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A Lorentzian cure for Euclidean troubles*

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There is strong evidence coming from Lorentzian dynamical triangulations that the unboundedness of the gravitational action is no obstacle to the construction of a well-defined non-perturbative path integral. In a continuum approach, a similar suppression of the conformal divergence comes about as the result of a non-trivial path-integral measure.

1. Lorentzian path integrals for gravity

The progress of the last few years has established the method of Lorentzian dynamical triangulations (LDT) as a serious candidate for a non-perturbative theory of quantum gravity in four dimensions. In this approach one tries to define quantum gravity as the continuum limit of a statistical sum over Lorentzian dynamically triangulated space-times [1,2]. The transition amplitudes G with respect to discrete proper time t are given by sums

$$G(\tau_1, \tau_2, t) = \sum_{T, \partial T = \tau_1 \cup \tau_2} \frac{1}{C_T} e^{iS(T)} \qquad (1)$$

over inequivalent Lorentzian triangulations T with three-dimensional spatial boundary triangulations τ_1 and τ_2, weighted by the gravitational Einstein action of T in Regge form, and including a discrete symmetry factor C_T.

Expression (1) looks unmanageable at first, but can be converted into a perfectly well-defined real state sum by means of a non-perturbative Wick rotation which maps each Lorentzian triangulation uniquely to a Euclidean one,

$$T^{\text{lor}} \mapsto T^{\text{eu}}, \qquad (2)$$

and the associated amplitudes to real Boltzmann weights according to

$$e^{iS^{\text{lor}}(T^{\text{lor}})} \mapsto e^{-S^{\text{eu}}(T^{\text{eu}})}. \qquad (3)$$

What is more, the Wick-rotated version of the sum (1) over infinitely many Lorentzian triangulated manifolds *converges* for sufficiently large (positive) bare cosmological constant.

It has already been shown in space-time dimensions two and three that the Lorentzian approach is inequivalent to the older method of Euclidean dynamical triangulations, whose starting point is a set of Euclidean and not of Lorentzian space-times. Last year's plenary talk at Lattice 2000 on the subject [3] contained a more detailed account of why in higher dimensions the Lorentzian approach seems to be preferable.

2. The conformal-factor problem

Any non-perturbative path-integral approach to gravity using complex weights e^{iS} *must* answer the question of how it achieves convergence, and

any path integral using Boltzmann weights e^{-S} in dimension $d \geq 3$ *must* address the question of how it deals with the unboundedness of the gravitational action.

The latter problem arises through the unusual behaviour of the conformal mode $\lambda(x)$ which determines the local scale factor of the metric $g_{\mu\nu}(x)$. Isolating its contribution to the kinetic part of the gravitational action, one finds that it appears with the *wrong* sign, rendering the action unbounded from below. This is most easily seen by applying a conformal transformation (*not* usually a gauge transformation) to the Einstein action

$$S = k \int d^d x \sqrt{g}(R + \ldots) \qquad (4)$$

in dimension $d \geq 3$. Under $g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{\lambda} g_{\mu\nu}$, one may write the resulting new action as

$$S' = \int d^d x \sqrt{g'}(-(\partial_0 \lambda)^2 + \ldots). \qquad (5)$$

The corresponding Euclidean weight factor is therefore of the form

$$e^{-S'} = e^{\int (\dot{\lambda}^2 + \ldots)}, \qquad (6)$$

which can grow without bound, thus spelling potential disaster for the Euclidean path integral. Perturbatively or in simple minisuperspace models, this problem is "fixed" by adopting some prescription of how the functional λ-integration is to be deformed into the complex λ-plane. The problem is that such prescriptions are ad-hoc, non-unique and cannot in any obvious way be translated to a non-perturbative context.

This raises the question of how the non-perturbative LDT approach – which after Wick-rotating operates with weights e^{-S} – overcomes this difficulty. First, it turns out that as a result of the discretization, the LDT action is not unbounded below, but has a minimum for fixed discrete space-time volume N_d. For example, in $d = 3$ the Euclidean action is

$$S = N_3 \left(\frac{k_0}{4} \frac{N_{2,2}}{N_3} + (k_3 - \frac{k_0}{4}) \right), \qquad (7)$$

and is therefore minimized when the ratio $\tau := N_{2,2}/N_3$, $0 \leq \tau \leq 1$, of the number of so-called 2-2 tetrahedral building blocks (c.f. [2]) to the total

number N_3 of tetrahedra is minimal. One can construct explicit triangulations of arbitrary size for which $\tau \simeq 0$. They correspond to space-times with seemingly large fluctuations of the conformal factor in proper time, so that neighbouring spatial slices approximately decouple.

What must then be investigated *dynamically* is whether in the continuum limit the *expectation value* $\langle \tau \rangle$ stays at its minimum (indicative of a dominance of the kinetic conformal term) or is bounded away from zero. Our numerical analysis in $d = 3$ [4] has shown that $\langle \tau \rangle > 0$ for *all* finite values of the inverse gravitational coupling k_0! (Note that this is not true in the Euclidean case where for sufficiently large k_0, the path integral is peaked at configurations with minimal action.) This means that although "sick" configurations with large conformal excitations are present in the path integral, they are entropically suppressed and play no role in the continuum limit of the Lorentzian approach.

It is very interesting to understand whether and how this non-perturbative result can possibly be understood from a continuum point of view. Such an explanation has been provided recently by a continuum path-integral calculation in proper-time gauge [5]. Without attempting to evaluate the full path integral (which is pretty much impossible because of its non-Gaussian nature), the authors concentrate on the integral over the conformal factor. This can actually be done after borrowing an assumption on the renormalization of the propagator from a previous successful application in two-dimensional gravity [6]. One finds that the conformal kinetic term in the action is cancelled by a corresponding contribution from the effective measure, coming from a Faddeev-Popov determinant which arises during gauge-fixing. Therefore, in agreement with arguments from a canonical treatment of gravity, the conformal mode is *not* a propagating degree of freedom. As in the discrete case, it is not just the action contribution, but also the non-trivial path integral measure that plays a decisive role in the argument.

In an attempt to study the cancellation mechanism for the conformal mode explicitly for a Lorentzian dynamically triangulated model, var-

ious reduced cosmological models in 2+1 dimensions are being investigated ([7] and work in progress). One of the simplest discrete models one can consider has *flat* tori T^2 as its spatial slices at constant integer-t. For simplicity, we can choose to obtain such tori by identifying the opposite boundaries of a strip of length l_t and width m_t of a regular triangulation of the flat two-plane so that the associated Teichmüller parameter τ_t is purely imaginary ($\tau_t = im_t/l_t$).

The Einstein action associated with a "space-time sandwich" $[t, t+1]$ is given by

$$S[t, t+1] = \alpha(l_t + l_{t+1})(m_t + m_{t+1})$$
$$-\beta(l_t - l_{t+1})(m_t - m_{t+1}), \qquad (8)$$

where the (positive) couplings α and β are functions of the bare gravitational and cosmological coupling constants. We see that even this simple model suffers potentially from a conformal factor problem, due to the presence of the difference term in (8), leading to a weight factor

$$e^{-S} \sim e^{\beta l \dot m}. \qquad (9)$$

Clearly this term is maximized by having spatial slices of minimal length and width alternating with slices of maximal l and m. In line with our previous argument we expect that a cancellation of this conformal divergence can only be achieved if we allow for sufficiently many interpolating 3d geometries in between the slices of integer-t, so that there is a chance of "entropy winning over energy". One can show that for the most restrictive way of interpolating (requiring that also the sections at *half*-integer times should be flat) this does not happen. There is not enough entropy and the model does not possess a continuum limit.

3. Summary

Any gravitational path integral that uses Euclidean weight factors must address the question of how it deals with the potential problems caused by the unboundedness of the gravitational action. Since this situation arises in Lorentzian dynamical triangulations after one has performed the Wick rotation, it is of great interest to understand the role of the conformal divergence in this context.

We have presented strong evidence from both discrete and continuum approaches that in a proper, non-perturbative formulation the conformal sickness seen in perturbative or symmetry-reduced treatments of the gravitational path integral is absent. This can happen because the kinetic conformal term in the action is compensated by non-trivial contributions coming from the path-integral measure.

This rather satisfactory result shows that there is nothing wrong in principle with path-integral formulations of quantum gravity. It is in line with expectations from canonical formulations where at least at the classical level the conformal mode is unphysical and non-propagating. It also suggests that problems in making minisuperspace path integrals well-defined may have their root in the "lack of entropy" as a result of imposing too stringent symmetry assumptions.

In the case of dynamical triangulations, starting from discrete geometries of *Lorentzian* signature seems to be crucial for achieving a cancellation of the conformal divergence. This result gives us further confidence in the method of Lorentzian dynamical triangulations as a pathway to a theory of four-dimensional quantum gravity.

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