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Exceptional non-renormalization properties and OPE analysis of chiral four-point functions in $\mathcal{N}=4$ SYM$_4$

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Abstract

We show that certain classes of apparently unprotected operators in $\mathcal{N}=4$ SYM$_4$ do not receive quantum corrections as a consequence of a partial non-renormalization theorem for the 4-point function of chiral primary operators. We develop techniques yielding the asymptotic expansion of the 4-point function of CPOs up to order $O(\lambda^2)$ and we perform a detailed OPE analysis. Our results reveal the existence of new non-renormalized operators of approximate dimension 6.

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1 Introduction

The supersymmetric $\mathcal{N}=4$ Yang-Mills theory in four dimensions (SYM$_4$) has recently attracted a lot of attention, primarily as the prototype example of the AdS/CFT correspondence [1]-[3]. Additionally, it has been gradually realized that SYM$_4$ by itself constitutes an interesting quantum field theoretic model where some unexpected properties emerge.

Perhaps the most interesting local operators in the theory are the chiral or analytic operators forming short multiplets of the superconformal group $SU(2,2|4)$ (see the classification in [4]). An important class of these [5, 6] can be written in terms of the $\mathcal{N}=4$ on-shell superfields W^i (i is an index of the irrep **6** of the R-symmetry group $SU(4) \sim SO(6)$) as $tr(W^{\{i_1}...W^{i_k\}})$. The conformal dimensions of short operators as well as their two- and three-point correlation functions are protected from perturbative corrections [7]–[11], therefore they are well-suited quantities for tests of the AdS/CFT correspondence. Other classes of operators in $\mathcal{N}=4$ SYM$_4$ include operators dual to massive string modes that decouple at strong coupling (e.g. the Konishi multiplet) [2] and operators dual to multi-particle supergravity states whose strong coupling anomalous dimensions are non-zero.

The renormalization properties of gauge invariant operators in $\mathcal{N}=4$ SYM$_4$ are to a large extent determined by superconformal invariance and unitarity [6]. A powerful test for the various predictions regarding the operator algebra is the study of 4-point functions, which encode all the relevant dynamical information through vacuum operator product expansions (OPEs). Recently the 4-point function of the chiral primary operators (CPOs), which are the lowest scalar components of the short superfield $tr(W^{\{i}W^{j\}})$ comprising the Yang-Mills stress-energy tensor multiplet, were computed in perturbation theory up to two loops (to order λ^2) [12]-[16].

On the gravity side the calculation of the 4-point function of CPOs via the AdS/CFT correspondence is highly complicated because one first has to establish the relevant part of the supergravity action for scalar fields corresponding to these CPOs. For the massless dilaton and axion fields the action is already known [17] and with the development of the powerful integration technique over the AdS space [18] the complete results for the 4-point functions became available [19]. However, the dilaton and axion fields correspond to descendants of CPOs, which rather complicates the corresponding CFT analysis [20, 21]. With the evaluation of the quartic supergravity couplings for scalar fields corresponding to CPOs [22] the computation of the 4-point functions of the lowest-weight CPOs in the supergravity approximation has been recently completed in [23].

The CPOs do not form a ring structure with respect to the OPE, i.e. in general their OPE contains fields acquiring non-zero anomalous dimensions. A partial OPE analysis of the 4-point functions of CPOs was performed in [15, 16, 24, 25] and the anomalous dimensions of certain operators were found both at weak and strong coupling. The results of these papers show agreement with the general considerations of [6] based on superconformal invariance and unitarity. However, the careful analysis of [24, 25] led to a surprise: the OPE of two lowest-weight CPOs contains operators whose anomalous dimensions vanish both at weak and at strong coupling, although they are apparently not protected by unitarity. Such an unexpected result indicates the existence of new non-renormalization theorems in $\mathcal{N}=4$ SYM that may be a consequence of the dynamics of the gauge theory rather than its kinematics.

The superconformal properties of the $\mathcal{N}=4$ SYM are accounted for very clearly by formulating the theory in $\mathcal{N}=2$ harmonic superspace [26]. In this formulation the analogues of the $\mathcal{N}=1$ chiral matter superfields obey the constraint of G-analyticity while their equations of motion take the form of H-analyticity. In a recent paper [27] it was shown that superconformal covariance and the requirements of G- and H-analyticity combined with Intriligator's insertion formula [28], constrain the 4-point correlation functions of the lowest-weight CPOs (a priory given by *two* arbitrary functions of conformal variables) to depend on a *single* function F, which in addition obeys constraints from crossing symmetry. This function comprises all possible quantum corrections (perturbative and instanton) to the free-field result.

In the present paper we show that non-renormalization of some operators, that are not in general protected by unitarity restrictions [24, 25], follows from the partial non-renormalization theorem of [27]. A typical example which we study in some detail is a scalar operator $O^{\mathbf{20}}$ of conformal dimension 4 transforming in the irrep **20** of $SO(6)$. In free-field theory it can be represented by a "double trace" operator $:\mathrm{tr}(\phi^i\phi^j)\mathrm{tr}(\phi^k\phi^l):$, where ϕ^i are the six scalars of N=4 Yang-Mills and the $SO(6)$-indices are projected onto the **20**. In free-field theory this operator saturates the unitarity bound of the A') series of UIRs in the classification of [6] but in an interacting theory it can in principle acquire an anomalous dimension.

To find implications of the partial non-renormalization for the OPE of short operators we use Conformal Partial Wave Amplitude (CPWA) analysis, a subject well developed in the past [29]-[33] and recently revitalized in the context of the AdS/CFT duality [34, 20, 35, 24, 25, 36, 37]. In a conformally invariant theory the OPE of two scalar fields is decomposed in terms of conformal blocks of traceless symmetric tensors. Each of them realizes an irreducible representation of the conformal group. The CPWA can be viewed as the contribution of the conformal block of a tensor field to the conformally covariant 4-point function. For the 4-point functions considered here, due to the universality of the quantum correction function F, the projections onto different irreps of the R-symmetry group are related to each other. Matching these relations against the CPWA expansion of the various projections we are able to demonstrate the absence of quantum corrections to some operators in the **20** and the **105**.

Apart from the non-renormalized operators just discussed, there exist other operators that do not receive quantum corrections [24]. However, their non-renormalization properties are encoded in the explicit form of the function F whose non-perturbative expression is currently unavailable. The function F contains information about both protected and unprotected operators, but the latter are mixed in perturbation theory. To solve for the operator mixing one has two possibilities. Firstly, one may compute the weak coupling 4-point functions of other fields appearing in the OPE of CPOs and then find and diagonalize the corresponding mixing matrix. Secondly, one may exploit the partial knowledge of F in different regimes. Here we employ the second possibility to trace new non-renormalized operators.

Since the two-loop ($O(\lambda^2)$) 4-point function is known [14, 16], we can use it to extract the corresponding OPE expansion. In view of comparison with a sum of CPWAs of different tensors it is therefore desirable to represent this function as a certain series expansion valid in the asymptotic region of conformal variables where we study the OPE. We solve this problem by using an analytic regularization that allows one to reduce the two-loop function to the function related with a one-loop diagram. Our approach is different from the one in [16]. Using the CPWA analysis of the one-loop, two-loop and strong coupling 4-point functions of CPOs we then demonstrate the vanishing of the anomalous dimension for the scalar of naive dimension 6 transforming in the **20** of $SO(6)$.

The paper is organized as follows. In Section 2 we start by recalling the structure of the 4-point functions of the lowest-weight CPOs and describe the partial non-renormalization theorem of [27]. Employing CPWA analysis we show that the absence of quantum corrections to $O^{\mathbf{20}}$ and to the rank-$2k$ tensors of dimension $4+2k$ is a direct consequence of the partial non-renormalization of the 4-point functions of CPOs. In Section 3 we derive a series representation (with logs) for the two-loop 4-point function of CPOs suitable for the study of the OPE. Some results of the CPWA analysis relevant for further non-renormalization issues are presented in Section 4. The technical details are relegated to two Appendices.

2 Partial non-renormalization of the 4-point function of CPOs

In the notation of [24, 25], the 4-point function of the lowest dimension canonically normalized CPOs is

$$
\begin{aligned}
\langle O^{I_1}(x_1)O^{I_2}(x_2)O^{I_3}(x_3)O^{I_4}(x_4)\rangle \;&=\; a_1(x)\,\delta^{I_1I_2}\delta^{I_3I_4} + a_2(x)\,\delta^{I_1I_3}\delta^{I_2I_4} \\
+a_3(x)\,\delta^{I_1I_4}\delta^{I_2I_3} + b_2(x)\,C^{I_1I_2I_3I_4} &+ b_1(x)\,C^{I_1I_3I_2I_4} + b_3(x)\,C^{I_1I_3I_4I_2}\,, \qquad (2.1)
\end{aligned}
$$

where $I_1, \ldots, I_4 = 1, 2, .., 20$ are indices of the irrep **20** of $SO(6)$ and the various C-tensors in (2.1) were defined in [24]. Here a_i and b_i are given by simple propagator factors times functions of the two biharmonic ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\,. \qquad (2.2)$$

In the sequel we will also use the variable $Y = 1 - \frac{v}{u}$.

The Bose symmetry (equivalently crossing symmetry) of the 4-point function (2.1) implies that only one of the a_i's and one of the b_i's are independent. A further restriction on the structure of the 4-point function is imposed by the partial non-renormalization theorem of [27] which states that all six coefficient functions in (2.1) are expressed in terms of *one and only one* arbitrary function of two variables $F(v, Y)$:

$$
\begin{aligned}
a_1 &= \frac{1}{x_{12}^4 x_{34}^4}\left[1 + uF(v,Y)\right] & (2.3)\\
a_2 &= \frac{1}{x_{12}^4 x_{34}^4}\left[u^2 + u^2\,F(v,Y)\right] & (2.4)\\
a_3 &= \frac{1}{x_{12}^4 x_{34}^4}\left[v^2 + vu\,F(v,Y)\right] & (2.5)\\
b_1 &= \frac{1}{x_{12}^4 x_{34}^4}\left[\frac{4}{N^2}\,vu + (vu^2 - u^2 - vu)F(v,Y)\right] & (2.6)\\
b_2 &= \frac{1}{x_{12}^4 x_{34}^4}\left[\frac{4}{N^2}\,v + (v - vu - u)F(v,Y)\right] & (2.7)\\
b_3 &= \frac{1}{x_{12}^4 x_{34}^4}\left[\frac{4}{N^2}\,u + (\frac{u^2}{v} - u^2 - u)F(v,Y)\right]\,. & (2.8)
\end{aligned}
$$

Here the F-independent terms correspond to the disconnected (a_i) and connected (b_i) parts in the free amplitude (Born approximation). The function $F(v,Y) \equiv \mathcal{F}(u, u/v)$ encodes all *quantum corrections* and obeys the crossing symmetry relations [27]:

$$\mathcal{F}(u, u/v) = \mathcal{F}(u/v, u) = \frac{v}{u}\mathcal{F}(v, v/u)\,. \qquad (2.9)$$

Our prime interest will be to understand the implications of this partial non-renormalization theorem for the OPE of chiral operators. To this end we will discuss the OPE for $x_{12}^2\,, x_{34}^2 \to 0$, or equivalently $u\,, v\,, Y \to 0$.

The product of two CPOs $O^{I_1}(x_1)O^{I_2}(x_2)$ decomposes under the R-symmetry group $SU(4)$ as

$$\mathbf{20} \times \mathbf{20} = \mathbf{1} + \mathbf{20} + \mathbf{105} + \mathbf{84} + \mathbf{15} + \mathbf{175}\,. \tag{2.10}$$

To label the different operators appearing in the operator product expansion we use the notation $O^{\mathbf{irrep}}_{\Delta,l}$, where Δ describes the free-field conformal dimension of the operator, l is its Lorentz spin and **irrep** denotes the corresponding representation of $SU(4)$.

By analyzing the 4-point function of chiral operators at strong coupling [23] it was found in [24] that there exist an operator $O^{\mathbf{20}}_{4,0}$ and a tower of rank-$2k$ tensors $O^{\mathbf{105}}_{4+2k,2k}$ which do not acquire anomalous dimension. In [25] the same phenomenon was observed at the one-loop ($O(\lambda)$) level. These operators do not belong to short superconformal representations and thus the standard protection mechanism [6] does not apply to them. The absence of quantum corrections should be interpreted as a dynamical rather then a kinematical effect.

In this section we demonstrate these new non-renormalization properties without making use of perturbative arguments. They are, in fact, a simple consequence of the general non-perturbative form (2.3)-(2.8) of the amplitude. The method we use to extract information about the content of the operator algebra is Conformal Partial Wave Amplitude (CPWA) analysis [29]-[33].

The correlator (2.1) (or any of its projections on the irreps (2.10)) can be written as an expansion of the type

$$\langle O(1)O(2)O(3)O(4)\rangle = \sum_{\Delta,l} a_{\Delta,l}\mathcal{H}_{\Delta,l}(x_{1,2,3,4})\,. \tag{2.11}$$

Here $\mathcal{H}_{\Delta,l}(x_{1,2,3,4})$ denotes the CPWA for the exchange of a symmetric traceless tensor of rank l and of (possibly anomalous) dimension Δ. The coefficients $a_{\Delta,l}$ are to be found by matching the explicit form of the left-hand side of eq. (2.11) to that of the CPWAs. The latter were obtained in [35] and are given in Appendix A, equation (4.1), in terms of the variables v, Y suitable for the study of the OPE in the direct channel. Here we only list some basic facts about these CPWAs needed for our argument.

Let us split the dimension of the exchanged operator $\Delta = \Delta_0 + h$, where Δ_0 is an integer and $-1 \le h < 1$. Then the CPWA is a double series of the type

$$\mathcal{H}_{\Delta,l} = \frac{1}{x_{12}^4\, x_{34}^4}\, v^{\frac{h}{2}} \sum_{n,m=0}^{\infty} c^{\Delta,l}_{nm} v^n Y^m\,. \tag{2.12}$$

Note that the factor $v^{\frac{h}{2}}$ is a fractional power of v, which will allow us to treat CPWAs with different h as functionally independent. As we show in Appendix A all monomials in this expansion obey

$$T \equiv 2n + m \geq \Delta_0 \,. \tag{2.13}$$

The "ordering parameter" T proves very helpful when comparing power expansions of the type (2.12).

The terms in the series (2.12) with $T_{\rm min} = \Delta_0$ are of the form (see Appendix A, (4.1)) $v^{\frac{1}{2}(\Delta_0 - l)} Y^l$, $v^{\frac{1}{2}(\Delta_0 - (l-2))} Y^{(l-2)}, \ldots$ down to $v^{\frac{1}{2}(\Delta_0 - 1)} Y$ or $v^{\frac{1}{2}\Delta_0}$ depending on whether l is even or odd. This means that we can choose Δ_0 even(odd) for even(odd) spins. Further, from the unitarity bound $\Delta \geq 1$ (if $l = 0$) we deduce that the entire range of scalar dimensions can be covered choosing $\Delta_0 = 2, 4, 6, \ldots$ and $-1 \leq h < 1$. If $l > 0$ the unitarity bound becomes $\Delta \geq 2 + l$, so we start at $\Delta_0 = 2 + l$ (restricting $0 \leq h < 1$) and further $\Delta_0 = 4 + l, 6 + l, \ldots$ with $-1 \leq h < 1$.

The main question we are addressing here concerns the exchange operator $O^{\mathbf{20}}_{4,0}$ for which $\Delta_0 = 4$. According to eq. (2.13), the corresponding CPWA has $T_{\rm min} = 4$. In order to find out whether such a CPWA can appear in a given projection of the amplitude, we have to consider all the CPWAs with $T_{\rm min} \leq 4$. Within a class of equal fractional power $v^{(h/2)}$ these CPWAs are[6]

- $\Delta_0 = 2$ scalar: $v + \ldots$
- $\Delta_0 = 3$ vector: $vY + \ldots$
- $\Delta_0 = 4$ scalar: $v^2 + \ldots$ rank 2 tensor: $v^2 - vY^2 + \ldots$

where only the terms with $T_{\rm min}$ are shown.

Let us now try to match a sum of such CPWAs with the 4-point amplitude in the form (2.1), (2.3)-(2.8) predicted by the partial non-renormalization theorem of [27]. Since we are only interested in anomalous dimensions which come from the quantum (F) terms in (2.3)-(2.8), we can drop the Born terms. Accordingly, when expanding the quantum terms we can neglect CPWAs with integer dimension. We begin by projecting the amplitude (2.1) onto the various $SU(4)$ irreps (2.10):

- Projection on the $\mathbf{1}$:

$$-\frac{1}{x_{12}^4\, x_{34}^4}\,(20 - 20Y - \frac{16}{3}v + \frac{10}{3}Y^2 + \frac{8}{3}vY + \frac{1}{3}v^2)\,\Phi(v,Y) \tag{2.14}$$

[6] We do not specify the normalization factors of the CPWAs, it is just assumed that they are non-singular for the range of dimensions and spins under consideration.

- Projection on the **15**:

$$-\frac{1}{x_{12}^4\,x_{34}^4}\,(-4Y+2Y^2+vY)\,\Phi(v,Y) \tag{2.15}$$

- Projection on the **20**:

$$\frac{1}{x_{12}^4\,x_{34}^4}\,(-\frac{5}{3}v+\frac{5}{3}Y^2+\frac{5}{6}vY+\frac{1}{6}v^2)\,\Phi(v,Y) \tag{2.16}$$

- Projection on the **84**:

$$-\frac{1}{x_{12}^4\,x_{34}^4}\,(-3v+\frac{3}{2}vY+\frac{1}{2}v^2)\,\Phi(v,Y) \tag{2.17}$$

- Projection on the **105**:

$$\frac{1}{x_{12}^4\,x_{34}^4}\,v^2\,\Phi(v,Y) \tag{2.18}$$

- Projection on the **175**:

$$\frac{1}{x_{12}^4\,x_{34}^4}\,vY\,\Phi(v,Y) \tag{2.19}$$

where we have set

$$\Phi(v,Y)=\frac{v\,F(v,Y)}{(1-Y)^2}\ .$$

Note that the polynomial prefactors have been T-ordered.

Consider the projections on the singlet and on the **20**. Both of them are supposed to have CPWA expansions of the type (2.11):

$$-\frac{1}{x_{12}^4\,x_{34}^4}\,(20-20Y-\frac{16}{3}v+\frac{10}{3}Y^2+\frac{8}{3}vY+\frac{1}{3}v^2)\,\Phi(v,Y)=\sum_{\Delta,l}a^{\mathbf{1}}_{\Delta,l}\mathcal{H}_{\Delta,l} \tag{2.20}$$

and

$$\frac{1}{x_{12}^4\,x_{34}^4}\,(-\frac{5}{3}v+\frac{5}{3}Y^2+\frac{5}{6}vY+\frac{1}{6}v^2)\,\Phi(v,Y)=\sum_{\Delta,l}a^{\mathbf{20}}_{\Delta,l}\mathcal{H}_{\Delta,l} \tag{2.21}$$

Since the function $\Phi(v,Y)$ is the same in both of these equations, we can eliminate it and obtain the consistency condition

$$(\frac{5}{3}v-\frac{5}{3}Y^2-\frac{5}{6}vY-\frac{1}{6}v^2)\sum_{\Delta,l}a^{\mathbf{1}}_{\Delta,l}\mathcal{H}_{\Delta,l}=(20-20Y-\frac{16}{3}v+\frac{10}{3}Y^2+\frac{8}{3}vY+\frac{1}{3}v^2)\sum_{\Delta,l}a^{\mathbf{20}}_{\Delta,l}\mathcal{H}_{\Delta,l} \tag{2.22}$$

Recall the form (2.12) of the CPWA, which contains a term $v^{h/2}$. Different fractional powers of v are functionally independent, hence the last equation splits into classes of different h. It is enough to investigate the problem for a given h.

We want to know whether the CPWA $\mathcal{H}_{4+h,0}$, corresponding to an anomalous dimension ($h \neq 0$) for the operator $O^{\mathbf{20}}_{4,0}$ can appear in the right-hand side of (2.22). Let us first assume that $h > 0$. This CPWA has $T_{\min} = 4$, therefore we can keep only terms with $T \leq 4$ on both sides of (2.22). In the left-hand side we have a polynomial with $T \geq 2$, so we need only keep the lowest CPWA $\mathcal{H}_{2+h,0} \sim v + \dots$ ($T \geq 2$). In the right-hand side the polynomial has $T \geq 0$, so we should include several CPWAs:

$$(\frac{5}{3}v - \frac{5}{3}Y^2 + \dots)\,[a^{\mathbf{1}}_{2+h,0}(v + \dots) + \dots] = \tag{2.23}$$

$$(20 + \dots)\left[a^{\mathbf{20}}_{2+h,0}(v + \dots) + a^{\mathbf{20}}_{3+h,1}(vY + \dots) + a^{\mathbf{20}}_{4+h,0}(v^2 + \dots) + a^{\mathbf{20}}_{4+h,2}(v^2 - vY^2 + \dots) + \dots\right] .$$

Clearly, the left-hand side has $T \geq 4$, so the first two terms in the right-hand side with $T < 4$ have no match. The crucial point now is that the polynomial $v^2 - vY^2$ in the left-hand side exactly matches the tensor term in the right-hand side. Therefore we conclude that

$$a^{\mathbf{20}}_{2+h,0} = a^{\mathbf{20}}_{3+h,1} = a^{\mathbf{20}}_{4+h,0} = 0\,. \tag{2.24}$$

It remains to consider the case when $h < 0$. In this case the unitarity bound prevents the CPWAs $\mathcal{H}_{3+h,1}$ and $\mathcal{H}_{4+h,2}$ from occurring in the right-hand side of eq. (2.23). Thus, up to order $T = 4$ there is no possible match and this case has to be ruled out.

The vanishing of a coefficient for a CPWA means that there is no operator with anomalous part of the dimension h, for any given value of h. In other words, the scalars of dimension 2,4 and the vector at dimension 3 remain non-renormalized. The operator $O^{\mathbf{20}}_{2,0}$ is itself a CPO belonging to a short multiplet of $SU(4)$ and its non-renormalization is well-known. The absence of a vector in this channel can be explained by parity. As to $O^{\mathbf{20}}_{4,0}$, we now see that its non-renormalization is a consequence of the particular structure of the 4-point function dictated by the superconformal invariance and the dynamics of $\mathcal{N} = 4$ SYM$_4$.

The tensor of approximate dimension 4 can be interpreted as the operator $\mathcal{K}_{4,2}$ from the Konishi multiplet.

Let us briefly comment on the irrep **105**. Here the polynomial factor is v^3. We have pointed out above, that the lowest order of the CPWA of an operator of free-field dimension Δ_0 and spin l contains a term $v^{(\Delta_0 - l)/2}Y^l$. It follows that any operator in the **105** receiving quantum corrections has $\Delta_0 - l \geq 6$, yielding a tower of non-renormalized operators $O_{4+2k,2k}$.

Obviously, there should exist other operators which do not receive quantum corrections; for example descendants of the operators discussed above. However, at present we do not see an easy way of unraveling their non-renormalization properties on the general grounds of the representation (2.3)-(2.8). In the next section we show that some other non-renormalized operators exist and they can be traced by using the knowledge of the function $F(v,Y)$ in different regimes.

3 Series representation of the conformal 4-point functions

In perturbation theory the function $F(v,Y)$ assumes the form of a series as

$$F(v,Y) = \frac{1}{N^2}(\tilde{\lambda}F^{(1)}(v,Y) + \tilde{\lambda}^2 F^{(2)}(v,Y) + \dots) + \mathcal{O}\left(\frac{1}{N^4}\right), \tag{3.1}$$

where $\tilde{\lambda} = \frac{g_{YM}^2 N}{(2\pi)^2}$ is the t'Hooft coupling. In the following we study only the leading terms in $1/N^2$.

The first two terms in the expansion (3.1) were computed in [12]-[14] by using the $\mathcal{N}=2$ harmonic superspace technique and in [15], [16] by means of the $\mathcal{N}=1$ superspace formalism. They are given by

$$F^{(1)}(v,Y) = -2\frac{v}{u}\Phi^{(1)}\left(v,\frac{v}{u}\right). \tag{3.2}$$

and

$$\begin{aligned} F^{(2)}(v,Y) &= \frac{1}{u}\Phi^{(2)}\left(\frac{1}{v},\frac{1}{u}\right) + \Phi^{(2)}\left(\frac{u}{v},u\right) + \frac{v}{u}\Phi^{(2)}\left(\frac{v}{u},v\right) \\ &+ \frac{v}{4u^2}(u+v+uv)\left(\Phi^{(1)}\left(v,\frac{v}{u}\right)\right)^2. \end{aligned} \tag{3.3}$$

Here the functions $\Phi^{(1,2)}$ admit representations in terms of the one- and two-loop box integrals respectively, and they are the first two elements of an infinite series of conformally covariant "multi-ladder" functions introduced in [38, 39].

The symmetry properties of the function $\Phi^{(1)}$ are

$$\Phi^{(1)}\left(u,\frac{u}{v}\right) = \frac{v}{u}\Phi^{(1)}\left(v,\frac{v}{u}\right), \qquad \Phi^{(1)}\left(v,\frac{v}{u}\right) = \frac{1}{v}\Phi^{(1)}\left(\frac{1}{v},\frac{1}{u}\right). \tag{3.4}$$

One can easily see that by virtue of (3.4) the functions (3.2) and (3.3) obey the symmetry relations (2.9).

Since the two-loop correlation function admits a representation in terms of the two integrals $\Phi^{(1)}$ and $\Phi^{(2)}$, each of them being covariant under conformal mappings, it is tempting to suggest that higher loop correlation functions can be as well represented as certain polynomials of all possible multi-ladder integrals that can be composed from field propagators at this level. To study then the OPE we require the behavior of the correlation functions in the asymptotic region, where, say, $x_{12}^2 \sim 0$ and $x_{34}^2 \sim 0$. Furthermore, to develop an efficient technique for constructing the field algebra at higher loops we face the difficult problem of finding an asymptotic expansion of these integrals in terms of conformally invariant variables valid in the relevant asymptotic region. Here we demonstrate that this problem may be overcome by using the method of analytic regularization [38] of the L-loop ladder diagram $\Phi^{(L)}$ that allows one to find the latter in terms of the sum of diagrams related to $\Phi^{(L-1)}$. Applying this procedure recurrently one will be subsequently left with a function related to a one-loop diagram.

For the sake of clarity we consider here only the case of the function $\Phi^{(2)}$ for which we obtain a series representation (with logs) in terms of conformal variables v and Y. In the next section this representation will be used to verify some predictions about the structure of the field algebra of chiral operators at two loops. Below we often use notation $y = 1 - Y = v/u$.

Following [38] we introduce a function

$$\Phi(v, y|\delta) = \int \frac{d\lambda ds}{(2\pi i)^2} \Gamma(-\lambda)\Gamma(-s)\Gamma(-\lambda-\delta)\Gamma(-s-\delta)\Gamma^2(1+\lambda+s+\delta) v^\lambda y^s \,. \tag{3.5}$$

The integration contours run sufficiently close to the imaginary axis to separate the ascendant and descendent sets of poles. The s-integral is convergent for $|y| < 1$ and $|\arg y| < \pi$. Using this integral representation one notices that the function $\Phi(v, y|\delta)$ is a particular example of a general family of $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$-functions (see A1)[7] which describe contribution of the scalar AdS graphs to the 4-point function [24] of chiral operators computed in AdS$_5$ supergravity. Precisely one has the following relation:

$$\Phi\left(v, \frac{v}{u}|\delta\right) = D_{1-\delta,1,1,1+\delta}(v, Y). \tag{3.6}$$

Representation of this type is rather useful since it allows one to establish a relation between Φ considered as a function of the conformal variables in the crossed channels and D-functions as functions of v, Y which parametrize the direct channel. In the Appendix B we show that the following formulae are valid:

$$\Phi\left(u, \frac{u}{v}|\delta\right) = \left(\frac{v}{u}\right)^{1+\delta} D_{1-\delta,1,1+\delta,1}(v, Y) \,, \qquad \Phi\left(\frac{1}{v}, \frac{1}{u}|\delta\right) = v^{1+\delta} D_{1-\delta,1+\delta,1,1}(v, Y) \,. \tag{3.7}$$

It is worth pointing out that the sum of parameters Δ_i of the D-functions we meet here is equal to four, which is the dimension of a space-time. This merely reflects the fact that in our situation D-functions coincide with the well-known star-integrals (the "box" diagram in momentum space). Evaluating the

[7] The D-functions we use here coincide with the $\bar{D}$-functions (without normalization factor) introduced in [24].

integral (3.5) one gets the following formula in terms double series in v, Y variables:

$$\begin{aligned}\Phi(v,Y|\delta) &= \sum_{m,n=0}^{\infty} \frac{Y^m}{m!}\frac{v^n}{(n!)^2}\left[\frac{\Gamma(1+\delta)\Gamma(-\delta)}{\Gamma(1+n+\delta)}\frac{\Gamma^2(1+n)\Gamma^2(1+n+m+\delta)}{\Gamma(2+2n+m+\delta)}\right.\\ &+ \left. v^{-\delta}\frac{\Gamma(1-\delta)\Gamma(\delta)}{\Gamma(1+n-\delta)}\frac{\Gamma^2(1+n-\delta)\Gamma^2(1+n+m)}{\Gamma(2+2n+m-\delta)}\right], \qquad (3.8)\end{aligned}$$

which converges in a neighborhood of $v = 0$, $Y = 0$. In the limiting case $\delta = 0$ one recovers from (3.5) the Mellin-Barnes integral for $\Phi^{(1)}(v, Y) = \Phi(v, Y|0)$. Taking the limit $\delta \to 0$ in (3.8) produces the following asymptotic expansion for $\Phi^{(1)}(v, Y)$:

$$\begin{aligned}\Phi^{(1)}(v,Y) &= \sum_{n,m=0}^{\infty} \frac{v^n Y^m}{(n!)^2 m!}\frac{\Gamma^2(1+n)\Gamma^2(1+n+m)}{\Gamma(2+2n+m)}\\ &\times \left[-\ln v + 2\psi(2+2n+m) - 2\psi(1+n+m)\right]. \qquad (3.9)\end{aligned}$$

This representation was extensively used in [25] to study the OPE of chiral operators at one loop.

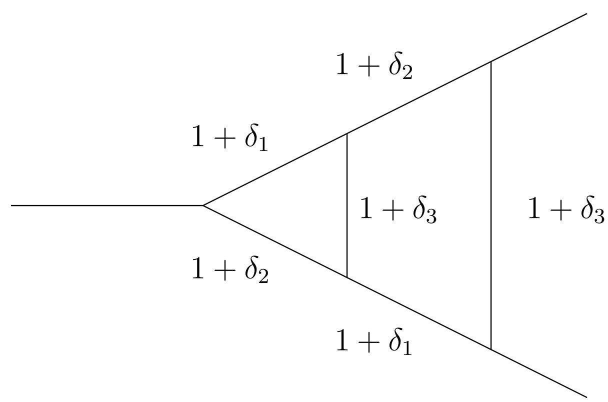

Figure 1: Regularized ladder diagram related to the function $\Phi^{(2)}(v, y)$.

The idea of [38] to compute the integral $\Phi^{(2)}(v, y)$ is to introduce a special analytic regularization of the corresponding two-loop ladder diagram and use the "uniqueness" method to reduce it to the function $\Phi(v, y|\delta)$ naturally related to a one-loop ladder diagram. The analytic regularization in question consists in replacing the powers in denominators by $1 + \delta_i$ obeying a condition $\delta_1 + \delta_2 + \delta_3 = 0$. After the computation a limit $\delta_i = 0$ is applied. In this way one finds the following formula [8]

$$\Phi^{(2)}(v,y) = \frac{1}{2}\left[3\partial_\delta^2\Phi(v,y|\delta) - (\ln^2 v + \ln v \ln y + \ln^2 y + \pi^2)\Phi^{(1)}(v,y)\right], \qquad (3.10)$$

[8]The appearance of an additional π^2 term in comparison with [38] is related to the particular series representation for $\Phi(x, y|\delta)$ that we use.

where the derivative is evaluated at $\delta = 0$.

With the help of formulae (3.7) we can also find representations for function $\Phi^{(2)}$ depending this time on the variables describing the crossed channels:

$$\Phi^{(2)}\left(u, \frac{u}{v}\right) = \frac{y}{2}\left[3\partial_\delta^2 (y^\delta D_{1-\delta,1,1+\delta,1}) - (\ln^2 v - 3\ln v \ln y + 3\ln^2 y + \pi^2)\Phi^{(1)}(v,y)\right] \tag{3.11}$$

and

$$\Phi^{(2)}\left(\frac{1}{v}, \frac{1}{u}\right) = \frac{v}{2}\left[3\partial_\delta^2 (v^\delta D_{1-\delta,1+\delta,1,1}) - (3\ln^2 v - 3\ln v \ln y + \ln^2 y + \pi^2)\Phi^{(1)}(v,y)\right]. \tag{3.12}$$

One may further simplify the latter expressions by using the fact that the first derivative of the function $\Phi(v,y|\delta)$ computed at $\delta = 0$ is not independent but can be rather expressed via $\Phi^{(1)}(v,y)$. To compute the derivatives it is convenient to use the Mellin-Barnes representation for $\Phi(v,y|\delta)$. Indeed, from (3.5), one can see that under the following shift of integration variables $\lambda \to \lambda - \delta/2$ and $s \to s - \delta/2$ the function $\Phi(v,y|\delta)$ acquires a form

$$\begin{aligned}\Phi(v,y|\delta) &= (vy)^{-\delta/2} \int \frac{d\lambda ds}{(2\pi i)^2}\Gamma(-\lambda+\delta/2)\Gamma(-s+\delta/2)\Gamma(-\lambda-\delta/2)\Gamma(-s-\delta/2) \\ &\times \Gamma^2(1+\lambda+s)v^\lambda y^s\,.\end{aligned} \tag{3.13}$$

Clearly, viewed as a series in the δ-variable, the integrand does not have a linear term. This fact allows one to derive an identity [38]:

$$\partial_\delta \Phi^{(1)}(v,y) = -\frac{1}{2}\ln vy\ \Phi^{(1)}(v,y)\,. \tag{3.14}$$

Similarly, by using the corresponding Mellin-Barnes representations for the remaining D-functions in (3.7) (see Appendix B) we obtain the formulae

$$\partial_\delta D_{1-\delta,1,1+\delta,1} = -\frac{1}{2}\ln v\,\Phi^{(1)}\,, \qquad \partial_\delta D_{1-\delta,1+\delta,1,1} = -\frac{1}{2}\ln y\,\Phi^{(1)}, \tag{3.15}$$

where the derivatives are taken at $\delta = 0$ and we omit the arguments.

Now performing the differentiation in (3.11), (3.12) and using expressions (3.15) we arrive at

$$\Phi^{(2)}\left(u, \frac{u}{v}\right) = \frac{y}{2}\left[3\partial_\delta^2 D_{1-\delta,1,1+\delta,1} - (\ln^2 v + \pi^2)\ \Phi^{(1)}(v,y)\right], \tag{3.16}$$

$$\Phi^{(2)}\left(\frac{1}{v}, \frac{1}{u}\right) = \frac{v}{2}\left[3\partial_\delta^2 D_{1-\delta,1+\delta,1,1} - (\ln^2 y + \pi^2)\ \Phi^{(1)}(v,y)\right]. \tag{3.17}$$

Since our main interest is the function $F^{(2)}(v,Y)$ describing the 4-point function of chiral operators at two loops, we combine the formulae above to get a quantity

$$
\begin{aligned}
\mathcal{S} &= \frac{1}{u}\Phi^{(2)}\left(\frac{1}{v},\frac{1}{u}\right)+\Phi^{(2)}\left(\frac{u}{v},u\right)+\frac{v}{u}\Phi^{(2)}\left(\frac{v}{u},v\right) \\
&= \frac{y}{2}\Bigg[3\partial_\delta^2(D_{1-\delta,1,1,1+\delta}+D_{1-\delta,1,1+\delta,1}+D_{1-\delta,1+\delta,1,1}) \qquad (3.18)\\
&\qquad -(2\ln^2 v+\ln v\ln y+2\ln^2 y+3\pi^2)\Phi^{(1)}(v,y)\Bigg].
\end{aligned}
$$

The remaining step consists in evaluating the Mellin-Barnes integrals for other D-functions involved in (3.18) with subsequent differentiation of the resulting series. In this way we arrive at a formula suitable for the study of the OPE of chiral operators in the direct channel.

It is worth emphasizing that $\ln^3 v$ terms cancel in the final expression for $\mathcal{S}$. This should not come as a surprise, otherwise one could see the presence of $\ln^3 v$-terms in the 4-point function. Such terms would contradict the general OPE expansion in this order of perturbation theory. Below we present explicit expressions for the $\ln^2 v$ and $\ln v$ terms of the function $\mathcal{S}$:

$$
\begin{aligned}
\mathcal{S}_{\ln^2}(v,Y) &= (1-Y)\sum_{n,m=0}^{\infty}\frac{Y^m}{m!}\frac{v^n}{(n!)^2}\frac{\Gamma^2(1+n)\Gamma^2(1+n+m)}{\Gamma(2+2n+m)}\times \qquad (3.19)\\
&\times\left(-\frac{1}{4}\ln(1-Y)-\psi(1+n+m)+\psi(2+2n+m)\right)\ln^2 v
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{S}_{\ln}(v,Y)=\frac{1-Y}{2}\sum_{n,m=0}^{\infty}\frac{Y^m}{m!}\frac{v^n}{(n!)^2}\frac{\Gamma^2(1+n)\Gamma^2(1+n+m)}{\Gamma(2+2n+m)}\Bigg[2\ln^2(1-Y)\\
&+\ln(1-Y)(6\psi(1+n)+5\psi(1+n+m)-5\psi(2+2n+m))+\pi^2\\
&-18\psi^2(1+n)-6\psi^2(2+2n+m)+12\psi(1+n)(\psi(1+n+m)+\psi(2+2n+m))\\
&-9\psi'(1+n)-3\psi'(1+n+m)+6\psi'(2+2n+m)\Bigg]\ln v \qquad (3.20)
\end{aligned}
$$

The non-logarithmic terms are more involved and not essential for our further study. Substituting (3.19), (3.20) into (3.3) and using representation (3.9) for $\Phi^{(1)}$ we obtain a series representation for $F^{(2)}$ suitable for the further OPE analysis.

4 OPE analysis at two loops

In this section we employ the expansion of the functions $F^{(2)}$ found in the previous section to study the operator algebra of chiral operators at two loops. Our prime interest will be to confirm the non-renormalization properties of certain lower-dimensional operators occurring in the operator algebra of chiral operators as well as to compute the two-loop anomalous dimensions of some other multiplets.

As was discussed above the non-renormalization property of $O^{\mathbf{20}}_{4,0}$ does not rely on a specific form of the function F. However, non-renormalization of higher-dimensional operators, in particular of descendants of this operator, can not be unravel without involving an explicit form of F. If we restrict our attention, say, to $F^{(1)}$ or $F^{(2)}$, then due to the problem of the operator mixing, an information we get from the OPE analysis is in general not enough to deduce the individual properties of mixed operators. However, combining $F^{(1,2)}$ with the knowledge of the F^{str}-function at strong coupling we will be able to trace the perturbative behavior of some of these operators. This happens due to the fact that the Yang-Mills multiplets dual to string states become infinitely massive at strong coupling and do not show up in the corresponding OPE, whose content is then given by non-renormalized operators and operators dual to multi-particle gravity states.

According to (2.12) a CPWA of any tensor contains a multiplier $v^{h/2}$, where h is treated as an anomalous dimension. It is then can be decomposed as

$$h = h^{(1)} + h^{(2)} + \dots , \tag{4.1}$$

where $h^{(1)}$, $h^{(2)}$ are anomalous dimensions at order $\tilde{\lambda}$, $\tilde{\lambda}^2$ and so on. Thus, in perturbation theory a term $v^{\frac{1}{2}h}$ is a origin of logarithmic terms of the form

$$v^{\frac{1}{2}h} = 1 + \frac{1}{2}h^{(1)} \ln v + \left(\frac{1}{2}h^{(2)} \ln v + \frac{1}{8}(h^{(1)})^2 \ln^2 v \right) + ... \tag{4.2}$$

Here the terms in the brackets occur at order $\tilde{\lambda}^2$ and should be matched with logarithmic terms in the 4-point function originating from $F^{(2)}$. In particular, the coefficients of the $\ln v$-terms encode a new information about two-loop anomalous dimensions, while the ones of $\ln^2 v$-terms depend on one-loop anomalous dimensions having been already found from $F^{(1)}$. Keeping track of the latter terms is an important consistency a check that perturbative expansion of the 4-point function fits the corresponding expansion of a sum of CPWAs.

In order not to overload the discussion with formulae, we consider only the lower-dimensional structure of the OPE for irreps **1**, **20** and **105**. We also do not write down explicitly the non-logarithmic terms in the 4-point function but simply present the relevant results where appropriate. In the following we also assume that for any operator T the ratio C_{OOT}/C_T of the normalization constants occurring in the corresponding three- and two-point functions with CPOs is kept equal to its free-field value. A correction to a coupling dependent constant $C_{OOT}(\tilde{\lambda})$ is introduced in the following way

$$C_{OOT}(\tilde{\lambda}) = C_{OOT}(1 + C^{(1)} + C^{(2)} + ...),$$

where C_{OOT} stands for the free-field value and $C^{(i)}$ describes an i-loop correction. Below we use the CPWAs normalized as in Section 2 with an exception of the CPWA of $T_{4,2}$, which we multiply by $-1/4$ to have an agreement with [25].

Singlet

The operators of approximate dimension up to 4 emerging in the singlet projection have been already discussed in [25]. These are the Konishi scalar $\mathcal{K}_{2,0}$, the Konishi tensor $\mathcal{K}_{4,2}$, the conserved stress-energy tensor $T_{\mu\nu}$, a tensor $\Xi_{4,2}$ being a lowest component of a new supersymmetry multiplet and scalar operators of dimension 4. In particular, with the chosen normalization of chiral operators the free-field normalization constants are found to be

$$\frac{C^2_{OO\mathcal{K}}}{C_{\mathcal{K}}} = \frac{4}{3N^2}\,, \qquad \frac{C^2_{OO\mathcal{K}_{4,2}}}{C_{\mathcal{K}_{4,2}}} = \frac{16}{63N^2}\,, \qquad \frac{C^2_{OO\Xi_{4,2}}}{C_{\Xi_{4,2}}} = \frac{16}{35N^2}\,. \tag{4.3}$$

Projecting the two-loop 4-point function on the singlet we find for the leading terms the following result

$$-\frac{1}{N^2\, x_{12}^4 x_{34}^4}\left[8v + \frac{5}{2}vY + \frac{53}{18}vY^2 + ...\right]\tilde{\lambda}^2 \ln v\,.$$

Here the term proportional v receives a contribution only from $\mathcal{K}_{2,0}$, for which we have $C^{(1)} = -3\tilde{\lambda}$, $h^{(1)} = 3\tilde{\lambda}$. Thus, comparison with the corresponding term in the CPWA allows one to find the two-loop anomalous dimension of the Konishi field

$$h^{(2)}_{\mathcal{K}} = -3\tilde{\lambda}^2\,. \tag{4.4}$$

The two-loop anomalous dimension of the Konishi field has been previously calculated in [16] by a different method and the result obtained there agrees with ours.

The term vY occurs only due to the Konishi field and does not provide us any new information. We therefore consider next the term vY^2, which receives contribution from the Konishi field as well as the tensors $\mathcal{K}_{4,2}$ and $\Xi_{4,2}$. By matching the coefficient of vY^2 with contributions of the corresponding CPWA's we find the anomalous dimension of Ξ at two loops:

$$h^{(2)}_{\Xi} = -\frac{25}{6}\tilde{\lambda}^2. \tag{4.5}$$

Similarly to the dimension of the Konishi field, the anomalous dimension of Ξ is negative.

In addition to the above discussed operators, the terms v^2 and $v^2 Y$ not indicated in (4.4) receive contributions from the scalar operators of dimension 4. In [25] we assumed that the free-field double trace operator $O^{\mathbf{1}}$ undergoes a splitting into a sum of operators O_i. However, despite having at our disposal the result for the two-loop 4-point function, the relatively big number of mixed operators (≥ 3) does not suffice to find their individual anomalous dimensions and free-field normalization constants.

Finally, analyzing the leading non-logarithmic terms in the 4-point function one obtains the following results for the two-loop corrections to the ratio of the normalization constants for $\mathcal{K}$:

$$C_{\mathcal{K}}^{(2)} = \frac{3}{2}(7 + 3\zeta(3))\frac{\tilde{\lambda}^2}{N^2} \, . \tag{4.6}$$

Irrep **20**

Here we show that the content of the operator algebra formed by operators up to approximate dimension 6 and transforming in the irrep **20** of $SO(6)$ can be depicted as follows:

Δ	spin		
2	$O_{2,0}$		
4	$O_{4,0}$	$\mathcal{K}_{4,2}$	
6	$O_{6,0}, \mathcal{K}_{6,0}, \Xi_{6,0}$	$T_{6,2}$	$\Xi_{6,4}$

The operator $O_{2,0}$ is the CPO and $O_{4,0}$ is an operator whose non-renormalization property was discussed in Section 2. Below we demonstrate that in addition to these operators there exist a scalar $O_{6,0}$ with vanishing anomalous dimension. We will also see that a free-field scalar $T_{6,0}$ splits in perturbation theory into the sum of three operators belonging to different representations of supersymmetry.

As was already shown in [25] (*c.f.* Section 2) the lowest-dimensional operator in irrep **20** that receives anomalous dimension is the second rank tensor Konishi tensor $\mathcal{K}_{4,2}$ with the free-field ratio $C^2_{OO\mathcal{K}_{4,2}}/C_{\mathcal{K}_{4,2}} = \frac{80}{9N^2}$. Extending the free-field and the one-loop analysis of [25] to dimension 6 operators, it is not difficult to show that a tensor $T_{6,4}$ has the one-loop anomalous dimension $h^{(1)} = \frac{25}{6}\tilde{\lambda}$, i.e. it is the same as for the tensor $\Xi_{4,2}$ occurring in the singlet projection. Thus, $T_{6,4} \equiv \Xi_{6,4}$ belongs to the Ξ-multiplet. With our convention for normalization of CPWAs its free-field ratio of the normalization constants is

$$\frac{C^2_{OO\Xi_{6,4}}}{C_{\Xi_{6,4}}} = \frac{4}{21N^2} \, . \tag{4.7}$$

To proceed it is useful to recall the strong coupling result [24] for the 4-point function of chiral operators projected onto **20**. For the first few leading terms we get

$$\frac{1}{N^2\, x_{12}^4 x_{34}^4}\left[\frac{40}{3}vF_1(Y)+v^2\left(\frac{26}{9}+\frac{26}{9}Y+\frac{119}{45}Y^2\right)+\frac{2}{15}v^3-\frac{4}{3}v^2\ln v\left(Y^2-v-\frac{3}{2}vY\right)\right], \tag{4.8}$$

where we have written out explicitly both logarithmic and non-logarithmic terms. Here a function $F_1(Y)=-Y^{-1}\ln(1-Y)$ provides a complete Y-contribution of the CPWA of a dimension 2 scalar that is the chiral operator $O_{2,0}^{\mathbf{20}}$ itself. Such a structure of the v-term allows one to conclude that all "single-trace" rank-l operators of dimension $2+l$ decouple at strong coupling, $\Xi_{6,4}$ among them [24]. From (4.8) one may see that the coefficient of the $\ln v$-term matches exactly the leading terms of the CPWA of a tensor $T_{6,2}$, in particular, this coefficient does not receive contribution from the CPWA of a scalar $T_{6,0}$. Thus, we have two options: either $T_{6,0}$ is non-renormalized or it is absent in the strong coupling OPE. Let us show that the first option is realized. To this end we study the non-logarithmic terms in (4.8).

We represent the $1/N^2$ corrections to a normalization constant in the usual way as e.g. $C_{\Delta\,,l}=C_{\Delta\,,l}\left(1+C^{(1)}_{\Delta\,,l}\right)$, where $C_{\Delta\,,l}$ on the r.h.s. is a leading term in $1/N^2$ and $C^{(1)}_{\Delta\,,l}$ is a next $1/N^2$-correction. In particular, the v^2Y^2-term contains the contribution from CPWAs of $O_{2,0}$, $O_{4,2}$, and of $T_{6,2}$. The contribution of the CPWA of $T_{6,0}$ is absent since it starts from v^3. Thus, matching the v^2Y^2-terms we find

$$\frac{C^2_{OOT_{6,2}}}{C_{6,2}}C^{(1)}_{6,2}=\frac{2}{45N^2}\,. \tag{4.9}$$

In a similar way studying the contribution of CPWA's to the v^3-term in (4.8) and taking into account (4.9) we find

$$\frac{C^2_{OOT_{6,0}}}{C_{6,0}}C^{(1)}_{6,0}=-\frac{4}{9N^2}\,. \tag{4.10}$$

Thus, we clearly see that scalar $T_{6,0}$ is present in the strong coupling OPE but does not receive any anomalous dimension.

To get more insight we consider the projection of the free-field 4-point function onto irrep. **20**:

$$\frac{1}{x_{12}^4 x_{34}^4}\left[\frac{40}{3N^2}vF_1(Y)+v^2\left(2+\frac{2}{3N^2}+\left(2+\frac{2}{3N^2}\right)Y+\left(3+\frac{2}{3N^2}\right)Y^2+...\right)\right]. \tag{4.11}$$

Note that the higher v-terms are absent. The terms v^2Y^2 and v^3 get contributions from the CPWAs of the CPO, $\mathcal{K}_{4,2}$, $T_{6,2}$, $O_{4,2}$ and $\Xi_{6,4}$ and this allows us to find the free-field values of the normalization constants

$$\frac{C^2_{OOT_{6,2}}}{C_{6,2}} = \frac{6}{5} + \frac{1}{15N^2}, \qquad \frac{C^2_{OOT_{6,0}}}{C_{6,0}} = \frac{2}{3} - \frac{1}{9N^2}. \tag{4.12}$$

Thus, we see that $1/N^2$ strong coupling corrections to the constants of $T_{6,0}$ and $T_{6,2}$ do not coincide with their free-field counterparts. This means that operators $T_{6,0}$ and $T_{6,2}$ undergo a splitting at weak coupling into a sum of operators with different perturbative behavior of anomalous dimensions. In particular the operator $T_{6,0}$ should contain in the split a non-renormalized operator.

Extension of the one-loop analysis performed in [25] to the operators of dimension 6 allows us to establish the following relations

$$\sum_i \frac{C^2_{OOT^i_{6,0}}}{C^i_{6,0}} (h^i_{6,0})^{(1)} = \frac{11}{9}\frac{\tilde{\lambda}}{N^2}, \qquad \sum_i \frac{C^2_{OOT^i_{6,2}}}{C^i_{6,2}} (h^i_{6,2})^{(1)} = \frac{5}{9}\frac{\tilde{\lambda}}{N^2}, \tag{4.13}$$

where we have taken into account that above discussed operators split at one-loop.

Finally we can use the whole power of our formulae to extract the one-loop anomalous dimensions by looking at the $\ln^2 v$ terms in the two-loop 4-point function projected on the irrep **20**. For the first few leading terms we find

$$\frac{1}{N^2}\left[\frac{5}{2}v(Y^2 - v - vY) - v^2\left(\frac{5}{2} + \frac{5}{2}Y + \frac{205}{108}Y^2\right) - \frac{73}{108}v^3\right]\tilde{\lambda}^2 \ln^2 v\,, \tag{4.14}$$

where the first term is distinguished to emphasize the contribution of the CPWA of the tensor $\mathcal{K}_{4,2}$. The essence of our analysis are the following equations:

$$\sum_i \frac{C^2_{OOT^i_{6,0}}}{C^i_{6,0}} [(h^i_{6,0})^{(1)}]^2 = \frac{124}{27}\frac{\tilde{\lambda}^2}{N^2}, \qquad \sum_i \frac{C^2_{OOT^i_{6,2}}}{C^i_{6,2}} [(h^i_{6,2})^{(1)}]^2 = \frac{205}{27}\frac{\tilde{\lambda}^2}{N^2}. \tag{4.15}$$

Consider now $T_{6,0}$ and make *an assumption* that in perturbation theory this operator splits into three operators, one $O_{6,0}$ is non-renormalized, the second, $\mathcal{K}_{6,0}$, is from the Konishi multiplet and the third one, $\Xi_{6,0}$, is an operator whose anomalous dimension that we are going to find[9]. The free-field normalization constant corresponding to $O_{6,0}$ should be the same as we have found from the strong coupling result, i.e.

$$\frac{C^2_{OOO_{6,0}}}{C_{O_{6,0}}} = \frac{2}{3} - \frac{4}{9N^2}. \tag{4.16}$$

Subtracting it from the free-field result (4.12) we are left with the sum of the constants of the operators $\mathcal{K}_{6,0}$ and $\Xi_{6,0}$:

$$\frac{C^2_{OO\mathcal{K}_{6,0}}}{C_{\mathcal{K}_{6,0}}} + \frac{C^2_{OO\Xi_{6,0}}}{C_{\Xi_{6,0}}} = \frac{1}{3N^2}. \tag{4.17}$$

[9]We denote this operator by Ξ since as will become clear in a moment it belongs to the Ξ-multiplet.

This equation together with (4.13) and (4.15) provides a system of three equations for three unknown variables that are normalization constants and the anomalous dimension of $\Xi_{6,0}$. Solving the system we obtain

$$\frac{C^2_{OO\mathcal{K}_{6,0}}}{C_{\mathcal{K}_{6,0}}} = \frac{1}{7N^2}\,, \qquad \frac{C^2_{OO\Xi_{6,0}}}{C_{\Xi_{6,0}}} = \frac{4}{21N^2}\,, \qquad h^{(1)}_{\Xi_{6,0}} = \frac{25}{6}\tilde{\lambda} \tag{4.18}$$

which clearly shows that $\Xi_{6,0}$ belongs to the Ξ-multiplet. As to $T_{6,2}$, the corresponding analysis is complicated by the fact that this operator(s) is present at strong coupling with a finite anomalous dimension and the information we can extract from the weak/strong 4-point functions is not enough to establish its split components.

Irrep **105**

As was shown in [24] the rank-$2k$ tensors $O_{4+2k,2k}$ and $O_{6+2k,2k}$ transforming in the irrep **105** are non-renormalized in the strong coupling limit. As we have seen in Section 2 the non-renormalization property of $O_{4+2k,2k}$ is a (non-perturbative) consequence of the non-renormalization theorem of [27]. The strong coupling behavior of the normalization constant of $O_{6+2k,2k}$ indicates however that a free-field theory operator $T_{6+2k,2k}$ splits is perturbation theory into a sum of operators, therefore the unraveling of its non-renormalized component $O_{6+2k,2k}$ requires the explicit knowledge of the function $F(v,Y)$.

Here, restricting our attention to the dimension 6 operators and assuming that there exist $O_{6,0}$ and $O_{6,2}$ that are non-renormalized, we reveal the corresponding weak coupling content of the operator algebra. The subsequent treatment does not involve the knowledge of the two-loop 4-point function and it relies only on the free-field, the one loop and the strong coupling considerations.

Analyzing the free-field 4-point function we find the free-field couplings

$$\frac{C^2_{OOO_{4,0}}}{C_{4,0}} = 2 + \frac{4}{N^2}\,, \quad \frac{C^2_{OOT_{6,2}}}{C_{6,2}} = \frac{6}{5} + \frac{2}{5N^2}\,, \quad \sum_i \frac{C^2_{OOT^i_{6,0}}}{C^i_{6,0}} = \frac{2}{3} - \frac{2}{3N^2}\,. \tag{4.19}$$

Here $O_{4,0}$ is an operator belonging to the short multiplet whose non-renormalization property is well-known. For $T_{6,0}$ we assume a perturbative splitting.

At strong coupling we find however a non-renormalized operator $O_{6,0}$ with the $1/N^2$ correction to the normalization constants: $-2/N^2$. Thus, $C^2_{OOO_{6,0}}/C_{O_{6,0}} = \frac{2}{3} - \frac{2}{N^2}$, i.e. it is different from (4.19). We assume that this difference is due to the fact that $T_{6,0}$ splits in perturbation theory into a sum of *two* operators: $O_{6,0}$ and another operator $\mathcal{K}_{6,0}$ with a free-field value of the ratio $C^2_{OO\mathcal{K}_{6,0}}/C_{\mathcal{K}_{6,0}} = \frac{4}{3N^2}$. With this assumption we can now analyze the one-loop 4-point function and determine the one-loop anomalous dimension of $\mathcal{K}_{6,0}$ that turns out to be $h^{(1)}_{\mathcal{K}_{6,0}} = 3\tilde{\lambda}$. Thus, $\mathcal{K}_{6,0}$ belongs to the Konishi multiplet.

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Appendix A

Truncation property of the CPWA

The CPWA for the exchange of a tensor arbitrary non-integer dimension Δ and spin l between two pairs of scalar fields was calculated in [35]. We state their result for the special case of space-time dimension $d = 4$ and dimension of the outer scalar operators $\tilde{\Delta} = 2$. The overall normalization factor $\beta_{\tilde{\Delta};\Delta,l}$ is omitted since the Γ-functions in it cancel in this case.

$$\mathcal{H}_{\Delta,l} = \frac{1}{(x_{12}^2\, x_{34}^2)^2} \sum_{n,m=0}^{\infty} \frac{v^n\, Y^m}{n!\, m!} \sum_{M=0}^{l} \frac{c_l^{(M)}}{2^M\, c_l^{(l)}} \sum_{n_i=0}^{M} (-1)^{n_1+n_3} \frac{M!}{n_1!\, n_2!\, n_3!\, n_4!} (1-Y)^{n_2} \times$$
$$v^{\frac{1}{2}(\Delta-M)}\, \alpha(\delta_2)\alpha(\delta_4)\alpha(\Delta) \frac{(\delta_1)_n\, (2-\delta_2)_n\, (\delta_3)_{n+m}\, (2-\delta_4)_{n+m}}{(\Delta)_{2n+m}\, (\Delta - \frac{1}{2}d + 1)_n} \tag{4.1}$$

Here

$$\alpha(x) = \frac{\Gamma(2-x)}{\Gamma(x)}\,, \tag{4.2}$$

$$\begin{aligned} \delta_1 &= \tfrac{1}{2}(\Delta - M) + n_4 + n_1, & \delta_2 &= 2 - \frac{1}{2}(\Delta + M) + n_1 + n_2, \\ \delta_3 &= \tfrac{1}{2}(\Delta - M) + n_2 + n_3, & \delta_4 &= 2 - \frac{1}{2}(\Delta + M) + n_3 + n_4 \end{aligned} \tag{4.3}$$

and the summation over the n_i is such that $\sum n_i = M$.

We split the dimension of the exchanged operator as $\Delta = \Delta_0 - h$, where Δ_0 is an integer and $-1 \leq h < 1$ is the anomalous part of the dimension. The overall factor $v^{h/2}$ may be pulled out and is ignored in the following.

We set out to show that the lowest terms in the v, Y expansion of (4.1) are of the form $v^{\frac{1}{2}(\Delta_0 - k)}\, Y^k$, where $k = l,\, l-2,\, l-4,\, \ldots$, but $k \geq 0$. This requires proving the cancellation of some powers of Y

arising from $(1-Y)^{n_2}$. It suffices to consider each value of n, m, M separately, hence we can restrict our attention to the sum over n_i. There are three summations because $\sum n_i = M$. Define $N_4 = n_3 + n_4$. A substantial simplification is obtained by rewriting

$$\sum_{n_i=0}^{M} = \sum_{N_4=0}^{M} \sum_{n_1=0}^{M-N_4} \sum_{n_4=0}^{N_4} \tag{4.4}$$

since then δ_2, δ_4 depend only on M, N_4 but not on the remaining two counters n_1, n_4.

The proof can in fact be established for fixed N_4. Consider

$$S = \sum_{n_1=0}^{M-N_4} \sum_{n_4=0}^{N_4} (-1)^{n_1+n_3} \frac{M!}{n_1!\, n_2!\, n_3!\, n_4!} (1-Y)^{n_2} (\delta_1)_n (\delta_3)_{n+m}\,, \tag{4.5}$$

the other terms being constant for fixed n, m, M, N_4. Rearrange as

$$\begin{aligned} S &= (-1)^{N_4} \binom{M}{M-N_4} \sum_{n_1=0}^{M-N_4} (-1)^{n_1} \binom{M-N_4}{n_1} \sum_{n_4=0}^{N_4} (-1)^{n_4} \binom{N_4}{n_4} \\ &\times (1-Y)^{M-N_4-n_1} (\delta_1)_n (\delta_3)_{n+m} \qquad (4.6)\\ &= (-1)^{N_4} \binom{M}{M-N_4} \sum_{k=0}^{M-N_4} (-Y)^k \sum_{n_1=0}^{M-N_4-k} (-1)^{n_1} \binom{M-N_4}{n_1} \binom{M-N_4-n_1}{k} \\ &\times \sum_{n_4=0}^{N_4} (-1)^{n_4} \binom{N_4}{n_4} (\delta_1)_n (\delta_3)_{n+m} \qquad (4.7)\\ &= (-1)^{N_4} \binom{M}{M-N_4} \sum_{k=0}^{M-N_4} (-Y)^k \binom{M-N_4}{k} \\ &\times \left[\sum_{n_1=0}^{M-N_4-k} (-1)^{n_1} \binom{M-N_4-k}{n_1} \sum_{n_4=0}^{N_4} (-1)^{n_4} \binom{N_4}{n_4} (\delta_1)_n (\delta_3)_{n+m} \right]. \qquad (4.8) \end{aligned}$$

All terms in the last line apart from the Binomial coefficients depend only on the sum $N_1 = n_4 + n_1$. Using

$$\sum_{n+m=p} \binom{N}{n} \binom{M}{m} = \binom{M+N}{p} \tag{4.9}$$

we replace the double sum in the square bracket by

$$\sum_{p=0}^{M-k} (-1)^p \binom{M-k}{p} \left(\frac{1}{2}(\Delta - M) + p\right)_n \left([\frac{1}{2}(\Delta - M) + k] + [(M-k) - p]\right)_{n+m}. \tag{4.10}$$

It will be demonstrated below that this sum vanishes if $M-k > 2n+m$ and that is equals $(-1)^n\,(2n+m)!$ if $M-k = 2n+m$. The lowest power of Y occurring in S is therefore $Y^{(M-2n-m)}$ if $M \geq 2n+m$ and Y^0 if $M < 2n+m$.

Recall that the complete expression for the CPWA (4.1) includes

$$v^{(\frac{h}{2})}\, v^{(\frac{1}{2}(\Delta_0-M)+n)} Y^m \tag{4.11}$$

the product of which with $Y^{(M-2n-m)}+\dots$ yields

$$v^{(\frac{h}{2})}\, v^{(\frac{1}{2}(\Delta_0-(M-2n))}(Y^{(M-2n)}+\dots) \tag{4.12}$$

so that the lowest term is in fact of the type postulated above, since the Gegenbauer coefficients $c_l^{(M)}$ are non-vanishing only if M, l are both odd or both even.

Referring to the "total power" $T(v^{(\frac{h}{2})} v^n Y^m) \equiv 2n+m$ introduced above we find only terms with $T \geq \Delta_0$.

Last, if $M < 2n+m$ we find

$$v^{(\frac{h}{2})}\, v^{(\frac{1}{2}(\Delta_0-M)+n)} Y^m (Y^0+\dots)\,. \tag{4.13}$$

The lowest of these terms comes with $T = \Delta_0 - M + 2n + m > \Delta_0$; they are never leading.

It remains to prove the vanishing of (4.10) for $M-k > 2n+m$. One may check by explicit calculation that

$$\begin{aligned} &(X+p)_a\,(Y+P-p)_b - (X+(p+1))_a\,(Y+P-(p+1))_b = \\ &b\,(X+p)_a\,((Y+1)+(P-1)-p)_{b-1} - a\,((X+1)+p)_{a-1}\,(Y+(P-1)-p)_b \end{aligned} \tag{4.14}$$

Using Pascal's triangle:

$$\begin{aligned} &\sum_{p=0}^{P}(-1)^p \binom{P}{p}\,(X+p)_a(Y+P-p)_b = \sum_{p=0}^{P-1}(-1)^p \binom{P-1}{p} \\ &\times\Big(b\,(X+p)_a\,((Y+1)+(P-1)-p)_{b-1} - a\,((X+1)+p)_{a-1}\,(Y+(P-1)-p)_b\Big)\,. \end{aligned} \tag{4.15}$$

On iterating this step the sum vanishes if $P > a+b$ and is equal to $(-1)^a P\,!$ if $P = a+b$.

Appendix B

Analytic continuation of $\Phi(v, y|\delta)$

Here we discuss the problem of the analytic continuation of the function $\Phi(v, y|\delta)$ to the conformal variables describing the crossed channels.

Recall the Mellin-Barnes representation for the $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$-functions [24]:

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(v,y) = \int \frac{d\lambda ds}{(2\pi i)^2}\Bigg[\Gamma(-\lambda)\Gamma(-s)\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3-\Delta_4}{2}-\lambda) \tag{4.1}$$
$$\times\Gamma(\frac{\Delta_1+\Delta_3-\Delta_2-\Delta_4}{2}-s)\Gamma(\frac{\Delta_2+\Delta_3+\Delta_4-\Delta_1}{2}+s+\lambda)\Gamma(\Delta_4+s+\lambda)v^{\lambda}\left(\frac{v}{u}\right)^s\Bigg].$$

Comparing this formula with (3.5) one obtains (3.6). On the other hand, the D-function has an integral representation in terms of Schwinger parameters (see e.g. [24]), so that for the case under consideration one gets

$$\Phi\left(v,\frac{v}{u}|\delta\right) = 2\int dt_1dt_2dt_3dt_4\ t_1^{-\delta}t_4^{\delta}exp\Bigg[-t_1t_2-t_1t_3-t_1t_4-t_2t_3-\frac{v}{u}t_2t_4-vt_3t_4\Bigg].$$

From here we immediately read off a representation for Φ in a crossed channel, e.g.,

$$\Phi\left(u,\frac{u}{v}|\delta\right) = 2\int dt_1dt_2dt_3dt_4\ t_1^{-\delta}t_4^{\delta}exp\Bigg[-t_1t_2-t_1t_3-t_1t_4-t_2t_3-ut_2t_4-\frac{u}{v}t_3t_4\Bigg].$$

Note that under the following rescaling of integration variables

$$t_1\to\lambda t_1,\quad t_2\to\frac{1}{\lambda}t_2,\quad t_3\to\frac{1}{\lambda}t_3,\quad t_4\to\frac{1}{\lambda}t_4 \tag{4.2}$$

the integral takes the form

$$\begin{aligned}\Phi\left(u,\frac{u}{v}|\delta\right) &= \frac{2}{(\lambda^2)^{1+\delta}}\int dt_1dt_2dt_3dt_4\ t_1^{-\delta}t_4^{\delta} \\ &\times\ exp\Bigg[-t_1t_2-t_1t_3-t_1t_4-\frac{1}{\lambda^2}t_2t_3-\frac{u}{\lambda^2}t_2t_4-\frac{u}{\lambda^2 v}t_3t_4\Bigg].\end{aligned}$$

Now we may choose $\lambda^2=\frac{u}{v}$ and perform the change of variables $t_2\leftrightarrow t_3$ and then $t_3\leftrightarrow t_4$. Finally, by using the Mellin-Barnes representation for the integral on the r.h.s., we arrive at the first formula in (3.7). The second formula in (3.7) is proved in an analogous manner.

The functions $D_{1-\delta,1,1+\delta,1}$ and $D_{1-\delta,1+\delta,1,1}$ have the following Mellin-Barnes representation

$$\begin{aligned}D_{1-\delta,1,1+\delta,1}(v,Y) &= v^{-\delta/2}\int\frac{d\lambda ds}{(2\pi i)^2}\Gamma^2(-s)\Gamma(-\lambda+\delta/2)\Gamma(-\lambda-\delta/2)\Gamma(1+\lambda+s+\delta/2) \\ &\times\ \Gamma(1+\lambda+s-\delta/2)v^{\lambda}y^s \qquad (4.3)\\ D_{1-\delta,1+\delta,1,1}(v,Y) &= y^{-\delta/2}\int\frac{d\lambda ds}{(2\pi i)^2}\Gamma^2(-\lambda)\Gamma(-s+\delta/2)\Gamma(-s-\delta/2)\Gamma(1+\lambda+s+\delta/2) \\ &\times\ \Gamma(1+\lambda+s-\delta/2)v^{\lambda}y^s \qquad (4.4)\end{aligned}$$

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