

## Editor's Note

### Relativistic Hydrodynamics

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*If you were to ask me what I have contributed to the theory of relativity, I believe that I could claim to have emphasized its geometrical aspects.*

J. L. Synge, 1972

The general theory of relativity may be viewed as the completion of classical macroscopic field physics. Since this theory identifies gravitation with some aspects of the metric of spacetime, and because the metric and its connection enter all parts of physics as basic prerequisites, the task arose to adapt all branches of classical physics to the generalized spacetime structure, and to investigate whether the modifications lead to new, possibly observable consequences. This holds, in particular, for hydrodynamics, the significance of which in this context is enhanced by the following facts. Due to an elementary argument by Max von Laue [1], relativistic causality implies that any extended body has infinitely many degrees of freedom, and the results of Karl Schwarzschild [2] and of subsequent authors show that Einstein's gravitational field equation is incompatible with the representation of bodies as points endowed with a positive mass. Therefore, in general relativity bodies such as stars and planets have to be modelled, at least in principle, in terms of hydro- or elastomechanics. In addition, the development of high-energy astrophysics shows that large-scale flows of matter with relativistic speeds in relativistic gravitational fields do occur in nature. Prime examples are supernovae, jets associated with active galactic nuclei, accretion flows

around and into black holes, and exotic fluid-like media such as quintessence. Relativistic hydrodynamics is also needed to study the structure and stability of stars.

Synge's paper reprinted below represents the first "systematic attempt to develop a hydrodynamical theory in general relativity," apart from an early investigation by L. P. Eisenhart [3] whose main results are included in Synge's work. Synge does not take into account the gravitational field equation; he uses only the covariant conservation law for the energy tensor of matter in a given, generally curved spacetime (test fluid approximation).

The paper begins (chapters I, II) with kinematical definitions and some immediate consequences, related to a congruence of timelike world lines interpreted as streamlines of a fluid. Two remarks may be picked out: The (by now well-known) geometrical characterization of irrotational motion (theorem II) and the Frenet–Serret formulas for a timelike curve, (2.4), which are applied in several places of the paper and which also have been used by Synge in his delightful book on general relativity [4] to translate simple kinematical facts into the language of general relativity. (Strangely enough, the term 4-acceleration is never used though the concept appears frequently, of course. Also, the rates of strain and shear needed, e.g. to discuss Born-type rigid motions [5], have been introduced by Synge only in [4].)

In chapter III Synge introduces the energy tensor. An answer to the question he raises there—to find conditions ensuring that the eigenvalues of a symmetric tensor with respect to a Lorentz-metric are all real—was given by Synge himself<sup>1</sup> in [6]: If  $T_{ij}\lambda^i\lambda^j > 0$  for all causal vectors  $\lambda^i$ , then the eigenvalues  $(-\rho, p_1, p_2, p_3)$  of  $T_j^i$  are real. (Synge's assumption is slightly stronger than Stephen Hawking's weak energy condition [7], but it implies neither the strong nor the dominant condition.) Synge then specializes to the usual perfect fluid energy tensor, he writes down the "Euler" equations implied by  $T^i_{j;i} = 0$ , and deduces some consequences without assuming a pressure-density relation. His theorem XV shows, e.g., that streamlines contained in a hypersurface of constant pressure are geodesics of that hypersurface with respect to its induced metric, a fact which applies in particular to a free boundary of a fluid body.

The longest and in my view most interesting part of the paper, chapter IV, deals with barotropic motions of perfect fluids in which the energy-density  $\rho$  depends on the pressure  $p$ ,  $\rho = f(p)$ . Following Eisenhart, Synge introduces in (9.1) what he calls the index function  $F(p)$ , the basic tool for the subsequent work. He characterizes the streamlines as the geodesics of the "fluid metric"  $\overline{ds}^2 = F^2 ds^2$ , and continues to deduce relativistic generalizations of classical theorems by Kelvin and Helmholtz on vortices. He does this in his characteristic

<sup>1</sup>In this note I use Synge's notation explained in his paper; so that a comma denotes *covariant* differentiation.

style, using spacetime diagrams to illustrate the contents of the theorems as well as the proofs. The chapter ends with setting up a simple metric (16.8) for irrotational flows. It is a pleasure to read this exposition, it does not need further comments.

In the last chapter, Synge specializes the metric to a flat one, perhaps because at that time he was not familiar with Killing vectors and Lie derivatives. He first deals with hydrostatics, deriving a generalization of Bernoulli's theorem for stationary fluid motions, and then treats the propagation of disturbances in a fluid. In this chapter the assumptions are unnecessarily restrictive and the formulations are somewhat awkward. The rather straightforward, but from the point of view of physics important generalization of stationary hydrodynamics to curved spacetime was provided by A. Lichnerowicz in [8].

Synge's "most remarkable result," the equation (21.11) on the speed of propagation of small disturbances in an irrotational fluid at rest, has also been generalized to General Relativity by Lichnerowicz [8].

The conclusion on the sound velocity (21.12), drawn by both authors, and their remarks on incompressibility (here and in sec. 8 and footnote) suffer from the fact that thermodynamics is not taken into account. As has been shown later [9], the result (21.11) is valid under more general assumptions, provided the derivative  $dp/d\rho$  is taken at fixed specific entropy, whether or not the flow is barotropic.

Some information about later developments related to relativistic hydrodynamics is contained in references [9–16].

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## Short Biography

John Lighton Synge, F.R.S. was born in Dublin on 23rd March, 1897. He was educated in St. Andrew's College and entered Trinity College, Dublin University in 1915. He graduated B.A. (1919), M.A. (1922) and Sc.D. (1926).

He was Assistant Professor of Mathematics in the University of Toronto (1920–25) and returned to Trinity College as Professor of Natural Philosophy (1925–30). It was at this time that he published a paper "On the Geometry of Dynamics" [*Phil. Trans. R. Soc.* **A226** (1926), 31–106] in which he obtained the equation of geodesic deviation on a Riemannian manifold and simultaneously this important equation was derived by Levi-Civita on a pseudo-Riemannian manifold. He also edited, with A. W. Conway, F.R.S. of University College Dublin, the first volume of the collected works of Hamilton on geometrical optics. This was an enterprise which had a strong influence subsequently on his own research in mechanics and optics.

He returned to the University of Toronto as Professor of Applied Mathematics (1930–43). He subsequently became chairman of the Mathematics Department in Ohio State University (1943–46) and Head, Mathematics Department at Carnegie Institute of Technology, Pittsburgh (1946–48) before coming back to Dublin to establish his "school of relativity" in the Dublin Institute for Advanced Studies (1948–72).

This was a golden age for relativity generally and in particular in Dublin. Many notable figures in the subject came to study with or consult Synge, influenced by his emphasis on the geometry of space-time and his impressive insight and mastery of this most fundamental point of view. He created around him a wonderful spirit of enquiry accompanied by intellectual discipline ("as far as I am concerned, you cannot beat a good equation"). Out of this emerged, some profound results most notably perhaps, Felix Pirani's study of the physical significance of the Riemann tensor and Werner Israel's proof of the uniqueness of the static black hole. These researches carry the imprint of Synge's point of view par excellence.

He himself was prolific, publishing over 250 papers and 11 books covering Differential Geometry, Applied Mathematics and Relativity Theory. After officially retiring at 75 he continued his research with amazing vigour for another

twenty years. He died on 30th March, 1995 in Dublin having bequeathed his body to the Medical School in Trinity College.

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The most outstanding characteristic of J. L. Synge's approach to mathematical physics was his extraordinary geometrical insight. "I asked myself why some things bored me while others excited me intellectually, and I came to the conclusion that the exciting problems must contain two ingredients – geometry *and* physics," Synge said in his Boyle Medal Lecture. His taste is clearly visible in his four books (one non-technical) on relativity and over 70 papers on that subject, about one third of his impressive and widely varied output which covers, besides relativity, classical mechanics, elasticity, geometrical optics, gas dynamics, differential geometry and several other subjects including a few papers on the stresses in the periodontal membranes in human teeth.

The community of relativists owes to Synge the use of spacetime diagrams, the clarification of many concepts in relativity, in generality and in terms of illustrative, often amusing examples. In particular, he showed how to use, in differential geometry and in relativity, the equation of geodesic deviation. An outstanding achievement of Synge's was the first complete analytic extension of the Schwarzschild field. Remembering my own study of relativity and the change of style brought about in the fifties and sixties under the influence of Synge, I can testify that he succeeded "to make spacetime a real workshop for physicists, not a museum visited occasionally with a feeling of awe."

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