Perturbations of the Kerr spacetime in horizon penetrating coordinates

Manuela Campanelli[1], Gaurav Khanna[2,3], Pablo Laguna[2,4], Jorge Pullin[2], Michael P. Ryan[5]
1. Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik,
Am Mühlenberg 1, D-14476 Golm, Germany
2. Center for Gravitational Physics and Geometry, Department of Physics,
The Pennsylvania State University, 104 Davey Lab, University Park, PA 16802.
3. Natural Science Division,
Southampton College of Long Island University,
Southampton NY 11968.
4. Department of Astronomy and Astrophysics, The Pennsylvania State University,
525 Davey Lab, University Park, PA 16802.
5. Instituto de Ciencias Nucleares, UNAM
Apartado Postal 70-543 México 04510 D. F., México.

We derive the Teukolsky equation for perturbations of a Kerr spacetime when the spacetime metric is written in either ingoing or outgoing Kerr–Schild form. We also write explicit formulae for setting up the initial data for the Teukolsky equation in the time domain in terms of a three metric and an extrinsic curvature. The motivation of this work is to have in place a formalism to study the evolution in the "close limit" of two recently proposed solutions to the initial value problem in general relativity that are based on Kerr–Schild slicings. A perturbative formalism in horizon penetrating coordinates is also very desirable in connection with numerical relativity simulations using black hole "excision".

I. INTRODUCTION:

There is considerable current interest in studying the collision of two black holes, since these events could be primary sources of gravitational waves for interferometric gravitational wave detectors currently under construction. On the theoretical side, it is expected that the problem of colliding two black holes will be tackled by some combination of full numerical and semi-analytical methods. The first three dimensional collisions of black holes are starting to be numerically simulated, albeit with very limited resolution and grid size. Long time stability of the codes is also an issue. It is therefore of interest to have at hand approximate results which in certain regimes could be used to test the codes. Among such approximation methods is the "close limit" approximation [1] in which the spacetime of a black hole collision is represented as a single distorted black hole. This approximation has been used successfully to test codes for the evolution of black hole space-times that are axisymmetric and tests are under way for "grazing" inspiralling collisions [2]. Another realm of application of perturbative calculations is to provide "outer boundaries" and to extend the reach of Cauchy codes into the radiation zone far away from the black holes, as was demonstrated in [3]. Finally, perturbative codes can be used after a full non-linear binary black hole code has coalesced the holes to continue the evolution in a simple and efficient fashion as was demonstrated in [4,5].

The perturbative approach requires specifying both a background metric and a coordinate system when performing calculations. For evolutions in the time domain such as the ones we are considering, one also has to specify an initial slice of the spacetime. In all perturbative evolutions performed up to now the background spacetime has either been the Schwarzschild solution in ordinary coordinates or the Kerr spacetime in Boyer–Lindquist coordinates. These backgrounds are adequate for instance, for the study of the evolution of "close limits" of the Bowen–York [6] and "puncture" [7] families of initial data, which reduce in the "close limit" to those background space-times.

There have been two recent proposals for alternative families of initial data that have some appealing features [8,9]. Both these proposals are based on the use of the Kerr–Schild form of the Schwarzschild (or Kerr) solutions to represent each of the black holes in the collision. Some of these solutions do not have an obvious close limit in which they yield a single black hole, although the close limit can be arranged with a simple modification of the original proposal. In the case in which the close limit exists, the initial data appear as a perturbation of the Schwarzschild or Kerr spacetime, but in Kerr–Schild coordinates. If one wishes to evolve perturbatively the spacetime, this requires having the perturbative formalism set up on a background spacetime in Kerr-Schild coordinates. To our knowledge, this has never been done in the past.

The use of Kerr–Schild coordinates appears quite desirable in the context of numerical evolutions of black holes. The coordinates penetrate the horizon of the holes without steep gradients in the metric components. This makes them amenable to the numerical technique of singularity excision, which is sometimes viewed as the key to long term binary black hole evolutions such as the ones needed for gravitational wave data analysis purposes [10].

Since the Kerr–Schild coordinates are horizon-penetrating, developing a perturbative formalism in these coordinates could, in principle, allow the study of perturbations arbitrarily close to the horizon and even inside the horizon. The traditional perturbative formalisms, based on Schwarzschild and Boyer–Lindquist (in the Kerr case) coordinates cannot achieve this. Having a perturbative formalism that works close to the horizon is desirable in the context of the recently introduced "isolated horizons" [11]. One could exhibit perturbatively the validity of several new results that are emerging in such context.

In this paper we will develop perturbative equations in Kerr–Schild coordinates, taking advantage of the fact that the Teukolsky formalism is coordinate invariant. We will end by constructing a perturbative equation that is well behaved inside, across and outside the horizon.

The organization of this paper is as follows. In section II we review the Teukolsky formalism (including the setup of initial data for Cauchy evolution). In section III we derive the Teukolsky equation in horizon penetrating Eddington–Finkelstein coordinates and display a numerical evolution of it. In an appendix we discuss the equation in outgoing Eddington–Finkelstein coordinates.

II. THE TEUKOLSKY EQUATION

Fortunately, the perturbative formalism due to Teukolsky [12] is amenable to a reasonably straightforward (but computationally non-trivial) change of background. The Teukolsky formalism is based on the observation that the Einstein equations written in the Newman–Penrose formalism naturally decouple in such a way that one obtains an equation for the perturbative portion of the Weyl spinor. In the notation and conventions of Teukolsky (which in turn follows those of Newman and Penrose), the resulting equation (in vacuum) is,

$$[(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)\,(D + 4\epsilon - \rho) - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)\,(\delta - \tau + 4\beta) - 3\psi_2]\,\psi_4 = 0, \tag{1}$$

where the quantities in brackets are computed using the background geometry and ψ_4 is a first order quantity in perturbation theory. A similar equation can be derived for the ψ_0 component of the Weyl tensor. We will not concentrate on this equation, however, since it has not proven as useful for evolutions in the time domain [13].

If we now particularize to a background spacetime given by the Kerr metric in Boyer–Lindquist coordinates,

$$ds^2 = (1 - 2Mr/\Sigma)\,dt^2 + \left(4Mar\sin^2\theta/\Sigma\right)dtd\varphi - \left(\frac{\Sigma}{\triangle}\right)dr^2 - \Sigma d\theta^2 - \sin^2\theta\left(r^2 + a^2 + 2Ma^2r\sin^2\theta/\Sigma\right)d\varphi^2, \tag{2}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\triangle = r^2 - 2Mr + a^2$ (and should not be confused with the Newman–Penrose quantity $\Delta = n^\mu\partial_\mu$) and considers the Kinnersley tetrad,

$$l^\mu = \left[\left(r^2 + a^2\right)/\triangle, 1, 0, a/\triangle\right], \tag{3}$$

$$n^\mu = \left[r^2 + a^2, -\triangle, 0, a\right]/(2\Sigma), \tag{4}$$

$$m^\mu = \left[ia\sin\theta, 0, 1, i/\sin\theta\right]/(\sqrt{2}(r + ia\cos\theta)), \tag{5}$$

for which the background Newman–Penrose quantities are,

$$\rho = -1/(r - ia\cos\theta), \tag{6}$$

$$\beta = -\rho^*\cot\theta/(2\sqrt{2}), \tag{7}$$

$$\pi = ia\rho^2\sin\theta/\sqrt{2}, \tag{8}$$

$$\tau = -ia\rho\rho^*\sin\theta/\sqrt{2}, \tag{9}$$

$$\mu = \rho^2\rho^*\triangle/2, \tag{10}$$

$$\gamma = \mu + \rho\rho^*(r - M)/2, \tag{11}$$

$$\alpha = \pi - \beta^*, \tag{12}$$

$$\psi_2 = M\rho^3. \tag{13}$$

and the differential operators $D = l^\mu\partial_\mu$, $\Delta = n^\mu\partial_\mu$, $\delta = m^\mu\partial_\mu$, then the resulting equation is the well known Teukolsky equation,

$$
\left[\frac{\left(r^2+a^2\right)^2}{\triangle} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\triangle}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\triangle} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}
$$
$$
-\triangle^2\frac{\partial}{\partial r}\left(\frac{1}{\triangle}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + 4\left[\frac{M(r^2-a^2)}{\triangle} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t}
$$
$$
+4\left[\frac{a(r-M)}{\triangle} + \frac{i\cos\theta}{\sin\theta}\right]\frac{\partial\psi}{\partial\varphi} + (4\cot^2\theta+2)\psi = 0, \tag{14}
$$

where $\psi = (r - ia\cos\theta)^4\psi_4$. As was discussed in references [14], one can write the initial data for the above equation in terms of the perturbative three metric and extrinsic curvature. The formulae for the Teukolsky function and its time derivative are,

$$
\psi_4 = -\left[R_{ijkl} + 2K_{i[k}K_{l]j}\right]_{(1)} n^i m^j n^k m^l + 8\left[K_{j[k,l]} + \Gamma^p_{j[k}K_{l]p}\right]_{(1)} n^{[0}m^{j]}n^k m^l \tag{15}
$$
$$
-4\left[R_{jl} - K_{jp}K^p_l + KK_{jl}\right]_{(1)} n^{[0}m^{j]}n^{[0}m^{l]},
$$
$$
\partial_t\psi_4 = N^\phi_{(0)}\partial_\phi(\psi_4) - n^i m^j n^k m^l\left[\partial_0 R_{ijkl}\right]_{(1)} \tag{16}
$$
$$
+8n^{[0}m^{j]}n^k m^l\left[\partial_0 K_{j[k,l]} + \partial_0\Gamma^p_{j[k}K_{l]p} + \Gamma^p_{j[k}\partial_0 K_{l]p}\right]_{(1)}
$$
$$
-4n^{[0}m^{j]}n^{[0}m^{l]}\left[\partial_0 R_{jl} - 2K^p_{(l}\partial_0 K_{j)p} - 2N_{(0)}K_{jp}K^p_q K^q_l\right.
$$
$$
\left.+K_{jl}\partial_0 K + K\partial_0 K_{jl}\right]_{(1)}
$$
$$
+2\{\psi_4(l_i\Delta - m_i\bar\delta)N^{i\ (0)} + \psi_3(n_i\bar\delta - \bar m_i\Delta)N^{i\ (0)}\},
$$

where

$$
\psi_3 = -\left[R_{ijkl} + 2K_{i[k}K_{l]j}\right]_{(1)} l^i n^j \bar m^k n^l + 4\left[K_{j[k,l]} + \Gamma^p_{j[k}K_{l]p}\right]_{(1)}(l^{[0}n^{j]}\bar m^k n^l - n^{[0}\bar m^{j]}l^k n^l) \tag{17}
$$
$$
-2\left[R_{jl} - K_{jp}K^p_l + KK_{jl}\right]_{(1)}(l^{[0}n^{j]}\bar m^0 n^l - l^{[0}n^{j]}n^0\bar m^l),
$$

$N_{(0)} = (g^{tt}_{\rm kerr})^{-1/2}$ is the zeroth order lapse, n^i in these equations should be taken to be related to that of the original tetrad as $n^0 = N_{(0)}n^0_{\rm orig}, n^i = n^i_{\rm orig} + N^{i\ (0)}n^0$. Latin indices run from 1 to 3, and the brackets are computed to only first order (zeroth order excluded). The derivatives involved in the above expressions can be computed in terms of the initial data on the Cauchy hypersurface as,

$$
\partial_0 K = N_{(0)}K_{pq}K^{pq} - \nabla^2 N_{(0)}, \tag{18}
$$

$$
\partial_0 R = 2K^{pq}\partial_0 K_{pq} + 4N_{(0)}K_{pq}K^p_s K^{sq} - 2K\partial_0 K, \tag{19}
$$

$$
\partial_0 R_{ijkl} = -4N_{(0)}\left\{K_{i[k}R_{l]j} - K_{j[k}R_{l]i} - \frac{1}{2}R\left(K_{i[k}g_{l]j} - K_{j[k}g_{l]i}\right)\right\} \tag{20}
$$
$$
+2g_{i[k}\partial_0 R_{l]j} - 2g_{j[k}\partial_0 R_{l]i} - g_{i[k}g_{l]j}\partial_0 R + 2K_{i[k}\partial_0 K_{l]j} - 2K_{j[k}\partial_0 K_{l]i},
$$

and,

$$
\partial_0 K_{ij} = N_{(0)}\left[\bar R_{ij} + KK_{ij} - 2K_{ip}K^p{}_j - N^{-1}_{(0)}\bar\nabla_i\bar\nabla_j N_{(0)}\right]_{(1)}. \tag{21}
$$

Remarkably, the above formulae are coordinate independent! Therefore the only adjustment needed to specify initial data for the evolution equations we will derive in the next two sections is to insert the appropriate background quantities in the above formulae.

III. THE TEUKOLSKY EQUATION IN KERR–SCHILD COORDINATES: INGOING EDDINGTON FINKELSTEIN COORDINATES

The initial data proposed by [8,9] is constructed using ingoing Eddington–Finkelstein (IEF) coordinates (strictly speaking, since the initial data might include net angular momentum, one is really talking about the generalization of IEF coordinates to the rotating case, commonly referred to as Kerr coordinates). This is in part due to the fact that these families of initial data are currently being evolved using a numerical code where the black holes are treated using the "excision" technique. This technique requires coordinates that penetrate the horizon, such as the IEF ones. The IEF coordinates $(\tilde{V}, r, \theta, \varphi)$ for the Kerr metric are defined through a redefinition of the time coordinate of the Boyer–Lindquist coordinates as,

$$\tilde{V} = t + r^* \tag{22}$$

$$\tilde{\varphi} = \varphi + \int \frac{a}{\triangle} dr \tag{23}$$

where r^* is the natural generalization to the Kerr case of the usual Schwarzschild "tortoise" coordinate, and is defined by,

$$r^* = \int \frac{r^2 + a^2}{r^2 - 2Mr + a^2} dr. \tag{24}$$

The codes currently being used to evolve the initial data in IEF are written in terms of a coordinate $\tilde{t} = \tilde{V} - r$. We will therefore derive the Teukolsky equation in the $(\tilde{t}, r, \theta, \tilde{\varphi})$ coordinates. The Kerr metric in these coordinates,

$$\begin{aligned} ds^2 = {} & (1 - 2Mr/\Sigma)\, d\tilde{t}^2 - (1 + 2Mr/\Sigma)\, dr^2 - \Sigma d\theta^2 - \sin^2\theta \left(r^2 + a^2 + 2Ma^2 r \sin^2\theta/\Sigma\right) d\tilde{\varphi}^2 \\ & - (4Mr/\Sigma)\, d\tilde{t} dr + \left(4Mra \sin^2\theta/\Sigma\right) d\tilde{t} d\tilde{\varphi} + 2a \sin^2\theta\, (1 + 2Mr/\Sigma)\, d\tilde{r} d\tilde{\varphi}, \end{aligned} \tag{25}$$

In addition to changing coordinates, it is immediate to see that one needs also to change tetrads. The usual Kinnersley tetrad is singular at the horizon, and therefore leads to a Teukolsky equation that is singular. However, it is easy to fix this problem by just rescaling l^μ by a factor of $\triangle$ and dividing n^μ by Δ. This does not change the orthogonality properties of the tetrad, but makes it well defined [1]. Therefore we re-derive the Teukolsky equation using as new tetrad vectors,

$$l^\mu = [\triangle + 4Mr, \triangle, 0, 2a] \tag{26}$$

$$n^\mu = [\frac{1}{2\Sigma}, -\frac{1}{2\Sigma}, 0, 0] \tag{27}$$

This redefinition of the tetrad vectors changes the values of the Newman–Penrose scalars. The new values are,

$$\epsilon = r - M \tag{28}$$

$$\gamma = \mu = -\frac{1}{2}\frac{r + ia\cos\theta}{\Sigma^2} \tag{29}$$

$$\rho = -(r + ia\cos\theta)\frac{\triangle}{\Sigma} \tag{30}$$

with $\alpha, \beta, \pi, \tau, \psi_2$ remain unchanged. The resulting Teukolsky equation reads,

$$\begin{aligned} & \frac{1}{2}\frac{\triangle + 4Mr}{\Sigma}\frac{\partial^2\psi_4}{\partial\tilde{t}^2} - \frac{1}{2}\frac{\triangle}{\Sigma}\frac{\partial^2\psi_4}{\partial r^2} - 2\frac{Mr}{\Sigma}\frac{\partial^2\psi_4}{\partial r\partial\tilde{t}} - \frac{a}{\Sigma}\frac{\partial^2\psi_4}{\partial r\partial\tilde{\varphi}} - \frac{1}{2}\frac{1}{\Sigma}\frac{\partial^2\psi_4}{\partial\theta^2} - \frac{1}{2}\frac{1}{\sin^2\theta\Sigma}\frac{\partial^2\psi_4}{\partial\tilde{\varphi}^2} \\ & - \left\{\frac{3a^2(r-M)\cos^2\theta + r(7r^2 + 4a^2 - 11Mr) + 4ia\cos\theta\triangle}{\Sigma^2}\right\}\frac{\partial\psi_4}{\partial r} \\ & - \left\{\frac{r^2(2r + 11M) + a^2(2r - 3M)\cos^2\theta - 2ia(r^2 + 7Mr + a^2\cos^2\theta)\cos\theta}{\Sigma(r - ia\cos\theta)^2}\right\}\frac{\partial\psi_4}{\partial t} \\ & - \left\{\frac{(r^2 + 8a^2 - 9a^2\cos^2\theta)\cos\theta + 2iar(4 - 5\cos^2\theta)}{2\Sigma(r - ia\cos\theta)^2\sin\theta}\right\}\frac{\partial\psi_4}{\partial\theta} \\ & - \left\{\frac{4ar(\sin^2\theta - \cos^2\theta) - 2i(r^2 + 2a^2 - 3a^2\cos^2\theta)\cos\theta}{\Sigma(r - ia\cos\theta)^2\sin^2\theta}\right\}\frac{\partial\psi_4}{\partial\tilde{\varphi}} \\ & + \left\{\frac{24Mr\sin^2\theta - 19r^2 + (21r^2 - 9a^2 + 7a^2\cos^2\theta)\cos^2\theta - 2ia[(7r - 6M)\cos^2\theta - (5r - 6M)]\cos\theta}{\Sigma(r - ia\cos\theta)^2\sin^2\theta}\right\}\psi_4 = 0. \end{aligned} \tag{31}$$

[1] We wish to thank Steve Fairhurst and Badri Krishnan for suggesting this rescaling.

This equation can be used to study a variety of different things, including boundary conditions at the excision region (inside the horizon), perturbations inside and near the horizon, and most significantly, we could use this implementation to compare and perhaps even continue evolutions from full numerical codes that use Kerr–Schild coordinates.

The Penetrating Teukolsky Code (PTC) evolves the following equation, where $\psi = (r - ia\cos(\theta))^4\psi_4$ is the Teukolsky function:

$$
\begin{aligned}
&(\Sigma + 2Mr)\frac{\partial^2\psi}{\partial\tilde{t}^2} - \triangle\frac{\partial^2\psi}{\partial r^2} - 6(r-M)\frac{\partial\psi}{\partial r} \\
&-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\tilde{\varphi}^2} - 4Mr\frac{\partial^2\psi}{\partial\tilde{t}\partial r} - 2a\frac{\partial^2\psi}{\partial r\partial\tilde{\varphi}} \\
&+\left(\frac{4i\cot\theta}{\sin\theta}\right)\frac{\partial\psi}{\partial\tilde{\varphi}} \\
&-(4r + 4ia\cos\theta + 6M)\frac{\partial\psi}{\partial\tilde{t}} + 2(3\cot^2\theta - \csc^2\theta)\psi = 0.
\end{aligned}
\tag{32}
$$

To implement this equation numerically, we break the above equation down into a 2+1 dimensional one, using a decomposition of the Teukolsky function into angular modes, $\psi = \Sigma\psi_m e^{im\tilde{\varphi}}$. The Teukolsky equation for each m mode now looks like the following:

$$
\begin{aligned}
&(\Sigma + 2Mr)\frac{\partial^2\psi_m}{\partial\tilde{t}^2} - \triangle\frac{\partial^2\psi_m}{\partial r^2} - (2aim + 6r - 6M)\frac{\partial\psi_m}{\partial r} \\
&-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_m}{\partial\theta}\right) - 4Mr\frac{\partial^2\psi_m}{\partial\tilde{t}\partial r} - (4r + 4ia\cos\theta + 6M)\frac{\partial\psi_m}{\partial\tilde{t}} \\
&+(4\cot^2\theta - 2 + m^2\csc^2\theta - 4m\cot\theta\csc\theta)\psi_m = 0.
\end{aligned}
\tag{33}
$$

We use Lax-Wendroff technique to numerically implement these simplified set of equations exactly as done in [13]. A full discussion of the applications of this code will be presented elsewhere. Here we just highlight in the figures how the code indeed evolves perturbations inside and outside the horizon as expected.

IV. CONCLUSIONS

We have set up the black hole perturbation framework in Kerr–Schild type coordinates. This framework will be useful for comparisons with fully nonlinear numerical codes currently being implemented that run naturally in Kerr–Schild coordinates. We have also discussed the numerical implementation of the perturbative evolution equation and how to set up initial data in terms of the initial value data that will be available for black hole collisions. Implementation of this framework for numerical computations is essentially complete.

This formalism allows us to study in a natural way perturbations close to the horizon and may also be of interest to study and test in a concrete fashion several attractive properties of "isolated horizons" [11]. This formalism allows us to make several predictions about quantities defined with notions intrinsic to the black hole (like a concept of local mass and angular momentum), and their evolution. Several attractive formulae, for instance relating the ADM mass to the "local horizon mass" and the "radiation content" can be worked out. Having a perturbative formalism that operates correctly near and on the horizon will allow us to test the validity of these formulae. This issue is currently under study.

V. ACKNOWLEDGMENTS

We wish to thank Darío Núñez for pointing out several typos in an earlier version of this paper. This work was supported in part by grants, NSF-INT-9722514, NSF-PHY-9423950, NSF-PHY-9800973, NSF-PHY-9800970, NSF-PHY-9800973, and by funds of the Pennsylvania State University. M.C. holds a Marie Curie Fellowship (HPMF-CT-1999-00334).

[1] J. Pullin, Prog. Theor. Phys. Suppl. **136**, 107 (2000).
[2] G. Khanna *et al.*, Phys. Rev. Lett. **83**, 3581 (1999)
[3] A. M. Abrahams *et al.* [Binary Black Hole Grand Challenge Alliance Collaboration], Phys. Rev. Lett. **80**, 1812 (1998)
[4] A. M. Abrahams, S. L. Shapiro and S. A. Teukolsky, Phys. Rev. **D51**, 4295 (1995)
[5] J. Baker, B. Bruegmann, M. Campanelli, C. Lousto, Class. Quantum Gravity, **17**, L149 - L156 (2000).
[6] J. Bowen, J. York, Phys. Rev. **D21**, 2047 (1980).
[7] S. Brandt and B. Bruegmann, Phys. Rev. Lett. **78**, 3606 (1997).
[8] R. A. Matzner, M. F. Huq and D. Shoemaker, Phys. Rev. **D59**, 024015 (1999).
[9] N. T. Bishop, R. Isaacson, M. Maharaj and J. Winicour, Phys. Rev. **D57**, 6113 (1998).
[10] S. Brandt *et al.*, Phys. Rev. Lett. **85**, 5496 (2000).
[11] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski and J. Wisniewski, Phys. Rev. Lett **85**, 3564 (2000).
[12] S. Teukolsky, Ap. J. **185**, 635 (1973).
[13] W. Krivan, P. Laguna, P. Papadopoulos and N. Andersson, Phys. Rev. **D56**, 3395 (1997).
[14] M. Campanelli, C. O. Lousto, J. Baker, G. Khanna and J. Pullin, Phys. Rev. **D58**, 084019 (1998). M. Campanelli, C. O. Lousto, J. Baker, G. Khanna and J. Pullin, Phys. Rev. **D62**, 069901(E), (2000);

Appendix:The Teukolsky equation in Outgoing Eddington–Finkelstein coordinates:

The outgoing Eddington–Finkelstein coordinate form of the Kerr metric might be useful to evolve in a perturbative fashion the outgoing wave zone exterior, say, to a tube in which a Cauchy evolution code is used, as in the spirit of [3]. The outgoing Eddington–Finkelstein coordinates for the Kerr metric are obtained by introducing a time variable,

$$\tilde{U} = t - r^* \tag{34}$$

$$\tilde{\varphi} = \varphi - \int \frac{a}{\triangle} dr \tag{35}$$

where r^* is defined as in (24).

As in the previous section, one presumably wishes to write a Cauchy evolution code for the resulting perturbative equation. It is therefore appropriate to introduce a time coordinate,

$$\tilde{t} = \tilde{U} + r, \tag{36}$$

and consider the Teukolsky equation in $(\tilde{t}, r, \theta, \tilde{\varphi})$ coordinates. The Kerr metric in these coordinates is the same as (25) except an opposite sign in the $drd\tilde{t}$ term.

The Newman–Penrose scalars do not change. The differential operators do. The Kinnersley tetrad in the new coordinates is,

$$l^\mu = [1, 1, 0, 0] \tag{37}$$

$$n^\mu = [\frac{\triangle}{2\Sigma}\left(1 + \frac{4Mr}{\triangle}\right), -\frac{\triangle}{2\Sigma}, 0, \frac{a}{\Sigma}] \tag{38}$$

$$m^\mu = [ia\sin\theta, 0, 1, \frac{i}{\sin\theta}]/(\sqrt{2}(r + ia\cos\theta)), \tag{39}$$

The resulting Teukolsky equation in these coordinates is given by,

$$\begin{aligned}(\Sigma + 2Mr)\frac{\partial^2\psi}{\partial\tilde{t}^2} - \triangle\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\tilde{\varphi}^2} + 4Mr\frac{\partial^2\psi}{\partial\tilde{t}\partial r} + 2a\frac{\partial^2\psi}{\partial r\partial\tilde{\varphi}}& \\ -2\left(2r + M + 2ia\cos\theta\right)\frac{\partial\psi}{\partial\tilde{t}} + 2\left(r - M\right)\frac{\partial\psi}{\partial r} - \cot\theta\frac{\partial\psi}{\partial\theta} + 4i\frac{\cos\theta}{\sin^2\theta}\frac{\partial\psi}{\partial\tilde{\varphi}} + \left(2 + 4\cot^2\theta\right)\psi = 0,&\end{aligned} \tag{40}$$

which can be rewritten in a form more reminiscent of the ordinary Teukolsky equation as,

$$\begin{aligned}&(\Sigma + 2Mr)\frac{\partial^2\psi}{\partial\tilde{t}^2} - \frac{\partial}{\partial r}\left(\triangle\frac{\partial\psi}{\partial r}\right) + 4(r - M)\frac{\partial\psi}{\partial r} \\ &-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\tilde{\varphi}^2} - 2a\frac{\partial^2\psi}{\partial\tilde{\varphi}\partial r} - 4Mr\frac{\partial^2\psi}{\partial\tilde{t}\partial r} \\ &-2\left(2r + M + 2ia\cos\theta\right)\frac{\partial\psi}{\partial\tilde{t}} + 4i\frac{\cos\theta}{\sin^2\theta}\frac{\partial\psi}{\partial\tilde{\varphi}} + 2(1 + 2\cot^2\theta)\psi = 0,\end{aligned} \tag{41}$$

where again $\psi = (r - ia\cos\theta)^4\psi_4$, and we see that the main differences with the equation we derived in the previous section are: a) the sign of the terms with the mixed $r, \tilde{\varphi}$ and $r, \tilde{t}$ derivatives, b) the first order derivative in $\tilde{t}$ and $\tilde{\varphi}$ terms.

As in the previous cases, the generic initial data formulae we presented are still valid in this case.

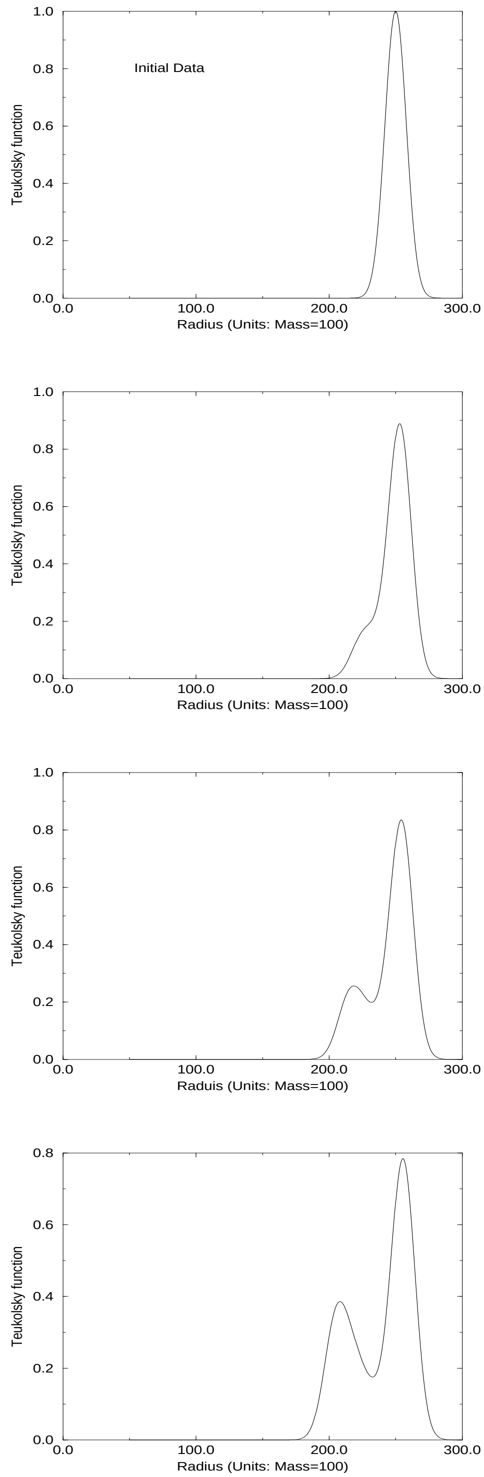
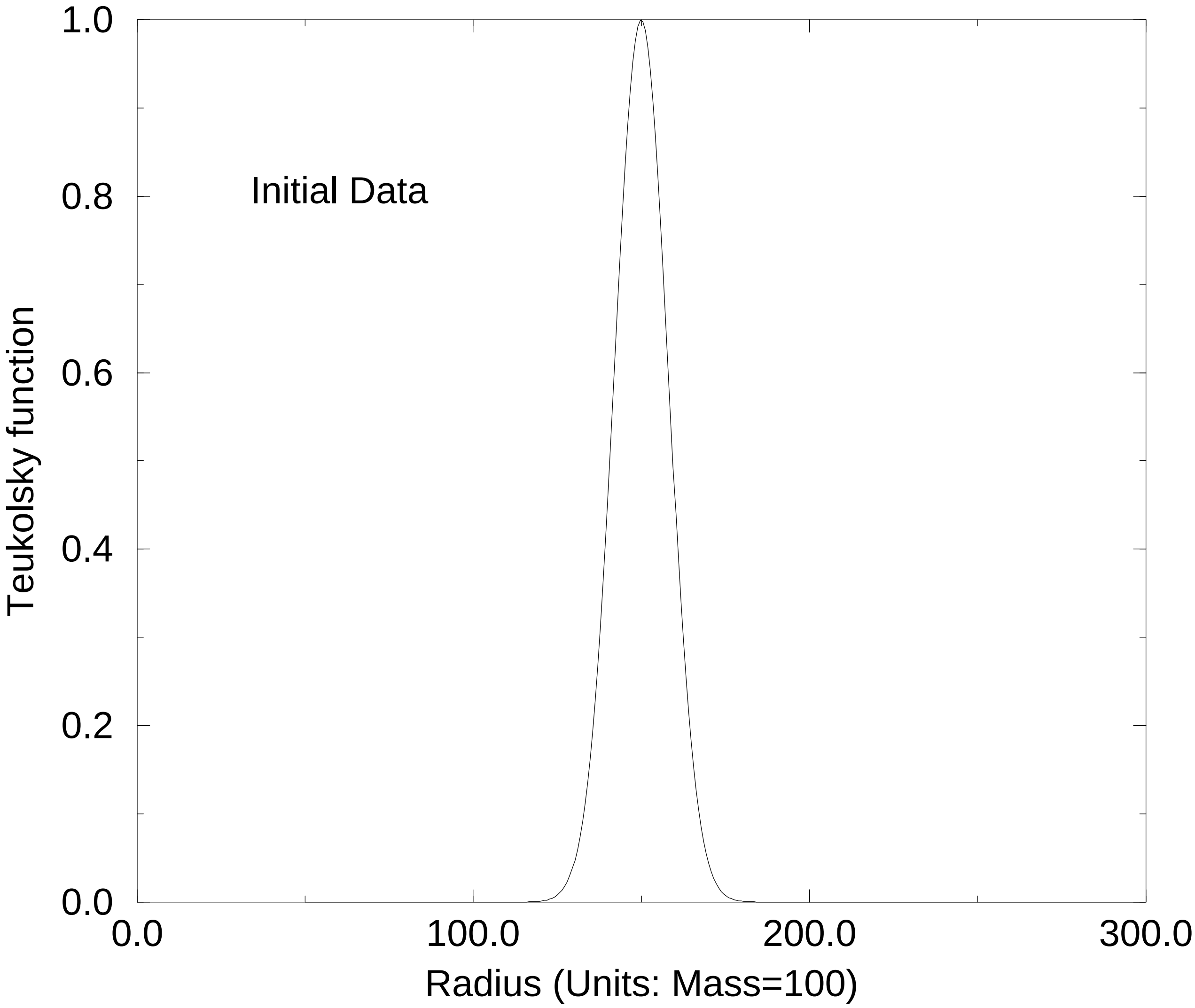
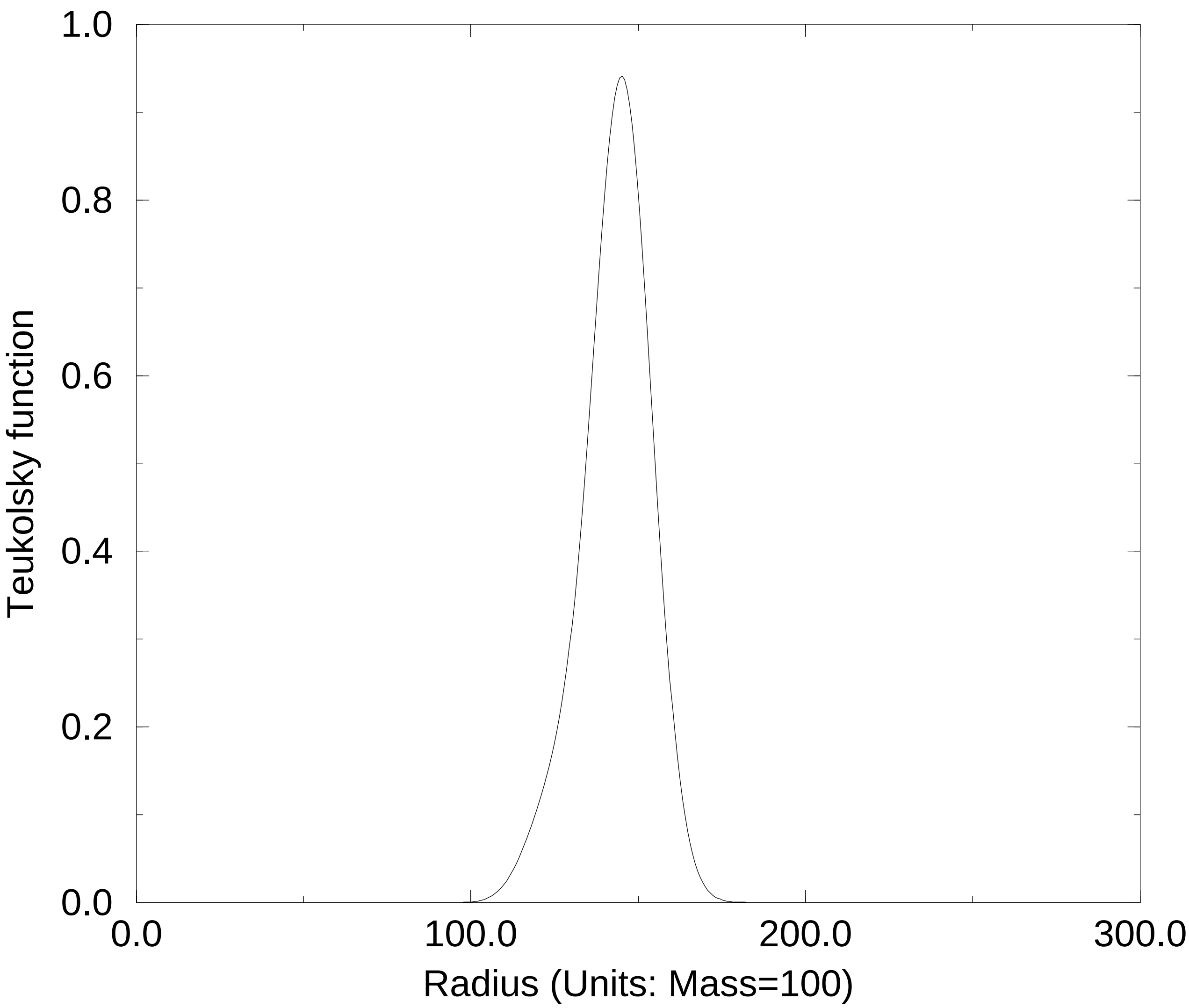
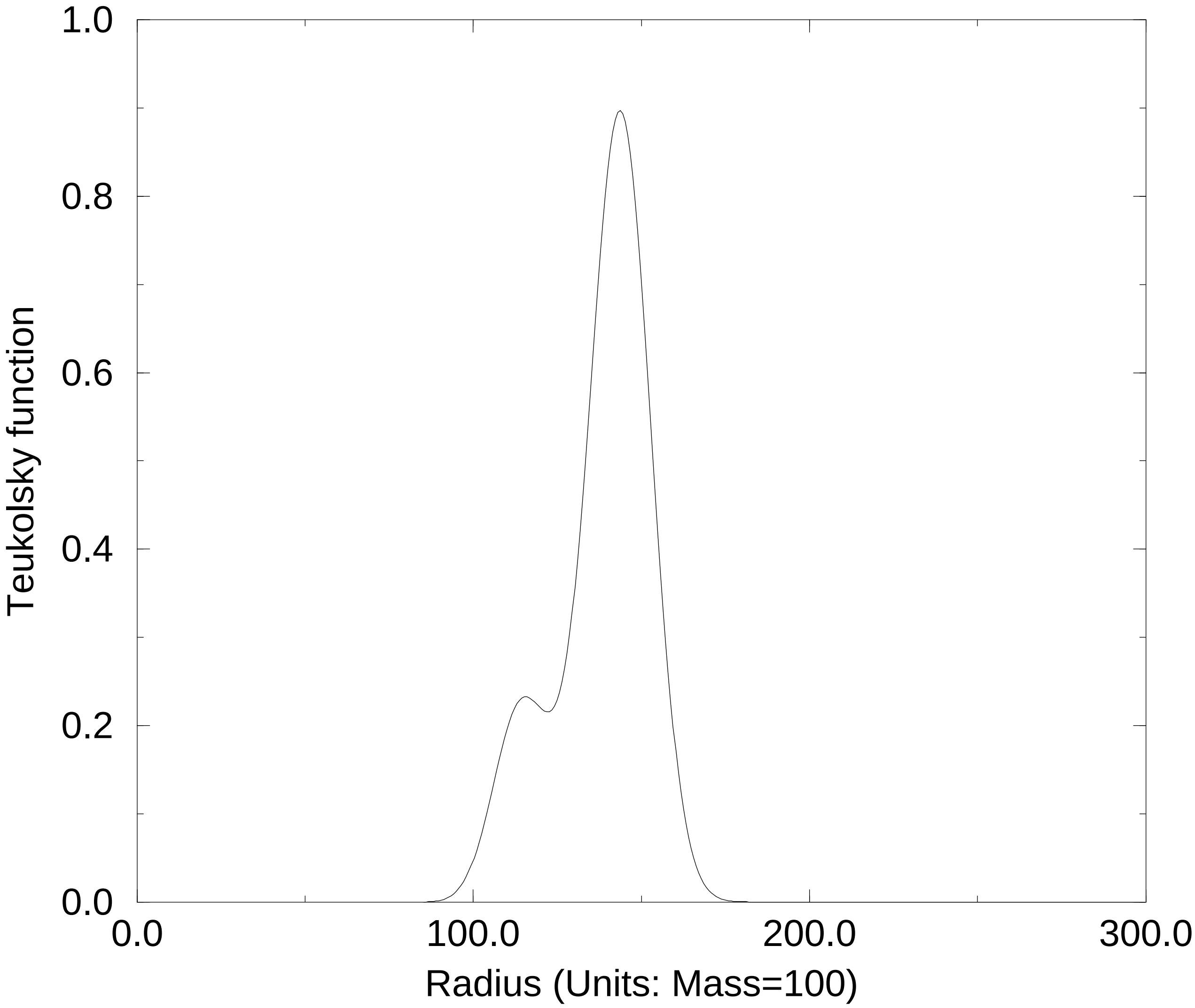


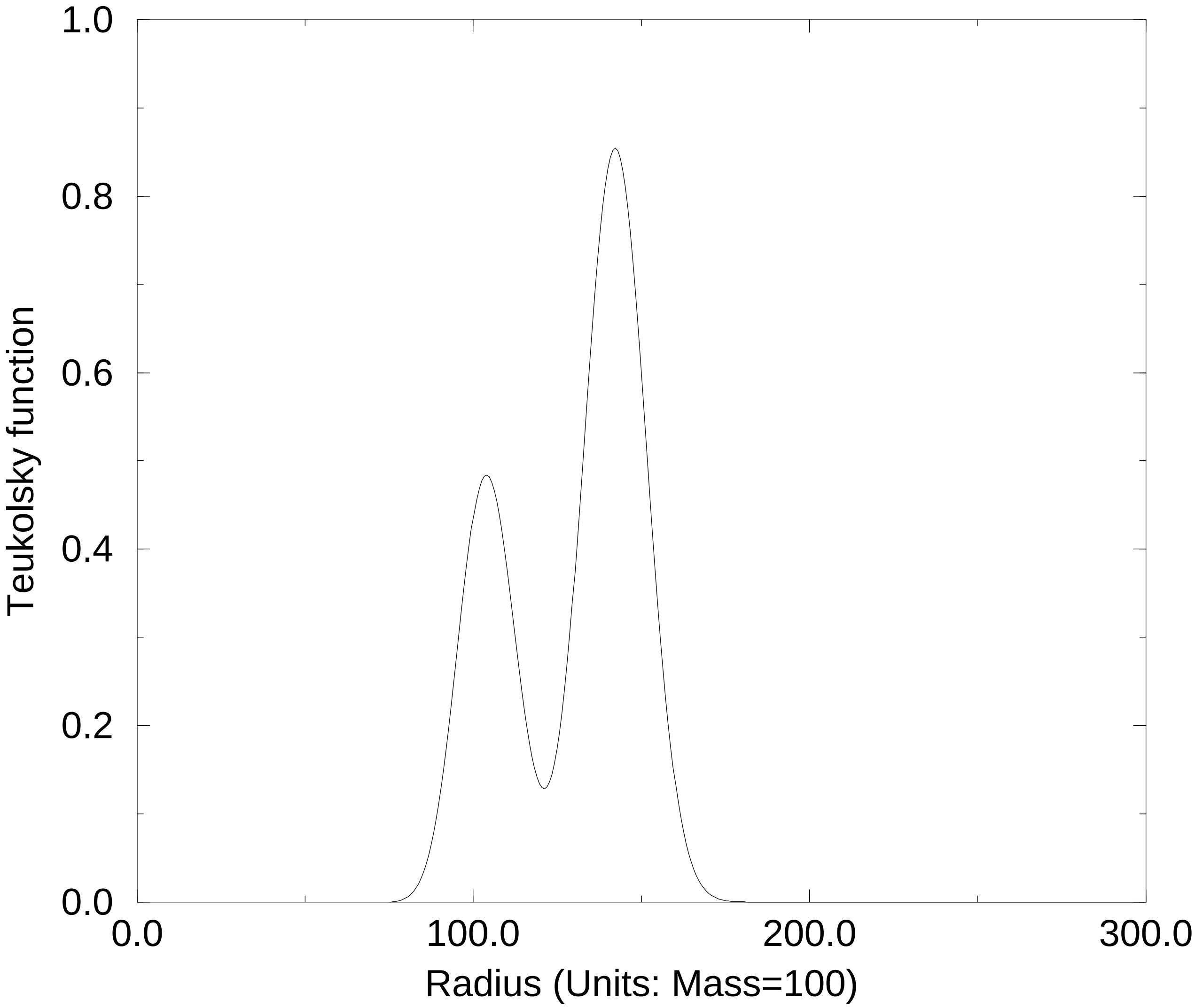
FIG. 1. Numerical evolution of the horizon penetrating Teukolsky equation. The horizon is at $r = 200$. We give as initial data a Gaussian pulse outside the horizon. The background black hole has a mass of 100 units, and zero spin. Shown above are snapshots of the evolution of the $m = 0$ mode at $t = 30$, 40, and 50 units. The pulse splits into two pulses, one infalling towards the singularity at $r = 0$, moving smoothly through the horizon. The other pulse moves rightwards and eventually escapes to scri. Notice that the speed of light does not appear as a $t = r$ motion in these coordinates!


1.0
0.8
0.6
0.4
0.2
0.0
Teukolsky function
Initial Data
0.0
100.0
200.0
300.0
Radius (Units: Mass=100)


1.0
0.8
0.6
0.4
0.2
0.0
Teukolsky function
0.0
100.0
200.0
300.0
Radius (Units: Mass=100)



1.0
0.8
0.6
0.4
0.2
0.0
0.0
100.0
200.0
300.0
Teukolsky function
Radius (Units: Mass=100)


1.0
0.8
0.6
0.4
0.2
0.0
0.0
100.0
200.0
300.0
Teukolsky function
Radius (Units: Mass=100)