Open Strings in Simple Current Orbifolds

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Abstract

We study branes and open strings in a large class of orbifold backgrounds using microscopic techniques of boundary conformal field theory. In particular, we obtain factorizing operator product expansions of open string vertex operators for such branes. Applications include branes in $\mathbb{Z}_2$ orbifolds of the SU(2) WZW model and in the D-series of unitary minimal models considered previously by Runkel.

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1 Introduction

The study of branes at orbifold singularities has a long history. Such investigations were motivated mainly by the fact that orbifolds of the form $\mathbb{C}^n/\Gamma$ can be used to locally model singularities in Calabi-Yau spaces. Beyond such examples, more general orbifold constructions are an important ingredient in conformal field theory (CFT) model building, i.e. in the construction of exactly solvable closed and open string backgrounds. In particular, Gepner's construction of string theories on Calabi-Yau spaces involves some orbifold-like projection. This suggests to analyse strings and branes in orbifolds more general than $\mathbb{C}^n/\Gamma$ and with orbifold actions that may not admit an interpretation as a geometric symmetry of the target space.

It is not possible to give a complete account here of all the previous results related to branes in orbifolds. Much of the work was devoted to orbifold constructions in flat space (see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). The basis for most of these developments were laid in [1] which uses earlier ideas originating from [12, 11]. Open string theory in more general conformal field theory orbifolds was pioneered by Sagnotti and collaborators starting from [12] (see also e.g. [13, 14]). Important contributions were made later by Behrend et al. [15, 16] and by Fuchs et al. [17, 18, 19]. The latter extends the simple current techniques that were developed for closed strings in [20, 21] to the case of open strings (see also [22, 23]). Work on open strings in Gepner models highlighting the orbifold aspects includes [24, 25, 26, 27, 28].

All the contributions we have listed so far focus on the couplings of open strings to the branes which is closely related to the spectra of open strings that can stretch between branes in orbifold spaces. The *operator product expansions* of open string vertex operators (boundary fields) in an orbifold of a non-trivial boundary conformal field theory, however, were addressed first in [29] for minimal models and then later in [30] for A-type branes in Gepner models. We note that such results on the couplings of open strings are a necessary prerequisite if one wants to extend the studies of brane effective actions in orbifolds [1] to general backgrounds.

In this work we address the issue of boundary operator product expansions for so-called *simple current orbifolds* of the extension type. As was observed in [25, 30], their treatment follows very closely the strategies known from orbifolds of a flat background, i.e. they involve lifting the theory from the orbifold to the covering space. The latter is described by conformal field theory constructions that go back to Cardy [31] and we will review them along with some background material on simple current symmetries in Section 2. After this preparation we enter the central section of this work which contains our main results (3.6,3.9) on boundary operator product expansions. Section 4 is devoted to applications. We start by discussing a $\mathbb{Z}_2$ orbifold of the SU(2) WZW model. In this case, our algebraic results can be interpreted geometrically. Finally, we reconsider the example of the D_{even}-series of unitary minimal models which is contained in [29] and we show that our formulas provide a very elegant construction of the solution.

2 Boundary CFT on the covering space

As we mentioned in the introduction, our strategy for dealing with branes on orbifold spaces follows the usual procedure in which the whole theory is lifted to a covering space. We shall assume that the latter can be solved using the standard microscopic techniques which go back to the work of Cardy [31]. The aim of this section is to provide a brief account on this theory.

Suppose we are given some rational bulk conformal field theory with chiral algebra $\mathcal{A} = \bar{\mathcal{A}}$ and a modular invariant partition function of the form

$$Z(q,\bar{q}) \;=\; \sum_j \; \chi_j(q)\; \chi_{\bar{\jmath}}(\bar{q}) \quad . \tag{2.1}$$

Here j runs through the sectors of the right moving chiral algebra $\mathcal{A}$ and each of these sectors j comes paired with a unique sector $\bar{\jmath}$ of the left moving chiral algebra $\bar{\mathcal{A}} = \mathcal{A}$. As usual, $\chi_j(q)$ denotes the character of the sector j.

The construction of boundary theories involves picking some automorphism Ω of the chiral algebra. This appears in the boundary conditions to describe how left- and right movers are glued along the boundary. Any such automorphism Ω induces a map ω acting on sectors j of the chiral algebra. Cardy's analysis of boundary conditions applies whenever the partition function (2.1) is (Ω)-diagonal in the sense that $\omega(j)^\vee = \bar{\jmath}$. Here, $j^\vee$ denotes the sector conjugate to j, i.e. the unique label with the property that its fusion product with j contains the vacuum representation 0 of the chiral algebra. We shall assume that Ω is chosen such that the modular invariant partition function (2.1) of the covering theory is Ω-diagonal.

Under this condition, Cardy provides us with a list of boundary theories. Their number agrees with the number of sectors of $\mathcal{A}$. We will use labels $I, J, K, \ldots$ to distinguish between boundary conditions and sectors but it should be kept in mind that small and capital letters run through the same index set. The spectrum of open strings that stretch between the branes which are associated with the labels I and J is given by

$$Z_{IJ}(q) \;=\; \sum_j N_{Ij}{}^J \;\chi_j(q) \quad . \tag{2.2}$$

Obviously, this tells us how the state space $\mathcal{H}_{IJ}$ of the boundary theory is built up from sectors of the chiral algebra. For a much more detailed explanation of these results the reader is referred to [32].

There is a version of the state-field correspondence in boundary conformal field theory that assigns a boundary field to each state in $\mathcal{H}_{IJ}$. Hence we can read off from (2.2) that the boundary primary field ψ_j appears with multiplicity $N_{Ij}{}^J$ in the boundary theory. The operator product expansion for two such primary fields is given by

$$\psi_i^{LM}(x_1)\;\psi_j^{MN}(x_2) \;=\; \sum_k \;(x_1-x_2)^{h_k-h_i-h_j}\;\psi_k^{LN}(x_2)\;F_{M,k}\big[\begin{smallmatrix} i & j \\ L & N\end{smallmatrix}\big] \;+\ldots \tag{2.3}$$

for $x_1 < x_2$. Here, F stands for the fusing matrix of the chiral algebra $\mathcal{A}$. It is defined as a linear transformation that relates two different orthonormal bases in the space of conformal blocks (see [33]) and it can be visualized as shown in Figure 1. For simplicity we shall assume that the fusion rules obey $N_{Ij}{}^J \leq 1$ so that the vertices carry no additional labels.

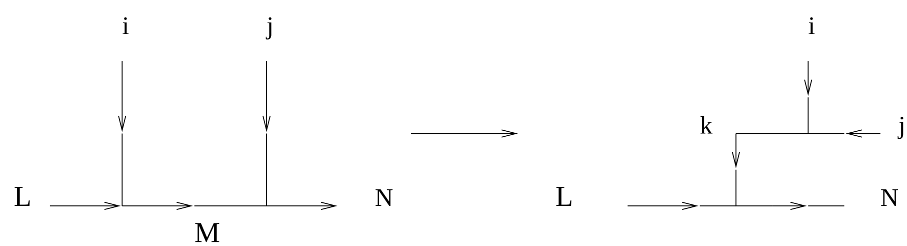

Figure 1: Graphical description of the fusing matrix

The formula (2.3) was originally found for minimal models by Runkel [38] and extended to more general cases in [34, 35, 36, 15]. Note that our conventions for the fusing matrix (see Figure 1) differ slightly from the ones used in e.g. [35, 36]. With the external legs being oriented as shown in Figure 1, non of the six labels in the fusing matrix needs to be conjugated.

In string theory, boundary operator products describe the scattering of two open strings which are stretched between the branes L, M and M, N, respectively, into an open string that stretches between L and N.

Note that for the relation between the coefficients of the boundary OPE and the fusing matrix it is crucial that boundary conditions and boundary fields are labeled with elements from the same set. This is no longer true for models with a 'non-diagonal' (in the sense specified above) bulk modular invariant partition function. We shall see below how this can affect the boundary operator product expansions. Examples of boundary OPEs for non-diagonal modular invariants were studied in [29, 30] and they are the main subject of this work.

The formulas (2.2,2.3) provide a complete solution of the open string sector on the covering space (for branes of gluing-type Ω). Before moving on to the orbifold theory let us briefly study the symmetry properties of the solution with respect to the group action that we plan to divide out. We shall assume that this orbifold group is generated by *simple currents* of the conformal field theory. To describe this more precisely, we need some new notation. Primaries (or the associated conformal families) of a conformal field theory form a set $\mathcal{J}$. Within this set $\mathcal{J}$ there can be non-trivial elements $g \in \mathcal{J}$ such that the fusion product of g with any other $j \in \mathcal{J}$ gives again a single primary $g \times j = gj \in \mathcal{J}$. Such elements g are called *simple currents* and the set $\mathcal{C}$ of all these simple currents forms an abelian subgroup $\mathcal{C} \subset \mathcal{J}$. The product in $\mathcal{C}$ is inherited from the fusion product of representations. From now on, let us fix some subgroup $\Gamma \subset \mathcal{C}$.

Through the fusion of representations, the index set $\mathcal{J}$ comes equipped with an action $\Gamma \times \mathcal{J} \to \mathcal{J}$ of the group Γ on labels $j \in \mathcal{J}$. Under this action, $\mathcal{J}$ may be decomposed into orbits. The space of these orbits will be denoted by $\mathcal{J}/\Gamma$ and we use the symbol $[j]$ to denote the orbit represented by $j \in \mathcal{J}$. These orbits may have *fixed points*, i.e. there can be labels $j \in \mathcal{J}$ for which the following stabilizer subgroup $\mathcal{S}_j \subset \Gamma$

$$\mathcal{S}_j \;=\; \{\; g \in \Gamma \;\mid\; g \cdot j \;=\; j \;\} \tag{2.4}$$

is nontrivial. Up to isomorphism, the stabilizer subgroups depend only on the orbits $[j]$ not on the choice of a particular representative $j \in [j]$, i.e. $\mathcal{S}_j = \mathcal{S}_{[j]}$.

The last object we have to introduce is the *charge* $\hat{Q}_g(j)$ of a primary j with respect to the simple current g. It is obtained from the following special matrix elements

$$\Omega\big({}_k{}^j{}_i\big) \;=\; B_{ji}\left[\begin{smallmatrix} j & i \\ 0 & k \end{smallmatrix}\right]$$

of the braiding matrix $B = B^{(+)}$ (see [33] for details) by specialization to simple currents $i = g$,

$$(-1)^{\hat{Q}_g(j)} := \Omega\left({}_{gj}{}^{j}{}_{g} \right) \quad . \tag{2.5}$$

Note that this specifies $\hat{Q}_g(j)$ up to an even integer, i.e. $\hat{Q}_g(j) \in \mathbb{R}/2\mathbb{Z}$. The charge $\hat{Q}_g(j)$ is related to the more standard *monodromy charge* $Q_g(j) \in \mathbb{R}/\mathbb{Z}$ by the prescription $Q_g(j) := \hat{Q}_g(j) \bmod 1$. We note that the monodromy charge can be computed from the conformal dimensions of the involved fields through the expression

$$Q_g(j) \; = \; h_j + h_g - h_{gj} \mod 1 \quad .$$

In case the simple currents have integer conformal weight (but not only then, see Section 4.2 for an example), the monodromy charge $Q_g(j)$ depends only on the equivalence class $[j]$ of $j \in J$. An orbit $[j]$ is said to be invariant, if $Q_g([j]) = Q_g(j) = 0$ for all $g \in \Gamma$.

We are finally in a position to formulate the symmetry properties of the open string sector for the theory on the covering space. To this end we introduce the following action of the simple current group Γ on boundary fields

$$g(\, \psi_i^{IJ}(x)\,) \; := \; (-1)^{-\hat{Q}_g(i)} \, \psi_i^{gI\,gJ}(x) \quad . \tag{2.6}$$

Here we use that in the Cardy theory the boundary labels I, J are taken from the set $\mathcal{J}$ which comes equipped with the action of Γ that we described above. Using the following symmetry property of the fusing matrix [30]

$$F_{gM,k}\left[\begin{smallmatrix} i & j \\ gL & gN \end{smallmatrix}\right] \; = \; F_{M,k}\left[\begin{smallmatrix} i & j \\ L & N \end{smallmatrix}\right](-1)^{\hat{Q}_g(i)+\hat{Q}_g(j)-\hat{Q}_g(k)} \tag{2.7}$$

one can show that the operator product expansions (2.3) respect the action of the simple current group Γ on boundary fields. In this sense, Γ describes a symmetry of the theory on the covering space.

3 Boundary CFT on the orbifold

Our goal now is to discuss D-branes on an orbifold of the original conformal field theory. Geometrically, one would like to understand these branes on the orbifold space through D-branes on the covering space. In such an approach, a brane on the orbifold gets represented by several pre-images on the covering space which are mapped onto each other by the action of the orbifold group. As discussed in [25, 30], there is a large class of cases in which these geometric ideas carry over to the construction of branes in exactly solvable conformal field theories.

To begin with, let us give a precise formulation of our main assumption on the partition function $Z^{\rm orb}(q)$ of the bulk theory that we want to study. We assume that there is an orbifold group Γ within the group of all simple currents such that $Z^{\rm orb}$ is of the form (see e.g. [21])

$$Z^{\rm orb}(q,\bar q) \;=\; \sum_{[j],Q_\Gamma([j])=0} |\,\mathcal{S}_{[j]}\,|\,|\sum_{j'\in[j]} \chi_{j'}(q)\,|^2 \quad . \tag{3.1}$$

Note that this partition function does not have the simple form (2.1) so that Cardy's theory for the classification and construction of D-branes does not apply directly.

An orbifold theory with bulk partition function of the form (3.1) possesses consistent boundary theories which are assigned to orbits $[I]$ of labels I that parametrize the boundary theories of the parent CFT. The open string spectra associated with a pair of such branes on the orbifold are given by

$$Z^{\rm orb}_{[I][J]}(q) \;=\; \sum_{g,k} N_{I\;k}^{\;gJ}\chi_k(q) \quad . \tag{3.2}$$

This agrees precisely with the prediction from the geometric picture of branes on orbifolds. In fact, the I, J can be considered as geometric labels specifying the position of the brane on the covering space. To compute spectrum of two branes $[I]$ and $[J]$ of the orbifold theory, we lift $[I]$ to one of its pre-images I on the covering space and include all the open strings that stretch between this fixed brane I on the cover and an arbitrary pre-image gJ of the second brane $[J]$.

It is important to notice that in many cases the boundary conditions $[I]$ can be further resolved, i.e. there exists a larger set of boundary theories such that $[I]$ can be written as a superposition of boundary theories with integer coefficients. This happens whenever the stabilizer subgroup $\mathcal{S}_{[I]}$ is non-trivial. In the absence of discrete torsion, the elementary branes resolving the boundary condition $[I]$ are labeled by characters $a, b, c, \dots$ of $\mathcal{S}_{[I]}$. [1] Geometrically, this corresponds to the fact that the Chan-Paton factors of branes at orbifold fixed points can carry different representations of the stabilizer subgroup.

[1]More general possibilities including discrete torsion have been discussed in [3, 4, 6, 5, 8, 9]. The extension to general conformal field theory backgrounds can be found in [28]

To spell out the spectrum of open strings stretching between two such resolved branes $[I]_a$ and $[J]_b$ we need some more notation. Let $H = H_{[I][J]} \subset \Gamma$ denote the subgroup $\mathcal{S}_{[I]} \cap \mathcal{S}_{[J]}$ of our symmetry group Γ. The characters $e_a^{[I]} : \mathcal{S}_{[I]} \to \mathrm{U}(1)$ and $e_b^{[J]} : \mathcal{S}_{[J]} \to \mathrm{U}(1)$ restrict to the common subgroup H so that the following numbers are well defined

$$d^{a}{}_{b}^{k} \; := \; \frac{1}{|H|} \sum_{h \in H} e_a(h)\,(-1)^{\hat{Q}_h(k)}\,e_b(h^{-1}) \quad . \tag{3.3}$$

With the chosen normalization, d takes values in the set $\{0,1\}$. The partition functions for the resolved branes are given by (see [16, 17, 18, 15] for formulas that deal with general backgrounds)

$$Z^{\mathrm{orb}}_{[I]_a[J]_b}(q) \; = \; \frac{1}{|\mathcal{S}_{[I]} \cdot \mathcal{S}_{[J]}|} \sum_{g,k} N_{I}{}^{gJ}_{k}\; d^{a}{}_{b}^{k}\; \chi_k(q) \quad . \tag{3.4}$$

The normalization ensures that the coefficients appearing in front of the characters $\chi_k(q)$ are integer. To see this one should note that

$$N_{I}{}^{gJ}_{k} \neq 0 \quad \Rightarrow \quad N_{I}{}^{g_1 g g_2^{-1} J}_{k} \neq 0 \quad \text{for all} \quad g_1 \in \mathcal{S}_{[I]}\ ,\ g_2 \in \mathcal{S}_{[J]} \quad .$$

This means that every term in eq. (3.2) comes with multiplicity

$$|\mathcal{S}_{[I]} \cdot \mathcal{S}_{[J]}| \; = \; |\mathcal{S}_{[I]}|\,|\mathcal{S}_{[J]}|\,|H|^{-1}$$

where the order of the subgroup H appears because H is isomorphic to the kernel of the multiplication map $(g_1, g_2) \mapsto g_1 g_2^{-1} \in \Gamma$. Let us remark that the partition functions (3.4) of the resolved branes sum up to the partition function (3.2) of the projected boundary states. This follows easily from the property $\sum_{a,b} d^{a}{}_{b}^{k} = |\mathcal{S}_{[I]} \cdot \mathcal{S}_{[J]}|$ of the constants d. Let us note that the expression (3.4) follows the usual intuition that was developed in the context of orbifolds of the form $\mathbb{C}^n/\Gamma$ (see e.g. [1, 5, 10]). In fact, the formula guarantees that an open string mode that transforms according to the representation $\hat{Q}_h(k) : h \mapsto \hat{Q}_h(k)$ of the subgroup H appears in the spectrum of strings stretching between the branes $[I]_a$ and $[J]_b$ if and only if the representation e_a of the first brane $[I]_a$ together with the representation $\hat{Q}(k)$ add up to the representation e_b of the second brane $[J]_b$ ('conservation of charges').

Restricting at first to unresolved D-branes, we will now give explicit expressions for the operator products of boundary fields. Before writing them down, let us have another look at eq. (3.2) and observe that for fixed I, J, k there can be several group elements $g \in \Gamma$ such that $N_{I\,k}^{gJ} \neq 0$. We denote the associated subspace of Γ by

$$\Gamma(\,_I{}^k{}_J\,) \;=\; \{\, g \in \Gamma \,|\, N_{I\,k}^{gJ} \neq 0 \,\} \quad . \tag{3.5}$$

While the size of $\Gamma(\,_I{}^k{}_J\,)$ depends only on k and the orbits $[I], [J]$, the subsets $\Gamma(\,_I{}^k{}_J\,)$ are selected depending on the choice of representatives $I \in [I]$ and $J \in [J]$. If we shift these representatives along their orbits, the $\Gamma(\,_I{}^k{}_J\,)$ shift according to

$$\Gamma(\,_I{}^k{}_{gJ}\,) \;=\; g^{-1}\,\Gamma(\,_I{}^k{}_J\,) \quad \text{and} \quad \Gamma(\,_{gI}{}^k{}_J\,) \;=\; g\,\Gamma(\,_I{}^k{}_J\,) \quad .$$

The group elements $g \in \Gamma(\,_I{}^k{}_J\,)$ label fields in the boundary theory that describes open strings stretching between the unresolved branes $[I]$ and $[J]$. We claim that they possess the following operator product expansions,

$$\Psi^{[L][M]}_{i,g_1}\;\Psi^{[M][N]}_{j,g_2} \;=\; \sum_k \;\Psi^{[L][N]}_{k,g_{12}}\;(-1)^{-\hat{Q}_{g_1}(j)}\,F_{g_1M,k}\big[\,{}^{i}_{L}\;{}^{j}_{g_{12}N}\,\big] \;+\; \dots \tag{3.6}$$

where we suppressed the obvious dependence on world-sheet coordinates. L, M, N are representatives of the orbits $[L], [M], [N]$, and the group element g_{12} on the right hand side is given by

$$g_{12} \;=\; g_1\,g_2 \;\in\; \Gamma(\,_L{}^k{}_N\,) \;=\; \Gamma(\,_L{}^i{}_M\,)\cdot\Gamma(\,_M{}^j{}_N\,) \quad .$$

Consistency of the boundary operator product expansion (3.6) requires that the coefficients on the right hand side satisfy certain sewing constraints that were first formulated by Lewellen [37] (see also [14, 38, 29]). For the case at hand, these are checked in the Appendix A.2.

Let us pause here for a moment and add two comments which can provide some insight into the formula (3.6). First, one should observe that the operator product expansions we propose mimic some kind of crossed product construction. [2] The formal similarities become most obvious if we think of the fields in the orbifold theory as a product of the form $\Psi^{[I][J]}_k \cdot g$ where the notation separates the dependence on the element $g \in \Gamma(\,_I{}^k{}_J\,)$ from the field. In multiplying such composite objects, we

[2]The relation between crossed products and orbifolds is well known in string theory (see e.g. [39] and references therein).

would begin by moving the g_1 from the first field through the second field. The factor $\exp(-\pi i \hat{Q}_{g_1}(j))$ encodes the non-trivial transformation law (2.6) of the second field under the action of g_1. Following rel. (2.3), multiplication of the two fields gives rise to a fusing matrix on the right hand side of eq. (3.6) and the resulting field comes with a factor $g_{12} = g_1 g_2$. Even though all this discussion is very symbolic, it captures nicely the basic ingredients of the formula (3.6). We will make the connection with the crossed product more precise when we discuss the first example below.

In the special case that all the involved stabilizer groups $\mathcal{S}$ are trivial, we can obtain the expansions (3.6) of the orbifold theory directly from the relations (2.3) on the covering space. In fact, under the assumption of trivial stabilizers, the new boundary fields in the orbifold theory can be constructed by averaging the boundary fields of the theory on the cover with respect to the group action, i.e.

$$\Psi^{[I][J]}_{i,g}(x) \ := \ \sum_{g' \in \Gamma} g'(\, \psi^{I\, gJ}_{i}(x)\,) \quad .$$

Here we have chosen representatives I, J of the orbits $[I], [J]$. It is then easy to recover eq. (3.6) from the corresponding expansions (2.3) on the cover.

When some of the boundary labels have non-trivial stabilizer groups, the boundary fields of the unresolved theory must be resolved according to the formula (3.4). As we shall show below, this is achieved by the expressions

$$\Psi^{[I]_a [J]_b}_{k,g} = \sum_{g_1 \in S_I} \sum_{g_2 \in S_J} \Psi^{[I][J]}_{k, g_1 g g_2^{-1}} (-1)^{-\hat{Q}_{g_1}(k)} e_a(g_1) e_b(g_2^{-1}) \quad . \tag{3.7}$$

Here, g runs through the set $\Gamma(\,{}_I{}^k{}_J\,)$ as above. Note, however, that the set $\Gamma(\,{}_I{}^k{}_J\,)$ carries an action of the subgroup $\mathcal{S}_I \cdot \mathcal{S}_J$. When we shift a field of the form (3.7) by $g_I \in \mathcal{S}_I$ and $g_J \in \mathcal{S}_J$ it behaves according to

$$\Psi^{[I]_a [J]_b}_{k, g_I g g_J^{-1}} \ = \ e_a(g_I^{-1}) e_b(g_J)\, \Psi^{[I]_a [J]_b}_{k,g} \quad . \tag{3.8}$$

After taking these relations into account, the space of boundary fields is labeled by elements in the coset space $\Gamma(\,{}_I{}^k{}_J\,)/\mathcal{S}_I \cdot \mathcal{S}_J$. Furthermore, it is easy to see that the expression (3.7) vanishes if $d^{a\,k}_{\ b} = 0$. These two observations together show that the space of fields is in agreement with the partition functions (3.4).

The expression for the boundary operator product expansions of the boundary fields (3.7) can now be calculated from the one for the unresolved fields. The result is

$$\Psi^{[I]_a[J]_b}_{i,g_1}\,\Psi^{[J]_c[K]_d}_{j,g_2} \;=\; \delta_{b,c} \sum_{g\in\mathcal{S}_J,k} \Psi^{[I]_a[K]_d}_{k,gg_1g_2}\, e_b(g)\,(-1)^{-\hat{Q}_{gg_1}(j)}\, F_{g_1J,k}\big[\begin{smallmatrix} i & j \\ I & gg_1g_2K\end{smallmatrix}\big] + \dots \quad , \tag{3.9}$$

where we have again neglected to spell out the obvious coordinate dependence. In Appendix A.3 it is shown that the coefficients on the right hand side of the above expression satisfy the appropriate sewing constraint. Our formulas (3.4,3.9) provide a complete solution of the open string sector in the orbifold theory.

4 Applications to WZW- and minimal models

In the final section we want to outline some simple examples of orbifolds that are covered by the general analysis of the previous sections. These include open strings on a $\mathbb{Z}_2$ orbifold of SU(2) and the so-called D_{even}-series of minimal models. In the former case, our results admit a nice geometric interpretation which will be discussed at the end of the first subsection.

4.1 The $\mathbb{Z}_2$ orbifold of the SU(2) WZW model

The first example that we are going to discuss is given by orbifolds in the SU(2) WZW model at level $k = 4n$ (see e.g. [40]). Let us start with the diagonal bulk partition function describing the theory before orbifolding. In our convention, the $k+1$ different sectors of the model will be labeled by $l = 0, 1, ..., k$ and the bulk partition function is given by

$$Z(q,\bar{q}) \;=\; \sum_{l=0}^{k} \mid \chi_l(q) \mid^2 \quad . \tag{4.1}$$

The fusion product of any two sectors l_1, l_2 can be computed using the standard rule

$$[l_1] \times [l_2] \;=\; [|l_1 - l_2|] + [|l_1 - l_2| + 2] + \dots + [\min(l_1 + l_2, 2k - l_1 - l_2)] \quad . \tag{4.2}$$

From this we can read off that the simple currents are given by $l = 0$ and $l = k$. These two sectors form the group $\Gamma = \mathbb{Z}_2 = \{0, k\}$ which we will use in the orbifold construction.

To specialize the general formula (3.1) to our example we need some preparation. Let us note first that the charges $\hat{Q}$ defined in (4.13) can be found from the usual expression for the braiding matrix of the WZW model (see e.g. [33]), which implies

$$\Omega\left(\begin{smallmatrix} & i & \\ l & & j \end{smallmatrix}\right) \;=\; (-1)^{\frac{1}{2}(i+j+l)}\, e^{\pi i(h_i+h_j-h_l)} \quad . \tag{4.3}$$

Here $h_l = l(l+2)/(4k+8)$ is the conformal dimension of the sector l at level k. Using these formulas for our simple currents $l = 0$ and $l = k$, we obtain

$$\hat{Q}_0(j) \;=\; 0 \qquad , \qquad \hat{Q}_k(j) \;=\; \frac{j}{2}. \tag{4.4}$$

Orbits of $\Gamma = \mathbb{Z}_2$ consists of pairs $\{l, k-l\}$ as long as $l \neq k/2$ and the sector $l = k/2$ leads to an orbit of length 1 with stabilizer group $\mathcal{S}_{[k/2]} = \mathbb{Z}_2$. An important condition for the general theory to apply is that the members of a Γ-orbit have the same monodromy charge $Q = \hat{Q}$ mod 1. For the orbits $[l] = \{l, k-l\}$ this amounts to $l/2 = (k-l)/2$ mod 1 which holds true since $k = 4n$ is even. Finally, the orbits $[l]$ have vanishing monodromy charge Q if $l = 2m$ and the orbit $[k/2] = [2n]$ of length 1 belongs to this set. All these observations are summarized in the following formula for the partition function of the orbifold theory,

$$Z^{\mathrm{orb}}(q,\bar{q}) \;=\; \sum_{l=0}^{2n-2} |\chi_l(q) + \chi_{4n-l}(q)|^2 + 2|\chi_{2n}(q)|^2 \tag{4.5}$$

where the summation is over even l only. Z^{orb} is known as the $\mathrm{D}_{\mathrm{even}}$ modular invariant of the SU(2) WZW model.

According to the general theory our orbifold model has branes labeled by $L = [0], [1], \ldots, [k/2-1]$ and two additional branes $[k/2]_\pm$ associated with the two characters e_+ and e_- of the stabilizer group $\mathcal{S}_{k/2} = \mathbb{Z}_2$. We will not discuss them in full detail here but rather restrict to the most interesting case which appears when the open strings have both ends on branes sitting at the fixed point. Before resolution, the partition function (3.2) for these open strings reads

$$Z^{\mathrm{orb}}_{[k/2][k/2]}(q) \;=\; \sum_{l=0}^{k} 2\,\chi_l(q) \tag{4.6}$$

where summations runs over even l. To split this Z^{orb} into the partition functions (3.4) for resolved branes $[k/2]_\pm$ we compute the associated symbol d. In a matrix notation

$d^l = (d^{a\,l}{}_b)$ it is given by

$$d^{4p} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad d^{4p+2} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $p = 0, \dots, n$. When inserted into eq. (3.4), the resolved partition functions become

$$Z^{\rm orb}_{[k/2]_\pm [k/2]_\pm}(q) \;=\; \sum_{p=0}^{n} \chi_{4p}(q) \quad , \quad Z^{\rm orb}_{[k/2]_\pm [k/2]_\mp}(q) \;=\; \sum_{p=1}^{n} \chi_{4p-2}(q) \quad . \tag{4.7}$$

These expressions are well known from previous studies of boundary conditions in the $\mathrm{D_{even}}$ theory (see e.g. [15]). Our formula (3.9) describes the operator products of boundary fields in this model and thereby completes the solution of the model.

It is quite instructive to relate the operator product expansions we found to the geometry of branes at the fixed point. Recall from [41] that branes on the SU(2) are localized along the k+1 'integer' conjugacy classes of SU(2)= S^3. The latter are 2-spheres centered around the origin $e \in$ SU(2) with the $(k/2)^{th}$ 2-sphere wrapping the equator of S^3. Since the non-trivial element of our orbifold group $\Gamma = \mathbb{Z}_2$ acts by *reflection* $g \to -g$ along the equator of S^3, the $(k/2)^{th}$ brane is located along the fixed surface of the group action, in agreement with our algebraic results above.

The algebra $Fun(S^2)$ of functions on the equatorial 2-sphere $S^2 \subset S^3$ is spanned by spherical harmonics Y^l_σ where $|\sigma| \leq l/2$ and l is an even integer (recall that we re-scaled all spins by a factor 2). This algebra inherits an involution ϑ from the reflection on S^3. It acts on spherical harmonics as $\vartheta(Y^l_\sigma) = (-1)^{l/2} Y^l_\sigma$. The involution ϑ and its square $\vartheta^2 = \mathrm{id}$ give rise to an action of $\mathbb{Z}_2$ on $Fun(S^2)$. Following a general construction, we can use this data to pass to the *crossed product* $Fun(S^2) \times_\vartheta \mathbb{Z}_2$. This amounts to extending the algebra of functions on S^2 by one additional element $\theta = \theta_k$ subject to the conditions $\theta^2 = 1$ and

$$\theta \; Y^l_\sigma \;=\; \vartheta(Y^l_\sigma)\; \theta \;=\; (-1)^{l/2}\, Y^l_\sigma \; \theta \quad .$$

Hence the new algebra is spanned by $Y^l_\sigma \theta_0 = Y^l_\sigma$ and $Y^l_\sigma \theta$, i.e. its basis contains two SU(2) multiplets for each even integer l. All this is very similar to the structure of the partition function (4.6). In fact, we see that the latter contains each representation of the SU(2) current algebra with multiplicity 2. For finite k, however, labels $l > k$ do not appear in the partition function. This truncation in the spectrum of boundary fields is related to the quantization of the spherical branes of the SU(2) WZW model

[34]. One can show that, whenever the level k is finite, all spherical branes on S^3 carry a non-vanishing B-field. [3]

The similarity between the crossed product geometry and the open string theory goes much further. For $k \to \infty$ we may identify the basis elements of the crossed products with the boundary primary fields according to

$$V[Y^l_\sigma \theta_g](x) \ = \Psi^{[k/2][k/2]}_{l,\sigma;g} \quad \text{for} \quad g \in \mathbb{Z}_2 \quad .$$

Here, σ labels different boundary fields associated with the ground states of the sector l of the WZW-model. Following the arguments in [34], it is easy to see that this identification preserves the multiplication, i.e. one finds

$$\Psi^{[k/2][k/2]}_{i,\sigma_1;g_1}(x_1)\, \Psi^{[k/2][k/2]}_{j,\sigma_2;g_2}(x_2) \ \stackrel{k\to\infty}{\to} \ V[Y^l_{\sigma_1}\theta_{g_1} Y^l_{\sigma_2}\theta_{g_2}](x_2) + \dots$$

In the spirit of [42, 34], this shows that the geometry of the unresolved equatorial brane is described by the crossed product $Fun(S^2) \times_\vartheta \mathbb{Z}_2$.

Passing on to the geometry of the resolved branes, it is rather easy to see that the elements $Y^l_\sigma \cdot (1 \pm \theta), l = 4m$, generate two sub-algebras of $Fun(S^2) \times_\vartheta \mathbb{Z}_2$. These elements correspond to primary boundary fields (3.7) for open strings which have both ends on the same resolved brane $[k/2]_\pm$. The rest of the basis in the crossed product, namely the elements $Y^l_\sigma \cdot (1 \pm \theta), l = 4m - 2$, are associated with open strings that stretch between two different resolved branes.

We conclude this section with a few remarks on the dynamics of branes in $\mathrm{SU}(2)/\mathbb{Z}_2$. Let us remark first that for a spherical brane with fixed label L the computation of the effective action carries over from [34]. This means that one still finds a linear combination of a Yang-Mills and a Chern-Simons term on a fuzzy S^2 to control the behavior of these branes in the limit $k \to \infty$. In particular, a stack of $D0$ branes at the origin is unstable and it can expand into a spherical brane. Following the reasoning of [43, 44], one can show that these (unresolved) branes contribute a term $\mathbb{Z}_{k/2+1}$ to the charge group of the background. In addition, the resolved branes at the equator carry one extra charge that can be measured through the coupling of the closed string states in the twisted sector when the level k becomes large.

[3] In the $k \to \infty$ theory, the B-field can be non-zero, but it vanishes on the equatorial 2-sphere.

4.2 D-branes in the D-series of minimal models

In this section we will analyze D-branes in minimal models. Our aim is to show that our general theory is applicable to branes in the D$_{even}$-series of minimal models. Their boundary operator product expansions were first analysed by Runkel in [29], but the resulting expressions where rather complicated and difficult to work with. For the case of the D$_{even}$-series, our formulas represent a considerable simplification.

Minimal models $M(p,p')$ are labeled by two integers p,p' [40]. Their chiral algebra is generated by Virasoro fields with central charge

$$c \;=\; 1 - 6\frac{(p-p')^2}{pp'} \quad . \tag{4.8}$$

The primary fields are labeled by pairs (r,s) with $1 \leq r \leq p'-1$ and $1 \leq s \leq p-1$. We compute their conformal weights through the formula

$$h_{r,s} \;=\; \frac{(pr-p's)^2-(p-p')^2}{4pp'} \quad . \tag{4.9}$$

Let us note that each sector of the model is represented by two pairs of the form (r,s). More precisely, the two labels

$$(r,s) \leftrightarrow (p'-r, p-s) \tag{4.10}$$

are associated with the same sector. This is consistent with the formula (4.9) for the conformal weights and it motivates to introduce the fundamental region $E(p,p')$ containing pairs (r,s) which satisfy $p's < pr$. By construction, $E(p,p')$ contains each sector exactly once. For unitarity it is needed that $p = p'+1$ and from now on we always assume this to be the case.

Let us proceed by investigating which simple currents are present in this model. This can be read off from the well known fusion rules

$$(r,s)\times(m,n) \;=\; \sum_{\substack{k=1+|r-m| \\ k+r+m=1\,mod\,2}}^{k_{max}} \quad \sum_{\substack{l=1+|s-n| \\ l+s+n=1\,mod\,2}}^{l_{max}} (k,l) \tag{4.11}$$

where

$$\begin{aligned} k_{max} &= \min(r+m-1, 2p'-(r+m+1)) \\ l_{max} &= \min(s+n-1, 2p-(s+n+1)) \end{aligned} \quad . \tag{4.12}$$

Using these formulas together with the identifications (4.10), it is rather easy to see that the simple currents are given by the labels $(1,1)$ and $(p'-1,1) \cong (1,p-1)$. Obviously, these two simple currents generate a group $\Gamma = \{(1,1),\ (p'-1,1)\} \cong \mathbb{Z}_2$. This is the group that we use in the orbifold construction. The orbits $[(r,s)]$ contain the sectors (r,s) and $(p'-r,s) \cong (r,p-s)$. A non-trivial stabilizer $\mathcal{S}_{(r,s)} = \mathbb{Z}_2$ appears if $r = p'/2$ or $s = p/2$.

Next, we shall construct the charge $\hat{Q}_g(j)$ of a sector j with respect to a simple current g. Using the definition of $\hat{Q}_g(j)$ from (4.13), and the relation $\Omega(\,{}_{gj}\,{}^{j}\,{}_{g}\,) = \exp \pi i(h_j + h_g - h_{gj})$, we can conclude that

$$\hat{Q}_g(j) \ = \ h_j + h_g - h_{gj} \quad \mathrm{mod}\ 2 \ \ . \tag{4.13}$$

Inserting the formula (4.9) for conformal weights and the labels for elements of the orbifold group $\Gamma = \{(1,1),\ (p'-1,1)\}$, the expression for $\hat{Q}$ reduces to

$$\begin{aligned} \hat{Q}_{(1,1)}(r,s) \ &= \ 0 \quad \mathrm{mod}\ 2 \\ \hat{Q}_{(p'-1,1)}(r,s) \ &= \ -rs + 1 + \frac{1}{2}(pr + p's - p - p') \quad \mathrm{mod}\ 2 \ \ . \end{aligned} \tag{4.14}$$

The condition $Q_\Gamma[(r,s)] = 0$ of vanishing monodromy charge is satisfied for odd r if p' is even and for odd s if p is even.

Let us now write down the relevant bulk partition functions. The A-series bulk partition function is given by the diagonal modular invariant

$$Z_{A_{p'-1},A_{p-1}} = \sum_{(r,s)\in E(p,p')} |\chi_{r,s}(q)|^2. \tag{4.15}$$

From this we can obtain the following D$_{even}$-series partition functions by orbifolding with the simple current group $\Gamma = \mathbb{Z}_2 = \{(1,1),(p'-1,1)\}$ (see e.g. [40]),

$$p' \ = \ 2(2m+1): \quad Z_{D_{p'/2+1},A_{p-1}} \ = \ \frac{1}{2} \sum_{\substack{(r,s)\in E(p,p') \\ r \quad odd}} |\chi_{r,s}(q) + \chi_{p'-r,s}(q)|^2, \tag{4.16}$$

$$p \ = \ 2(2m+1): \quad Z_{A_{p'-1},D_{p/2+1}} \ = \ \frac{1}{2} \sum_{\substack{(r,s)\in E(p,p') \\ s \quad odd}} |\chi_{r,s}(q) + \chi_{r,p-s}(q)|^2, \tag{4.17}$$

where m is an integer. Note that the unitarity condition $p = p' + 1$ implies that either p or p' is even but the conditions we have formulated above require that neither p nor p' is a multiple of 4. The partition functions of the D_{even}-series are precisely of the form (3.1). This can be seen using the properties of the simple currents, the field identification (4.10) and the discussion from the previous paragraph. To explain the pre-factors 1/2 we have a short look at the first partition function (4.16). If the first entry r in the label $j = (r, s)$ is odd then so is $p' - r$ in $gj = (p' - r, s)$. Consequently, each orbit $[j]$ with trivial stabilizer will contribute twice to the sum over $E(p, p')$. When $j = (r, s)$ has a non-trivial stabilizer subgroup, the argument is slightly modified: From (3.1), we infer that such a term will appear with a factor of 2. This is in agreement with (4.16), since $\chi_{r,s} + \chi_{p'-r,s} = 2\chi_{r,s}$ so that the character of a fixed point appears with a factor 4 before we multiply the whole partition function by 1/2. The same arguments apply to eq. (4.17).

As discussed previously, an orbifold theory with a bulk partition function of the type (3.1) possesses consistent boundary theories with partition functions given by formula (3.4),

$$Z^{\mathrm{orb}}_{[I]_a[J]_b}(q) \;=\; \sum_{k\in E(p,p')} n_{[I]_a}{}^{[J]_b}_{\;k}\,\chi_k(q) \quad , \tag{4.18}$$

where

$$n_{[I]_a}{}^{[J]_b}_{\;k} \;=\; \frac{1}{|\mathcal{S}_{[I]}\cdot\mathcal{S}_{[J]}|}\; d^{a\;k}_{\;b}\;\sum_{g\in\Gamma} N_{I\;k}^{\;gJ} \quad . \tag{4.19}$$

Since $|\mathcal{S}_{[I]}\cdot\mathcal{S}_{[J]}| \;=\; |\mathcal{S}_{[I]}|\,|\mathcal{S}_{[J]}|\,|H|^{-1}$, we conclude that $|\mathcal{S}_{[I]}\cdot\mathcal{S}_{[J]}| \;=1$ if neither $[I]$ nor $[J]$ has a non-trivial stabilizer subgroup. If at least one of them does, the correct factor is instead $|\mathcal{S}_{[I]}\cdot\mathcal{S}_{[J]}| \;=2$. Moreover, we have $d^{(r,s)} = 1$ if either $[I]$ or $[J]$ has trivial stabilizer. Otherwise, the labels a, b can assume two different values $a, b = \pm$ and using the same matrix notations as in the previous subsection the symbol d reads

$$d^{(r,s)} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{if} \quad s \;=\; 1 \bmod 4$$

$$d^{(r,s)} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{if} \quad s \;=\; 3 \bmod 4 \quad .$$

Here, we assumed that $p = 2(2m+1)$. The same expressions for d are used in case of $p' = 2(2m+1)$ but with conditions depending on r rather than s. Collecting the results for this discussion, we conclude that

$$
\begin{aligned}
(1) \quad & n_{[I]_a}{}^{[J]_b}_{\;k} \;=\; N_{I\,k}^{\;J} + N_{I\,k}^{\;gJ} \\
(2) \quad & n_{[I]_a}{}^{[J]_b}_{\;k} \;=\; N_{I\,k}^{\;J} \;=\; N_{I\,k}^{\;gJ} \\
(3) \quad & n_{[I]_a}{}^{[J]_b}_{\;k} \;=\; N_{I\,k}^{\;J} \;=\; d_b^{a\,k} \quad ,
\end{aligned}
\tag{4.20}
$$

where (1) is valid when neither stabilizer subgroup is non-trivial, (2) applies to the case when precisely one stabilizer subgroup is non-trivial and (3) describes the case of both stabilizer subgroups being non-trivial. We note that these spectra coincide with the partition functions that were studied by Runkel [29]. Hence, our general theory is indeed applicable to the D_{even}-series of minimal models for the modular invariants (4.16) and (4.17), resulting in rather simple formulas for the boundary operator product expansions of the associated boundary theories. Most importantly, this opens the way for studies of brane dynamics in the D_{even}-series of minimal models along the lines of [45].

5 Conclusions and outlook

In this work we provided the boundary operator product expansions (3.6,3.9) for a large class of orbifolds and illustrated the results in two non-trivial examples. Whereas the $\mathbb{Z}_2$ orbifold of the SU(2) WZW-model admits a nice geometric interpretation, there is no obvious geometry underlying the D_{even}-series of the minimal models. It is one of the advantages of boundary conformal field theory techniques that they do not require the existence of any geometric orbifold action.

The most important applications of our results are associated with the investigation of B-type branes in Gepner models. As proposed in [25], these branes should be analysed as A-type branes on the mirror where the latter is realized as an orbifold using the Greene-Plesser construction [46]. Together with the projection on integer charges, the Greene-Plesser orbifold group fits into the framework we have discussed here. To be precise, let us note that discrete torsion can occur for B-type branes in Gepner models [27, 28]. Based on the formula (3.6) for the unresolved branes, however, it is not difficult to include such cases into our formalism.

The boundary operator product expansions of the form (3.6, 3.9) may be used to calculate correlation functions of open string vertex operators for a very large class of branes in Gepner models. Computations of this type were initiated in [30] in the case of A-type branes and they can provide important insight into the super-potential of branes deep in the stringy regime. For B-type branes, this is particularly interesting since one expects that such data do not require corrections upon variation of the Kähler moduli. Hence, the conformal field theory data may be compared with geometric results in the large volume whenever the latter are available. Otherwise, conformal field theory predicts e.g. the dimension of moduli spaces for super-symmetric cycles. These issues are currently under investigation.

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A Appendix: Solution of the sewing constraints

Sewing constraints for the coefficients of operator product expansions are consistency conditions which are obtained by taking different limits in correlation functions [37, 38, 29]. In this section we wish to show that the solutions we propose do indeed satisfy the appropriate sewing constraints. We begin by making some preliminary considerations. Then we proceed to the cases at hand, the unresolved and the resolved branes in the orbifold theory.

A.1 Collection of relevant formulas

Let us begin by collecting a few relations which will prove to be useful below. First, we note that the charges $\hat{Q}_g(k)$ provide a representation of the simple current orbifold group Γ in the sense that

$$\hat{Q}_{g_1 g_2}(k) \;=\; \hat{Q}_{g_1}(k) + \hat{Q}_{g_2}(k) \quad , \quad \hat{Q}_{id}(k) \;=\; 0 \quad . \tag{A.1}$$

The latter equation is a consequence of the former. Moreover, from the work [33], we know that the fusing matrix F obeys the pentagon relation,

$$\sum_s F_{p_2,s}\left[\begin{smallmatrix} j & k \\ p_1 & b \end{smallmatrix}\right] F_{p_1,l}\left[\begin{smallmatrix} i & s \\ a & b \end{smallmatrix}\right] F_{s,r}\left[\begin{smallmatrix} i & j \\ l & k \end{smallmatrix}\right] \;=\; F_{p_1,r}\left[\begin{smallmatrix} i & j \\ a & p_2 \end{smallmatrix}\right] F_{p_2,l}\left[\begin{smallmatrix} r & k \\ a & b \end{smallmatrix}\right] \quad , \tag{A.2}$$

along with the following relation, expressing a symmetry of the fusing matrix F,

$$F_{p,i}\left[\begin{smallmatrix} j & k \\ n & l \end{smallmatrix}\right] F_{n,0}\left[\begin{smallmatrix} i & i \\ l & l \end{smallmatrix}\right] \; = \; F_{n,k}\left[\begin{smallmatrix} i & j \\ l & p \end{smallmatrix}\right] F_{p,0}\left[\begin{smallmatrix} k & k \\ l & l \end{smallmatrix}\right] \quad . \tag{A.3}$$

Recall that in our conventions, the label 0 corresponds to the vacuum representation. Completing the list of properties of the fusing matrix, we finally note the relation

$$F_{p,0}\left[\begin{smallmatrix} i & i^\vee \\ j & k \end{smallmatrix}\right] \; = \; 0 \quad \text{ if } j \neq k^\vee \tag{A.4}$$

To simplify notations we will assume that $j = j^\vee$ throughout this appendix. Proofs for the general case follow the same strategy but are a bit more tedious.

Based on the relations (A.1)-(A.4) we will now derive an equation that will serve as departing point for the proof of the sewing constraints. Before stating it, we need some more notation. Let us choose four elements

$$g_1 \in \Gamma\left({}_I{}^i{}_J \right) \; , g_2 \in \Gamma\left({}_J{}^j{}_K \right) \;\; , \quad g_3 \in \Gamma\left({}_K{}^k{}_L \right) \; , g_4 \in \Gamma\left({}_L{}^l{}_I \right) \tag{A.5}$$

and use the shorthand notation g_{12} for the product $g_1 g_2$ etc. Furthermore, we pick group elements $g_I \in \mathcal{S}_I$, $g_J \in \mathcal{S}_J$ etc. satisfying the relation

$$g_1 g_2 g_3 g_4 g_I g_J g_K g_L \; = \; \text{id} \quad . \tag{A.6}$$

After this preparation we want to show that the constants

$$C^{[I]_a [J]_b [K]_c}_{(i,g_1)(j,g_2)(k,g_{12}\tilde{g})} := F_{g_1 J,k}\left[\begin{smallmatrix} i & j \\ I & g_{12}\tilde{g}K \end{smallmatrix}\right] (-1)^{-\hat{Q}_{g_1 \tilde{g}}(j)} e_b(\tilde{g}) \tag{A.7}$$

obey the following system of equations,

$$\begin{aligned} & C^{[J]_b[K]_c[L]_d}_{(j,g_2)(k,g_3)(q,g_{23}g_K)} C^{[I]_a[J]_b[L]_d}_{(i,g_1)(q,g_{23}g_K)(l,g_{123}g_K g_J)} C^{[I]_a[L]_d[I]_a}_{(l,g_{123}g_K g_J)(l,g_4)(0,g_{123}g_4 g_K g_J g_L)} \\ & = \sum_p C^{[I]_a[J]_b[K]_c}_{(i,g_1)(j,g_2)(p,g_{12}g_J)} C^{[K]_c[L]_d[I]_a}_{(k,g_3)(l,g_4)(p,g_{34}g_L)} C^{[I]_a[K]_c[I]_a}_{(p,g_{12}g_J)(p,g_{34}g_L)(0,g_{12}g_{34}g_J g_L g_K)} F_{p,q}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \quad . \end{aligned} \tag{A.8}$$

To prove this relation we spell out its right-hand side (rhs) using (A.7) and the eq. (A.1),

$$\begin{aligned} \text{rhs} \; = \; & \sum_p F_{g_1 J,p}\left[\begin{smallmatrix} i & j \\ I & g_{12}g_J K \end{smallmatrix}\right] F_{g_3 L,p}\left[\begin{smallmatrix} k & l \\ K & g_{34}g_L I \end{smallmatrix}\right] F_{g_{12}g_J K,0}\left[\begin{smallmatrix} p & p \\ I & I \end{smallmatrix}\right] F_{p,q}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \\ & (-1)^{-\hat{Q}_{g_1}(j) - \hat{Q}_{g_3}(l) - \hat{Q}_{g_{12}}(p) - \hat{Q}_{g_J}(p) - \hat{Q}_{g_J}(j) - \hat{Q}_{g_L}(l) - \hat{Q}_{g_K}(p)} \quad . \end{aligned} \tag{A.9}$$

We now want to apply the pentagon relation. To this end, let us rewrite the first and third fusing matrix in the previous expression with the help of eqs. (A.3) and (2.7) for $g = g_{34}$, i.e.

$$\begin{aligned} F_{g_1J,p}\big[\begin{smallmatrix} i & j \\ I & g_{12}g_JK \end{smallmatrix}\big] F_{g_{12}g_JK,0}\big[\begin{smallmatrix} p & p \\ I & I \end{smallmatrix}\big] &= F_{g_{12}g_JK,i}\big[\begin{smallmatrix} j & p \\ g_1J & I \end{smallmatrix}\big] F_{g_1J,0}\big[\begin{smallmatrix} i & i \\ I & I \end{smallmatrix}\big] \\ &= F_{g_{1234}g_JK,i}\big[\begin{smallmatrix} j & p \\ g_{134}J & g_{34}I \end{smallmatrix}\big] F_{g_1J,0}\big[\begin{smallmatrix} i & i \\ I & I \end{smallmatrix}\big] (-1)^{\hat{Q}_{g_{34}}(i) - \hat{Q}_{g_{34}}(j) - \hat{Q}_{g_{34}}(p)} \quad . \end{aligned}$$

As for the second fusing matrix of expression (A.9), we shift it using (2.7) with $g = g_{IL}^{-1}$. Altogether, these changes bring (A.9) to a suitable form such that the pentagon relation (A.2) can be inserted to obtain

$$\begin{aligned} \text{rhs} \;=\; & F_{g_{1234}g_JK,q}\big[\begin{smallmatrix} j & k \\ g_{134}J & g_3g_{IL}^{-1}L \end{smallmatrix}\big] F_{g_3g_{IL}^{-1}L,i}\big[\begin{smallmatrix} q & l \\ g_{134}J & g_{34}I \end{smallmatrix}\big] F_{g_1J,0}\big[\begin{smallmatrix} i & i \\ I & I \end{smallmatrix}\big] \\ & (-1)^{\hat{Q}_{g_{IL}}(k) + \hat{Q}_{g_{IL}}(l) + \hat{Q}_{g_{34}}(i) - \hat{Q}_{g_{34}}(j) - \hat{Q}_{g_1}(j) - \hat{Q}_{g_3}(l) - \hat{Q}_{g_J}(j) - \hat{Q}_{g_L}(l)} \quad . \end{aligned} \tag{A.10}$$

The final step consists of yet another rewriting of all the fusing matrices. We shift the first fusing matrix using (2.7) with $g = g_2 g_{IKL}$, while the second is shifted with $g = g_{34}^{-1}$. The second and third fusing matrix can then be rewritten using (A.3). If we also take advantage of (A.1), the right-hand side is finally turned into

$$\begin{aligned} \text{rhs} \;=\; & F_{g_2K,q}\big[\begin{smallmatrix} j & k \\ J & g_{23}g_KL \end{smallmatrix}\big] F_{g_1J,l}\big[\begin{smallmatrix} i & q \\ I & g_{123}g_{JK}L \end{smallmatrix}\big] F_{g_{123}g_{JK}L,0}\big[\begin{smallmatrix} l & l \\ I & I \end{smallmatrix}\big] \\ & (-1)^{-\hat{Q}_{g_2}(k) - \hat{Q}_{g_1}(q) - \hat{Q}_{g_{123}}(l) - \hat{Q}_{g_{JK}}(l) - \hat{Q}_{g_K}(k) - \hat{Q}_{g_J}(q) - \hat{Q}_{g_L}(l)} \quad . \end{aligned} \tag{A.11}$$

This is identical to the left-hand side of (A.8), as can be seen using (A.7) and it therefore completes our derivation of the equation (A.8).

A.2 Sewing constraints for the unresolved case

In equation (3.6), we claim that the operator product expansion in the unresolved theory is

$$\Psi^{[L][M]}_{i,g_1}(x_1)\Psi^{[M][N]}_{j,g_2}(x_2) \;=\; \sum_k (x_1 - x_2)^{h_i + h_j - h_k} \Psi^{[L][N]}_{k,g_{12}}(x_2) C^{[L][M][N]}_{(i,g_1)(j,g_2)(k,g_{12})} + \ldots \tag{A.12}$$

where the coefficients are given by

$$C^{[L][M][N]}_{(i,g_1)(j,g_2)(k,g_{12})} \;=\; F_{g_1M,k}\big[\begin{smallmatrix} i & j \\ L & g_{12}N \end{smallmatrix}\big] (-1)^{-\hat{Q}_{g_1}(j)} \quad . \tag{A.13}$$

Here and in the following the elements g_i are taken from subsets of Γ that we have specified in (A.5).

The coefficients (A.13) must satisfy the following version of the sewing relations (see [37, 38, 29]),

$$\begin{aligned} & C^{[J][K][L]}_{(j,g_2)(k,g_3)(q,g_{23})} C^{[I][J][L]}_{(i,g_1)(q,g_{23})(l,g_{123})} C^{[I][L][I]}_{(l,g_{123})(l,g_4)(0,g_{123}g_4)} \\ & = \sum_p \; C^{[I][J][K]}_{(i,g_1)(j,g_2)(p,g_{12})} C^{[K][L][I]}_{(k,g_3)(l,g_4)(p,g_{34})} C^{[I][K][I]}_{(p,g_{12})(p,g_{34})(0,g_{12}g_{34})} F_{p,q}\big[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\big] \end{aligned} \tag{A.14}$$

If $g_{1234} \notin \mathcal{S}_I$ then both sides of the equation vanish because the third operator product coefficient on either side is zero. Hence we can assume that $g_{1234} \in \mathcal{S}_I$. We then derive the relation (A.14) from the master equation (A.8) by setting $g_J = g_K = g_L = e$ and $g_I = g_{1234}^{-1} \in \mathcal{S}_I$.

A.3 Sewing constraints for the resolved case

We turn now to the resolved case. Our claim is that the operator product expansions for resolved boundary fields are given by

$$\Psi^{[I]_a[J]_b}_{i,g_1}(x_1)\,\Psi^{[J]_b[K]_c}_{j,g_2}(x_2) \;=\; \sum_{k,\tilde{g}\in S_J} (x_1-x_2)^{h_i+h_j-h_k}\,\Psi^{[I]_a[K]_c}_{k,g_1g_2\tilde{g}}(x_2)\,C^{[I]_a[J]_b[K]_c}_{(i,g_1)(j,g_2)(k,g_{12}\tilde{g})} + \dots \tag{A.15}$$

where the coefficients are given by

$$C^{[I]_a[J]_b[K]_c}_{(i,g_1)(j,g_2)(k,g_{12}\tilde{g})} = F_{g_1J,k}\big[\begin{smallmatrix} i & j \\ I & g_{12}\tilde{g}K \end{smallmatrix}\big](-1)^{-\hat{Q}_{g_1\tilde{g}}(j)} e_b(\tilde{g}) \quad . \tag{A.16}$$

The associated sewing relation is obtained from evaluating products of four different boundary fields [37, 38, 29] in two different ways. Comparison of the contributions from the identity fields gives

$$\begin{aligned} & \sum_{\tilde{g},\tilde{g}',\tilde{g}''} C^{[J]_b[K]_c[L]_d}_{(j,g_2)(k,g_3)(q,g_{23}\tilde{g}'')} C^{[I]_a[J]_b[L]_d}_{(i,g_1)(q,g_{23}\tilde{g}'')(l,g_{123}\tilde{g}''\tilde{g})} C^{[I]_a[L]_d[I]_a}_{(l,g_{123}\tilde{g}''\tilde{g})(l,g_4)(0,g_{123}g_4\tilde{g}''\tilde{g}\tilde{g}')} \Psi^{[I]_a[I]_a}_{0,\tilde{g}\tilde{g}'\tilde{g}''} \;= \\ & \sum_{g,g',g''} \sum_p C^{[I]_a[J]_b[K]_c}_{(i,g_1)(j,g_2)(p,g_{12}g)} C^{[K]_c[L]_d[I]_a}_{(k,g_3)(l,g_4)(p,g_{34}g')} C^{[I]_a[K]_c[I]_a}_{(p,g_{12}g)(p,g_{34}g')(0,g_{12}g_{34}gg'g'')} \Psi^{[I]_a[I]_a}_{0,gg'g''} F_{p,q}\big[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\big] \end{aligned} \tag{A.17}$$

where $g, \tilde{g} \in \mathcal{S}_J$, $g', \tilde{g}' \in \mathcal{S}_L$ and $g'', \tilde{g}'' \in \mathcal{S}_K$. In spelling out these equations, we have decided to keep the identity field. This allows us to take care of the obvious linear relations (3.8) that exist between our boundary fields.

To take advantage of our relation (A.8), we choose to compare terms on the left-hand side and right-hand side such that $g = \tilde{g} = g_J$, $g' = \tilde{g}' = g_L$ and $g'' = \tilde{g}'' = g_K$ where we have indicated our choice of the group elements g_J, g_L, g_K at the same time. As in the unresolved case, the sewing relation holds trivially for $g_{1234} g_{IJK} \notin \mathcal{S}_I$. Otherwise, we can choose $g_I \in \mathcal{S}_I$ with the help of equation (A.6) and obtain the terms in our sewing relations (A.17) from the corresponding terms in equation (A.8).

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