Geometric Characterizations of the Kerr Isolated Horizon.

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Abstract

We formulate conditions on the geometry of a non-expanding horizon Δ which are sufficient for the space-time metric to coincide on Δ with the Kerr metric. We introduce an invariant which can be used as a measure of how different the geometry of a given non-expanding horizon is from the geometry of the Kerr horizon. Directly, our results concern the space-time metric at Δ at the zeroth and the first orders. Combained with the results of Ashtekar, Beetle and Lewandowski, our conditions can be used to compare the space-time geometry at the non-expanding horizon with that of Kerr to every order. The results should be useful to numerical relativity in analyzing the sense in which the final black hole horizon produced by a collapse or a merger approaches the Kerr horizon.

1 Introduction.

In a new quasi-local theory of black holes [1, 2, 3, 4], one considers situation in which the black hole has reached equilibrium although the exterior space-time still admits outgoing radiation. The black hole in equilibrium is described by a non-expanding horizon, i.e. a null cylinder Δ, generated by segments of null geodesics orthogonal to a space-like 2-surface diffeomorphic with a 2-sphere. The geometry of Δ relevant for extracting physics of Δ is defined by the space-time metric tensor $g_{ab|_\Delta}$ and its derivative $\mathcal{L}_n g_{ab|_\Delta}$ with respect to a transversal (also called radial) vector field n. (In the next section we recall the definition which does not depend on the choice of an n.) The geometry of Δ has local degrees of freedom even if we assume that the vacuum Einstein equations hold in a neighborhood of Δ. A priori the Kerr solution does not appear to play a special role in this context: the Kerr horizon is only an example of a non-expanding horizon. On the other hand, there are some heuristic arguments suggesting that a black hole formed in a physical process should converge, in some suitable sense, to the Kerr black hole [5]. To probe this important issue of the 'final state', one can begin with a preliminary question: what is condition on the geometry of a non-expanding horizon Δ that ensures that its geometry coincides with that of the Kerr horizon? We will analyze this question here.

This analysis will not unravel any unknown properties of the Kerr metric. Rather, our goal is to select a covariantly defined property of the Kerr horizon which uniquely distinguishes its geometry among all non-expanding horizons. Our conditions are local to Δ in contrast to characterizations of the Kerr solution that rely on global assumptions about space-time and can be easily checked in numerical simulations. The detailed calculations and proofs will appear in [6].

In Section 4 we explain how our local characterization can be used to compare the space-time metric tensor on a given non-expanding horizon with the Kerr solution to every order in an expansion with respect to a coordinate parametrizing incoming null geodesics transversal to Δ.

2 Isolated horizons.

Definitions

A *non-expanding horizon* is a null surface Δ in a 4-spacetime M such that

- Δ is generated by segments of null geodesics orthogonal to a space-like 2-sphere $\tilde{\Delta} \subset M$, and is diffeomorphic to $\tilde{\Delta} \times \mathbf{R}$;
- the expansion of every null vector field ℓ^a tangent to Δ vanishes;
- Einstein's equations hold on Δ and the stress-energy tensor T_{ab} is such that $-T^a{}_b\ell^b$ is future pointing for every null vector tangent to Δ.

It follows from the conditions above, that the degenerate metric tensor $q_{ab} = g_{\underleftarrow{ab}}$ is Lie dragged by ℓ^a,

$$\mathcal{L}_\ell q_{ab} \;=\; 0. \tag{1}$$

The parallel transport defined by the space-time connection along any curve contained in Δ preserves the tangent bundle $T(\Delta)$, and induces a connection $\mathcal{D}_a$ therein. The pair $(q_{ab}, \mathcal{D}_a)$ is referred to as *the geometry of* Δ. If a non-expanding horizon Δ admits a null vector field ℓ^a such that its flow $[\ell]$ is a symmetry of $(q_{ab}, \mathcal{D}_a)$, then we say that $(\Delta, [\ell])$ is an *isolated horizon*. We also assume that the restriction of the flow $[\ell]$ to every null geodesic is non-trivial.

An isolated horizon $(\Delta, [\ell], q_{ab}, \mathcal{D}_a)$ defined by the non-extremal future, outer (inner) event horizon of the Kerr metric, with ℓ^a the restriction to Δ of the Killing vector field which is null at Δ, will be called *the Kerr outer (inner) isolated horizon.*

The degrees of freedom.

The degrees of freedom in the geometry of an isolated horizon are: the intrinsic metric tensor q_{ab}, the *rotation 1-form potential* ω_a defined on Δ by the derivative of ℓ^a,

$$\mathcal{D}_b \ell^a \;=\; \omega_b \ell^a, \tag{2}$$

and the pull-back $R_{\underleftarrow{ab}}$ of the Ricci tensor constrained by [1]

$$R_{\underleftarrow{ab}} \ell^b = 0. \tag{3}$$

Owing to the definition of an isolated horizon, we have

$$\mathcal{L}_\ell \omega_a \;=\; \mathcal{L}_\ell R_{\underleftarrow{ab}} \;=\; 0. \tag{4}$$

The factor $\kappa^{(\ell)}$ in $\ell^a \mathcal{D}_a \ell^b \;=\; \kappa^{(\ell)} \ell^b$ is called *the surface gravity* of ℓ^b. It follows from the symmetry of the isolated horizon and from the vanishing of $R_{\underleftarrow{ab}} \ell^b$, that the *surface gravity* $\kappa^{(\ell)}$ is constant,

$$\kappa^{(\ell)} \;=\; \text{const}. \tag{5}$$

We call an isolated horizon *non-extremal* whenever

$$\kappa^{(\ell)} \neq 0, \tag{6}$$

and *extremal* otherwise.

An isolated horizon Δ will be called *vacuum isolated horizon* if the pull-back $R_{\underleftarrow{ab}}$ of the Ricci tensor vanishes on Δ. It will be useful to characterize the geometry of a non-extremal vacuum isolated horizon $(\Delta, [\ell])$ by two scalar invariants [4]. The first one is the Gauss curvature K of the 2-metric $\tilde{q}$ induced on any space-like 2-surface passing through a given point of Δ. The second one is the rotation scalar Ω defined by

$$d\omega \;=\; \Omega\ {}^2\!\epsilon, \tag{7}$$

where ${}^2\!\epsilon$ is the area 2-form[1] of Δ. The invariants are combined into a single, complex valued function, namely the component $\Psi_2 = C_{abcd}\ell^a n^b(\ell^c n^d - m^c\bar{m}^d)$ of the Weyl tensor, where m^a, n^a are any complex and, respectively, real null vector fields such that $n_a\ell^a = -1$, $m_a\bar{m}^a = 1$ and $m_a n^a = m_a\ell^a = 0$; we have

$$\Psi_2 \;=\; \frac{1}{2}(-K + i\Omega). \tag{8}$$

For every global, space-like cross-section $\tilde{\Delta}$ of Δ, Ψ_2 satisfies the following global constraint,

$$\int_{\tilde{\Delta}} \Psi_2\, {}^2\!\epsilon \;=\; -2\pi. \tag{9}$$

Every non-extremal isolated horizon geometry $(\Delta, q_{ab}, \mathcal{D}_a, [\ell])$ is determined[2], up to diffeomorphisms preserving the null generators of Δ by the pair of invariants $(K,\, \Omega)$. Owing to the symmetry (1, 4), the invariant Ψ_2 is constant along the null generators of Δ. Therefore, it defines a function on the sphere $\hat{\Delta}$ of the null geodesics of Δ. It follows from the results of [7], that given a 2-sphere $\hat{\Delta}$ equipped with a metric tensor $\hat{q}_{AB}$ and a function $\hat{\Omega}$ such that $\int_{\hat{\Delta}} \hat{\Omega}\, {}^2\hat{\epsilon} = 0$, there is a non-extremal vacuum isolated horizon whose invariants K and Ω correspond to $\hat{q}$ and $\hat{\Omega}$.

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[1] There is naturally defined area 2-form ${}^2\!\epsilon$ on each null 3-surface such that for every space-like 2-subsurface S' the integral $\int_{S'} {}^2\!\epsilon$ equals the area of S'.

[2] Isolated horizon is not assumed to be geodesically complete. Therefore, some extra information would be needed to know which finite segment of each null geodesics tangent to Δ is contained in Δ.

3 Geometric conditions distinguishing the Kerr isolated horizon

The vacuum, Petrov type D isolated horizons

The geometry of an isolated horizon $(\Delta, [\ell])$ can not be assigned a 'Petrov type' because it does not determine all of the components of the Weyl tensor on Δ. However, if we assume that all the components of the Ricci tensor and their first radial derivatives vanish on Δ, then, in the non-extremal case, the Bianchi identities and $(q_{ab}, \mathcal{D}_a)$ determine the evolution of the missing Weyl tensor component along Δ. Combined with the assumption that the Weyl tensor is of the Petrov type D on Δ the evolution equation reduces to a certain condition on the geometry of the isolated horizon. To write down the condition, introduce a complex, null vector field m tangent to Δ,

$$m^a m_a = 0, \quad \bar{m}^a m_a \ = \ 1, \tag{10}$$

such that it is tangent to some 2 sub-surfaces in Δ, that is such that

$$\mathcal{L}_{\bar{m}} m^a \ = \ (\alpha - \bar{\beta}) m^a - (\bar{\alpha} - \beta) \bar{m}^a. \tag{11}$$

We will denote the differential operator corresponding to the vector field m^a by δ,

$$\delta := m^a \partial_a. \tag{12}$$

The condition on the geometry of the isolated horizon reads

$$3\Psi_2 \bar{\delta}\bar{\delta}\Psi_2 + 3(\alpha - \bar{\beta})\Psi_2 \bar{\delta}\Psi_2 - 4(\bar{\delta}\Psi_2)^2 \ = \ 0. \tag{13}$$

If Ψ_2 vanishes at a point of Δ, then Weyl tensor is of the Petrov type III, N or 0. Therefore, we can assume that $\Psi_2 \neq 0$ and conclude that:

Lemma. *Suppose $(\Delta, [\ell])$ is a non-extremal isolated horizon, and (i) the Ricci tensor and its first radial derivative vanish on Δ, and (ii) the Weyl tensor is of the type D on Δ; then the invariant Ψ_2 of the geometry of Δ satisfies the following equation,*

$$(\bar{\delta} + \alpha - \bar{\beta})\bar{\delta}(\Psi_2^{-\frac{1}{3}}) \ = \ 0. \tag{14}$$

The converse statement requires an additional assumption that, there is an extension of the isolated horizon vector field ℓ^a to a neighborhood of Δ such that, the Weyl tensor $C^a{}_{abc}$ is Lie dragged by ℓ^a on Δ. Then, the conditions (i) and (14) of Lemma imply that the Weyl tensor is of the Petrov type D at Δ. The above equation (14) was derived in [6]. It is independent of the choice of a null frame m^a, $\bar{m}^a$, n^a, ℓ^a, provided ℓ^a is tangent to Δ and $\mathrm{Re}\, m^a, \mathrm{Im}\, m^a$ are surface forming at Δ. Notice also that, it involves the 4th order derivatives of the 2-metric q_{ab}, because

$$-2\mathrm{Re}\Psi_2 \;=\; \delta(\alpha-\bar{\beta})+\bar{\delta}(\bar{\alpha}-\beta)-2(\alpha-\bar{\beta})\overline{(\alpha-\bar{\beta})}. \tag{15}$$

(The condition (i) can be weakened [6]: not all of the Ricci tensor components have to satisfy (i).)

Definition *A non-extremal isolated horizon is vacuum, type D, whenever its geometry satisfies the condition (14).*

The conditions distinguishing the Kerr isolated horizon

We say that an isolated horizon $(\Delta, [\ell])$ admits an axial symmetry whenever it admits a vector field Φ^a tangent to Δ, whose all the orbits are closed, and such that[3]

$$\mathcal{L}_\Phi q_{ab} \;=\; 0, \;\; \mathcal{L}_\Phi \ell^a \;=\; 0, \;\; [\mathcal{L}_\Phi, \mathcal{D}_a] = 0. \tag{16}$$

Proposition *Suppose a non-extremal isolated horizon* $(\Delta, [\ell], q_{ab}, \mathcal{D}_a)$ *admits an axial symmetry group generated by a vector field* Φ^a*; then it is vacuum, type D if and only if the following equation is satisfied,*

$$d(\Psi_2^{-\frac{1}{3}}) \;=\; A_0 \Phi \lrcorner\, {}^2\epsilon, \tag{17}$$

where d is the exterior derivative on Δ and A_0 is a complex constant which turns out to be pure imaginary.

The condition (14) (as well as (17)) is a complex equation on the metric tensor q_{ab} and the rotation scalar Ω. We found all local solutions (q_{ab}, Ω) defined in some open subset of Δ. Imposing on Ψ_2 the globality condition

[3] For Φ to be a symmetry of the geometry, it is enough if the second equation is weakened to $\mathcal{L}_\Phi \ell^a = a_0 \ell^a$, a_0 being a constant. But in the case of an axial symmetry, $a_0 = 0$ necessarily.

(9) restricts the set of solutions to a two dimensional family $(q_{ab}^{(A,J)}, \Omega^{(A,J)})$, parametrized by two real parameters $A > 0, J \geq 0$. The parameters have a geometrical and physical meaning, namely their values are equal to the area and, respectively, the angular momentum [9] of the corresponding non-extremal isolated horizon $(\Delta^{(A,J)}, [\ell^{(A,J)}])$. Given an axi-symmetric, non-extremal vacuum, type D isolated horizon $(\Delta, [\ell])$, to find the corresponding values of (A, J), one has to use the following quasi-local formulas involving an arbitrary cross-section $\tilde{\Delta}$ of Δ,

$$A = A_\Delta = \int_{\tilde{\Delta}} {}^2\epsilon \tag{18}$$

$$J = |J_\Delta| = \frac{1}{4\pi} \left| \int_{\tilde{\Delta}} \phi \, \mathrm{Im}\, \Psi_2 \; {}^2\epsilon \right| \tag{19}$$

$$\tag{20}$$

where the function ϕ is defined up an additive constant as the generator of the vector field Φ, that is

$$\phi_{,a} := \Phi^a \; {}^2\epsilon_{ab} \tag{21}$$

Now, $(\Delta^{(A,J)}, [\ell^{(A,J)}])$ is the Kerr outer isolated horizon provided

$$\frac{A}{8\pi} > J, \tag{22}$$

and the Kerr inner horizon if

$$\frac{A}{8\pi} < J. \tag{23}$$

However, in the case when

$$\frac{A}{8\pi} = J \tag{24}$$

the pair $(q_{ab}^{(A,J)}, \Omega^{(A,J)})$ coincides with the metric tensor and the rotation scalar induced on the event horizon of the Kerr solution in the extremal case. On the other hand, the corresponding $(q_{ab}^{(A,8\pi A)}, \Omega^{(A,8\pi A)})$ is non-extremal by definition; let as call it a *special isolated horizon*.[4] We can conclude our results by the following Theorem.

[4]The geometry of every special isolated horizon Δ has a certain non-generic property [4, 8]: it admits a 2-dimensional family of null symmetries; each generator defines a distinct isolated horizon structure on Δ and exactly one of them is extremal.

Theorem *Suppose* $(\Delta, [\ell])$ *is a vacuum, axi-symmetric, non-extremal isolated horizon such that* $A_\Delta > 8\pi J_\Delta$ *(respectively,* $A_\Delta < 8\pi J_\Delta$*), where* A_Δ *is the area and* J_Δ *is the angular monentum of* Δ*. Then each of the following two properties implies that* $(\Delta, [\ell])$ *is the Kerr outer (inner) isolated horizon:*

(i) it is vacuum, type D; i.e is it satisfies (14)

(ii) it satisfies the condition (17)

Conversely, every non-extremal Kerr outer (inner) isolated horizon satisfies all the properties assumed above.

Remark 1. A priori it may happen, that the geometry $(q_{ab}, \mathcal{D}_a)$ of an isolated horizon $(\Delta, [\ell])$ admits an axial symmetry which does not commute with the null flow of $[\ell]$. Then, we can pick any $\ell^a \in [\ell]$ and average it with respect to the symmetry group. It can be shown by using the results of [4], that the result $\bar{\ell}$ is nontrivial on every null generator of Δ. If ℓ in non-extremal, then so is $\bar{\ell}$. More over, the condition, that $(\Delta, [\ell])$ be vacuum, type D is expressed by the invariants independent of $[\ell]$. Hence, if Δ admits two distinct non-extremal isolated horizon structures $[\ell]$ and $[\ell']$, then one of them is vacuum, type D if and only if so is the other one. In conclusion, if we drop the assumption $\mathcal{L}_\Phi \ell^a = 0$ from the definition of the axi-symmetry, then the Theorem still holds.

Remark 2. The assumption that the Weyl tensor is of the Petrov type D excludes the vanishing of Ψ_2 at any point. One could weaken this condition, and allow for the vanishing of Ψ_2. However, there are no such solutions defined globally on Δ.

Remark 3. The condition $A_\Delta > 8\pi J_\Delta$ in Theorem can be replaced by the following inequality to be satisfied at a point x_0 belonging to the symmetry axis,

$$\left|\mathrm{Re}[(\Psi_2)^{-\frac{1}{3}}(x_0)]\right| > \left|\mathrm{Im}[(\Psi_2)^{-\frac{1}{3}}(x_0)]\right|, \tag{25}$$

and the conclusion still holds.

4 Applications of the result

The results of the previous section are relevant for the comparison of the space-time metric tensor near a non-expanding horizon Δ with the Kerr metric. For that we need certain generalization of the Bondi coordinates [9].

Every null vector field ℓ^a tangent to Δ and every foliation of Δ with space-like 2-surfaces preserved by the flow $[\ell]$, defines uniquely another null vector field n^a orthogonal to the leaves of the foliation and such that $\ell^a n_a = -1$. Use the flow of n^a to extend the vector field ℓ^a to a neighborhood of Δ. It is not any longer null but for the simplicity let as denote it by the same letter ℓ^a. Then, it is true [10] that

$$\mathcal{L}_\ell g_{ab|\Delta} \;=\; 0. \tag{26}$$

Suppose that the vacuum Einstein equations hold in a neighborhood of a non-expanding horizon Δ and that ℓ^a is future pointing, it does not vanish in the future, and $\kappa^{(\ell)} = \text{const} > 0$. Then, every transversal derivative $(\mathcal{L}_n)^m\ g_{ab|\Delta}$, $m = 1, 2, ..., k, ...$ exponentially converges in future to some value, $(\mathcal{L}_n)^m\ g_{ab}^{(\infty)}$ say, as we move along each generator of Δ [9]. The values $(\mathcal{L}_n)^m\ g_{ab}^{(\infty)}$ are determined by the the geometry $(q_{ab}, \mathcal{D}_a)$ of Δ for every m. Suppose now, that $(q_{ab}, \mathcal{D}_a, [\ell])$ is axi-symmetric and that the conditions

$$(\bar{\delta} + \alpha - \bar{\beta})\bar{\delta}(\Psi_2^{-\frac{1}{3}})_{|\Delta} \quad = \quad 0, \tag{27}$$

$$A_\Delta \;-\; 8\pi J_\Delta \quad > \quad 0 \tag{28}$$

hold on Δ (the inequality can be replaced by (25)). Then, it follows from Theorem that, all the asymptotic values $(\mathcal{L}_n)^m\ g_{ab}^{(\infty)}$ of the transversal derivatives of the metric coincide with the corresponding derivatives of the Kerr solution. Notice finally, that the quantity

$$\mathcal{I} \;:=\; |(\bar{\delta} + \alpha - \bar{\beta})\bar{\delta}(\Psi_2^{-\frac{1}{3}})| \tag{29}$$

is independent of the choice of a null frame at Δ, provided ℓ is tangent to the null generators of Δ. Therefore, it is an invariant of the geometry $(q_{ab}, \mathcal{D}_a)$ of a non-expanding horizon. If $\mathcal{I}$ fails to be zero, then its value is a measure of the departure of the values of $(\mathcal{L}_n)^m g_{ab}^{(\infty)}$, $m = 0, 1, 2, ..., k...$ from those of the Kerr solution at the future, outer event horizon. These results should be useful to numerical relativity in analyzing the sense in which the final black hole horizon produced by a collapse or a merger approaches the Kerr horizon.

These results should be directly applicable to numerical simulations of black hole collisions to verify whether or not a Kerr horizon is produced at late times and, if it is not, to estimate how large the departure is. The criterion involves fields defined just on the 'world-tube' of apparent horizons and some of the existing codes can easily calculate them.

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