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Classification of image distortions in terms of Petrov types

Thoralf Chrobok
TU Berlin, Sekr. PN 7-1, 10623 Berlin, Germany.
tchrobok@itp.physik.tu-berlin.de

Volker Perlick
Albert Einstein Institute, 14476 Golm, Germany.
(Permanent address: TU Berlin, Sekr. PN 7-1, 10623 Berlin, Germany.
vper0433@itp.physik.tu-berlin.de)

Abstract

An observer surrounded by sufficiently small spherical light sources at a fixed distance will see a pattern of elliptical images distributed over the sky, owing to the distortion effect (shearing effect) of the spacetime geometry upon light bundles. In lowest non-trivial order with respect to the distance, this pattern is completely determined by the conformal curvature tensor (Weyl tensor) at the observation event. In this paper we derive formulae that allow one to calculate these distortion patterns in terms of the Newman-Penrose formalism. Then we represent the distortion patterns graphically for all Petrov types, and we discuss their dependence on the velocity of the observer.

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1 Introduction

The general-relativistic light deflection, i.e., the influence of the spacetime geometry on the paths of light rays, has the effect that, in general, the apparent shape of a distant object at the celestial sphere of an observer will be distorted. This distortion effect is directly related to the *shear* of light bundles, whereas the apparent size and the apparent brightness of images are related to the *expansion* of light bundles. By the well-known *Sachs equations* [1], the shear is an effect of the conformal curvature tensor (Weyl tensor), whereas the expansion is influenced by the Ricci tensor along the light rays. In the following we concentrate on the distortion effect, i.e. on the shear. In a pioneering paper, Kristian and Sachs [2] introduced a certain measure for the distortion effect, given in terms of a series expansion with respect to the distance between source and observer, and suggested using it as a cosmological observable. In lowest non-trivial order with respect to the distance the distortion is completely determined by the Weyl tensor at the observation event. It is then a natural question to ask to what extent the distortion is already determined by the Petrov type of the Weyl tensor. Up to now only some special aspects of this question have been addressed in the literature (see Penrose and Rindler [3], Volume II, Chapter 8); it is the purpose of this paper to give a more detailed discussion, including graphical representations of the distortion patterns on the sky for each Petrov type.

To gain understanding the distortion effect it is didactically helpful to begin the discussion with an over-idealized universe in which all galaxies are perfectly spherical. We shall assume that all light rays that reach the observer from a particular galaxy can be viewed as 'infinitesimally close to each other', i.e., that the laws of light propagation can be linearized around a central light ray. Then the image of each galaxy results by applying a linear map to a circle, i.e., it is an ellipse. By the term 'distortion pattern' we mean the distribution of those ellipses over the celestial sphere of the observer, for galaxies at a specific distance. (We are interested only in the shape, not in the size, of the ellipses.) Having fixed the distance, the distortion pattern is given by a non-negative scalar function on the celestial sphere that gives the eccentricity of the respective ellipse, and by a direction field on the celestial sphere that indicates the major axis of the respective ellipse. The relevant formulae for determining these fields, which are due to Kristian and Sachs [2], will be given in Section 2 below. We shall restrict our discussion to the case that only terms of lowest non-trivial order with respect to the distance are to be taken into account. (In this approximation it does not matter which of the various non-equivalent general-relativistic notions of 'distance' we use, see Section 2 below.) The distortion pattern is then completely determined by the Weyl tensor at the observation event, with the eccentricities depending quadratically on the distance and the direction field being independent of the distance. If the Weyl tensor is zero, then there is no distortion effect, i.e., the eccentricity is everywhere zero and the direction field is undetermined. If the Weyl tensor is not zero, then the distortion effect vanishes at exactly four (not necessarily distinct) points on the sky, corresponding to the four principal null directions of the Weyl tensor. Kristian and Sachs [2] concentrate on the magnitude of the distortion effect, i.e., on the eccentricities which they measure in terms of a function they denote by e. Penrose and Rindler [3], Volume II, Chapter 8, on the other hand, concentrate on the direction field which they call the 'fingerprint' of the Weyl tensor. It is our goal to discuss both the magnitude and the direction of the distortion effect; therefore, we assign to each point of the observer's celestial sphere a 'distortion length element' whose length gives the eccentricity and whose direction gives the major axis of the respective ellipse. In Section 2 we show how these quantities can be calculated with the help of the Newman-Penrose formalism. In Section 3 we discuss, in terms of pictures, the 'distortion length element field' and its dependence on the oberserver's velocity for each Petrov type.

If we want to link these deliberations with the real universe we have to face the problem that galaxies are not spherical. We may restrict to those galaxies which appear elliptical on the sky, to within reasonable approximation, and we may assume that the actual shape of those galaxies is approximately that of a rotational ellipsoid. The problem is that we do not know the actual eccentricity and the actual direction of the rotation axis for any individual galaxy. As long as this is true, it is in principle impossible to measure the distortion effect upon any individual galaxy. The only way out of this difficulty is to use statistical methods in order to measure the distortion effect upon sufficiently large samples of galaxies. This approach is based on the assumption that, on a sufficiently large scale, the rotation axes of galaxies are randomly distributed in the universe. (This is a reasonable working hypothesis, but it is not beyond any doubt. In a rotating universe, e.g., the angular momenta of galaxies are expected to be preferably aligned with the universal rotation axis.) The distortion pattern, as introduced above, should then be obtained by averaging galaxy ellipticities over sufficiently large fields in the sky. In this statistical sense, the distortion effect has actually been measured by several groups. In the following paragraphs we give a brief overview of the observational situation.

The first attempt of observing the distortion effect was done already in 1966 by Kristian [4] with a small number of galaxies in several clusters. The observation was unsuccessful which, together with some reasonable assumptions, yielded an upper limit for the magnetic part of the Weyl tensor. This null result was confirmed in 1983 by Valdes et al. [5] with a 300fold improvement in accuracy over Kristian, owing to a bigger number of galaxies and including galaxies at larger distances. The first successful observation of the distortion effect by statistical methods was reported in 1990 by Tyson et al. [6] who considered the distorting effect of rich clusters of galaxies upon background galaxies. The same mechanism – background galaxies distorted by the gravitational field of an intervening cluster – is usually considered as an explanation for the socalled *giant luminous arcs* which have been discovered in great number since 1986, beginning with Lynds and Petrosian [7] and Soucail et al. [8]. (In the latter case, the linear approximation is, of course, not applicable, i.e., it is not justified to view all light rays coming from the source to the observer as 'infinitesimally close' to some central light ray.) In this sense, the distortion effect produced by galaxy clusters was a well-established phenomenon by the mid-1990s. On the other hand, by this time all attempts to observe the distortion effect produced by large-scale structure had failed, see Mould et al. [9] for an unsuccessful attempt in 1994.

With the recent advancement of CCD mosaic cameras the observational situation has greatly changed. Now it is possible to observe 'great' parts ($\sim 30 \times 30$ square arc-minutes) of the sky with a large number of galaxies ($\sim 10^5$) in a single picture and to handle the contained information automatically. The unwrought data can be cleaned by several techniques described in [11, 12, 13, 14] which improves the significance of the effect. The results of these recent observations can be summarized in the following way. A non-zero distortion effect on a large scale has been measured by demonstrating a correlation of galaxy ellipticities in several selected fields in the sky [10, 11, 12, 13, 14], and there is evidence that this correlation decreases if the field size is increased [11, 12, 13, 14]. Image distortion by large-scale structure is often called *cosmic shear*, refering to the shear induced in light bundles on a cosmic scale. This term should *not* be confused with a hypothetical shear of the Hubble flow which occasionally is also called 'cosmic shear' or 'cosmological shear'.

For future perspectives of observations we refer to the Sloane Digital Sky Survey (see `http://www.sdss.org`) and to the Deep Lens Survey (see `http://dls.bell-labs.com`) which are both under way, but also to the proposed Dark Matter Telescope (see, e.g., Tyson, Wittman and Angel [15]). It does not seem to be overly optimistic to hope that in some years the observational material on the distortion effect covers large parts of the sky.

Observations of image distortions are usually evaluated with the help of the *weak-lensing formalism* which is based on early ideas of Gunn [16, 17] (also see Webster [18]) and was developed since 1990 by several authors including, e.g., Miralda-Escudé [19], Kaiser [20], van Waerbeke, Bernardeau and Mellier [21]; for a comprehensive review we refer to Bartelmann and Schneider [22]. This formalism uses Newtonian approximations in a weakly perturbed Friedmann-Robertson-Walker model. As the Weyl tensor of a Friedmann-Robertson-Walker model vanishes, the distortion effect in such a universe is produced by the perturbations alone. Among other things, the weak-lensing formalism provides a relation between the correlation function of ellipticities and the power spectrum of mass fluctuations, thereby relating the distortion effect to the distribution of (dark) matter. These results rely, of course, on the Newtonian approximations. The fact that the distortion effect is determined by the Weyl tensor, on the other hand, is purely geometric and quite general. In the weak-lensing formalism this central role of the Weyl tensor is somewhat disguised and partly overshadowed by the approximations. In particular, the effect of the magnetic part of the Weyl tensor is completely neglected because it has no Newtonian counterpart. The present paper is motivated, at least partly, by our desire to rediscuss the role of the Weyl tensor for image distortions in view of the recent observations. More precisely, we want to ask if the observed image distortion by large-scale structure can be used to gain some information about the Weyl tensor of the universe. We shall discuss this question, on the basis of the results derived in the body of this paper, in the conclusions.

Leaving aside all applications to cosmology, we feel that the patterns derived and discussed in Section 3 have some general value from a didactical point of view. They illustrate the image distortion not only in cosmological models but in all spacetimes of the respective Petrov type (in lowest non-trivial order with respect to the distance) and the dependence of this effect on the observer's velocity. This might be helpful for associating the Weyl tensor with some geometrical imagination.

2 Derivation of the distortion field

As already mentioned in the introduction, distortion as a cosmological observable was introduced in a pioneering paper by Kristian and Sachs [2]. In this section it is our goal to rewrite the basic equations by which the distortion field is described in terms of Newman-Penrose coefficients. In the following section we will then discuss these results for each Petrov type. If the reader is not familiar with the Newman-Penrose formalism and with the Petrov classification he or she may consult Chandrasekhar [23], Chapters 1.8 and 1.9, for a detailed introduction. We adopt the same sign and factor conventions as Chandrasekhar. In particular, we use the signature $(+,-,-,-)$.

In an arbitrary spacetime (i.e., a 4-dimensional Lorentzian manifold), we fix a point p and a pseudo-orthonormal frame $(\boldsymbol{E}_1,\boldsymbol{E}_2,\boldsymbol{E}_3,\boldsymbol{E}_4)$ at p. The latter may be expressed equivalently in terms of a null tetrad $(\boldsymbol{\ell},\boldsymbol{n},\boldsymbol{m},\bar{\boldsymbol{m}})$ via

$$
\begin{aligned}
E_1^a = \tfrac{1}{\sqrt{2}}\big(m^a+\bar{m}^a\big)\,, \quad E_2^a = \tfrac{i}{\sqrt{2}}\big(m^a-\bar{m}^a\big)\,,\\
E_3^a = \tfrac{1}{\sqrt{2}}\big(\ell^a-n^a\big)\,, \quad E_4^a = \tfrac{1}{\sqrt{2}}\big(\ell^a+n^a\big)\,,
\end{aligned}
\tag{1}
$$

cf. Chandrasekhar [23], Chapter 1.8. We interpret the timelike vector $\boldsymbol{E}_4$ as past-pointing so that $-\boldsymbol{E}_4$ may be viewed as the 4-velocity of an observer. Then the totality of lightlike geodesics issuing from p into the past is in natural one-to-one relation with the set of past-pointing lightlike initial vectors

$$
\begin{aligned}
k^a &= \sin\vartheta\big(\cos\varphi\, E_1^a+\sin\varphi\, E_2^a\big)+\cos\vartheta\, E_3^a+E_4^a\\
&= \tfrac{1}{\sqrt{2}}\big\{\sin\vartheta\big(e^{i\varphi}m^a+e^{-i\varphi}\bar{m}^a\big)+\big(1+\cos\vartheta\big)\ell^a+\big(1-\cos\vartheta\big)n^a\big\}\,,
\end{aligned}
\tag{2}
$$

where ϑ and φ have their usual range as standard coordinates on the 2-sphere. We may interpret ϑ and φ as coordinatizing the *celestial sphere* of the observer with 4-velocity $-\boldsymbol{E}_4$ at p. Every point on the past light cone of p can be written as $\exp(s\boldsymbol{k})$ with some parameter $s>0$ and some $\boldsymbol{k}$ given by (2); if we restrict to sufficiently small s, this representation is unique. Henceforth we shall use the affine parameter s as a measure for the *distance* from p, and it is our goal to discuss the distortion effect in lowest non-trivial order with respect to s. To within this approximation, s can be replaced with the *angular diameter distance* $D=s+O(s^2)$, with the *luminosity distance* $\hat{D}=s+O(s^2)$, or with any other reasonable measure of distance used in cosmology. The redshift z is related to s by an equation of the form $z=Hs+O(s^2)$ with a 'Hubble constant' H which, in general, is a non-constant function on the sky. Also, it is important to realize that z depends not only on the velocity of the observer but also on the velocity of the light source. (Writing cosmological relations in terms of power series was a basic idea introduced by Kristian and Sachs [2]. A discussion of the relations between s, D, $\hat{D}$ and z needed here can be found, e.g., in Hasse and Perlick [24].)

At each point of the celestial sphere, the two orthonormal vectors

$$
\begin{aligned}
\frac{\partial k^a}{\partial \vartheta} &= \tfrac{1}{\sqrt{2}} \Big\{ \cos\vartheta \big(e^{i\varphi} m^a + e^{-i\varphi} \bar{m}^a \big) - \sin\vartheta \big(\ell^a - n^a \big) \Big\} \,, \\
\frac{1}{\sin\vartheta} \frac{\partial k^a}{\partial \varphi} &= \tfrac{i}{\sqrt{2}} \big(e^{i\varphi} m^a - e^{-i\varphi} \bar{m}^a \big)
\end{aligned}
\tag{3}
$$

span the tangent space to the celestial sphere. (Here and in the following it goes without saying that one has to mind the familiar coordinate singularities at $\vartheta = 0$ and $\vartheta = \pi$.) Hence, every tangent vector to the celestial sphere is of the form

$$
\begin{aligned}
t^a &= u \, \frac{\partial k^a}{\partial \vartheta} \; + \; v \, \frac{1}{\sin\vartheta} \frac{\partial k^a}{\partial \varphi} \\
&= \tfrac{1}{\sqrt{2}} \Big\{ \big(u \cos\vartheta + iv \big) e^{i\varphi} m^a + \big(u \cos\vartheta - iv \big) e^{-i\varphi} \bar{m}^a - u \sin\vartheta \big(\ell^a - n^a \big) \Big\}
\end{aligned}
\tag{4}
$$

with $u, v \in \mathbb{R}$. Restricting u and v to a circle, $u^2 + v^2 \le r^2$, gives an 'infinitesimally thin' bundle of light rays with an initially circular cross-section. When following those rays into the past we will observe, in general, that the cross-section does not stay circular but becomes elliptical, owing to the shear produced in the bundle by the spacetime geometry. (Please note that, according to general relativity, the question of whether an 'infinitesimally thin' light bundle has a circular shape has an observer-independent answer; this was demonstrated already in 1932 by Kermack, McCrea and Whittaker [25], cf. Schneider, Ehlers and Falco [26], Section 3.4.1.) Correspondingly, if the cross-section of the bundle is supposed to be circular at some affine distance s, then we have to restrict u and v to a certain ellipse. The equations necessary to calculate the shape of this ellipse can be found in the paper by Kristian and Sachs [2]. Their analysis is based on the well-known fact, established in a fundamental paper by Sachs [1], that the shear of an infinitesimally thin bundle of light rays is governed by the Weyl tensor along the central ray. More precisely, their result reads as follows. To determine an infinitesimally thin bundle which has a circular cross-section at some affine parameter distance s from the event p, one has to consider a certain quadratic form $p_{ab} t^a t^b$ on the two-dimensional space of tangent vectors (4). In lowest non-trivial order with respect to s, p_{ab} is given as

$$
p_{ab} = - \tfrac{s^2}{6} C_{acbd} k^c k^d + O(s^3) \,, \tag{5}
$$

where C_{acbd} denotes the Weyl tensor at the event p, cf. eq. (28) in Kristian and Sachs [2]. If we neglect the $O(s^3)$-term, we can read from (5) that p_{ab} is (real-symmetric and) trace-free; so it has two real eigenvalues which differ by sign, say $\frac{s^2}{6}\varepsilon$ and $-\frac{s^2}{6}\varepsilon$ with $\varepsilon \ge 0$. To get a bundle with circular cross-section at s one has to restrict t^a to an ellipse whose major axis is the eigenspace of p_{ab} to the eigenvalue $-\frac{s^2}{6}\varepsilon$ and whose eccentricity is equal to $\frac{s^2}{6}\varepsilon$.

The special form of the $O(s^3)$-term in (5) will be of no interest to us since we will restrict to affine distances which are small enough to neglect this term. However, it is interesting to note that, if this term is written as a power series in s, the factor in front of s^{i+2} is a linear function of the i th covariant derivative of the Weyl tensor at p. Hence, if the Weyl tensor is covariantly constant, then the $O(s^3)$-term in (5) is zero. Correspondingly, if the Weyl tensor varies but little this term may be neglected even for values of s that are not small.

Now we restrict to light sources at a fixed affine distance s and we assume that, for this s, the $O(s^3)$-term in (5) can be neglected. Then the distortion effect is completely determined by the Weyl tensor at p. Knowing C_{acbd} allows to calculate the eigenvectors and eigenvalues

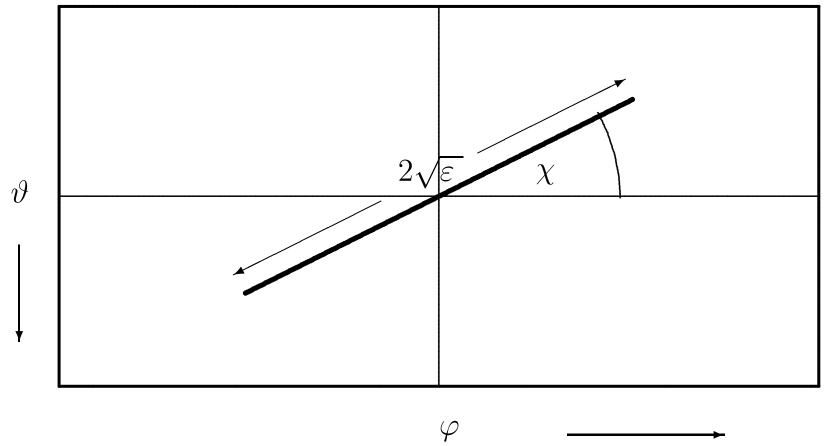


Figure 1: At each point of the celestial sphere, the distortion length element is determined by the non-negative number ε and the angle χ (mod π)

of (5), for all values of ϑ and φ (hidden in $k^c k^d$). The result may be graphically represented by assigning to each point of the celestial sphere a 'length element' whose direction indicates the eigenspace of p_{ab} to the eigenvalue $-\frac{s^2}{6}\varepsilon$ and whose length is a measure of $\frac{s^2}{6}\varepsilon$ (for fixed s). We find it convenient to choose the length elements proportional to $\sqrt{\varepsilon}$, see Figure 1. Using the notation of this figure, the 'distortion length element field' is determined by giving the number ε and the angle χ for each point of the celestial sphere. Kristian and Sachs [2] used the quantity $e = 1 + 2\varepsilon$ as a measure for the magnitude of the distortion effect. Penrose and Rindler [3], Volume II, Section 8, discussed the direction field spanned by our 'distortion length element field', and they called this the *fingerprint direction field* of the Weyl tensor.

If we exclude the trivial case that the Weyl tensor vanishes at p, then there are exactly four (not necessarily distinct) points on the celestial sphere at which ε is zero. These four points correspond to the four *principal null directions* of the Weyl tensor, and the question of how many principal null directions coincide leads to the *Petrov classification*, see Chandrasekhar [23], Chapter 1.9. In this paper it is our main goal to discuss the distortion field over the whole sky for the various Petrov types, thereby extending the results of Penrose and Rindler [3], Volume II, Section 8, who analyze the *fingerprint direction field* near a principal null direction. To that end it will be convenient to express the distortion field (i.e., ε and χ as functions of ϑ and φ) with the help of the Newman-Penrose coefficients

$$\begin{gathered}\Psi_0 = -C_{abcd}\ell^a m^b \ell^c m^d \,, \quad \Psi_1 = -C_{abcd}\ell^a n^b \ell^c m^d \,, \quad \Psi_2 = -C_{abcd}\ell^a m^b \bar{m}^c n^d \,, \\ \Psi_3 = -C_{abcd}\ell^a n^b \bar{m}^c n^d \,, \quad \Psi_4 = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d \,.\end{gathered} \tag{6}$$

Inserting (2) and (4) into (5) with the $O(s^3)$-term neglected, and using the well-known symmetry properties of the Weyl tensor, allows one to express the quadratic form $p_{ab}t^a t^b$ in terms of the five complex Newman-Penrose coefficients (6). This gives a rather long expression, but with some elementary algebra it can be conveniently rewritten as

$$p_{ab}t^a t^b = \tfrac{s^2}{6}\,\mathrm{Re}\big\{(u+iv)^2 \varepsilon e^{-2i\chi}\big\} \tag{7}$$

where $\varepsilon \geq 0$ is the modulus and -2χ (mod 2π) is the argument of the complex number

$$\begin{aligned}\varepsilon\, e^{-2i\chi} \;=\; & \tfrac{1}{2}\,\Psi_0\,(1+\cos\vartheta)^2\, e^{2i\varphi} + 2\,\Psi_1 \sin\vartheta\,(1+\cos\vartheta)\, e^{i\varphi} + \\ & 3\,\Psi_2 \sin^2\!\vartheta + 2\,\Psi_3 \sin\vartheta\,(1-\cos\vartheta)\, e^{-i\varphi} + \tfrac{1}{2}\,\Psi_4\,(1-\cos\vartheta)^2\, e^{-2i\varphi}\,.\end{aligned} \tag{8}$$

Eq. (7) can be rewritten in matrix notation as

$$p_{ab}t^a t^b = \tfrac{s^2}{6}\,\varepsilon\,(\,u\;\;v\,)\begin{pmatrix}\cos 2\chi & \sin 2\chi\\ \sin 2\chi & -\cos 2\chi\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}. \tag{9}$$

From this equation we can easily deduce that the eigenvalues of p_{ab} are $\frac{s^2}{6}\varepsilon$ and $-\frac{s^2}{6}\varepsilon$, and that the eigenspace to the negative eigenvalue is spanned by

$$t^a = -\sin\chi\,\frac{\partial k^a}{\partial\vartheta} + \cos\chi\,\frac{1}{\sin\vartheta}\,\frac{\partial k^a}{\partial\varphi}\,. \tag{10}$$

This shows that ε and χ, as defined by (8), have indeed the same meaning as introduced before, see Figure 1. In other words, if we know the five Newman-Penrose coefficients $\Psi_0, \dots, \Psi_4$ (i.e., if we know the components of the Weyl tensor with respect to our null frame), then (8) gives us the distortion field, i.e., ε and χ as functions of ϑ and φ.

The distortion field depends, of course, on the frame chosen. In terms of the pseudo-orthonormal frame $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{E}_4)$, which is related to the null frame $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}})$ via (1), the behavior of the distortion pattern under Lorentz transformations can be summarized in the following way. (We are interested only in *proper* Lorentz transformations, i.e., those which do not change the temporal or spatial orientation.) Changing the spatial vectors $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ and leaving the timelike vector $\boldsymbol{E}_4$ unaffected leads to a rigid rotation of the celestial sphere. Changing $\boldsymbol{E}_4$ leads to a conformal transformation of the celestial sphere, see Penrose and Rindler [3], Volume I, for a detailed discussion, and to a corresponding deformation of the distortion pattern. We want to discuss this dependence on the observer in some detail for the various Petrov types in the next section. As our analysis will be based on the representation of the distortion field in terms of Newman-Penrose coefficients, we shall need the behavior of those coefficients under proper Lorentz transformations. The relevant formulae are conveniently listed in the book by Chandrasekhar [23], Chapter 1.8, who makes use of the fact that all proper Lorentz transformations can be composed of (spacetime) *rotations of class I, II, and III*, defined respectively by

$$I:\ \boldsymbol{\ell}\mapsto\boldsymbol{\ell}\,,\ \boldsymbol{m}\mapsto\boldsymbol{m}+a\boldsymbol{\ell},\ \bar{\boldsymbol{m}}\mapsto\bar{\boldsymbol{m}}+\bar{a}\boldsymbol{\ell}\,,\ \boldsymbol{n}\mapsto\boldsymbol{n}+\bar{a}\boldsymbol{m}+a\bar{\boldsymbol{m}}+|a|^2\boldsymbol{\ell}\,, \tag{11}$$

$$II:\ \boldsymbol{n}\mapsto\boldsymbol{n}\,,\ \boldsymbol{m}\mapsto\boldsymbol{m}+b\boldsymbol{n}\,,\ \bar{\boldsymbol{m}}\mapsto\bar{\boldsymbol{m}}+\bar{b}\boldsymbol{n}\,,\ \boldsymbol{\ell}\mapsto\boldsymbol{\ell}+\bar{b}\boldsymbol{m}+b\bar{\boldsymbol{m}}+|b|^2\boldsymbol{n}\,, \tag{12}$$

$$III:\ \boldsymbol{\ell}\mapsto A^{-1}\boldsymbol{\ell},\ \boldsymbol{n}\mapsto A\boldsymbol{n}\,,\ \boldsymbol{m}\mapsto e^{i\Theta}\boldsymbol{m}\,,\ \bar{\boldsymbol{m}}\mapsto e^{-i\Theta}\bar{\boldsymbol{m}}\,, \tag{13}$$

with complex numbers a and b and real numbers $A>0$ and Θ (mod 2π). The transformation behavior of the Newman-Penrose coefficients is for rotations of class I:

$$\begin{gathered}\Psi_0\mapsto\Psi_0\,,\quad \Psi_1\mapsto\Psi_1+\bar{a}\Psi_0\,,\quad \Psi_2\mapsto\Psi_2+2\bar{a}\Psi_1+\bar{a}^2\Psi_0\,,\\ \Psi_3\mapsto\Psi_3+3\bar{a}\Psi_2+3\bar{a}^2\Psi_1+\bar{a}^3\Psi_0\,,\quad \Psi_4\mapsto\Psi_4+4\bar{a}\Psi_3+6\bar{a}^2\Psi_2+4\bar{a}^3\Psi_1+\bar{a}^4\Psi_0\,;\end{gathered} \tag{14}$$

for rotations of class II:

$$\begin{gathered}\Psi_0\mapsto\Psi_0+4b\Psi_1+6b^2\Psi_2+4b^3\Psi_3+b^4\Psi_4\,,\quad \Psi_1\mapsto\Psi_1+3b\Psi_2+3b^2\Psi_3+b^3\Psi_4\,,\\ \Psi_2\mapsto\Psi_2+2b\Psi_3+b^2\Psi_4\,,\quad \Psi_3\mapsto\Psi_3+b\Psi_4\,,\quad \Psi_4\mapsto\Psi_4\,;\end{gathered} \tag{15}$$

and for rotations of class III:

$$\begin{gathered}\Psi_0\mapsto A^{-2}e^{2i\Theta}\Psi_0\,,\quad \Psi_1\mapsto A^{-1}e^{i\Theta}\Psi_1\,,\quad \Psi_2\mapsto\Psi_2\,,\\ \Psi_3\mapsto Ae^{-i\Theta}\Psi_3\,,\quad \Psi_4\mapsto A^2e^{-2i\Theta}\Psi_4\,.\end{gathered} \tag{16}$$

3 Discussion of the distortion field

Eq. (8) gives the distortion field, i.e., ε and χ as functions of ϑ and φ, in an arbitrary frame. It is our goal to discuss (8) for each Petrov type. To that end we first choose a frame such that $\Psi_4 \neq 0$ which can always be achieved by a rotation of class I, see (14). (Here and in the following we exclude the trivial case where the Weyl tensor is zero, i.e., of Petrov type O.) We then apply a rotation of class III such that $\Psi_4 = 2$, see (16). In the resulting frame, (8) is equivalent to

$$\varepsilon\, e^{-2i\chi} = \prod_{\mu=1}^{4} \left(z_\mu \sqrt{1+\cos\vartheta}\; e^{i\varphi/2} - \sqrt{1-\cos\vartheta}\; e^{-i\varphi/2} \right) \tag{17}$$

where z_1, z_2, z_3, z_4 are the roots of the equation

$$z^4 + 2\,\Psi_3\, z^3 + 3\,\Psi_2\, z^2 + 2\,\Psi_1\, z + \tfrac{1}{2}\,\Psi_0 \;=\; 0\,. \tag{18}$$

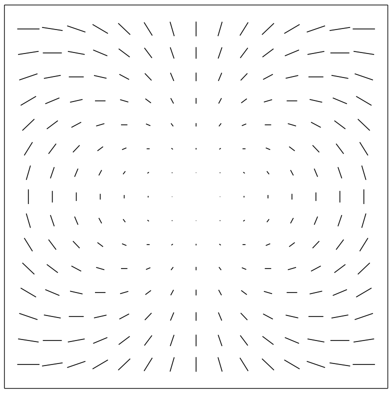
(To prove this, set the left-hand side of (18) equal to $(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ and verify that this yields the same equations for z_1, z_2, z_3, z_4 as equating the right-hand sides of (8) and (17).) From this representation we immediately read that the distortion field has exactly four (not necessarily distinct) zeros, corresponding to the four principal null directions of the Weyl tensor. In particular, we read that at the 'north pole' $\vartheta = 0$ there is a zero of multiplicity k if and only if

$$\begin{gathered} \Psi_i = 0 \quad \text{for } i = 0, \dots, k-1\,, \\ \Psi_k \neq 0\,. \end{gathered} \tag{19}$$

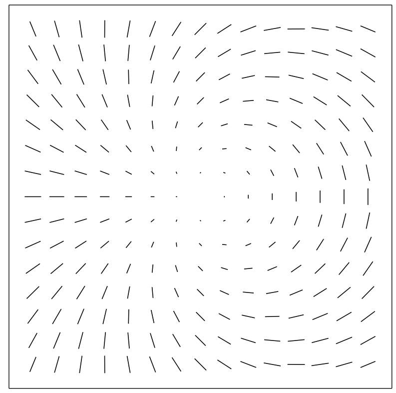
Thus, if we know that there is a principal null direction of multiplicity k, then we know that, in an appropriately chosen frame, the distortion pattern is given by (8) with (19) and $\Psi_4 = 2$. We shall call such a frame *adapted* to the principal null direction henceforth.

Before discussing the global features of the distortion patterns for each Petrov type, we briefly show that our previous results easily allow one to analyze the local structure of the distortion pattern near a principal null direction of multiplicity k, thereby reproducing the results of Penrose and Rindler [3], Volume II, Section 8. After inserting, for $k = 4, 3, 2, 1$, (19) and $\Psi_4 = 2$ into (8) we may analyze the resulting pattern near $\vartheta = 0$, see Figure 2. In each picture ϑ gives the distance from the center and φ is the azimuthal angle encircling the center; the length elements are proportional to $\sqrt{\varepsilon}$. By inspection we find that for $k = 4, 3, 1$ the pattern is uniquely determined, in a neighborhood of $\vartheta = 0$, up to diffeomorphisms, i.e., the differential-topological structure of the distortion pattern is unique. For $k = 2$, the pattern depends on whether or not, in the representation of (19), the argument of Ψ_2 is zero (mod 2π), i.e., Ψ_2 is real and positive. If the argument of Ψ_2 is zero (mod 2π), then the integral lines of the distortion field encircle the double zero, otherwise they are spiralling into the double zero, with a twist depending on the argument of Ψ_2. Moreover, from Figure 2 we read that the distortion field is generated by a continuous vector field near a principal null direction of multiplicity k if and only if k is even. In other words, only for $k = 4$ and $k = 2$ is it possible to assign an orientation to the distortion length elements in a consistent way. These features have already been discussed by Penrose and Rindler. In particular, the reader should compare Figure 8-3 in Penrose and Rindler [3], Volume II, with our Figure 2.

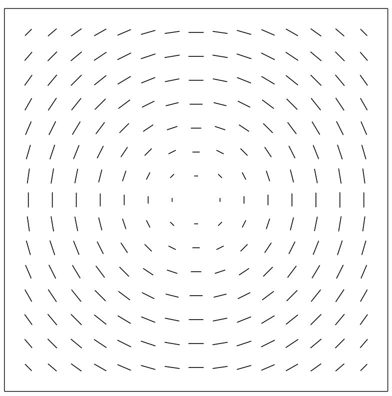
We shall now discuss the explicit and global structure of the distortion pattern for each Petrov type, rather than the local differential-topological structure near a principal null direction. For each Petrov type, we always start out with a frame adapted to a principal



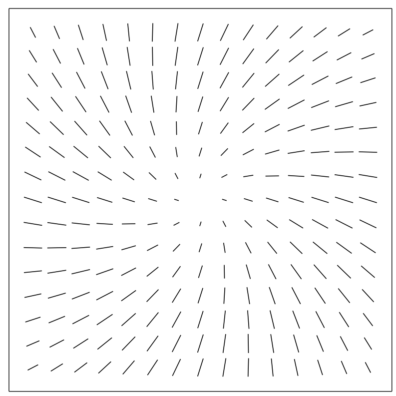
$k = 4$



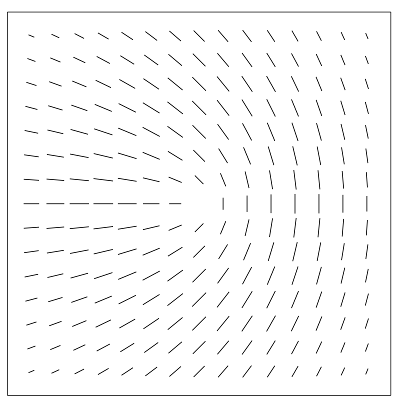
$k = 3$



$k = 2$ (Ψ_2 real and positive)



$k = 2$ (Ψ_2 not real and positive)



$k = 1$

Figure 2: Distortion pattern near a principal null direction of multiplicity k, cf. Penrose and Rindler [3], Volume II, Figure 8-3. This picture, like all the following ones, has been produced with MATHEMATICA. Choosing in all pictures the length elements proportional to $\sqrt{\varepsilon}$, rather than to ε, was motivated by the fact that otherwise their length would vary too strongly, in particular in the case that there is a principal null direction of multiplicity 4.

null direction of highest multiplicity, and we denote the Newman-Penrose coefficients in this particular frame by $\hat{\Psi}_0 = 0$, $\hat{\Psi}_1$, $\hat{\Psi}_2$, $\hat{\Psi}_3$, and $\hat{\Psi}_4 = 2$. We then apply an arbitrary rotation of class I followed by an arbitrary rotation of class III, see (14) and (16). Under such a transformation, which is the general form of a proper Lorentz transformation keeping the direction of $\boldsymbol{\ell}$ fixed, the Newman-Penrose coefficients change according to

$$\begin{gathered}\Psi_0 = 0\,, \quad \Psi_1 = A^{-1}\, e^{i\Theta}\, \hat{\Psi}_1\,, \quad \Psi_2 = \hat{\Psi}_2 + 2\,\bar{a}\,\hat{\Psi}_1\,, \\ \Psi_3 = A\, e^{-i\Theta}\,(\hat{\Psi}_3 + 3\,\bar{a}\,\hat{\Psi}_2 + 3\,\bar{a}^2\,\hat{\Psi}_1)\,, \\ \Psi_4 = A^2\, e^{-2i\Theta}\,(2 + 4\,\bar{a}\hat{\Psi}_3 + 6\,\bar{a}^2\,\hat{\Psi}_2 + 4\,\bar{a}^3\,\hat{\Psi}_1)\,,\end{gathered} \tag{20}$$

see (14) and (16). Inserting the new Newman-Penrose coefficients into (8) and allowing A to run over $\mathbb{R}^+$, Θ over $[0, 2\pi[$, and a over $\mathbb{C}$ will give us the distortion pattern in an arbitrary Lorentz frame, with the only restriction that we keep the zero of highest multiplicity at the north pole. Clearly, this is just a restriction on the choice of the spatial axes which is a matter of convenience only. Similarly, the effect of the angle Θ is trivial in the sense that it produces just a spatial rotation that can be compensated for by a transformation $\varphi \longmapsto \varphi + \text{const.}$

3.1 Type N

Type N is characterized by the fact that all four principal null directions coincide. In a frame adapted to this fourfold principal null direction we have $\hat{\Psi}_1 = \hat{\Psi}_2 = \hat{\Psi}_3 = 0$ (and, of course, $\hat{\Psi}_0 = 0$ and $\hat{\Psi}_4 = 2$). Inserting the transformed Newman-Penrose coefficients (20) into (8) gives us the distortion pattern in an arbitrary Lorentz frame (except for spatial rotations),

$$\varepsilon\, e^{-2i\chi} = A^2\, e^{-2i(\Theta+\varphi)}\,(1 - \cos\vartheta)^2\,, \tag{21}$$

see Figure 3. In the hatted frame (i.e., for $A = 1$ and $\Theta = 0$), (21) involves no parameter whatsoever. Hence, the distortion pattern for type N is universal in the sense that at any two points (in the same or in different spacetimes) where the Weyl tensor is of type N we can choose observers that see exactly the same distortion patterns. Changing to some other observer has the only effect of introducing a scaling factor A which is constant over the sky. (The angle Θ just produces a spatial rotation.) By choosing a sequence of observers whose 4-velocities approach the fourfold principal null direction, the distortion effect can be made arbitrarily large, $A \longrightarrow \infty$. By choosing a sequence of observers whose 4-velocities approach some other null direction, the distortion effect can be made arbitrarily small, $A \longrightarrow 0$.

3.2 Type III

Type III is characterized by the fact that three principal null directions coincide whereas the fourth is different. In a frame adapted to the threefold principal null direction we have $\hat{\Psi}_1 = \hat{\Psi}_2 = 0$, $\hat{\Psi}_3 \neq 0$ (and $\hat{\Psi}_0 = 0$, $\hat{\Psi}_4 = 2$). Inserting the transformed Newman-Penrose coefficients (20) into (8) yields

$$\varepsilon\, e^{-2i\chi} = 2\, A\, e^{-i(\Theta+\varphi)}\, \hat{\Psi}_3 \sin\vartheta\,(1 - \cos\vartheta) + A^2\, e^{-2i(\Theta+\varphi)}\,(1 + 2\bar{a}\hat{\Psi}_3)\,(1 - \cos\vartheta)^2\,, \tag{22}$$

see Figure 4. Whatever the (non-zero) value of $\hat{\Psi}_3$ may be, by choosing a, A and Θ appropriately we may give any two values we like to the coefficients $A\, e^{-i\Theta}\hat{\Psi}_3 \in \mathbb{C} \setminus \{0\}$ and

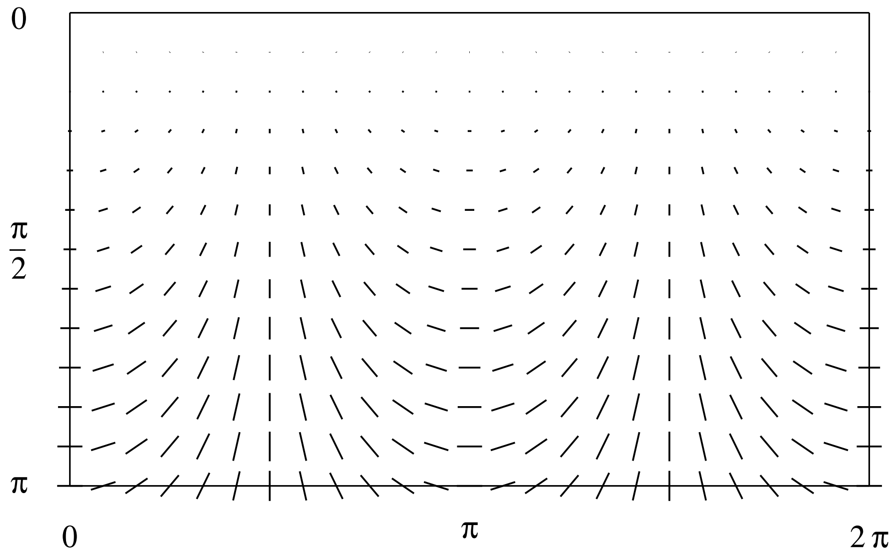


Figure 3: Distortion pattern for Petrov type N. In this picture, and in the following ones, the celestial sphere is given in Mercator projection, with φ as the horizontal and ϑ as the vertical variable. The picture shows the pattern given by (21) with $Ae^{-i\Theta} = 0.05$.

$A^2 e^{-2i\Theta}(1 + 2\bar{a}\hat{\Psi}_3) \in \mathbb{C}$ in (22). Thus, type III shows the same universality property as type N: For any two points where the Weyl tensor is of type III we can find observers that see exactly the same distortion patterns. However, the effect of a Lorentz transformation is now more complicated than for type N. Choosing a such that $1 + 2\bar{a}\,\hat{\Psi}_3 = 0$ gives us a frame such that the simple zero of the distortion pattern is at the south pole ($\vartheta = \pi$), i.e., at the antipodal point of the threefold zero. This restriction on the frame, which is equivalent to requiring that $\boldsymbol{E}_4 = \frac{1}{\sqrt{2}}(\boldsymbol{\ell} + \boldsymbol{n})$ lies in the plane spanned by the two different principal null directions, still allows to choose A (and Θ) arbitrarily; changing A has the same magnifying or demagnifying effect on the distortion pattern as for type N (and Θ, as always, produces a trivial spatial rotation). If we now choose a different value for a, thereby boosting the frame such that $\boldsymbol{E}_4$ is no longer in the plane spanned by the two different principal null directions, the simple zero of the distortion pattern moves away from the south pole. Hence, at a type-III spacetime point the observer can easily read from the distortion pattern whether his or her 4-velocity is in the plane spanned by the two different principal null directions.

3.3 Type D and type II

In a frame adapted to a principal null direction of multiplicity two the Newman-Penrose coefficients satisfy $\hat{\Psi}_1 = 0$, $\hat{\Psi}_2 \neq 0$ (and $\hat{\Psi}_0 = 0$, $\hat{\Psi}_4 = 2$). The remaining two principal null directions are given by the two non-zero solutions of equation (18) with the hatted coefficients, i.e.,

$$z_{3/4} = -\hat{\Psi}_3 \pm \sqrt{\hat{\Psi}_3^2 - 3\,\hat{\Psi}_2}\ . \tag{23}$$

If these two solutions coincide, the Weyl tensor is of type D, otherwise it is of type II. Hence, we have

$$\text{Type D}: \qquad 3\,\hat{\Psi}_2 = \hat{\Psi}_3^2 \tag{24}$$

$$\text{Type II}: \qquad 3\,\hat{\Psi}_2 \neq \hat{\Psi}_3^2. \tag{25}$$

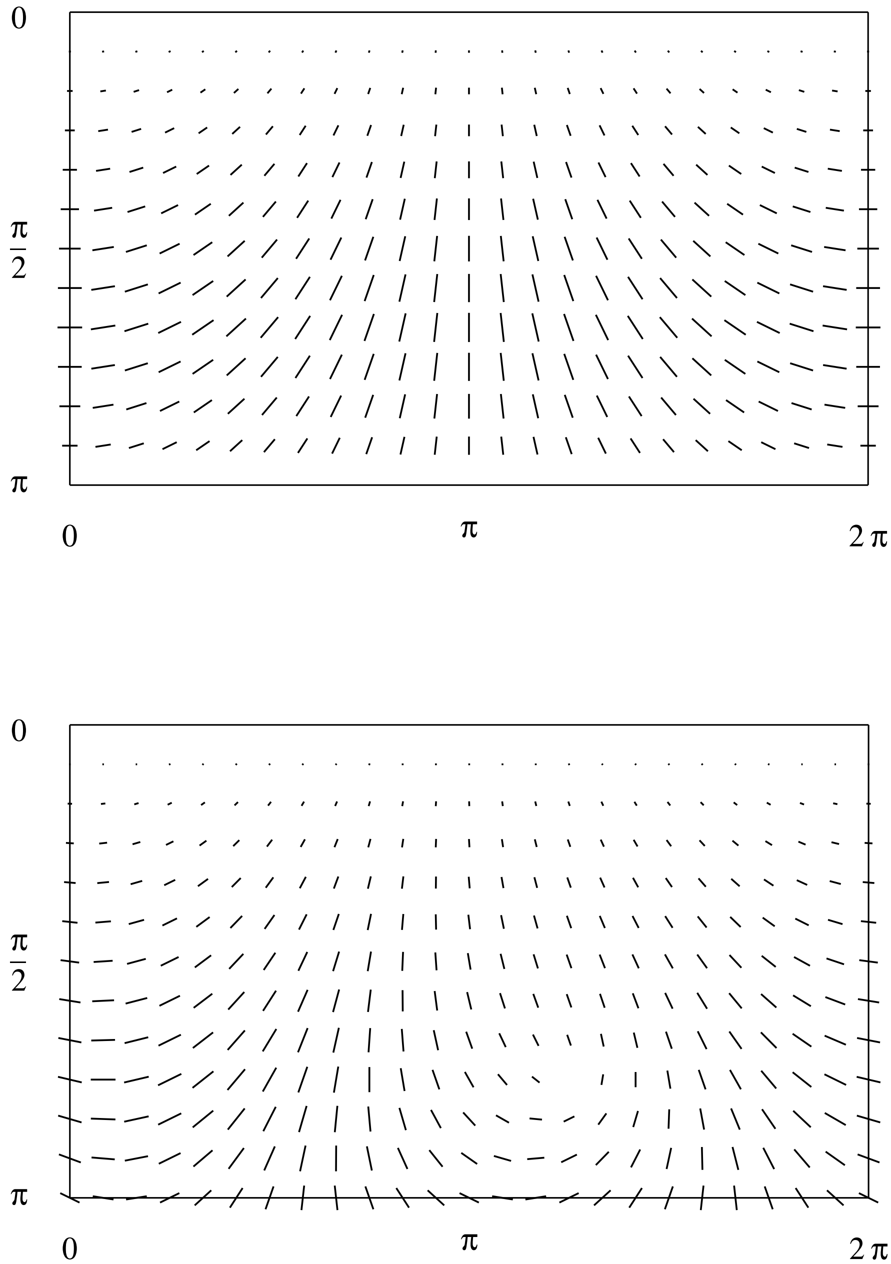


Figure 4: Distortion pattern for Petrov type III, seen by an observer whose 4-velocity lies in the plane spanned by the two different principal null directions (top) and by another observer (bottom). The first picture shows the pattern $\varepsilon e^{-2i\chi} = 0.007 \sin\vartheta \,(\, 1 - \cos\vartheta \,)$, the second one results by applying a rotation of class I with $a = 0.52 - i\,0.52$, followed by a rotation of class III with $Ae^{-i\Theta} = 0.56$. Here and in the following pictures the apparently strange choice of numerical values is motivated by our desire to have the zeros of the distortion pattern on (or close to) grid points.

Inserting the transformed Newman-Penrose coefficients (20) into (8) yields the distortion pattern for type D and type II in an arbitrary frame,

$$\begin{aligned}\varepsilon\, e^{-2i\chi} \;&=\; 3\,\hat{\Psi}_2 \sin^2\!\vartheta + 2\,A\,e^{-i(\Theta+\varphi)}\,(\hat{\Psi}_3 + 3\,\bar{a}\,\hat{\Psi}_2)\sin\vartheta\,(1-\cos\vartheta) + \\ &\qquad A^2\,e^{-2i(\Theta+\varphi)}\,(1+2\,\bar{a}\,\hat{\Psi}_3 + 3\,\bar{a}^2\,\hat{\Psi}_2\,)\,(1-\cos\vartheta)^2\,.\end{aligned} \tag{26}$$

We first discuss type D. With (24), (26) simplifies to

$$\varepsilon\, e^{-2i\chi} = \big(\hat{\Psi}_3 \sin\vartheta + A\,e^{-i(\Theta+\varphi)}\,(1+\bar{a}\,\hat{\Psi}_3)\,(1-\cos\vartheta)\big)^2, \tag{27}$$

see Figure 5 and Figure 6. From this equation we read that for type D the universality property known from types N and III is not satisfied: The distortion pattern depends on the (non-zero) value of $\hat{\Psi}_3$, i.e., there is a one-complex-parameter family of genuinely distinct type-D spacetime points. Even the differential-topological structure of the distortion pattern depends on $\hat{\Psi}_3$. If $\hat{\Psi}_3$ is real (i.e., if $\hat{\Psi}_2 = \frac{1}{3}\hat{\Psi}_3^2$ is real and positive), then the integral lines of the distortion field are closed curves encircling each of the two double zeros; if $\hat{\Psi}_3$ is non-real, those integral lines start at one double zero and terminate at the other. The effect of a Lorentz transformation is as follows. Choosing a such that $1+\bar{a}\,\hat{\Psi}_3 = 0$ gives us a frame such that the two double zeros are in antipodal positions at the sky. This is the case if and only if the vector $\boldsymbol{E}_4 = \frac{1}{\sqrt{2}}(\boldsymbol{\ell}+\boldsymbol{n})$ lies in the plane spanned by the two double principal null directions. Restricting the frames in this way leaves the freedom of choosing A and Θ arbitrarily; this, however, has no effect on the distortion pattern. In other words, all observers whose 4-velocity is in the plane spanned by the two double principal null directions see exactly the same pattern. Changing a, i.e., boosting to a frame such that $\boldsymbol{E}_4$ is not in the plane spanned by the two double principal null directions, has the effect that the second double zero moves away from the south pole. Hence, by observing the distortion pattern an observer can find out whether the Weyl tensor at the point of observation is of type D, and whether his or her 4-velocity is in the plane spanned by the two double principal null directions. Moreover, the differential-topological structure of the pattern alone allows to find out whether in a frame adapted to one of the two double principal null directions the Newman-Penrose coefficient $\hat{\Psi}_2 = \frac{1}{3}\hat{\Psi}_3^2$ is real and positive.

We now turn to type II, see Figure 7 and Figure 8. By choosing a such that $\hat{\Psi}_3 + 3\,\bar{a}\,\hat{\Psi}_2 = 0$ and choosing A and Θ such that $A^2\,e^{-2i\Theta}\,(3\,\hat{\Psi}_2 - \hat{\Psi}_3^2) = 9\hat{\Psi}_2^2$ we may boost to a frame such that $\boldsymbol{E}_4 = \frac{1}{\sqrt{2}}(\boldsymbol{\ell}+\boldsymbol{n})$ lies in the plane $\mathcal{H}$ spanned by the two simple principal null directions and the double principal null direction is spanned by the sum of $\boldsymbol{E}_4$ and a vector perpendicular to $\mathcal{H}$. These properties fix $\boldsymbol{E}_4$ uniquely. In this frame, the distortion field is given by

$$\varepsilon\, e^{-2i\chi} = 3\,\hat{\Psi}_2\,\big(\sin^2\!\vartheta + e^{-2i\varphi}\,(1-\cos\vartheta)^2\big)\,. \tag{28}$$

In this representation the two simple principal null directions are situated at the equator $\vartheta = \pi/2$ and opposite to each other. Similarly to the type-D case, there is a one-complex-parameter family of genuinely distinct type-II spacetime points, depending on the value of $\hat{\Psi}_2$. Again, the differential-topological structure of the distortion pattern depends on whether or not $\hat{\Psi}_2$ is real and positive. If this is true, then all the integral lines of the distortion field are closed curves encircling the double zero, with the exception of one particular integral line that connects the two simple zeros. If $\hat{\Psi}_2$ is not real and positive, then every integral line connects the double zero with one of the two simple zeros.

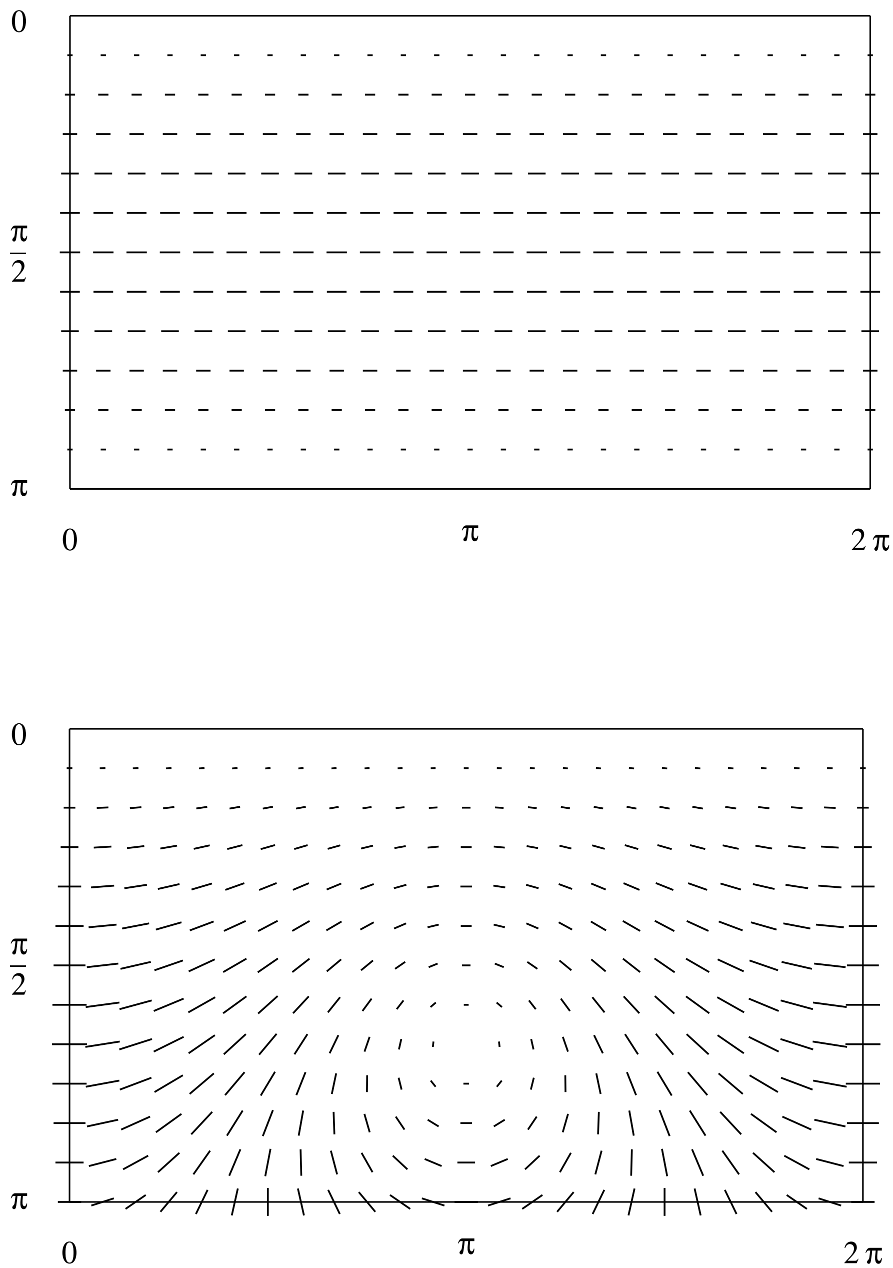


Figure 5: Distortion pattern for Petrov type D with $\hat{\Psi}_2 = \frac{1}{3}\hat{\Psi}_3^2$ real and positive, seen by an observer whose 4-velocity lies in the plane spanned by the two different principal null directions (top) and by another observer (bottom). We have chosen $\hat{\Psi}_3 = 0.08$. The second picture results from the first one by applying a rotation of class I followed by a rotation of class III with $\bar{a}Ae^{-i\Theta} = 0.58$.

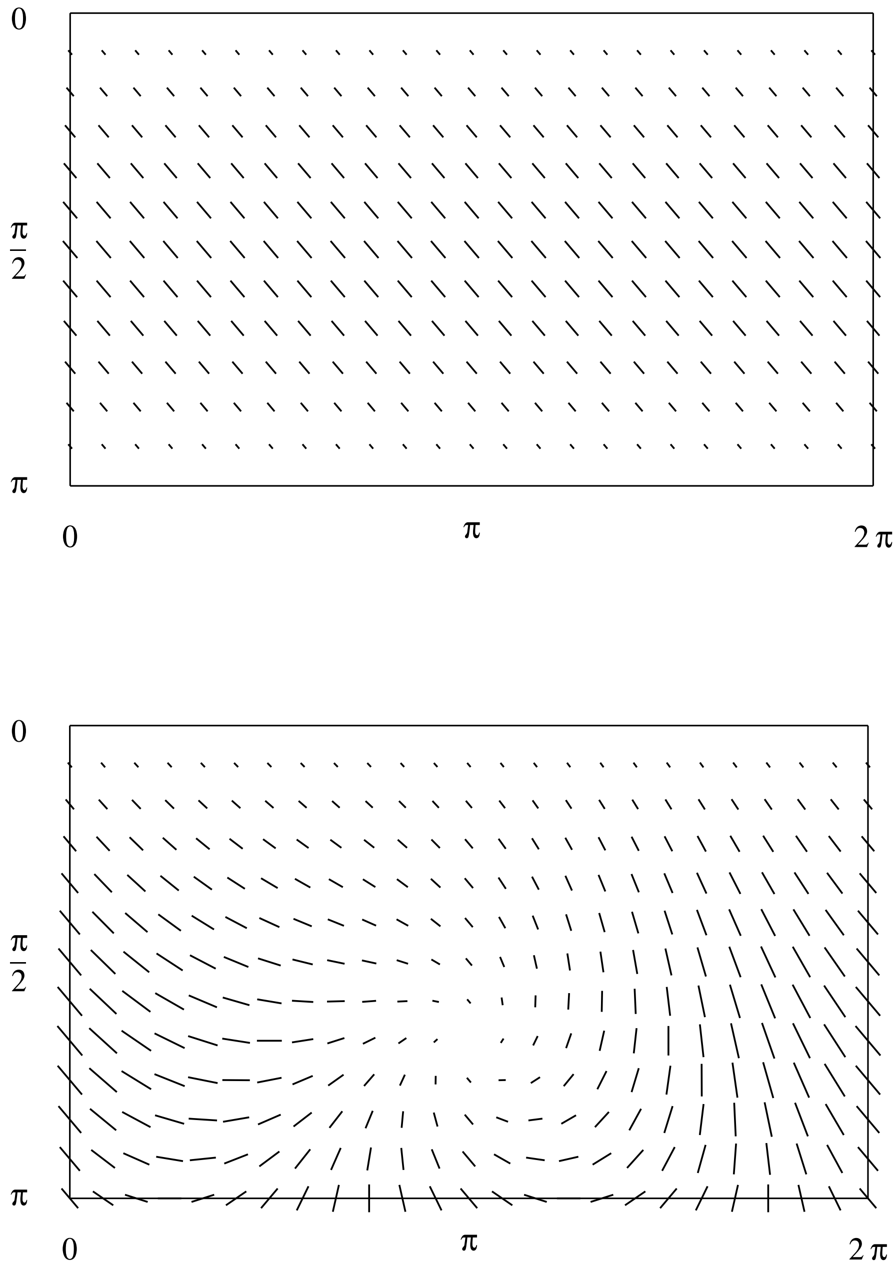


Figure 6: Distortion pattern for Petrov type D with $\hat{\Psi}_2 = \frac{1}{3}\hat{\Psi}_3^2$ not real and positive, seen by an observer whose 4-velocity lies in the plane spanned by the two different principal null directions (top) and by another observer (bottom). This time we have chosen $\hat{\Psi}_3 = 0.08e^{i\pi/4}$. As in Figure 5, the second picture results from the first one by applying a rotation of class I followed by a rotation of class III with $\bar{a}Ae^{-i\Theta} = 0.58$.

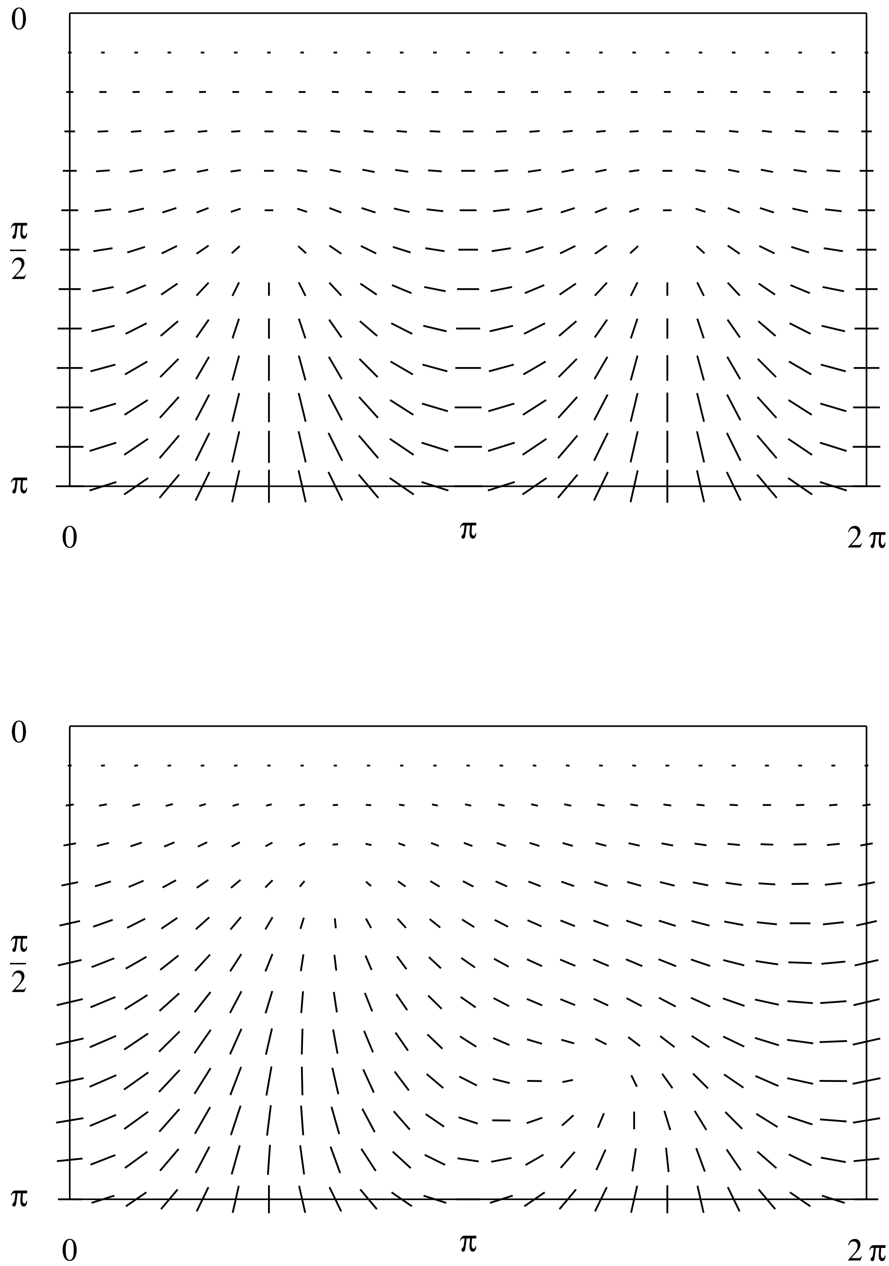
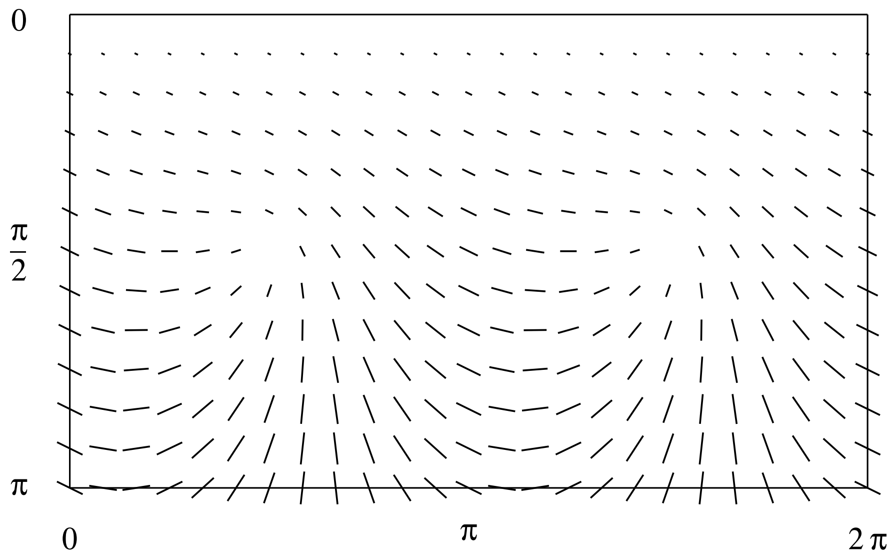


Figure 7: Distortion pattern for Petrov type II with $\hat{\Psi}_2$ real and positive, seen by an observer with the special 4-velocity such that the distortion pattern is given by (28) (top) and by another observer (bottom). We have chosen $\hat{\Psi}_2 = 0.001$. $\hat{\Psi}_3$ is arbitrary except for $3\hat{\Psi}_2 \neq \hat{\Psi}_3^2$. The second picture results from the first one by applying a rotation of class I with $a = 0.32 + i\,0.73$, followed by a rotation of class III with $Ae^{-i\Theta} = 0.93 + i\,0.33$.



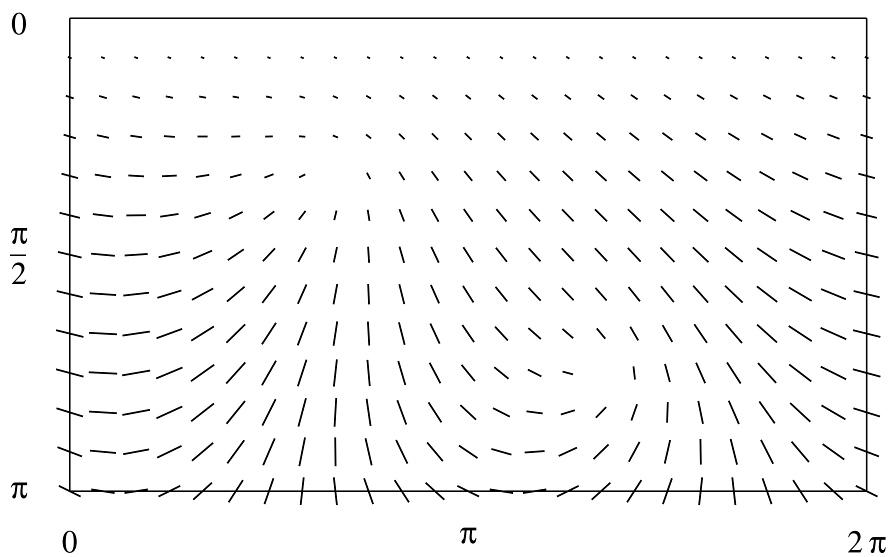


Figure 8: Distortion pattern for Petrov type II with $\hat{\Psi}_2$ not real and positive, seen by an observer with the special 4-velocity such that the distortion pattern is given by (28) (top) and by another observer (bottom). This time we have chosen $\hat{\Psi}_2 = 0.001e^{i\pi/4}$. Again, $\hat{\Psi}_3$ is arbitrary, except for $3\hat{\Psi}_2 \neq \hat{\Psi}_3^2$, and the second picture follows from the first one by applying a rotation of class I with $a = 0.32 + i\,0.73$, followed by a rotation of class III with $Ae^{-i\Theta} = 0.93 + i\,0.33$.

3.4 Type I

Type I is characterized by the fact that all four principal null directions are distinct. In a frame adapted to one of them we have $\hat{\Psi}_1 \neq 0$ (and $\hat{\Psi}_0 = 0$, $\hat{\Psi}_4 = 2$). With the transformed Newman-Penrose coefficients (20) we find the following expression for the distortion pattern

$$\begin{aligned} \varepsilon\, e^{-2i\chi} \; = \; & 2\, A^{-1}\, e^{i(\Theta+\varphi)}\, \hat{\Psi}_1 \sin\vartheta\,(1+\cos\vartheta) + 3(\hat{\Psi}_2 + 2\,\bar{a}\,\hat{\Psi}_1)\sin^2\!\vartheta \, + \\ & 2\, A\, e^{-i(\Theta+\varphi)}\,(\hat{\Psi}_3 + 3\,\bar{a}\,\hat{\Psi}_2 + 3\,\bar{a}^2\,\hat{\Psi}_1)\sin\vartheta\,(1-\cos\vartheta) \, + \\ & A^2\, e^{-2i(\Theta+\varphi)}\,(1 + 2\,\bar{a}\,\hat{\Psi}_3 + 3\,\bar{a}^2\,\hat{\Psi}_2 + 2\,\bar{a}^3\,\hat{\Psi}_1)(1-\cos\vartheta)^2 \, . \end{aligned} \tag{29}$$

We may choose a such that $1 + 2\,\bar{a}\,\hat{\Psi}_3 + 3\,\bar{a}^2\,\hat{\Psi}_2 + 2\,\bar{a}^3\,\hat{\Psi}_1 = 0$, thereby moving the second zero to the south pole $\vartheta = \pi$. With a fixed that way, we may choose A and Θ such that $A^{-2}\, e^{2i\Theta}\hat{\Psi}_1 = (\hat{\Psi}_3 + 3\,\bar{a}\,\hat{\Psi}_2 + 3\,\bar{a}^2\,\hat{\Psi}_1)$, thereby moving the remaining two simple zeros in symmetric positions, $\vartheta_3 + \vartheta_4 = \pi$ and $\varphi_3 + \varphi_4 = 2\pi$. The resulting frame is characterized by the fact that $\boldsymbol{E}_4 = \frac{1}{\sqrt{2}}(\boldsymbol{\ell} + \boldsymbol{n})$ lies in the plane $\mathcal{H}$ spanned by two of the simple principal null directions and the other two simple principal null directions are spanned by vectors of the form $\boldsymbol{E}_4 + \boldsymbol{X} + \boldsymbol{Y}$ and $\boldsymbol{E}_4 + \boldsymbol{X} - \boldsymbol{Y}$ where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are perpendicular to $\boldsymbol{E}_4$ and $\boldsymbol{X}$ is perpendicular to $\mathcal{H}$. These properties characterize $\boldsymbol{E}_4$ uniquely, except for the possibility of permutating the four principal null directions. In this frame, the distortion pattern takes the form

$$\varepsilon\, e^{-2i\chi} = Q \sin\vartheta\,\big((1+\cos\vartheta)\, e^{i\varphi} + (1-\cos\vartheta)\, e^{-i\varphi}\big) + P\sin^2\!\vartheta \tag{30}$$

where $Q = 2\, A^{-1} e^{i\Theta}\, \hat{\Psi}_1 \;\in\; \mathbb{C}\setminus\{0\}$ and $P = 3\,(\,\hat{\Psi}_2 \,+\, 2\,\bar{a}\,\hat{\Psi}_1\,) \;\in\; \mathbb{C}$ are parameters that characterize the spacetime point. Thus, there is a two-complex-parameter family of type-I spacetime points which are genuinely distinct in the sense that their distortion patterns are not related by a Lorentz transformation (i.e., by a conformal transformation of the celestial sphere). However, the distortion patterns of type-I spacetime points are always related by diffeomorphism, i.e., the differential-topological properties of the distortion pattern are the same for all values of Q and P. The top part of Figure 9 shows the distortion pattern given by (30) for some values of Q and P; the bottom part demonstrates the effect of a Lorentz transformation on this pattern. Two real parameters are necessary for fixing the positions of the zeros in the first picture. The other two real parameters in Q and P fix the overall scaling factor and the twist of the pattern.

As an aside, we mention that there is always a frame in which the four simple zeros on the sky form a *disphenoid*, i.e., a tetrahedron such that opposite edges have equal length, see Penrose and Rindler [3], Volume II, Section 8. However, with our convention of keeping one of the zeros at the north pole the distortion field in this particular frame is given by a rather awkward expression.

4 Conclusions

In this paper we have given a new formula, eq. (8), for the distortion effect on light sources around an observer in an arbitrary spacetime. This formula, which is exact to within lowest non-trivial order with respect to the distance between light sources and observer, comes about by rewriting classical results of Kristian and Sachs [2] in terms of the Newman-Penrose formalism. On the basis of this formula we have discussed the distortion patterns graphically for each Petrov type. We feel that the resulting pictures have some didactic value

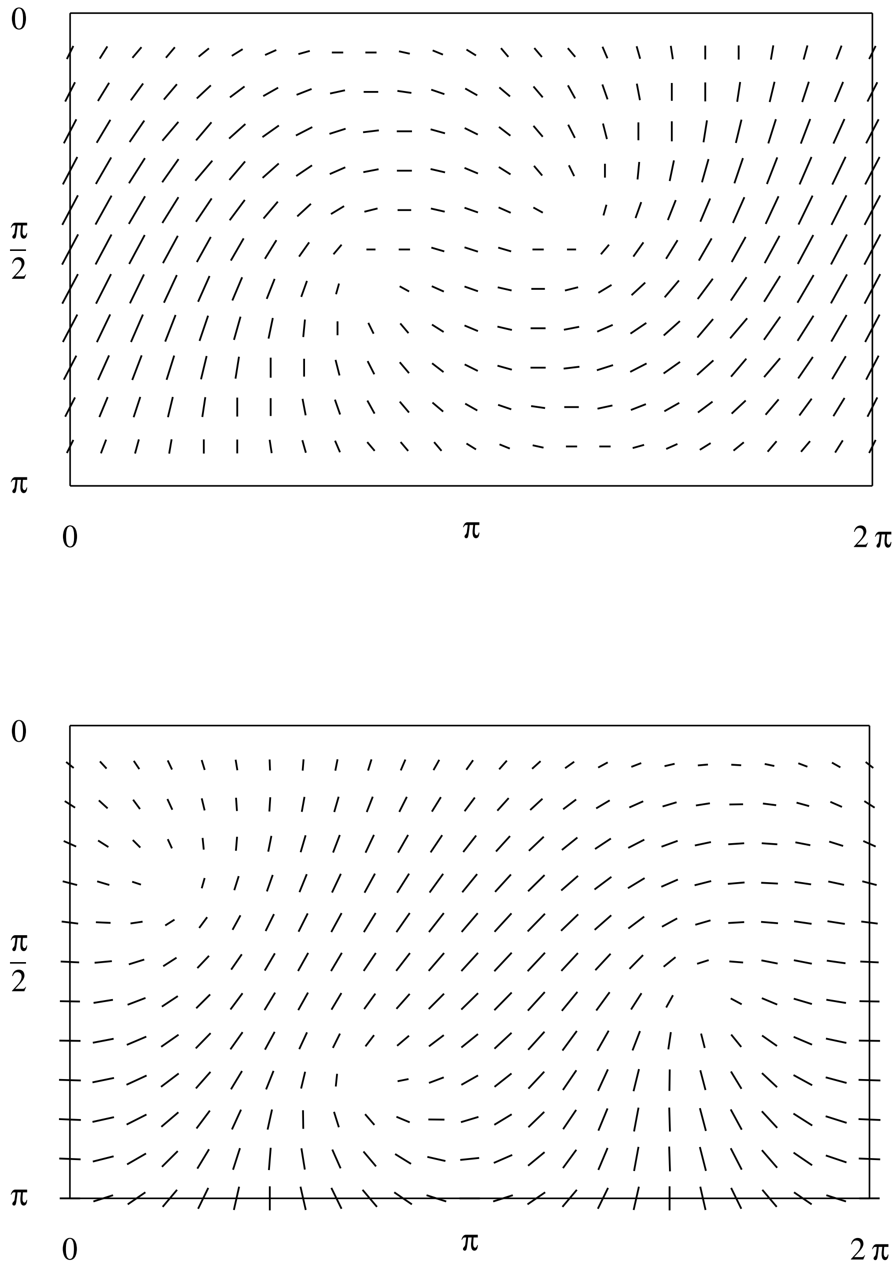


Figure 9: Distortion pattern for Petrov type I, seen by an observer with the special 4-velocity such that the distortion pattern is given by (30) (top) and by another observer (bottom). We have chosen $P = -0.0018 - i\,0.0056$ and $Q = -0.0022 - i\,0.0035$. The second picture results from the first one by applying a rotation of class I with $a = -0.25 + i\,0.14$, followed by a rotation of class III with $Ae^{-i\Theta} = -1.37 + i\,0.41$.

as visualizing the effect of the Weyl tensor on light rays. In this respect our results extend the work by Penrose and Rindler [3] on the 'fingerprint direction field' of the Weyl tensor insofar as (i) we discuss not only the direction but also the magnitude of the distortion, (ii) our patterns are metrically correct on the whole celestial sphere, not just differential-topologically near a principal null direction, (iii) we discuss the dependence on the observer's velocity.

As a special application, it is an interesting question to ask whether the distortion patterns derived in this paper have some relevance in view of cosmology. As mentioned already in the introduction, image distortion produced by large-scale structure has been observed recently [10, 11, 12, 13, 14] by statistically evaluating galaxy ellipticities in selected fields on the sky. Following the original ideas of Kristian and Sachs [2], we want to ask whether observations of this kind can be used to determine the Weyl tensor 'here and now' in an appropriately smoothed universe. For discussing this question we subdivide our celestial sphere into fields of some specified size and we specify a maximal distance up to which galaxies are to be observed. By averaging galaxy ellipticities over each field and assigning the result to the field center we get the distortion pattern of a smoothed universe. Here the smoothing procedure refers to averaging the Weyl tensor over conic regions, with the observer at the tip of the cone, which is not directly related to averaging the matter density. If the field size and/or the maximal distance has been chosen small, then the distortion effect in each conic region is essentially determined by local inhomogeneities. In other words, even in the smoothed universe the Weyl tensor will have a fairly strong variation, so the $O(s^3)$-terms in (5), which involve covariant derivatives of the Weyl tensor, cannot be neglected. For this reason we cannot expect that the resulting pattern coincides with one of the patterns derived in the preceding section. However, it does make sense to compare the observed distortion pattern with the ones derived in the preceding section if we increase the field size and the maximal distance sufficiently such that the distortion effect of local inhomogeneities becomes irrelevant. There are three possible outcomes.

First, the distortion effect may vanish over the whole sky, for field size and maximal distance chosen appropriately. This would indicate that, at this level of averaging, the Weyl tensor is zero. This is what we expect if we rely on the assumption that our universe differs from a Robertson-Walker cosmos only by local 'lumps' whose effect on light rays averages to zero. The present observations are in accordance with this assumption, but one has to keep in mind that they cover only a very small portion of the sky. When supported by future observations that cover large parts of the sky, this can be viewed as an independent confirmation of the standard cosmology assumptions. It is remarkable that this line of thought is based on geometry only, i.e., Einstein's field equation is not used.

Second, the distortion pattern may become similar to one of the patterns derived in the preceding section, for some field size and some maximal distance. This would indicate that in our standard cosmology model the Robertson-Walker background has to be replaced by some other background model, namely by one with a non-trivial Petrov type. In this case we could read from the distortion pattern not only the Petrov type of the Weyl tensor 'here and now' (in the smoothed universe) but also the position of our 4-velocity vector in relation to the principal null directions.

Third, the distortion effect may be non-zero but different from all the patterns derived in the preceding section, for any field size and any maximal distance. In this case there would be the following possible explanations. (i) The galaxies inside one field have non-randomly distributed actual axes, even for large field size and large maximal distance. This could be taken as indicating a prefered direction in the universe such as given, e.g., by a cosmic rotation or by a cosmic magnetic field. (ii) Even in the smoothed universe the derivative terms of the Weyl tensor are so large that the $O(s^3)$ terms in (5) cannot be neglected. (iii) In some fields the average distance of the galaxies is larger than in other fields. As long as the $O(s^3)$ terms can be neglected, this has an effect only on the magnitude but not on the direction of the distortion effect, i.e., it has an effect on ε, as a function on the celestial sphere, but not on χ.

In this sense, observing the distortion effect can provide us with some information on the Weyl tensor in the universe, based on geometry arguments alone. This line of thought is meant as complementary to the ongoing efforts of using distortion observations for tracing the matter distribution in the universe, based on the approximative assumptions of the weak-lensing formalism.

Two questions are left open for future work. The first one is whether there are some classes of spacetimes, e.g. Bianchi models, for which the distortion patterns can be calculated *exactly*, not just to within lowest order with respect to the distance. The second one is whether distortion patterns can be linked to the matter distribution without making approximative assumptions of the weak-lensing kind. It is true that, quite generally, the Weyl tensor is that part of the curvature which is *not* determined, at a particular event, by the energy-momentum tensor at that event via Einstein's field equation. However, the divergence of the Weyl tensor is related to derivatives of the energy-momentum tensor by the equation ${C^{abcd}}_{;d} = T^{c[a;b]} - \frac{1}{3} g^{c[a} T^{;b]}$ which follows from the Bianchi identity and Einstein's field equation (see Kundt and Trümper [27] or Ellis [28]). Maybe this equation can be used, at least in special classes of spacetimes, for gaining some information about the energy-momentum tensor from distortion patterns. In this connection recent work by Frittelli, Kling and Newman [29, 30] is of some interest. In these articles the authors discuss image distortion in arbitrary spacetimes without using approximative assumptions of the weak-lensing kind and also without using series expansions of the Kristian-Sachs type.

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