle = \int_\Gamma f(\zeta)\psi(\zeta)\,\mathrm{d}\zeta$ with some contour $\Gamma \subset \mathbb{C}$, $\Gamma \not\subseteq \mathbb{R}$ (see, e.g., [25, II., 1.4 (2)]) are not included, due to the different topologies of $\mathcal{Z}'$ and $\mathbb{C}$. Those functionals would contradict the little later established Fourier transformation, because integrals along such contours do not equal generally the according integrals along the real axis.

Since we have the same topological conditions as by the above embeddings, we briefly state the final chain

$$\mathcal{D} \Subset \mathcal{Z}|_{\mathbb{R}} \Subset \mathcal{S} \Subset \mathcal{S}' \Subset \mathcal{Z}'|_{\mathbb{R}} \Subset \mathcal{D}'. \tag{45}$$

5.3.8. *The Fourier Transformation in $\mathcal{Z}'$ and $\mathcal{D}'$*

We are now ready to extend the isomorphism between $\mathcal{D}$ and $\mathcal{Z}$ via $\mathcal{F}_{\mathbb{C}}$ and $\mathcal{F}$ on their dual spaces.

DEFINITION 12.

1. $\mathcal{F}_{\mathbb{C}} : \mathcal{D}' \to \mathcal{Z}' : f \mapsto \widetilde{f}$. For $f \in \mathcal{D}'$ define $\widetilde{f} \in \mathcal{Z}'$:
   $\langle \widetilde{f} \mid \phi \rangle := \langle f \mid \widetilde{\phi} \rangle \quad$ for all $\phi \in \mathcal{D}$.
2. $\mathcal{F} : \mathcal{D}' \to \mathcal{Z}'|_{\mathbb{R}} : f \mapsto \widehat{f}$
   $\widehat{f} = \widetilde{f}|_{\mathbb{R}} \quad$ for all $f \in \mathcal{D}'$.
3. $\mathcal{F} : \mathcal{Z}'|_{\mathbb{R}} \to \mathcal{D}' : g \mapsto \widehat{g}$. For $g \in \mathcal{Z}'|_{\mathbb{R}}$ define $\widehat{g} \in \mathcal{D}'$:
   $\langle \widehat{g} \mid \varphi \rangle := \langle g \mid \widehat{\varphi} \rangle \quad$ for all $\varphi \in \mathcal{Z}|_{\mathbb{R}}$.

Remark 9. Obviously, (3) is well defined: $\widehat{\varphi} \in \mathcal{D}$ and the completeness of $\mathcal{D}'$ cause the right-hand side to be a well-defined $\mathcal{D}'$-element.

To see that the right-hand side in (1) is a well-defined $\mathcal{Z}'$-element, one needs a further Paley–Wiener theorem (see, e.g., [19, 7.23]). This, briefly, yields $\langle f \mid \widetilde{\phi} \rangle \in \mathcal{Z}'$ for all $f \in \mathcal{D}'$ with compact support. The embedding in $\mathcal{Z}'$ follows again from the completeness of $\mathcal{D}'$ and $\mathcal{Z}'$, respectively.

5.3.9. *Examples*
The reader may easily verify the following $\mathcal{F}_{\mathbb{C}}$-transforms:

1. Let $\delta_\tau(t) := \delta(t-\tau)$ and $\zeta = \xi + i\eta$. Then

   $$\widetilde{\delta}_\tau(\zeta) = (2\pi)^{-1/2}\,\mathrm{e}^{-i\,\tau\,\zeta} = \mathrm{e}^{-\tau\,\eta}\widehat{\delta}_\tau(\xi),$$

   consistent with $\widetilde{\delta}_\tau(\zeta)|_{\mathbb{R}} = \widehat{\delta}_\tau(\xi)$, as well as with the $\mathcal{F}$-translation rule.
2. From $\widetilde{f^{(k)}}(\zeta) = (-i)^k(t^k f)\tilde{}$ and $f \equiv 1$ we get

   $$(t^k)\tilde{} = \sqrt{2\pi}\,i^k\,D^k\delta(\zeta).$$

3. Important is

   $$(\mathrm{e}^{at})\tilde{} = \sqrt{2\pi}\,\delta(\zeta + ia), \quad a \in \mathbb{C}.$$

   This result follows from Lemma 11 and the $\mathcal{F}_{\mathbb{C}}$-transform of the power series of e^{at}:

   $$(\mathrm{e}^{at})\tilde{} = \sqrt{2\pi}\,\sum_{k=0}^{\infty}\frac{(ia)^k}{k!}\,D^k\delta(\zeta).$$

4. Consequently,

   (a) $(\sin(at))\tilde{} = i\pi\,[\delta(\zeta + a) - \delta(\zeta - a)]$,
   (b) $(\cos(at))\tilde{} = \pi\,[\delta(\zeta + a) + \delta(\zeta - a)]$,
   (c) $(\sinh(at))\tilde{} = \pi\,[\delta(\zeta + ia) - \delta(\zeta - ia)]$,
   (d) $(\cosh(at))\tilde{} = \pi\,[\delta(\zeta + ia) + \delta(\zeta - ia)]$.

5.4. Functional Calculus on $\mathcal{D}'$

We have now got a sufficiently large space for the extension of $\mathcal{A}$. We conclude firstly

$$f \in \mathcal{D}' \Rightarrow a(\omega)\,\widehat{f}(\omega) \in \mathcal{Z}'|_{\mathbb{R}}.$$

Since $\mathcal{F}^{-1}(\mathcal{Z}'|_{\mathbb{R}}) = \mathcal{F}(\mathcal{Z}'|_{\mathbb{R}}) = \mathcal{D}'$, we have finally established

$$\mathcal{A} := \mathcal{F}^{-1}a(\omega)\mathcal{F}$$

as a well-defined (pseudo-)differential operator on $\mathcal{D}'$.

5.4.1. *Examples*
We may now add the most required $\mathcal{D}'$-examples to the short list of fractional derivatives in $\mathcal{S}'$, given in Section 5.2.8. From the Fourier transforms in Section 5.3.9 the reader may easily conclude

1. $D^q\,\mathrm{e}^{at+b} = a^q\,\mathrm{e}^{at+b}$.
2. $D^q(t\,\mathrm{e}^{at+b}) = \mathrm{e}^{at+b}(a^q t + qa^{q-1}) \quad (a \neq 0)$.

3. $D^q(t^m\,\mathrm{e}^{at+b}) = \mathrm{e}^{at+b}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(q+1)}{\Gamma(q+1-k)}a^{q-k}t^{m-k} \quad \binom{m\in\mathbb{N}}{a\neq 0}$.
4. $D^q(\mathrm{e}^{at}\sin(\sigma t)) = \mathrm{e}^{at}r^q\sin(\sigma t + q\vartheta) \quad (\sigma > 0)$,
   where $r = \sqrt{a^2+\sigma^2}$; $\tan(\vartheta) = |a|/\sigma$.

We end the theoretical part by looking for the kernel of $\mathcal{A}$.

5.5. The Kernel of $\mathcal{A}$

The handling (Section 2) can now totally be justified, if we verify the unstable case (Section 2.4.4). This is to calculate the homogeneous $\mathcal{D}'$-solutions of the considered differential equation (29). We start from the basic proposition:

PROPOSITION 13. *Let $f \in \mathcal{Z}'$ and $f(\zeta) \neq 0$ in some neighbourhood Ω of $a \in \mathbb{C}$. Then, for $k \in \mathbb{N}$*

$$f(\zeta)(\zeta-a)^k \equiv 0 \text{ in } \Omega \Leftrightarrow f|_\Omega(\zeta) = \sum_{m=0}^{k-1} c_m\, D^m\delta|_\Omega(\zeta-a),$$

where c_m are arbitrary complex constants.

Proof. From Section 5.2.8, item (4) and Proposition 12 '$\Leftarrow$' becomes clear as well as higher orders of differentiation than $k-1$ will not do (a fractional one is ruled out by Section 5.2.8, (2)). '$\Rightarrow$' requires moreover finite point measures in a which in $\mathcal{Z}'|_\Omega$ are represented by convergent series of integer-ordered derivatives of $\delta|_\Omega$ (note that $g(\zeta)\delta(\zeta-a) = g(a)\delta(\zeta-a)$). □

Note that the only possible $\mathbf{L}_2$-solution $f(\zeta) \equiv 0$ is included.

Thus we may now easily answer the above question and represent the homogeneous solutions of Equation (29).

THEOREM 14. *Let $A(s)$ denote the symbol of $\mathcal{A}$. Let the principal branch zeros of $A(s)$ be $s_k = \sigma_k \pm i\nu_k$ with multiplicity m_k. Then the real solutions of*

$$\mathcal{A}x(t) = 0$$

are given by all $x(t)$:

$$x(t) = \sum_k \mathrm{e}^{\sigma_k t}(p_{m_k}(t)\sin(\nu_k t) + q_{m_k}(t)\cos(\nu_k t)),$$

where p_{m_k}, q_{m_k} are arbitrary real polynomials of degree $\leq m_k - 1$.

Proof. We have to solve $a(\omega)\widehat{x}(\omega) = 0$ or $A(s)\widehat{x}(-is) = 0$. Let $\mathbb{C} = \bigcup_k \Omega_k$, such that Ω_k are disjunct neighbourhoods of the s_k. Proposition 13 holds in all Ω_k. Thus $\mathbb{C}$ can be covered by all those $\widehat{x}|_{\Omega_k}$. If we interpret $D^m\delta$ as analytic continuation of $D^m\delta|_{\Omega_k}$ on $\mathbb{C}$ we arrive at

$$\widehat{x}(-is) = \sum_k \sum_{m=1}^{m_k-1} c_{k,m} D^m\delta(s-s_k).$$

Inverse Fourier transform yields

$$x_{\mathbb{C}}(t) = \sum_k r_{m_k}(t)\, \mathrm{e}^{is_k t},$$

where r_{m_k} are $\mathbb{C}$-polynomials of degree $\leq m_k - 1$. Thus the theorem is valid. □

Remark 10. Note that the trivial ($\mathbf{L}_2$-)solution is included as well as all non-oscillatory ones. Note further that the shape of the kernel is the same as in the integer case. Thus the examples in Section 5.4.1, items (1)–(4) are consistent with this theorem.

6. Conclusion

The applications (Section 4) demonstrate the applicability of the described handling (Section 2) which is later on justified in the theory (Section 5). If fractional derivatives occur only with respect to one variable (mostly the time t), the method can be applied directly to linear partial differential equations with constant coefficients. We have clarified why we suggest to solve the resulting integrals via residue calculus. The efforts to get the solutions this way is not much higher than the suggested methods in classical models. Only the integral for the relaxation K_{rel} is additional. (For stationary oscillations it can even be dropped.) Structures and properties are easily obtainable and become much clearer than from representations by special functions.

An important feature of the presented functional calculus approach is its mathematical stringency. The frequently used fractional calculus seems – particularly concerning some confusions about initial value problems together with memory effects (see [5]) – less transparent. So some authors use different definitions of fractional derivatives for different problems, even if the mathematical model is the same. Since the functional calculus approach includes the whole past, no such adaptions are necessary. Together with its simplicity (including the 'integer' case) it provides a powerful tool in hereditary physics.

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