



ELSEVIER

14 June 2001

Physics Letters B 509 (2001) 323–330

PHYSICS LETTERS B

www.elsevier.nl/locate/npe

Hyperbolic Kac–Moody algebras and chaos in Kaluza–Klein models

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Received 14 March 2001; accepted 25 March 2001

Editor: P.V. Landshoff

Abstract

Some time ago, it was found that the never-ending oscillatory chaotic behaviour discovered by Belinskii, Khalatnikov and Lifshitz (BKL) for the generic solution of the vacuum Einstein equations in the vicinity of a spacelike (“cosmological”) singularity disappears in spacetime dimensions $D \equiv d + 1 > 10$. Recently, a study of the generalization of the BKL chaotic behaviour to the superstring effective Lagrangians has revealed that this chaos is rooted in the structure of the fundamental Weyl chamber of some underlying hyperbolic Kac–Moody algebra. In this Letter we show that the same connection applies to pure gravity in any spacetime dimension ≥ 4 , where the relevant algebras are AE_d . In this way the disappearance of chaos in pure gravity models in $D \geq 11$ dimensions becomes linked to the fact that the Kac–Moody algebras AE_d are no longer hyperbolic for $d \geq 10$. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

A remarkable result in theoretical cosmology has been the construction, by Belinskii, Khalatnikov and Lifshitz (BKL), of a generic solution to the 4-dimensional vacuum Einstein equations in the vicinity of a spacelike (“cosmological”) singularity [1]. This solution exhibits a never-ending oscillatory behaviour of the mixmaster type [2,3] with strong chaotic properties. Some time ago, it was found that the BKL analysis for pure gravity leads to completely different qualitative features in spacetime dimensions $D \geq 11$ [4,5]. Namely, for those dimensions, the generic solution

to the vacuum Einstein equations ceases to exhibit chaotic features, but is instead asymptotically characterized by a monotonic Kasner-like solution (for a review, see [6]). The critical dimension $D = 11$ was discovered by a straightforward but lengthy procedure, with no direct interpretation. Another system for which chaos is known to disappear is the pure gravity-dilaton system in all spacetime dimensions [7,8].

More recently [9–11], the BKL analysis was extended to the supergravity Lagrangians in 10 [12,13] and 11 dimensions [14] that emerge as the low energy limits of the superstring theories (IIA, IIB, I, HO, HE) and M theory, respectively. Contrary to what happens for the gravity-dilaton system in 10 dimensions or pure gravity in 11 dimensions, the chaotic oscillatory behaviour was found to be generic in all superstring and

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M-theory models thanks to the p -forms present in the field spectrum [9]. It was furthermore proved that this chaos was rooted in the structure of the fundamental Weyl chamber of some Kac–Moody algebra [11]. More precisely, reformulating the asymptotic analysis of the dynamics as a billiard problem ‘*a la* Chitre–Misner [15,16], it was shown that the never ending oscillatory BKL behaviour could be described as a relativistic billiard within a simplex in 9-dimensional hyperbolic space. The reflections on the faces of this billiard were shown to generate a Coxeter group, which was then identified with the Weyl group of the hyperbolic Kac–Moody algebras E_{10} for the type IIA, IIB, and M theories, and BE_{10} for the type I, HO, HE theories (for background on Kac–Moody algebras and notations, see the textbooks [17,18]). In this way, a relation was established between the fact that the billiard has finite volume, and hence chaotic dynamics, and the hyperbolicity of the underlying indefinite Kac–Moody algebras E_{10} and BE_{10} .

In this Letter, we reexamine the case of pure gravity in arbitrary space–time dimension $D \equiv d + 1$ in the light of these results. We demonstrate that the asymptotic dynamics (for $t \rightarrow 0$, at any point in space) can again be viewed as a billiard in the fundamental Weyl chamber of an indefinite Kac–Moody algebra, which is now $AE_d \equiv A_{d-2}^{\wedge\wedge} \equiv A_{d-2}^H$. This algebra is the “overextended” [19] or “canonical hyperbolic” extension [17] of the (finite dimensional) Lie algebra A_{d-2} ; its associated Dynkin diagram is obtained by attaching, at the affine node, one more node to the Dynkin diagram of the affine algebra $A_{d-2}^{(1)} \equiv A_{d-2}^{\wedge}$ and is displayed in Fig. 1. The algebra $AE_3 \equiv A_1^{\wedge\wedge} \equiv A_1^H$ has been particularly studied in [20] and was related in [21] to $D = 4$ (super-)gravity. Note that the double-line (in the conventions of [17]) in its Dynkin

diagram can be viewed as the formal limit of the loop of AE_d as $d \rightarrow 3$. [It is interesting to remark that the Weyl group of AE_3 is $PGL_2(\mathbb{Z})$, which is arithmetic [20,22].] Furthermore we show very explicitly how the occurrence of chaotic behaviour is correlated to the hyperbolicity of the underlying Kac–Moody algebra. More specifically, the algebras AE_d are hyperbolic (in the sense defined in Section 3 below) for $d < 10$, whence pure gravity in dimensions $4 \leq D \leq 10$ is chaotic, whereas chaos disappears in dimensions $D \geq 11$ in accord with the fact that the algebras AE_d are no longer hyperbolic for $d \geq 10$.

The very existence of a connection between the BKL dynamics and indefinite Kac–Moody algebras is already remarkable in itself. For the generic Einstein system with matter couplings, one can always define a billiard that describes the asymptotic dynamics, but in general, this billiard will not exhibit any noticeable regularity properties. In particular, the faces of this billiard need not intersect at angles which are submultiples of π , and consequently the associated reflections will not generate a Coxeter (discrete) reflection group in general; *a fortiori*, the billiard need not be the fundamental Weyl chamber of any Kac–Moody algebra. The hyperbolic Kac–Moody algebra E_{10} (and DE_{10}) was already conjectured in [19,23] to be a hidden symmetry of maximal supergravity reduced to one dimension. The results of [11] and of this Letter indeed support the idea that hyperbolic Dynkin diagrams play a key role in the massless bosonic sectors of supergravity and superstring theory. But we should emphasize that the Kac–Moody algebras do not appear in the present BKL analysis as symmetry algebras with associated Noether charges. They underlie nevertheless the dynamics through their Weyl group, in the sense that the dynamics can be described in terms of “Weyl words” $W_{i_1} W_{i_2} \dots$ made out of the “letters” W_i generating the Weyl reflections.

It is amazing to see the chaos being controlled by the U-duality group G of the toroidal compactification to 3 dimensions via its overextension $G^{\wedge\wedge}$. Recently, it has been shown [24] that both $G = SO(8, 8)$ and $G = SO(8, 9)$ are the U-duality groups of anomaly-free string models; in fact, other $SO(8, 8 + n)$ groups can be realized beyond the heterotic $SO(8, 24)$. A possible explanation for the universality of BE_{10} will be given there as well.

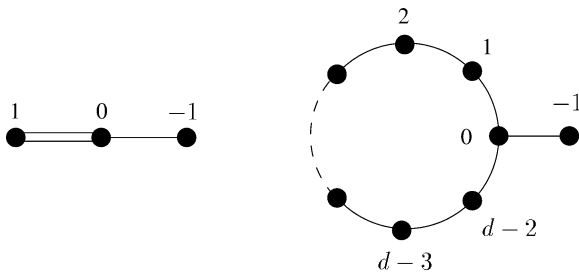


Fig. 1. Dynkin diagram of AE_d (with AE_3 on the left).

2. Gravitational billiard in $d + 1$ dimensions

We first review how Einstein’s theory gives rise to a “gravitational billiard” as one approaches a cosmological singularity; for more details, see [11]. As usual, we assume that the singularity is at $t \rightarrow 0^+$, where t is the proper time in a Gaussian coordinate system adapted to the singularity. In fact, it is convenient to use a time coordinate $\tau \sim -\log t$ such that $\tau \rightarrow +\infty$ as $t \rightarrow 0^+$ [1,2]. In the asymptotic limit, the metric takes the form

$$ds^2 = -(N\sqrt{g}d\tau)^2 + \sum_{\mu=1}^d \exp[-2\beta^\mu(\tau, x^i)](\omega^\mu)^2, \tag{2.1}$$

where the time dependence of the spatial one-forms $\omega^\mu \equiv e_i^\mu(x^j, \tau) dx^i$ ($i = 1, \dots, d$) can be neglected with respect to the time-dependence of the scale functions β^μ . In (2.1), N is the (rescaled) lapse $\sqrt{-g_{00}/g}$, where $g = \exp(-2\sum_{\mu=1}^d \beta^\mu)$ is the determinant of the spatial metric in the frame $\{\omega^\mu\}$. We assume $d \geq 3$ (i.e., $D \geq 4$) since pure gravity in $D = 3$ spacetime dimensions has no local degrees of freedom.

The central feature that enables one to investigate the equations of motion in the vicinity of a spacelike singularity is the asymptotic decoupling of the dynamics at the different spatial points [1]. The remaining effect of the spatial gradients can be accounted for by potential terms for the local scale factors β^μ . Therefore, we focus from now on a specific spatial point and drop reference to the spatial coordinates x^i . In the limit $\tau \rightarrow +\infty$, the dynamics for the scale factors β^μ is governed by the action

$$S[\beta^\mu(\tau), N(\tau)] = \int d\tau \left[\frac{G_{\mu\nu}}{N} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} - N V(\beta^\mu) \right], \tag{2.2}$$

where $G_{\mu\nu}$ is the metric defined by the Einstein–Hilbert action in a d -dimensional auxiliary space \mathcal{M}_d spanned by the “coordinates” β^μ , which must not be confused with physical space–time. This metric is flat and of Minkowskian signature $(-, +, +, \dots, +)$; explicitly, it reads

$$G_{\mu\nu} V^\mu W^\nu = \sum_{\mu=1}^d V^\mu W^\mu - \left(\sum_{\mu=1}^d V^\mu \right) \left(\sum_{\nu=1}^d W^\nu \right). \tag{2.3}$$

We shall also need the inverse metric $G^{\mu\nu}$

$$G^{\mu\nu} \theta_\mu \psi_\nu = \sum_{\mu=1}^d \theta_\mu \psi_\mu - \frac{1}{d-1} \left(\sum_{\mu=1}^d \theta_\mu \right) \left(\sum_{\nu=1}^d \psi_\nu \right). \tag{2.4}$$

In (2.2), the potential V is a sum of sharp wall potentials,

$$V = \sum_i V_i, \quad V_i = \Theta_\infty(-2w_i(\beta)), \tag{2.5}$$

where Θ_∞ vanishes for negative argument and is (positive) infinite for positive argument.¹ The functions $w_i(\beta)$ are homogeneous linear forms, viz.

$$w_i(\beta) = w_{i\mu} \beta^\mu, \tag{2.6}$$

where the covectors $w_{i\mu}$ will be given explicitly below.

Varying the rescaled lapse N yields the Hamiltonian constraint

$$G_{\mu\nu} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} + V = 0, \tag{2.7}$$

where we have set $N = 1$ (i.e., $dt = -\sqrt{g}d\tau$) after taking the variation, since this gauge choice simplifies the formulas (note that this implies indeed $\tau \sim -\log t$ since $\sqrt{g} \sim t$ [1,2]). The dynamics is also subject to the spatial diffeomorphism (momentum) constraints, but these affect the spatial gradients of the initial data and need not concern us here.

We stress that the action (2.2) is *not* obtained by making a dimensional reduction to one dimension of the D -dimensional Einstein–Hilbert action assuming some internal d -dimensional group manifold. Rather, the action (2.2), or, more precisely, the sum over all spatial points of copies of (2.2), supplemented by the momentum constraints, is supposed to yield the asymptotic dynamics in the limit $t \rightarrow 0^+$ for generic

¹ Of course, the factor 2 in the argument of Θ_∞ in (2.5) could be dropped ($\Theta_\infty(\lambda x) = \Theta_\infty(x)$ for $\lambda > 0$), but we keep it in order to emphasize that the walls come with a natural normalization linked to the fact that they initially appear as Toda walls $\sim \exp(-2w_i(\beta))$ [11]. These exponential walls become sharp in the Chitre–Misner limit [15,16], generalized to higher dimensions [11,25].

inhomogeneous solutions [1]. We should mention that the derivation of (2.2) from the Einstein–Hilbert action involves a number of steps that have not been rigorously justified so far. Nevertheless, there is now a wealth of supporting evidence for the BKL analysis, both of analytical and of numerical type [26,27].

Let us study the dynamics of the billiard ball whose motion is described by the functions $\beta^\mu = \beta^\mu(\tau)$. From (2.5) we immediately see that the interior region of the billiard is defined by the inequalities $w_i(\beta) \geq 0$, and that its walls are coincident with the hyperplanes $w_i(\beta) = 0$. Away from the walls, the Hamiltonian constraint becomes

$$G_{\mu\nu} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} = 0. \tag{2.8}$$

Thus the ball travels freely at the speed of light on straight lines until it hits one of the walls and gets reflected. The change of the velocity $v^\mu \equiv \dot{\beta}^\mu$ after a collision on the wall $w_i(\beta) = 0$ is given by a geometric reflection in the corresponding wall hyperplane [5,11]

$$v^\mu \rightarrow v'^\mu = (W_i(v))^\mu \\ \equiv v^\mu - 2 \frac{w_{i\nu} v^\nu}{w_{i\rho} w_i^\rho} w_i^\mu \quad (\text{no sum over } i), \tag{2.9}$$

where $w_i^\mu \equiv G^{\mu\nu} w_{i\nu}$ are the contravariant components of w_i . For a timelike wall (whose normal vector is spacelike), the reflection is an orthochronous Lorentz transformation; hence the velocity remains null and future-oriented. Let C^+ denote the future light cone with vertex at the origin ($\beta^\mu = 0$) where the walls intersect. In the asymptotic regime under study, the initial point from which one starts the motion has positive value of the timelike combination $\sum_{\mu=1}^d \beta^\mu$ of the coordinates; therefore, since the walls $w_i(\beta) = 0$ are all timelike — see below — the ball worldline remains within C^+ [11].

The confinement of the billiard motion to the forward light cone enables one to project, if one so wishes, the piecewise linear motion of the ball in the Minkowski space \mathcal{M}_d onto the upper sheet \mathcal{H}_{d-1} of the unit hyperboloid:

$$\mathcal{H}_{d-1} : G_{\mu\nu} \beta^\mu \beta^\nu = -1, \quad \sum_{\mu=1}^d \beta^\mu > 0. \tag{2.10}$$

A projection is in fact physically necessary in order to take into account the gauge redundancy (time-

reparametrization invariance) and its associated Hamiltonian constraint. One of the β^μ 's does not correspond to an independent degree of freedom. The projection to the upper hyperboloid \mathcal{H}_{d-1} corresponds to viewing the $d - 1$ coordinates of \mathcal{H}_{d-1} as the physical degrees of freedom and $\sum_{\mu=1}^d \beta^\mu$ (or a function of it) as the “time” (see, e.g., [28]). For practical purposes, however, it is also convenient to keep the redundant description in terms of which the evolution is piecewise linear. We shall switch back and forth between the two descriptions. Note that the linear motion of β^μ projects to a geodesic motion on hyperbolic space \mathcal{H}_{d-1} , so the problem is equivalent, in the limit under consideration, to a billiard in hyperbolic space.

We now wish to describe in more detail the convex (half) cone \mathcal{W}^+ defined by the simultaneous fulfillment of all the conditions $w_i(\beta) \geq 0$, to which the motion of the billiard ball is also confined. There are altogether two types of walls. Setting $n \equiv d - 2$, they are

1. Symmetry walls [11]

$$w_i(\beta) = \beta^i - \beta^{i-1} \\ (i = 2, \dots, n \equiv d - 2), \tag{2.11}$$

$$w_0(\beta) = \beta^{d-1} - \beta^{d-2}, \tag{2.12}$$

$$w_{-1}(\beta) = \beta^d - \beta^{d-1}. \tag{2.13}$$

2. Gravitational wall [4]

$$w_1(\beta) = 2\beta^1 + \sum_{i=2}^{d-2} \beta^i \quad (d \geq 4) \tag{2.14}$$

(for $d = 3$, $w_{-1} = \beta^3 - \beta^2$, $w_0 = \beta^2 - \beta^1$ and $w_1 = 2\beta^1$).

There is a total of d walls, which are all timelike since the associated wall forms (normal vectors) w_i ($i = -1, 0, 1, \dots, n$) are spacelike in any spacetime dimension:

$$G^{\mu\nu} w_{i\mu} w_{i\nu} = 2 \quad (i \text{ fixed}).$$

The walls therefore intersect the upper light cone C^+ .

The qualitative dynamics of the billiard can be understood in terms of the relative positions of C^+ and \mathcal{W}^+ . Two cases are possible:

1. \mathcal{W}^+ is contained within C^+ (i.e., all vectors of \mathcal{W}^+ are timelike or null);

2. \mathcal{W}^+ is not entirely contained within \mathcal{C}^+ (i.e., there are not only timelike and null but also spacelike vectors in \mathcal{W}^+).

In the first case, the walls define a generalized, finite-volume simplex in hyperbolic space \mathcal{H}_{d-1} (generalized because some vertices can be at infinity, which occurs when some edges of the cone \mathcal{W}^+ are light-like²). As the walls are timelike, the ball will undergo an infinite number of collisions because, moving at the speed of light, it will always catch up with one of the walls. The only exception, of measure zero, occurs when the ball moves precisely parallel to a lightlike edge of the billiard (there is always at least one such edge). As we shall see in the next section, the dihedral angles of the wall are all submultiples of π , so that the reflections on the sides of the billiard generate a discrete group of isometries of hyperbolic space. Similarly to what happens in the superstring case [11], the projected dynamics on \mathcal{H}_{d-1} is then chaotic (Anosov flow) according to general theorems on the geodesic motion on finite-volume manifolds with constant negative curvature.

In the second case, some walls intersect outside \mathcal{C}^+ and the billiard on \mathcal{H}_{d-1} has infinite volume. The ball undergoes a finite number of collisions until its motion is directed toward a region of \mathcal{W}^+ that lies outside \mathcal{C}^+ . It then never catches a wall anymore because it cannot leave \mathcal{C}^+ : no “cushion” impedes its motion. The dynamics on \mathcal{H}_{d-1} is non-chaotic and the spacetime metric asymptotically tends to a generalized Kasner metric, corresponding to an uninterrupted geodesic motion of the ball.

The question of chaos vs. regular motion is thereby reduced to determining whether it is case 1 or case 2 that is realized. We discuss this in the next section by relating the “wall cone” \mathcal{W}^+ to the fundamental Weyl chamber of a certain indefinite Kac–Moody algebra.

3. Hyperbolic Kac–Moody algebras and chaos

In this section, we show that the reflections (2.9) can be identified with the fundamental Weyl reflections of the indefinite Kac–Moody algebra AE_d , and therefore

² The edges of \mathcal{W}^+ are the (one-dimensional) intersections of $d - 1$ distinct faces of \mathcal{W}^+ .

that the cone \mathcal{W}^+ can be identified with the fundamental Weyl chamber of AE_d . To do that, we need to compute the dihedral angles between the walls. A direct calculation shows that the Gram matrix

$$A_{ij} \equiv G^{\mu\nu} w_{i\mu} w_{j\nu} \quad \text{for } i, j = -1, 0, 1, \dots, n \quad (3.1)$$

of the scalar products of the wall forms is given by

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \quad \text{for } d = 3 \quad (3.2)$$

and

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad \text{for } d > 3. \quad (3.3)$$

In both cases, the wall forms have same length $\sqrt{2}$. As in [11], we identify them with the simple roots of a Kac–Moody algebra. To emphasize the identifications “wall forms = simple roots”, we shall henceforth switch to a new notation and denote the wall forms w_i by r_i . We shall also denote the Cartan subalgebra of the Kac–Moody algebra by \mathfrak{h} and its dual (space of linear forms on \mathfrak{h} , i.e., the “root space”) by \mathfrak{h}^* . Thus,

$$w_i \equiv r_i \in \mathfrak{h}^*. \quad (3.4)$$

We recall that the root space \mathfrak{h}^* is endowed with a bilinear form, which we identify with the bilinear form defined by the (contravariant) metric $G^{\mu\nu}$ given above,

$$r_i \cdot r_j \equiv G^{\mu\nu} r_{i\mu} r_{j\nu}. \quad (3.5)$$

Since the roots have all same length squared 2, the algebra is “simply-laced” and the Gram matrix A_{ij} computed in (3.2) and (3.3) is also the Cartan matrix a_{ij} ,

$$a_{ij} \equiv \frac{2r_i \cdot r_j}{r_i \cdot r_i}, \quad (3.6)$$

i.e., $A_{ij} = a_{ij}$. We then recognize the first matrix as the Cartan matrix of the Kac–Moody algebra AE_3 , while the second matrix is the Cartan matrix of the Kac–Moody algebra AE_d ($d > 3$). This is what justifies the identifications (3.4) and (3.5). The roots r_0, \dots, r_n

form the closed ring of the Dynkin diagram, r_0 is the (affine) root closing the ring, and r_{-1} is the overextended root connected to r_0 .

Once the wall forms are identified with the simple roots of a Kac–Moody algebra, the space \mathcal{M}_d in which the dynamics of the scale factors takes place becomes identified with the Cartan subalgebra \mathfrak{h} of AE_d . The cone \mathcal{W}^+ defining the billiard is given by the conditions

$$\langle r_i, \beta \rangle \geq 0 \quad \text{for all } i = -1, 0, 1, \dots, n \quad (3.7)$$

where $\langle r_i, \beta \rangle$ denotes the pairing between a form $r_i \in \mathfrak{h}^*$ and a vector $\beta \in \mathfrak{h}$. The cone \mathcal{W}^+ is then just the fundamental Weyl chamber [17,18], as was anticipated by our notations. It is striking to note that the finite dimensional germ A_{d-2} of the hyperbolic algebra AE_d is nothing but the Ehlers symmetry of the toroidal compactification of the original gravity from $d + 1$ to 3 dimensions [29]. The reduction to two dimensions brings the affine extension and the final elimination of all spatial coordinates increases the rank further to d [23].

The above Cartan matrices are indecomposable. They are also of indefinite, Lorentzian type since the metric $G_{\mu\nu}$ in \mathfrak{h} is of Lorentzian signature. A Cartan matrix with these properties is said to be of hyperbolic type if any subdiagram obtained by removing a node from its Dynkin diagram is either of finite or affine type [17]. The concept of hyperbolicity is particularly relevant here because it is a general result that the fundamental Weyl chamber \mathcal{W}^+ of a hyperbolic Kac–Moody algebra is contained within the light cone \mathcal{C}^+ ; the Weyl cell is then a (generalized) simplex of finite volume. Furthermore, for hyperbolic KM algebras the closure of the Tits cone, defined as the union of the fundamental Weyl chamber and all its images under the Weyl group, is just \mathcal{C}^+ ([17], Section 5.10).

As already mentioned, the Kac–Moody algebras AE_d are hyperbolic for $d \leq 9$. We will now verify by explicit computation that their associated fundamental Weyl chambers are indeed contained in the forward light cone. The location of the fundamental Weyl chambers in the general case is most conveniently (and most easily) analyzed by means of the fundamental weights $\Lambda_j \in \mathfrak{h}^*$. The latter are defined by

$$r_i \cdot \Lambda_j \equiv G^{\mu\nu} r_{i\mu} \Lambda_{j\nu} = \delta_{ij}, \quad (i, j = -1, 0, 1, \dots, n \equiv d - 2). \quad (3.8)$$

Let us also introduce the coweights $\Lambda_i^\vee \in \mathfrak{h}$, i.e., the contravariant vectors associated with the forms Λ_i with components $(\Lambda_i^\vee)^\mu \equiv G^{\mu\nu} \Lambda_{i\nu}$. Because the fundamental Weyl chamber \mathcal{W}^+ is defined by the conditions $\langle r_i, \beta \rangle \geq 0$, we have

$$\mathcal{W}^+ = \left\{ \beta \in \mathcal{M}_d \equiv \mathfrak{h} \mid \beta = \sum_{i=-1}^n a_i \Lambda_i^\vee, \right. \\ \left. a_i \in \mathbb{R}, a_i \geq 0 \right\}. \quad (3.9)$$

The (one-dimensional) edges of \mathcal{W}^+ are obtained by setting all a_j except one to zero, which gives the vectors Λ_i^\vee . The question of determining whether the fundamental Weyl chamber is contained in the forward light cone or not is thus reduced to a simple computation of the norms of the fundamental weights.

To get the fundamental weights, we observe that if the root r_{-1} is dropped, the associated Cartan matrix reduces to the Cartan matrix of affine $sl(n + 1)$. The affine null root is given by

$$\delta = r_0 + r_1 + \dots + r_n. \quad (3.10)$$

It obeys $\delta^2 \equiv \delta \cdot \delta = 0 = r_j \cdot \delta$ for all $j = 0, 1, \dots, n$ (but $r_{-1} \cdot \delta = -1$). The fundamental weights for the subalgebra A_n are defined by

$$r_i \cdot \lambda_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, n. \quad (3.11)$$

They are explicitly given by

$$\lambda_j = \frac{n - j + 1}{n + 1} [r_1 + 2r_2 + \dots + jr_j] \\ + \frac{j}{n + 1} [(n - j)r_{j+1} + (n - j - 1)r_{j+2} \\ + \dots + r_n] \quad (3.12)$$

with norm

$$\lambda_j^2 = \frac{j(n - j + 1)}{n + 1} > 0 \quad (3.13)$$

(note that $r_0 \cdot \lambda_j = -1$ for all $j = 1, \dots, n$). One then finds for the fundamental weights³ of AE_d

$$\Lambda_{-1} = -\delta, \quad \Lambda_0 = -r_{-1} - 2\delta, \\ \Lambda_j = \Lambda_0 + \lambda_j \quad \text{for } j = 1, \dots, n. \quad (3.14)$$

Their norms (with $\Lambda^2 \equiv \Lambda \cdot \Lambda \equiv G^{\mu\nu} \Lambda_\mu \Lambda_\nu$) are easily computed:

$$\Lambda_{-1}^2 = 0, \quad \Lambda_0^2 = -2, \\ \Lambda_j^2 = -2 + \frac{j(n-j+1)}{n+1}. \quad (3.15)$$

Note that Λ_{-1} is always lightlike, and Λ_0 is timelike for all n . It is furthermore elementary to check that

$$\Lambda_j^2 \leq 0 \quad \text{for all } j \text{ if } n \leq 7 \quad (3.16)$$

with equality only for $n = 7$ and $j = 4$. For $n \geq 8$ there is always at least one spacelike fundamental weight Λ_j ; e.g., for $n = 8$ we have

$$\Lambda_4^2 = \Lambda_5^2 = \frac{2}{9} > 0. \quad (3.17)$$

The above calculation then tells us that for $n \leq 7$ (i.e., for AE_d with $d \leq 9$) the fundamental Weyl chamber is contained in the forward light cone with one edge touching the light cone (two edges for $n = 7$). For $n \geq 8$ there is at least one spacelike edge, so the Weyl chamber contains timelike, lightlike and spacelike vectors. This is, then, the Kac–Moody theoretic understanding of the fact that the asymptotic solution of the vacuum Einstein equations in the vicinity of a spacelike singularity exhibits the never-ending oscillatory behaviour of the BKL type in spacetime dimensions ≤ 10 , while this ceases to be the case for $D \geq 11$ [4].

To conclude this Letter, we would like to stress once more that the emergence of a Kac–Moody algebra is not automatic for the gravitational systems under consideration. For instance, the billiard associated with the Einstein–Maxwell system in D space-time dimensions has the same symmetry walls (2.11),

³ In general case with the highest root $\theta = \sum_j m_j r_j$, we have $r_0 \cdot \lambda_j = -m_j$ and the fundamental weights are given by

$$\Lambda_{-1} = -\delta, \quad \Lambda_0 = -r_{-1} - 2\delta, \quad \Lambda_j = m_j \Lambda_0 + \lambda_j.$$

An alternative representation is $\Lambda_i = \sum_j (a^{-1})_{ij} r_j$, where $(a^{-1})_{ij}$ is the inverse Cartan matrix.

(2.12), (2.13), but the gravitational wall (2.14) is replaced by the (asymptotically dominant) electric wall $w_1(\beta) = \beta^1$. This wall is orthogonal to all symmetry walls, except w_2 (w_0 for $d = 3$) with which it makes an angle α given by $\cos \alpha = \sqrt{(d-1)/2(d-2)}$. This dihedral angle is generically not a submultiple of π and the associated group of reflections is not a discrete group, with two notable exceptions: (i) α is equal to zero for $D = 4$, where electric and gravitational walls coincide (though the wall forms are normalized differently), and (ii) the angle α is equal to $\pi/6$ for the case $D = 5$, whose study was advocated in [10] in the context of homogeneous models. Taking into account that the wall form w_1 has norm squared equal to $(d-2)/(d-1) = 2/3$, one gets in that case the Cartan matrix

$$a_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 2 & -3 \\ 0 & -1 & -1 & 2 \end{pmatrix}. \quad (3.18)$$

The underlying Kac–Moody algebra is the canonical hyperbolic extension of the exceptional Lie algebra G_2 (hyperbolic algebra number 16 in Table 2 of [30]). One Einstein–Maxwell theory in 5 dimensions is particularly interesting because it is the bosonic sector of simple supergravity in 5 dimensions, which shares many similarities with $D = 11$ supergravity, such as the cubic Chern–Simons term for the vector field [31]. The relevance of the exceptional group G_2 to that theory was pointed out in [29,32]. This system, as well as pure gravity or superstring models and M-theory, for which one does get the Weyl group of a Kac–Moody algebra, are thus rather exceptional [11].

Acknowledgements

T.D. thanks Victor Kac for informative communications. M.H. and H.N. are grateful to the Institut des Hautes Etudes Scientifiques for its kind hospitality. The work of M.H. is partially supported by the “Actions de Recherche Concertées” of the “Direction de la Recherche Scientifique — Communauté Française de Belgique”, by IISN–Belgium (convention 4.4505.86) and by the European Commission programme HPRN-CT-2000-00131 in which he is associated to K.U. Leuven. The work of H.N. was supported in part by the

European Union under Contract No. HPRN-CT-2000-00122. The work of B.J. was supported in part by the European Union under Contract No. HPRN-CT-2000-00131.

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