On the existence of global solutions for T^3-Gowdy spacetimes with stringy matter

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Abstract

We show a global existence theorem for Einstein-matter equations of T^3-Gowdy symmetric spacetimes with stringy matter. The areal time coordinate is used. It is shown that this spacetime has a crushing singularity into the past. From these results we can show that the spacetime is foliated by compact hypersurfaces of constant mean curvature.

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1 Introduction

The singularity theorem states that generic spacetimes have a singularity (spacetimes with incomplete geodesics). If the singularity can be seen, i.e. there exists a *naked singularity*, the predictability is violated. Thus, the strong cosmic censorship conjecture proposed by Penrose is the most important (and an unsolved) problem in classical general relativity. Roughly speaking, the conjecture states that naked singularities should not evolve from regular initial data. However, this statement remains elusive. Then, a sharp formulation that seems provable is given as follows:

Conjecture 1 *[AL99, CD, KN, ME] Let M be a partial Cauchy surface and Φ_i are matter fields on M. Then for generic data sets (M, h, k, Φ_i), where h and k are the first and second fundamental forms of M, the maximal Cauchy development of (M, h, k, Φ_i) is equal to the maximal extension of (M, h, k, Φ_i).* □

To answer this problem we must prove (1) a global existence theorem for the Einstein-matter equations with a *suitable* time coordinate and (2) inextendibility of the maximal Cauchy development of initial data. This paper is mainly related to the above problem (1). It is very difficult to answer the conjecture in general by using present mathematical techniques. Then, it is necessary to make some simplifications, which are symmetry assumptions or restriction on initial data.

For the case of small initial data without symmetry assumptions, partial results for this problem are provided by Christodoulou-Klainerman theorem which states that any asymptotically flat initial data set which is sufficiently close to the trivial one has a complete maximal future development [CK, KN]. Recently, Andersson and Moncrief have shown that for a vacuum data set which is sufficiently close to the data in a spatially compact (local) open Friedmann-Robertson-Walker spacetime, the maximal Cauchy development is causally geodesically complete in the expanding direction [AL99] (see also Ref. [R02]).

For the case of large initial data with symmetry assumptions, some results have been obtained. (Hereafter, models we consider are restricted to spatially compact spacetimes.) The first result for inhomogeneous cosmological spacetimes was proved by Moncrief [M81]. He proved a global existence theorem for T^3-Gowdy spacetimes (which are vacuum spacetimes with $U(1) \times U(1)$ symmetry and whose spatial topology is T^3) in the areal coordinate (defined in section 2). This result has been generalized to the case with non-vanishing twist [BCIM] or with Vlasov matter [AH]. A result on global existence in the areal time has also been obtained for hyperbolic symmetric spacetimes with Vlasov matter [ARR].

Although the areal time coordinate is well-chosen one in the sense that it is geometrically defined, this coordinate choice strongly depends on spacetime symmetry. The most attractive (and independent on spacetime symmetry) time coordinate is the constant mean curvature (CMC) one. CMC foliations can avoid a crushing singularity which is one where there is a foliation on a neighborhood of the singularity whose mean curvature tends uniformly to infinity as the singularity is approaches [ES]. It was shown from Hawking's singularity theorem that the crushing singularity is either a true singularity or a boundary of maximal Cauchy development, i.e., a Cauchy horizon. Thus, the existence of CMC foliations is closely related to conjecture 1. Indeed, it has been conjectured as follows:

Conjecture 2 *[ME] Every maximally extended, globally hyperbolic spacetime can be foliated by CMC hypersurfaces.* □

A global existence theorem in the CMC time coordinate for T^3-Gowdy spacetimes was proved in Ref. [IM]. Recently, this result has been generalized to the case of local $U(1) \times U(1)$ symmetric spacetimes with Vlasov or wave-map matter [R97, H02b]. Results on global existence in the CMC time have been shown for hyperbolic symmetric spacetimes with Maxwell field or Vlasov matter [H02a, ARR].

Note that the choice of matter models is serious. For some matter models (Vlasov matter, Maxwell field and wave map), there are global existence theorems as above. Contrary, a global non-existence result have been obtained for the Einstein-dust system [IR]. Therefore, it would be worth to investigate global existence problems for several matter models, in particular, systems of *nonlinear* and *fundamental fields* equations.

The purpose of the present paper is generalization of the above results, that is, to show global existence theorems for the T^3-Gowdy symmetric spacetimes with stringy matter fields. From the unified theoretical point of view, there are many reasons to believe that the distinction between fundamental fields (i. e. gravitational and matter fields) interactions is impossible in asymptotic regions (e. g. near singularities)

of spacetimes and the most consistent theory along these line is superstring/M-theory. Therefore, the matter fields we will consider are Maxwell-dilaton-axion fields which arise naturally from low-energy effective superstring theory [ON]. This system is nonlinear even if the background spacetime is flat since there is a dilaton coupling. One of the main results of this paper is a global existence theorem for such system in the areal time coordinate (Theorem 1) and another of them is one in the CMC time coordinate (Theorem 2).

In section 2, we will review T^3-Gowdy symmetric spacetimes in the Einstein-Maxwell-dilaton-axion (EMDA) system and derive the Einstein-matter equations. In section 3, we shall show a global existence theorem for the system in the areal time coordinate. In section 4, it is shown that a global existence theorem for the system in the CMC time coordinate. In section 5, we discuss on inextendibility of the spacetime.

2 T^3-Gowdy symmetric spacetimes in the Einstein-Maxwell-dilaton-axion system

The action of the EMDA theory [STW] is

$$S = S_G + S_M, \tag{1}$$

$$S_G = \int d^4x \sqrt{-g} \left[- {}^{(4)}R \right], \tag{2}$$

$$S_M := \int d^4x \sqrt{-g} \left[e^{-2a_M\phi} F^2 + 2 \left(\nabla\phi \right)^2 + \frac{1}{3} e^{-4a_A\phi} H^2 \right] = \int d^4x \mathcal{L}_M, \tag{3}$$

where g is the determinant of a 4-dimensional spacetime metric g_{ab}, ${}^{(4)}R$ is the Ricci scalar for g_{ab}, F is the Maxwell field, ϕ is the dilaton field, $H = dB \equiv -\frac{1}{2} e^{4a\phi} * d\kappa$ is the three-index antisymmetric tensor field dual to the axion field κ, and a_M and a_A are coupling constants. For simplicity, the Chern-Simon term is neglected. Varying the action (1) with respect to the functions, we get the Einstein-matter equations.

The metric of Gowdy symmetric spacetimes [GR] is given by

$$ds^2 = e^{\lambda(t,\theta)/2} t^{-1/2} (-dt^2 + d\theta^2) + R(t,\theta) \left[e^{-Z(t,\theta)} \left(dy + X(t,\theta)dz \right)^2 + e^{Z(t,\theta)} dz^2 \right]. \tag{4}$$

Gowdy symmetric spacetimes have two twist free spacelike Killing vectors $\partial/\partial y$ and $\partial/\partial z$. Properties of the metric (4) depend on whether ∇R is timelike, spacelike or null. When the metric (4) describes a cosmological model, i.e. ∇R is globally timelike and the spatial topology is T^3 (periodic in θ), one can take the function $R(t,\theta) = t$ without loss of generality by Gowdy's corner theorem if the spacetime is vacuum [GR, M81]. This fact be seen from Einstein's equations,

$$G_{tt} - G_{\theta\theta} = \ddot{R} - R'' = 0, \tag{5}$$

where dot and prime denote t and θ derivatives, respectively. In the EMDA system, equation (5) is not satisfied generically. However, in the case that the Maxwell field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ has only the following components [MP],

$$F_{ty} = \dot{\omega}(t,\theta), \quad F_{\theta y} = \omega'(t,\theta), \quad F_{tz} = \dot{\chi}(t,\theta), \quad F_{\theta z} = \chi'(t,\theta),$$

and $\phi = \phi(t,\theta)$ and $\kappa = \kappa(t,\theta)$, equation (5) is satisfied. As similar as the T^3-vacuum case, one can take the function $R(t,\theta) = t$ without loss of generality. In this gauge choice, the spacetimes have spacelike singularities at $t = 0$. We call this gauge the *areal time coordinate* since R is proportional to the geometric area function of the orbit of the isometry group.

We should mention on another important choice for the Maxwell field. In [WIB], it was taken that ω and χ are zero but one of the other components of the Maxwell field is non-zero. This spacetime is called "magnetic Gowdy spacetime" and concerns the complementary case with ours.

In this coordinate we obtain the Einstein-matter equations as follows:

$$\dot{\lambda} - t[\dot{Z}^2 + Z'^2 + e^{-2Z}(\dot{X}^2 + X'^2)] = 4tT_{tt}, \tag{6}$$

where

$$\begin{aligned}
T_{tt} &= (\dot{\phi}^2 + \phi'^2) + \frac{1}{4}e^{4a_A\phi}(\dot{\kappa}^2 + \kappa'^2) \\
&+ \frac{1}{t}e^{-2a_M\phi}[e^{-Z}\{(X\dot{\omega} - \dot{\chi})^2 + (X\omega' - \chi')^2\} + e^Z(\dot{\omega}^2 + \omega'^2)] \\
&= T_{\theta\theta},
\end{aligned} \tag{7}$$

$$\lambda' - 2t[\dot{Z}Z' + e^{-2Z}\dot{X}X'] = 8tT_{t\theta}, \tag{8}$$

where

$$T_{t\theta} = \dot{\phi}\phi' + \frac{1}{4}e^{4a_A\phi}\dot{\kappa}\kappa' + \frac{1}{t}e^{-2a_M\phi}\{e^{-Z}(X\dot{\omega} - \dot{\chi})(X\omega' - \chi') + e^Z\dot{\omega}\omega'\}, \tag{9}$$

$$\begin{aligned}
t^2\ddot{Z} + t\dot{Z} - t^2Z'' &+ t^2e^{-2Z}(\dot{X}^2 - X'^2) \\
&- 2te^{-2a_M\phi}[-e^{-Z}\{(X\dot{\omega} - \dot{\chi})^2 - (X\omega' - \chi')^2\} + e^Z(\dot{\omega}^2 - \omega'^2)] = 0,
\end{aligned} \tag{10}$$

$$t^2\ddot{X} + t\dot{X} - t^2X'' - 2t^2(\dot{X}\dot{Z} - X'Z') - 4te^{-2a_M\phi}e^Z[(\dot{\omega}^2 - \omega'^2)X - (\dot{\omega}\dot{\chi} - \omega'\chi')] = 0, \tag{11}$$

$$\begin{aligned}
t^2\ddot{\phi} + t\dot{\phi} - t^2\phi'' &- \frac{a_A}{2}t^2e^{4a_A\phi}(\dot{\kappa}^2 - \kappa'^2) \\
&+ a_Mte^{-2a_M\phi}[e^{-Z}\{(X\dot{\omega} - \dot{\chi})^2 - (X\omega' - \chi')^2\} + e^Z(\dot{\omega}^2 - \omega'^2)] = 0,
\end{aligned} \tag{12}$$

$$t^2\ddot{\kappa} + t\dot{\kappa} - t^2\kappa'' + 4a_At^2(\dot{\phi}\dot{\kappa} - \phi'\kappa') = 0, \tag{13}$$

$$\begin{aligned}
\ddot{\chi} - \chi'' - (\dot{Z} + 2a_M\dot{\phi})\dot{\chi} + (Z' + 2a_M\phi')\chi' &+ (2X\dot{Z} - \dot{X})\dot{\omega} - (2XZ' - X')\omega' \\
&+ e^{-2Z}X\{\dot{X}(X\dot{\omega} - \dot{\chi}) + X'(X\omega' - \chi')\} = 0,
\end{aligned} \tag{14}$$

$$\ddot{\omega} - \omega'' + (\dot{Z} - 2a_M\dot{\phi})\dot{\omega} - (Z' - 2a_M\phi')\omega' + e^{-2Z}\{\dot{X}(X\dot{\omega} - \dot{\chi}) + X'(X\omega' - \chi')\} = 0, \tag{15}$$

where $T^{ab} := \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g_{ab}}$ is the energy-momentum tensor. Hereafter, we call the above system the T^3-*Gowdy symmetric EMDA* system.

Note that the metric function λ is decoupled with other functions. The function appears only in the Hamiltonian and momentum constraints (6) and (8). Then, we can calculate the metric function λ by evaluating the integral of λ' from $-\pi$ to $-\pi$ after obtaining other functions from the evolution equations (10)-(15).

3   Global existence theorem in the areal coordinate

For simplicity, we will assume that the initial data are C^∞ on T^3. In this circumstance we can show the following global existence theorem.

Theorem 1 *Let $(\mathcal{M}, g, \phi, \kappa, A)$ be the maximal globally hyperbolic development of the initial data for the Gowdy symmetric EMDA system. Then, $\mathcal{M}$ can be foliated by areal coordinate with $t \in (0, \infty)$.*

Proof of Theorem 1: The local existence and uniqueness of smooth solutions of the partial differential equation system (10)-(15) follows from standard results for hyperbolic system [AS, CBG, FR, HL]. Then, it is sufficient to verify that for any globally hyperbolic subset of the (t, θ) cylinder on which they exist as a solution to (10)-(15), the functions $(Z, X, \phi, \kappa, \chi, \omega)$ and their first and second derivatives are uniformly bounded [MA]. To do this, we will use *light cone estimate* [AH, BCIM, M81, M97].

Let us now define the quadratic forms $\mathcal{G}$ and $\mathcal{H}$ by

$$
\begin{aligned}
\mathcal{G} \;\; := \;\; & \frac{1}{2}t\left[\dot{Z}^2 + Z'^2 + e^{-2Z}\left(\dot{X}^2 + X'^2\right)\right] \\
& + 2e^{-2a_M\phi}\left[e^{-Z}\{(X\dot{\omega} - \dot{\chi})^2 + (X\omega' - \chi')^2\} + e^{Z}(\dot{\omega}^2 + \omega'^2)\right] \\
& + 2t\left(\dot{\phi}^2 + \phi'^2\right) + \frac{1}{2}e^{4a_A\phi}t\left(\dot{\kappa}^2 + \kappa'^2\right),
\end{aligned}
\tag{16}
$$

and

$$
\mathcal{H} := t\left[\dot{Z}Z' + e^{-2Z}\dot{X}X'\right] + 4e^{-2a_M\phi}\{e^{-Z}(X\dot{\omega} - \dot{\chi})(X\omega' - \chi') + e^{Z}\dot{\omega}\omega'\} + 4t\dot{\phi}\phi' + te^{4a_A\phi}\dot{\kappa}\kappa'.
\tag{17}
$$

Deriving $\mathcal{G}$ and $\mathcal{H}$ with respect to t and θ and using the evolution equations (10)-(15), after long calculation, we have the following inequalities.

$$
\begin{aligned}
\sqrt{2}\partial_\eta(\mathcal{G} + \mathcal{H}) \;\; &= \;\; \frac{1}{2}\left[-\dot{Z}^2 + Z'^2 + e^{-2Z}(-\dot{X}^2 + X'^2)\right] + 2\left[-\dot{\phi}^2 + \phi'^2 + \frac{1}{4}e^{4a_A\phi}(-\dot{\kappa}^2 + \kappa'^2)\right] \\
&:= \;\; J \leq \frac{1}{t}\mathcal{G},
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\sqrt{2}\partial_\xi(\mathcal{G} - \mathcal{H}) \;\; &= \;\; \frac{1}{2}\left[-\dot{Z}^2 + Z'^2 + e^{-2Z}(-\dot{X}^2 + X'^2)\right] + 2\left[-\dot{\phi}^2 + \phi'^2 + \frac{1}{4}e^{4a_A\phi}(-\dot{\kappa}^2 + \kappa'^2)\right] \\
&:= \;\; L \leq \frac{1}{t}\mathcal{G},
\end{aligned}
\tag{19}
$$

where $\partial_\xi := \frac{1}{\sqrt{2}}(\partial_t + \partial_\theta)$ and $\partial_\eta := \frac{1}{\sqrt{2}}(\partial_t - \partial_\theta)$.

At first, let us consider the future (expanding) direction. Integrating these equations (18) and (19) along null paths starting at $(\hat{\theta}, \hat{t})$ and ending at the initial t_0-surface (i.e. from the future into the past) and adding them we have

$$
\begin{aligned}
\mathcal{G}(\hat{\theta}, \hat{t}) \;\; = \;\; & \mathcal{G}(\hat{\theta} + t_0 - \hat{t}, t_0) + \mathcal{G}(\hat{\theta} - t_0 + \hat{t}, t_0) \\
& + \mathcal{H}(\hat{\theta} + t_0 - \hat{t}, t_0) - \mathcal{H}(\hat{\theta} - t_0 + \hat{t}, t_0) \\
& + \int_{t_0}^{\hat{t}}[J(\hat{\theta} + s - \hat{t}, s) + L(\hat{\theta} - s + \hat{t}, s)]ds.
\end{aligned}
\tag{20}
$$

Next, we take supremums over all values of θ on the both sides of the equation (20). Then, we have

$$
\begin{aligned}
\sup_\theta \mathcal{G}(\theta, \hat{t}) \;\; \leq \;\; & 2\sup_\theta \mathcal{G}(\theta, t_0) + 2\sup_\theta \mathcal{H}(\theta, t_0) \\
& + \int_{t_0}^{\hat{t}}\frac{1}{s}\sup_\theta \mathcal{G}(\theta, s)ds,
\end{aligned}
\tag{21}
$$

where we used the estimates (18) and (19). We can apply the following lemma to (21).

Lemma 1 (Gronwall's lemma [HL]) *Suppose that $\phi(t)$, $a(t)$, $b(t) \geq 0$. If*

$$
\phi(t) \leq a(t) + \int_{t_0}^{t} b(s)\phi(s)ds,
$$

then,

$$
\phi(t) \leq a(t) + \int_{t_0}^{t} a(r)b(r)\exp\int_{r}^{t} b(s)ds\,dr.
$$

$\square$

By Lemma 1, we get the following inequality

$$\sup_\theta \mathcal{G}(\theta, \hat{t}) \le 2[\sup_\theta \mathcal{G}(\theta, t_0) + \sup_\theta \mathcal{H}(\theta, t_0)] \exp(\int_{t_0}^{\hat{t}} \frac{1}{s} ds). \tag{22}$$

From equation (22), since $\exp(\int_{t_0}^{\hat{t}} \frac{1}{s} ds)$ is bounded, we get the desired bound on $| \dot{Z} |$, $| Z' |$, $| e^{-Z} \dot{X} |$, $| e^{-Z} X' |$, $| e^{-a_M \phi - \frac{1}{2} Z}(X \dot{\omega} - \dot{\chi}) |$, $| e^{-a_M \phi - \frac{1}{2} Z}(X \omega' - \chi') |$, $| e^{-a_M \phi + \frac{1}{2} Z} \dot{\omega} |$, $| e^{-a_M \phi + \frac{1}{2} Z} \omega' |$, $| \dot{\phi} |$, $| \phi' |$, $| e^{2a_A \phi} \dot{\kappa} |$ and $| e^{2a_A \phi} \kappa' |$ for all $t \in (t_0, \infty)$.

Once we have bounds on the first derivatives of Z and ϕ, it follows that Z and ϕ are bounded for all $t \in (t_0, \infty)$ as well since

$$u(t, x) = u(t_0, x) + \int_{t_0}^t \partial_s u(s, x) ds,$$

where $u(t, x)$ is a function. Then, we have bounds on $\dot{X}$, X', $X \dot{\omega} - \dot{\chi}$, $X \omega' - \chi'$, $\dot{\omega}$, ω', $\dot{\kappa}$ and κ'. Consequently, we obtain bounds on X, ω and κ and furthermore, we have bounds on $\dot{\chi}$ and χ'. Finally, we have bounds on χ. Thus, it has shown the boundedness of the zeroth and first derivatives of all the function except for λ.

Next, we must show bounds on the second derivatives of the functions. There is a well-known general fact that in order to ensure the continued existence of a solution of a system of semilinear wave equations it is enough to bound the first derivative pointwise (see Theorem 1.1 of Chapter III and page 42 of Ref. [AS]). Our system is a special case of that. Then, we have boundedness of the higher derivatives.

The same argument can be applied into the past (contracting) direction. By the constraint equations (6) and (8), boundedness for the function λ is also shown. Indeed, we have the following equations:

$$\dot{\lambda} = P \qquad \text{and} \qquad \lambda' = Q. \tag{23}$$

Since P and Q are bounded as we have already seen, $\dot{\lambda}$ and λ' are bounded. Then, we have shown that the first derivatives of λ must be bounded uniformly for all $0 < t < \infty$. Consequently, λ itself also is bounded. Concerning the second derivative of λ, we can apply the argument of Alinhac [AS] again. Then, we have uniform C^2 bounds on all of the functions for all $t \in (0, \infty)$.

Finally, we must demand that the function λ is compatible with the periodicity in θ. The following argument is similar with Moncrief's one [M81]. This is true if $\lambda(t, -\pi) = \lambda(t, \pi)$ over the interval of existence. Integrating equation (8) for the interval $\theta \in [-\pi, \pi]$, we have a constraint

$$\int_{-\pi}^{\pi} d\theta \{ t[\dot{Z} Z' + e^{-2Z} \dot{X} X'] - 4t T_{t\theta} \} = 0. \tag{24}$$

This constraint condition need only be imposed on the initial Cauchy surface since this integral is conservation on any time interval if all other functions satisfy the periodicity condition. This fact follows from the constraint equations (6) and (8):

$$\frac{\partial}{\partial t} \{ t[\dot{Z} Z' + e^{-2Z} \dot{X} X'] - 4t T_{t\theta} \} = \frac{\partial}{\partial \theta} \{ \frac{t}{2}[\dot{Z}^2 + Z'^2 + e^{-2Z}(\dot{X}^2 + X'^2)] - 2t T_{tt} \}. \tag{25}$$

Thus, we have completed the proof of Theorem 1. $\qquad\qquad\square$

4 Existence of constant mean curvature foliations

For vacuum T^3-Gowdy spacetimes it has been proved that conjecture 2 is valid [IM]. In that paper, the important idea is that one can apply the estimates for the functions in the areal time coordinate to show the existence of CMC foliations. Using the same argument with Ref. [IM], we can show a global existence theorem in the CMC time coordinate for the T^3-Gowdy symmetric EMDA system.

For the Gowdy symmetric spacetime the Einstein-matter equations imply that

$$-trK(t) = e^{-\frac{\lambda}{4}} t^{\frac{1}{4}} \left[\frac{1}{4} \dot{\lambda} + \frac{3}{4t} \right] = e^{-\frac{\lambda}{4}} t^{\frac{1}{4}} \mathcal{B}_t, \tag{26}$$

where $-trK(t)$ is mean curvature of a compact Cauchy surface S_t and

$$\mathcal{B}_t := \frac{3}{4t} + \frac{t}{4}\left[\dot{Z}^2 + Z'^2 + e^{-2Z}(\dot{X}^2 + X'^2)\right] + tT_{tt}. \tag{27}$$

From the positivity (see equation (7)) and bounds on the functions of right-hand-side of equation (27), we have the following estimate.

$$\inf_{S_t}(-trK(t)) \geq Ct^{-\frac{3}{4}}, \tag{28}$$

for some constant C and for all $0 < t < t_0$. This means that the spacetime has a crushing singularity as $t \to 0$ since the mean curvature goes to infinity uniformly [ME, M81].

Cosmological spacetimes imply that spatially compact, globally hyperbolic spacetimes satisfying the strong energy condition [BR]. Gerhardt has shown that in cosmological spacetimes if there exists a foliation whose mean curvature tends uniformly to infinity, there is a CMC foliation with same property [GC]. Then, the past region of the initial hypersurface S_{t_0}, $D^-(S_{t_0})$, is covered by CMC hypersurfaces.

Next, we must show that $D^+(S_{t_0})$ also can be covered by CMC hypersurfaces. To do this the following lemma is useful. Note that, from the previous argument, we have one initial CMC hypersurface Σ_{τ_0} in $D^-(S_{t_0})$.

Lemma 2 *Suppose that $(D^+(\Sigma_{\tau_0}), g, \phi, \kappa, \chi)$ be a maximally extended, globally hyperbolic development of the initial data Σ_{τ_0} of T^3-Gowdy symmetric spacetime in the EMDA system. Then, $(D^+(\Sigma_{\tau_0}), g, \phi, \kappa, \chi)$ admit a unique, monotonic CMC foliation i_τ which covers $(D^+(\Sigma_{\tau_0}), g, \phi, \kappa, \chi)$.*

Proof of Lemma 2: The argument is very similar with one of the proof of Lemma 1 of Ref. [IM]. It is well known that there is a unique, monotonic, local CMC foliation of $(D^+(\Sigma_{\tau_0}), g, \phi, \kappa, \chi)$ defined near the initial hypersurface Σ_{τ_0} [MT] and that the spacelike Killing fields must be tangent to CMC hypersurfaces [AL99].

First, we shall show that this foliation is uniformly spacelike. If $\Sigma_\tau = i_\tau(T^3)$ is a CMC hypersurface in the local foliation, there is a smooth function $h_\tau : S^1 \to R^+$ such that Σ_τ is defined in the coordinates (t, θ, y, z) by $t = h_\tau(\theta) \in [t_0, t_1]$ and h_τ satisfies the following equation:

$$-\frac{d}{d\theta}\left[\frac{e^{\frac{\lambda}{4}}t^{-\frac{1}{4}}h_\tau h'_\tau}{(1 - (h'^2_\tau))^{1/2}}\right] = e^{\frac{\lambda}{4}}t^{-\frac{1}{4}}h_\tau\left[(1 - (h'^2_\tau))^{1/2}(\frac{1}{4}\dot{\lambda} + \frac{3}{4t}) + trK e^{\frac{\lambda}{4}}t^{-\frac{1}{4}}\right]\Big|_{t=h_\tau(\theta)}, \tag{29}$$

where $h'_\tau := \frac{dh_\tau}{d\theta}$, $\lambda = \lambda(h_\tau(\theta), \theta)$ and $-trK$ is the (constant) mean curvature of the embedded hypersurface. If $|h'_\tau| < 1$, Σ_τ is spacelike since the induce metric γ_τ on Σ_τ is

$$\gamma_\tau = e^{\frac{\lambda}{2}}t^{-\frac{1}{2}}(1 - (h'^2_\tau))d\theta^2 + t\left[e^{-Z(t,\theta)}(dy + X(t,\theta)dz)^2 + e^{Z(t,\theta)}dz^2\right]. \tag{30}$$

Integrating equation (29) from θ_0 to θ_1, we get

$$\left|\frac{e^{\frac{\lambda}{4}}t^{-\frac{1}{4}}h_\tau h'_\tau}{(1 - (h'^2_\tau))^{1/2}}\right|\Big|_{\theta=\theta_1} \leq \int_0^{2\pi} d\theta\left[e^{\frac{\lambda}{4}}t^{-\frac{1}{4}}h_\tau\left[(1 - (h'^2_\tau))^{1/2}\mathcal{B}_t + |trK| e^{\frac{\lambda}{4}}t^{-\frac{1}{4}}\right]\right]\Big|_{t=h_\tau(\theta)}, \tag{31}$$

where θ_1 is arbitrary and we can choose that $h'_\tau(\theta_0) = 0$. From the boundedness of the right-hand-side of equation (31) and the fact that $h_\tau(\theta) \in [t_0, t_1] \subset (0, \infty)$, we have that $\frac{h'_\tau}{(1-(h'^2_\tau))^{1/2}}$ is bounded by a constant and then, the estimate $|h'_\tau| < 1$ is obtained. Thus, the CMC hypersurfaces are uniformly spacelike.

The above argument holds for any $[t_0, t_1] \subset (0, \infty)$. Then, the local foliation i_τ can be extended to $t \to \infty$. Next, we shall show that the leaves Σ_τ cannot approach the boundary at $t \to \infty$ without foliating a full neighborhood of the boundary. Since $\theta \in [0, 2\pi)$ and $|h'_\tau| < 1$, we have

$$0 \leq \sup_{i_\tau(T^3)}(t) - \inf_{i_\tau(T^3)}(t) \leq 2\pi.$$

Then, the hypersurface $i_\tau(T^3)$ cannot approach the boundary at $t \to \infty$ without foliating a full neighborhood of the boundary. (See also the proof of theorem 6.2 of [ARR].) □

Lemma 2 states that future Cauchy development of the initial Cauchy surface is cover by CMC foliations. Combining the result from Gerhardt's theorem, which states that past Cauchy development of the initial Cauchy surface is cover by CMC foliations, we have the following theorem which supports the validity of conjecture 2.

Theorem 2 *The spacetime can be covered by hypersurfaces of CMC, $-trK(t) \in (0, \infty)$.* □

5 Discussion

The remaining open question to prove the conjecture 1 is to prove the inextendibility of the spacetime. This can be divided two questions which are: (1) Does the Kretschmann scalar of the spacetime blow up tend to singularities? (2) Is the spacetime geodesic complete into the future?

For question (1), we have a very useful tool to analyze the nature of singularities, that is, the *Fuchsian algorithm* developed by Kichenassamy and Rendall [KR]. It has been shown that T^3-Gowdy spacetimes have an asymptotically velocity-terms dominated (AVTD) singularity *in general in the sense that a family of solutions depends on the maximal number of arbitrary functions.* Recently this result has been generalized to the case of the Einstein-scalar system without symmetry assumptions [AR], of the EMDA system with Gowdy symmetry [NTM] and of the D-dimensional Einstein-dilaton-p-form system without symmetry assumptions [DHRW]. These spacetimes are inextendible beyond the singularity since the Kretschmann scalar blows up there. We can conclude the inextendibility *in general* if we can answer the following question: Is there an open set of initial data on a regular Cauchy surface whose singularity is AVTD? For the last question, we have only result for vacuum Gowdy spacetimes [CIM, RH]. We may extend the technique in their paper to our case.

Concerning question (2), Rein has obtained a related result for the Einstein-Vlasov system with hyperbolic symmetry [RG]. The spacetimes are shown to be causally geodesically complete in the future (expanding) direction if the data satisfy a certain size restriction.

If all of the above were proved, we would complete a proof of the strong cosmic censorship conjecture in the T^3-Gowdy symmetric EMDA system.

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