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A proper-time cure for the conformal sickness in quantum gravity

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Abstract

Starting from the space of Lorentzian metrics, we examine the full gravitational path integral in 3 and 4 space-time dimensions. Inspired by recent results obtained in a regularized, dynamically triangulated formulation of Lorentzian gravity, we gauge-fix to proper-time coordinates and perform a non-perturbative "Wick rotation" on the physical configuration space. Under certain assumptions about the behaviour of the partition function under renormalization, we find that the divergence due to the conformal modes of the metric is cancelled *non-perturbatively* by a Faddeev-Popov determinant contributing to the effective measure. We illustrate some of our claims by a 3d perturbative calculation.

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1 Path integrals for quantum gravity

It is the central aim of path integral formulations of quantum gravity to give a physical and mathematical meaning to the formal expression

$$Z = \int_{\frac{\text{Metrics(M)}}{Diff(M)}} \mathcal{D}[g_{\mu\nu}] \mathrm{e}^{iS}, \qquad S[g_{\mu\nu}] = \frac{1}{16\pi G_N} \int_M d^4x \sqrt{|\det g|}(R - 2\Lambda), \qquad (1)$$

for the gravitational propagator, subject to boundary conditions on the metric fields $g_{\mu\nu}(x)$. The earliest attempts to construct a Feynman propagator for gravity [1, 2] go back to a time when neither of the present authors had been born or, well, barely. The perturbation series for (1) around flat Minkowski space $\eta_{\mu\nu}$ is non-renormalizable and thus cannot serve as a *fundamental* definition of the theory. Assuming that a quantum theory of gravity does indeed exist, one is therefore forced to consider non-perturbative methods for constructing Z. However, a non-perturbative evaluation of (1) in the continuum meets with a number of well-known problems:

(i) explicit field coordinates on the physical configuration space $\frac{\text{Metrics(M)}}{\text{Diff(M)}}$ of diffeomorphism-equivalence classes of metrics $[g_{\mu\nu}]$ (the so-called *geometries*) must be found;

(ii) a measure $\mathcal{D}[g_{\mu\nu}]$ on the "space of paths" (the set of all d-dimensional space-time geometries interpolating between an initial and a final spatial geometry) must be given;

(iii) since we are dealing with a field theory, a regularization and renormalization – respecting the diffeomorphism symmetry of the gravitational theory – must be found.

Even if good candidates (i)-(iii) have been identified, we still expect difficulties with the evaluation of the non-perturbative integral since

(iv) the action is not quadratic in the fundamental metric fields;

(v) the integral is unlikely to converge because of the imaginary factor i in front of the Einstein action.

In ordinary quantum field theory on a fixed Minkowskian background, problem (v) is usually solved by rotating to imaginary "time", evaluating n-point functions in the Euclidean sector and invoking the Osterwalder-Schrader axioms. It is much less obvious how to proceed in gravity, where the metric field is a dynamical variable. A generic metric $g_{\mu\nu}$ has no geometrically distinguished notion of time t, and it is therefore unclear how to perform a Wick rotation of the form $t \mapsto \tau = it$.

This difficulty has motivated some researchers to change the configuration space of the theory, from Lorentzian space-time metrics $g_{\mu\nu}$ with signature $(-+++)$ to Euclidean metrics $g^{\text{eu}}_{\mu\nu}$ with signature $(++++)$, and simultaneously to replace the complex amplitudes $\exp iS[g_{\mu\nu}]$ by real Boltzmann weights $\exp -S[g^{\rm eu}_{\mu\nu}]$. It is important to realize that this substitution is ad hoc in the sense of replacing one physical problem by another one which – without a non-perturbative generalization of the Wick rotation – is a priori unrelated and potentially inequivalent.

Unfortunately, because of the so-called "conformal-factor problem", such a procedure does *still* not guarantee the convergence of the regularized path integral. This property is visualized most easily by decomposing the (Euclidean or Lorentzian) metric g into a product of a conformal factor and a metric $\bar{g}$ of constant curvature according to $g_{\mu\nu} = {\rm e}^{2\lambda}\bar{g}_{\mu\nu}$. Rewriting the Einstein action as a function of λ and $\bar{g}$, the kinematic term $\sim (\nabla_0\lambda)^2$ for the conformal field is seen to contribute with the wrong sign, making the action unbounded from below, and the functional λ-integration in the Euclidean case potentially divergent.

This "conformal sickness" has been known since the early days of the Euclidean path integral [3, 4]. Following the suggestion of performing a "conformal rotation" $\lambda \mapsto i\lambda$ [3] for asymptotically Euclidean metrics (and $\Lambda = 0$), the typical cure consists in a suitable integration over *complex* instead of real metrics $g_{\mu\nu}$. A place where Euclidean amplitudes are essential is the no-boundary proposal of Hartle and Hawking [5]. Extensive studies of cosmological models with compact slices have been conducted in search of definite prescriptions of complex integration contours, satisfying certain criteria of physicality and semi-classicality [6, 7]. For simple minisuperspace models such contours can be found, but it seems very difficult to come up with a prescription for selecting a contour *uniquely* which at the same time could claim some generality.

Other authors, again in a perturbative context, have insisted that the proper physical starting point for any analysis should be *Lorentzian* gravity. Either by working "backwards" from a continuum phase-space path integral in terms of reduced, physical variables [8] or by gauge-fixing the configuration-space path integral and properly including the ensuing Faddeev-Popov determinants [9], they have argued that the conformal divergence is spurious. These arguments highlight the potential importance of the measure $\mathcal{D}[g_{\mu\nu}]$ in (1), an issue to which we will return in due course.

Because of the ill-definedness of the perturbative path integral, the relevance of these considerations for a full theory of quantum gravity is not immediately clear. To our knowledge, the conformal problem of the path integral has not been addressed in a genuinely non-perturbative setting. This has to do with the general lack of regularizations for gravity within which this issue could be treated in a mathematically meaningful way. In addition, going beyond the perturbative case, a Wick rotation on the space of all metrics is needed if one believes – as we do – that the Lorentzian signature and the causal structure of space-time are of fundamental physical importance, and should therefore be built into any quantization from the outset.

Our interest in gravitational path integrals is motivated by the recent construction of a non-perturbative regularized path integral for gravity based on simplicial

Lorentzian geometries [10, 11, 12] (see [13] for recent reviews), a Lorentzian version of previously investigated so-called dynamically triangulated models. The model can be defined in any dimension d, possesses a well-defined notion of Wick rotation and a set of causality constraints reflecting the properties of the discrete Lorentzian structure.

This formulation of quantum gravity goes some way in addressing the list of problems mentioned earlier. In the spirit of Regge's old idea of describing "geometries without coordinates" [14], it is defined directly on the physical space of geometries. The (discretized) geometries are described in terms of the combinatorial data of how a set of flat d-dimensional simplicial building blocks (whose metric properties are encoded in their geodesic edge lengths) is glued together. This amounts to a definite prescription for (i)-(iii) above. The non-perturbative Wick rotation gets rid of the factor of i of problem (v), and the Wick-rotated path integral can be shown to converge for a suitable choice of bare coupling constants.

It is remarkable that a regularization for quantum gravity with such properties should exist and it is of great interest to understand whether the path integral can be evaluated explicitly, and simultaneously the diffeomorphism-invariant cut-off be removed to give rise to a well-defined continuum theory. Instead of performing Gaussian continuum integrals as in (iv), "solving the model" means the evaluation of the discrete combinatorial state sum over distinct gluings. This program can be carried out exactly by analytical methods in dimension $d = 2$ [10], yielding a well-defined propagator (1), in agreement with a (formal) continuum calculation in proper-time gauge [15].

These results are reassuring as far as the consistency of Lorentzian dynamically triangulated gravity is concerned, but more serious problems are expected to appear in higher dimensions, in the case of the conformal-factor problem for $d \geq 3$. Although the discrete model always possesses a phase where Z converges, this may be attributed to the effective curvature bounds inherent in the regularization. It does not necessarily exclude a dominance of the unphysical conformal mode in the state sum. One can indeed identify simplicial geometries (whose spatial volumes oscillate rapidly in proper time) with a large and negative Euclidean action. Nevertheless, it has been established by numerical simulations that for the 3d model there is a large range of the gravitational coupling constant where such modes do not play a role [16, 17]. This entails a win of "entropy over energy", that is, well-behaved geometries outnumber completely the potentially dangerous ones associated with conformal excitations.

It would be very significant if the same behaviour persisted in four space-time dimensions, since it would suggest a resolution of the conformal-factor problem *at the non-perturbative level*, where a quantum theory of gravity has a chance of existing. The question we will address in the present work is whether and how such a behaviour can be understood from a continuum point of view. The evidence from Lorentzian dynamical triangulations so far suggests that a crucial contribution in the cancellation of the conformal divergence must come from the path integral *measure.*

To imitate the discretized formulation as closely as possible, we will use a configuration space path integral in terms of metric fields $g_{\mu\nu}$. Our calculations will be done for $d=3,4$. In order to gauge-fix, we will work with "proper-time" (or "Gaussian") coordinates. This is motivated by the presence of a preferred notion of (discrete) proper time in the lattice model (although it should be pointed out that in this case there is no gauge-fixing – proper time is simply selected from the combinatorial data characterizing each geometry).

We do not expect to be able to perform the non-perturbative functional integrals explicitly (this is problem (iv) from above), but we will show that under certain plausible assumptions about the behaviour of the path integral under renormalization the conformal divergence is cancelled by a compensating term in the measure, arising as a Faddeev-Popov determinant during the gauge-fixing. Our treatment will concentrate on the conformal factor-dependence and will remain formal in the sense that we will not spell out the details of the regularization and renormalization. However, the results from the simplicial formulation make us confident that suitable diffeomorphism-invariant regularization schemes do indeed exist.

The cancellation mechanism we uncover is a non-perturbative version of the one found by Mazur and Mottola [9], and requires that $C<-\frac{2}{d}$ for the constant C appearing in the DeWitt measure, exactly the range where the DeWitt metric is indefinite. It leads us to conjecture that the "natural" measure given by the dynamical triangulations approach corresponds to a value of $C<-\frac{2}{d}$. This is quite plausible, given that the only distinguished value of C (inherent in the action and appearing explicitly in a canonical treatment of three- and four-dimensional gravity) is $C=-2$, which lies in the required range.

The contents of the remainder of our paper is as follows: in the next section we will separate out the gauge components of the metric tensor and discuss some properties of the proper-time gauge. We also introduce our conventions for various scalar products. In Sec.3, we explicitly isolate the negative-definite part of the action responsible for the conformal divergence. The Faddeev-Popov determinants associated with the variable changes on the space of metrics are computed in Sec.4. We then show that under certain assumptions on the renormalization behaviour of the state sum (borrowed from 2d Liouville gravity), a piece of these determinants exactly cancels the leading conformal divergence in the action. Sec.5 contains a *perturbative* evaluation of the complete proper-time path integral around a fixed constant-curvature torus metric in 3d, to illustrate the cancellation mechanism at work in a complete and explicit calculation. In the final Sec.6, we summarize our findings.

2 Implementing the proper-time gauge

Our first task will be to split the metric degrees of freedom into physical and gauge components, and to divide the gravity partition function by the (infinite) volume of the diffeomorphism group. We will work with d-dimensional space-times M, $d{=}3,4$, with topology ${}^{(d)}M = [0,1] \times {}^{(d\text{–}1)}\Sigma$, where Σ denotes a compact spatial manifold.

In an attempt to follow as closely as possible the discrete construction of [11, 12], we will represent the physical configuration space of geometries on M (i.e. the quotient space of space-time metrics $\mathcal{M}$ = Metrics(M) and space-time diffeomorphisms Diff(M)) by the space of metrics in "proper-time gauge"[3], which are of the block-diagonal form

$$g^{\rm pt}_{\mu\nu} = \begin{pmatrix} \epsilon & \vec{0} \\ \vec{0} & g_{ij} \end{pmatrix}, \quad \mu,\nu = 0,\dots,d{-}1, \quad i,j = 1,\dots,d{-}1. \tag{2}$$

Our "Wick rotation" consists in substituting $\epsilon = -1$ in the Lorentzian case by $\epsilon = +1$ in the Euclidean case (where we define $\sqrt{-1} := +i$). For the case that the spatial (d–1)-dimensional metric g_{ij} is time-independent – as for instance in the case of the flat Minkowski metric – this prescription is equivalent to an analytical continuation in proper time t. It is not straightforward to define an exact analogue of the discrete Wick rotation of [11, 12], which is given as an operation on discrete *geometries*, without the need to introduce any coordinates. In that case, one can nevertheless choose a coordinate system on each individual flat simplicial building blocks in which the metric tensor takes the form (2), and the discrete Wick rotation (up to a constant rescaling of proper time) corresponds to a sign flip of the (00)-component. Another property our Wick rotation shares with the discrete case is the fact that it maps real Lorentzian metrics to real Euclidean metrics. Note that unlike its discrete counterpart, our prescription $\epsilon \mapsto -\epsilon$ does not in general map solutions of Lorentzian gravity to Euclidean solutions. (For the dynamically triangulated Lorentzian models this is ensured in the sense that the two actions are mapped into each other.) However, this is no obstacle to our non-perturbative construction, where $\epsilon \mapsto -\epsilon$ simply gives us a 1-to-1 map from Lorentzian to Euclidean geometries. All computations are then performed in the Euclidean sector where – up to regularization – they are well-defined. We will not address the question of what is the most suitable way of "rotating back the result", since this will ultimately be dictated by the physical interpretation of the final, non-perturbative partition function (obvious candidates are an inverse flip $\epsilon{=}1 \mapsto \epsilon{=}{-1}$ or an analytic continuation in proper time).

One may wonder why we have not adopted a prescription of the form of an analytic continuation in time, $t \mapsto it$. The problem is that although such prescriptions "work" for a handful of metrics $g_{\mu\nu}$ with special symmetries (flat space, static solutions etc.), they do not exist in general. Firstly, a generic space-time does not have

[3] Note that this gauge was used by Leutwyler in one of the first path-integral treatments [2]. In the context of the *canonical* path integral, a similar proper-time gauge was employed in [18].

a physically preferred time-direction, and the prescription is clearly not invariant under diffeomorphisms. Secondly, if by some gauge choice one does distinguish a preferred system of coordinates (like the Gaussian coordinates we are using), the substitution $t \mapsto it$ will in general lead to *complex* metrics, defeating the purpose of making the non-perturbative path integral better defined.

Keeping track of the signature is particularly convenient in proper-time gauge and we will work throughout with factors of ϵ. The metric g_{ij} is taken to be positive definite. Locally on a space-time one can always find so-called "Gaussian normal coordinates" in which the metric tensor assumes the form (2), but in general one expects obstructions to introducing such coordinates globally.

As with any gauge choice the gauge must be *attainable* and *unique*, that is, any point $g_{\mu\nu} \in \mathcal{M}$ must lie on a gauge orbit that cuts the constraint surface $\mathcal{C}$ (in our case defined by the vanishing of the gauge condition, $g_{0\mu} - \epsilon\delta^0_\mu = 0$) exactly once. A necessary condition which is easier to prove is that any $g_{\mu\nu}$ in the vicinity of $\mathcal{C}$ can be uniquely decomposed into a configuration $g^{\rm pt}_{\mu\nu} \in \mathcal{C}$ and an infinitesimal diffeomorphism. This is demonstrated in appendix 1.

Potential difficulties with the global implementation of the proper-time gauge have to do with the focussing properties of time-like geodesics. Anti-de Sitter space in 3 and 4 dimensions is an example of a solution to the classical Einstein equations where proper-time coordinates do not cover the entire space-time. The 4d metric in these coordinates assumes the form

$$ds^2 = -dt^2 + \cos^2 t \left(d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right), \tag{3}$$

with coordinate singularities at $t = \pm\pi/2$, where the time-like geodesics orthogonal to the hypersurfaces $t = const$ converge to points, as is illustrated by the Penrose diagram in Fig. 1. This is therefore an example of a metric – albeit one with rather bizarre causality properties [19] – that cannot be reached from the constrained surface $\mathcal{C}$ by a diffeomorphism. (By contrast, no such problems occur for the *de Sitter* space, say.)

The existence of such configurations in an infinite-dimensional context is not surprising. For example, it is well-known from the Riemannian case [20] that configurations $g_{\mu\nu}$ with special symmetries must be excised from $\mathcal{M}$ to make quotient spaces of the kind $\mathcal{M}/Diff(M)$ well-defined. Similarly, for the purposes of the non-perturbative path integral we are only interested in capturing the properties of "generic" metrics, and not of "sets of measure zero". In our work we assume that the diffeomorphism orbits $f^* g^{\rm pt}_{\mu\nu}$ through metrics of the special form (2) are in a suitable, function-theoretic sense dense in the space $\mathcal{M}$ of all metrics.[4] Since we do not have a precise definition of a suitable quantum analogue of the space $\mathcal{M}/Diff(M)$

[4]As far as we can see, this assumption is not in contradiction with the well-known difficulty of using proper-time coordinates in an *initial-value* formulation of the classical Einstein equations: in this case, given initial data for the spatial metric g_{ij} and the extrinsic curvature K_{ij}, caustics in the proper-time coordinate system will generically develop at some finite (positive or negative) time if Tr $K \not\equiv 0$ initially (see, for example, [21]).

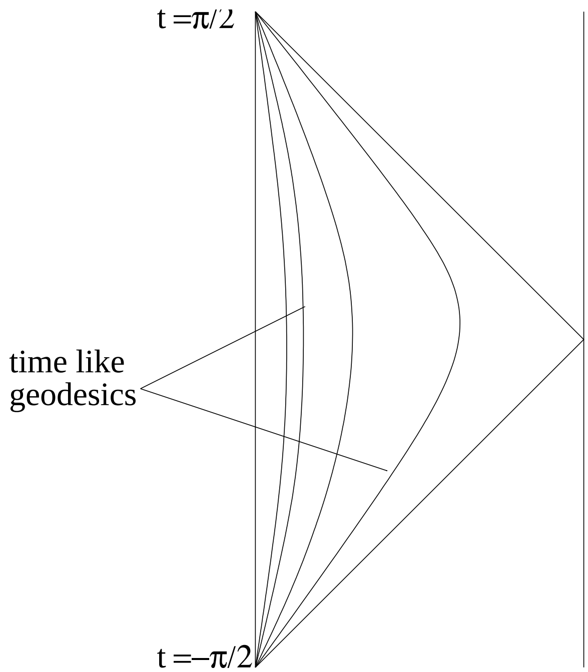

Figure 1: *Penrose diagram of anti-de Sitter space, illustrating the convergence of time-like geodesics.*

beyond formal continuum calculations, such an assumption can ultimately only be justified by the results of a properly regularized formulation of the theory.

We now must implement our gauge choice to isolate the physical degrees of freedom. This requires a change of coordinates $g_{\mu\nu} \mapsto (g^{\rm pt}_{\mu\nu}, f)$ on $\mathcal{M}$, where f labels space-time diffeomorphisms that map any boundaries of $\mathcal{M}$ into themselves. (We do not specify any detailed boundary conditions because our main argument will not depend on them.) Such a coordinate transformation must be accompanied by a Jacobian [2, 22], whose explicit functional form depends on the gauge condition imposed on the metric. We will determine this Jacobian in proper-time gauge using the methods of Mottola et al [9, 23] (see also [24] for a pedagogical introduction). We decompose an arbitrary tangent vector $\delta g_{\mu\nu}|_g \equiv h_{\mu\nu}|_g$ in a point $g \in \mathcal{M}$ according to

$$h_{\mu\nu} = h^{\rm pt}_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu =: h^{\rm pt}_{\mu\nu} + (L\xi)_{\mu\nu}, \tag{4}$$

where $h^{\rm pt}_{\mu\nu}$ is the gauge-invariant piece of $h_{\mu\nu}$ defined by

$$(F \circ h^{\rm pt})_\mu = F^\nu h^{\rm pt}_{\mu\nu} \equiv \delta^\nu_0 h^{\rm pt}_{\mu\nu} = h^{\rm pt}_{\mu 0} = 0, \tag{5}$$

and the vector field ξ generates an infinitesimal diffeomorphism of M. Note that we are not separating out the trace-free part of the metric at this stage. A natural scalar product for tangent vectors to $\mathcal{M}$ is given by

$$\langle h, h' \rangle^{\rm T}|_g = \int_M d^d x \, \sqrt{|\det g|}\; h_{\mu\nu} G^{\mu\nu\rho\sigma}_{(C)} h'_{\rho\sigma}, \tag{6}$$

where the "T" stands for "tensor" and we will from now on suppress the dependence on the base point $g \in \mathcal{M}$. The DeWitt metric is

$$G^{\mu\nu\rho\sigma}_{(C)} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} + Cg^{\mu\nu}g^{\rho\sigma}), \tag{7}$$

for an arbitrary real constant C. Similarly, the scalar product for vector fields on M is

$$\langle \xi, \xi' \rangle^{\mathrm{V}} = \int_M d^dx \, \sqrt{|\det g|} \, \xi_\mu g^{\mu\nu} \xi'_\nu, \tag{8}$$

and for scalars

$$\langle \omega, \omega' \rangle^{\mathrm{S}} = \int_M d^dx \, \sqrt{|\det g|} \, \omega\omega'. \tag{9}$$

Note in passing that Lorentz-invariance is not an issue in defining expressions like (6,8,9), since the metric manifold $(M, g_{\mu\nu})$ does not carry a global action of the Lorentz group unless g is the flat Minkowski metric. (It *would* be a requirement if we were considering a perturbative formulation around flat space.) On the other hand, diffeomorphism invariance of the entire path integral will be maintained throughout our construction.

The (base-point dependent) Jacobian $J_\epsilon[g_{\mu\nu}]$ is defined by

$$[\mathcal{D}h_{\mu\nu}]_\epsilon = J_\epsilon \, [\mathcal{D}h^{\mathrm{pt}}_{\mu\nu}]_\epsilon [\mathcal{D}\xi_\mu]_\epsilon, \tag{10}$$

and can be computed by assuming Gaussian normalization conditions of the form

$$\int [\mathcal{D}h_{\mu\nu}]_\epsilon \exp\left[-\frac{\sqrt{\epsilon}}{2}\langle h, h\rangle^{\mathrm{T}}_\epsilon\right] = 1, \quad \int [\mathcal{D}\xi_\mu]_\epsilon \exp\left[-\frac{\sqrt{\epsilon}}{2}\langle \xi, \xi\rangle^{\mathrm{V}}_\epsilon\right] = 1 \tag{11}$$

and similarly for $h^{\mathrm{pt}}_{\mu\nu}$. The diffeomorphism-invariance of the measure $[\mathcal{D}h_{\mu\nu}]$ has been shown in [23]. We have introduced the subscript ϵ for the measures and scalar products to indicate their dependence on the signature. Analogously, functional determinants of suitable operators $\mathcal{O}$ are computed according to

$$\int [\mathcal{D}h_{\mu\nu}]_\epsilon \exp\left[-\frac{\sqrt{\epsilon}}{2}\langle h, \mathcal{O}h\rangle^{\mathrm{T}}_\epsilon\right] = \frac{1}{\sqrt{\det \mathcal{O}}}. \tag{12}$$

The way the ϵ-dependence is to be interpreted in the functional integrals above is as follows. All computations are to be performed for the Euclidean value $\epsilon = 1$ and then continued. This continuation can be non-trivial in a relation like (12) only if the original operator $\mathcal{O}$ had an explicit ϵ-dependence.

The measure $[\mathcal{D}g_{\mu\nu}]$ for the full path integral (1) must be diffeomorphism-invariant and is usually assumed to be closely related to the tangent space measure $[\mathcal{D}h_{\mu\nu}]$. Since both the measure and the Einstein action are invariant under diffeomorphisms, we can factor out the volume of the d-dimensional diffeomorphism group and are left with the path integral (c.f. [9, 23])

$$Z^{(\epsilon)} = \int [\mathcal{D}g^{\mathrm{pt}}_{\mu\nu}]_\epsilon \; J_\epsilon[g^{\mathrm{pt}}] \; \mathrm{e}^{i\sqrt{-\epsilon}S_\epsilon[g^{\mathrm{pt}}]}. \tag{13}$$

The gravitational action in terms of the gauge-fixed metrics appearing in (13) is given by

$$S_\epsilon[g] = -\frac{\epsilon}{16\pi G} \int d^d x \sqrt{\det g_{ij}} \left({}^{(d-1)}R - 2\Lambda - \frac{\epsilon}{4} G^{ijkl}_{(-2)} (\partial_0 g_{ij})(\partial_0 g_{kl}) \right), \qquad (14)$$

where ${}^{(d-1)}R$ denotes the scalar curvature of the spatial metric g^{pt}_{ij}, and where we have now dropped the explicit superscript indicating proper-time gauge. The Jacobian has the form

$$J = \sqrt{\det{}_{\mathrm{V}} (F \circ F^\dagger)^{-1} (F \circ L)^\dagger (F \circ L)}, \qquad (15)$$

where L – defined in (4) – maps vectors to symmetric tensors, and F – the gauge condition according to (5) – maps symmetric tensors into vectors. We tacitly assume that zero eigenvectors have been removed in the computation of determinants like (15). Adjoints and determinants are defined with respect to the scalar product on d-vectors induced by the DeWitt metric (7) at $g = g^{\mathrm{pt}}$, namely,

$$\langle \vec{\eta}, \vec{\epsilon} \rangle = \int d^d x \sqrt{\det g^{\mathrm{pt}}_{ij}}\ \eta^*_\mu\ (g^{\mathrm{pt}})^{\mu\nu}\ \epsilon_\nu. \qquad (16)$$

3 Isolating the conformal divergence

As we have already described in the introduction, the Euclidean gravity path integral in $d \geq 3$ suffers from a "conformal sickness" arising because of the unboundedness of the action from below [3, 9]. It is straightforward to see that the same is true for the action (14) in proper-time gauge. To isolate the relevant kinetic terms, we decompose the time-derivatives according to

$$\begin{aligned} \partial_0 g_{ij} &= (\partial_0 g_{ij})^{\parallel} + (\partial_0 g_{ij})^{\perp} \\ &:= (1 - \tilde{G})_{ij}{}^{kl} (\partial_0 g_{kl}) + \tilde{G}_{ij}{}^{kl} (\partial_0 g_{kl}), \end{aligned} \qquad (17)$$

into a trace part and a trace-free part, where the projector $\tilde{G}$ onto the trace-free subspace is given by

$$\tilde{G}_{ij}{}^{kl} = \frac{1}{2} (\delta_i{}^l \delta_j{}^k + \delta_i{}^k \delta_j{}^l - \frac{2}{d-1}\ g_{ij}\ g^{kl}). \qquad (18)$$

The kinetic term in (14) can be rewritten as

$$\begin{aligned} G^{ijkl}_{(-2)} (\partial_0 g_{ij})(\partial_0 g_{kl}) &= -\frac{d-2}{d-1}\ (\partial_0 g_{ij})^{\parallel} g^{ij} g^{kl} (\partial_0 g_{kl})^{\parallel} + (\partial_0 g_{ij})^{\perp} g^{ik} g^{jl} (\partial_0 g_{kl})^{\perp} \\ &= -\frac{d-2}{d-1}\ \frac{(\partial_0 \det g)^2}{(\det g)^2} + (\partial_0 g_{ij})^{\perp} g^{ik} g^{jl} (\partial_0 g_{kl})^{\perp}. \end{aligned} \qquad (19)$$

The first term on the right-hand side is the negative definite trace-part. This is precisely the kinetic term of the conformal mode one isolates in perturbative expansions around Ricci-flat metrics, and which leads to the conformal divergence. The second term in (19), coming from the trace-free directions, is positive definite (c.f. [25]). In order to make the λ-dependence more explicit, we decompose the metric according to $g_{ij} = \mathrm{e}^{2\lambda}\bar{g}_{ij}$, where $\bar{g}_{ij}$ is a constant-curvature metric. This is always possible for the Riemannian metrics we are considering [26], but note that in our case g_{ij} has an additional time-dependence.

We will deal with the Jacobian accompanying the coordinate change $g_{ij} \mapsto (\lambda, \bar{g}_{ij})$ in the next section. The complete action (14) becomes

$$S_\epsilon = -\frac{\epsilon}{16\pi G}\int d^dx\sqrt{\det\bar{g}_{ij}}\,\Big(\mathrm{e}^{(d-3)\lambda}[\bar{R} + (d-2)(d-3)(\bar{\nabla}_i\lambda)(\bar{\nabla}^i\lambda)] +$$
$$\mathrm{e}^{(d-1)\lambda}\Big(-2\Lambda + \epsilon(d-1)(d-2)[\partial_0(\lambda + \frac{\log\det\bar{g}}{2(d-1)})]^2 - \frac{\epsilon}{4}(\partial_0\bar{g}_{ij})^\perp\bar{g}^{ik}\bar{g}^{jl}(\partial_0\bar{g}_{kl})^\perp)\Big)\Big), \tag{20}$$

and is unbounded below (for either signature) because of the "wrong" sign for the kinetic term in the shifted scaling parameter

$$\tilde{\lambda} = \lambda + \frac{1}{2(d-1)}\log\det\bar{g}. \tag{21}$$

This presents a potential problem for the Euclidean approach where the exponentiated action contains a term $\sim \mathrm{e}^{\gamma^2\int(\partial_0\tilde{\lambda})^2}$ which can become arbitrarily large for strongly oscillating conformal factors. For later convenience, we rewrite the action in terms of the shifted variable $\tilde{\lambda}$ as

$$S_\epsilon = -\frac{\epsilon}{16\pi G}\int d^dx\,\Big((\det\bar{g}_{ij})^{\frac{1}{d-1}}\mathrm{e}^{(d-3)\tilde{\lambda}}[\bar{R} + (d-2)(d-3)(\bar{\nabla}_i\tilde{\lambda})(\bar{\nabla}^i\tilde{\lambda})] +$$
$$\mathrm{e}^{(d-1)\tilde{\lambda}}\Big(-2\Lambda + \epsilon(d-1)(d-2)[\partial_0\tilde{\lambda}]^2 - \frac{\epsilon}{4}(\partial_0\bar{g}_{ij})(\bar{g}^{ik}\bar{g}^{jl} - \frac{1}{(d-1)}\bar{g}^{ij}\bar{g}^{kl})(\partial_0\bar{g}_{kl}))\Big)\Big), \tag{22}$$

where we have now written out the positive-definite kinetic term explicitly. Note also that S_ϵ is non-polynomial in both $\tilde{\lambda}$ and $\bar{g}_{ij}$. In three dimensions, the expression (22) simplifies further to

$$S_\epsilon^{d=3} = -\frac{\epsilon}{16\pi G}\int dt\Big(4\pi\chi + \int d^2x\ \mathrm{e}^{2\tilde{\lambda}}(-2\Lambda + 2\epsilon[\partial_0\tilde{\lambda}]^2$$
$$-\frac{\epsilon}{4}(\partial_0\bar{g}_{ij})(\bar{g}^{ik}\bar{g}^{jl} - \frac{1}{2}\bar{g}^{ij}\bar{g}^{kl})(\partial_0\bar{g}_{kl}))\Big), \tag{23}$$

with χ denoting the Euler characteristic of the two-dimensional spatial manifold.

From the point of view of the canonical formulation of gravity the presence of the conformal divergence is rather puzzling. In that case it is clear that the conformal factor is not a propagating degree of freedom, since it is canonically conjugate to a gauge variable and becomes fixed by imposing the Hamiltonian constraint. In metric path integrals of the kind we are considering, this property is not at all obvious. The natural place to look for a cancellation of the divergence is the path-integral measure, which is a central ingredient in any non-perturbative formulation. As we have mentioned earlier, this scenario seems to be realized in the non-perturbative approach based on piece-wise linear Lorentzian geometries [10, 11, 12], which is one of the few well-defined regularized path integrals available that do not rely on any fixed background geometry. Numerical investigations of the corresponding continuum theory in $d=3$ have shown that for sufficiently large bare Newton constant there is a phase whose ground state has a stable and extended geometry, without the large fluctuations indicative of conformal excitations [16].

This clearly non-perturbative effect has motivated us to re-examine the continuum gravitational path integral, to understand how such a cancellation may occur when the measure is properly taken into account. Having identified the explicit form of the conformal divergence in proper-time gauge, we will now look for potentially compensating terms in the measure, more precisely, relevant contributions in the form of Faddeev-Popov determinants.

4 Computing the measure and cancelling the divergence

Next we determine the measure contributions arising as a result of the coordinate transformations of the previous section, $g_{\mu\nu} \mapsto (g^{\rm pt}_{\mu\nu}, f)$ and $g^{\rm pt}_{\mu\nu} \equiv g_{ij} \mapsto (\bar{g}_{ij}, \tilde{\lambda})$.

The two functional determinants appearing in the Jacobian J in (15) are vector determinants. The operators in their arguments are formally self-adjoint, because they are of the form of products of operators with their adjoints. Computing the explicit operators, one finds

$$(F^\dagger \xi)_{\mu\nu} = \frac{1}{2}(\xi_\mu g_{\nu 0} + \xi_\nu g_{\mu 0}) - \frac{C}{dC+2}\, \xi_0 g_{\mu\nu} \tag{24}$$

for the adjoint of F, leading to the diagonal operator

$$(F \circ F^\dagger)_\mu{}^\nu = \frac{1}{2}\, \epsilon \delta_\mu{}^\nu + \frac{C(d-2)+2}{2(dC+2)}\, g_{\mu 0}\delta_0{}^\nu = \frac{\epsilon}{2} \begin{pmatrix} \frac{2(d-1)C+4}{dC+2} & \\ & 1_{d-1} \end{pmatrix}, \tag{25}$$

where 1_{d-1} denotes the $(d-1)$-dimensional unit matrix. We will be needing the determinant of the inverse of this operator which for later convenience we factorize into a scalar and a spatial $(d-1)$-dimensional vector determinant according to

$$\det{}_{\rm V}(F \circ F^\dagger)^{-1} = \det{}_{\rm S}\left(\frac{2+dC}{2+(d-1)C}\right) \det{}_{{\rm V}^{d-1}}(2\epsilon)\,. \tag{26}$$

The remaining terms in the Jacobian J depend on the operator

$$(F \circ L)_\mu{}^\nu = \delta_0{}^\nu \nabla_\mu + \delta_\mu{}^\nu \nabla_0 \tag{27}$$

together with its adjoint,

$$(F\circ L)^{\dagger\,\nu}_{\mu} = -g_{0\mu}\nabla^\nu - \Gamma^\nu_{0\mu} - \delta_\mu{}^\nu(\nabla_0 + \Gamma^\lambda_{0\lambda}). \tag{28}$$

Substituting in the expression $\nabla_0^\dagger = -(\nabla_0 + \Gamma^\mu_{0\mu})$ for the adjoint of ∇_0, we finally obtain

$$\begin{aligned}
[(F\circ L)^\dagger (F\circ L)]_\mu{}^\nu \;&=\; -g_{0\mu}\delta_0{}^\nu\nabla^\lambda\nabla_\lambda - \delta_0{}^\nu\Gamma^\lambda_{0\mu}\nabla_\lambda + \delta_0{}^\nu\nabla_0^\dagger\nabla_\mu \\
&\quad\; -g_{0\mu}\nabla^\nu\nabla_0 - \Gamma^\nu_{0\mu}\nabla_0 + \delta_\mu{}^\nu\nabla_0^\dagger\nabla_0.
\end{aligned} \tag{29}$$

Note that the determinant of this operator can be written as a product of two determinants of operators which are separately self-adjoint, namely,

$$\begin{aligned}
\det{}_{\mathrm{V}}(F\circ L)^\dagger (F\circ L) \;&=\; \det{}_{\mathrm{V}}(\nabla_0^\dagger\nabla_0)\;\det{}_{\mathrm{V}}((\nabla_0^\dagger)^{-1}(F\circ L)^\dagger(F\circ L)\nabla_0^{-1}) \\
&=\; \det{}_{\mathrm{V}}(\nabla_0^\dagger\nabla_0)\;\det{}_{\mathrm{V}}(F\circ L\circ\nabla_0^{-1})^\dagger(F\circ L\circ\nabla_0^{-1}) \\
&=:\; \det{}_{\mathrm{V}}(\nabla_0^\dagger\nabla_0)\;\det{}_{\mathrm{V}}(K^\dagger\circ K).
\end{aligned} \tag{30}$$

We have separated out the time derivatives since we are particularly interested in identifying terms that can cancel the divergence associated with the conformal kinetic terms in (20), (22). The Faddeev-Popov operator (29) contains terms of the same kind, coming from eigenfunctions $\rho_\nu(x)$ that are rapidly oscillating in time. In the region where $|\nabla_i\rho_\nu| \ll |\nabla_0\rho_\nu|$, this behaviour is captured by the factorized operator $(\nabla_0^\dagger\nabla_0)$. The factorization (30) will be used in the cancellation argument below.

We proceed similarly for the second Jacobian $\tilde J$, which comes from isolating the divergent mode $\tilde\lambda$ in the action (c.f. the discussion in Sec. 3). For the purposes of this section, it is not necessary to specify explicitly which variables $(g_{ij})^\perp$ are used on the remainder of the configuration space. Using the projectors $\tilde G$ and $1-\tilde G$ as in (17), (18), we decompose the tangent vectors as

$$\delta g_{ij} = (\delta g_{ij})^\parallel + (\delta g_{ij})^\perp, \tag{31}$$

where the trace part is given by

$$(\delta g_{ij})^\parallel = (1-\tilde G)_{ij}{}^{kl}\delta g_{kl} = \frac{1}{d-1} g_{ij}\;\delta\log\det g \equiv 2g_{ij}\;\delta\tilde\lambda \tag{32}$$

and $\tilde\lambda$ has already appeared in (21). The Jacobian $\tilde J$ is now defined through

$$\begin{aligned}
1 \;&=\; \int[\mathcal{D}\delta g^\parallel]_\epsilon\int[\mathcal{D}\delta g^\perp]_\epsilon\;\mathrm{e}^{-\frac{\sqrt{\epsilon}}{2}\langle\delta g,\delta g\rangle_C} \\
&=\; \tilde J\int[\mathcal{D}\delta\tilde\lambda]_\epsilon\int[\mathcal{D}\delta g^\perp]_\epsilon\;\mathrm{e}^{-\frac{\sqrt{\epsilon}}{2}(\langle\delta g^\parallel(\delta\tilde\lambda),\delta g^\parallel(\delta\tilde\lambda)\rangle_C + \langle\delta g^\perp,\delta g^\perp\rangle_C)},
\end{aligned} \tag{33}$$

where the scalar products are taken with respect to the DeWitt metric (7) restricted to the spatial components,

$$\langle\delta g,\delta g\rangle_C = \int d^dx\;\sqrt{\det g_{ij}}\;\delta g_{ij}\;G^{ijkl}_{(C)}\;\delta g_{kl}. \tag{34}$$

As in Sec. 2 above, we assume separate Gaussian normalizations for the two functional integrals, leading to

$$\tilde{J}^{-1} = \int [\mathcal{D}\delta\tilde{\lambda}]_\epsilon \ \mathrm{e}^{-\sqrt{\epsilon}(d-1)(2+(d-1)C)\int d^dx \ \sqrt{\det g}(\delta\tilde{\lambda})^2}, \tag{35}$$

and therefore

$$\tilde{J} = \sqrt{\det{}_{\mathrm{S}} \ 2(d-1)(2+(d-1)C)}. \tag{36}$$

We now collect all determinants to obtain

$$J \cdot \tilde{J} \ = \ \sqrt{\det{}_{\mathrm{S}}((2+dC)(d-1)2\epsilon\partial_0^\dagger\partial_0) \ \det{}_{\mathrm{V}^{d-1}}(2\epsilon\nabla_0^\dagger\nabla_0) \ \det{}_{\mathrm{V}}(K^\dagger \circ K)}, \tag{37}$$

where we now have decomposed also the vector determinant of the time derivatives into a scalar and a spatial vector piece. This combined Jacobian appears in our final form of the non-perturbative continuum path integral in proper-time gauge,

$$Z_\epsilon = \int [\mathcal{D}g_{ij}^\perp]_\epsilon \int [\mathcal{D}\tilde{\lambda}]_\epsilon \ J_\epsilon \tilde{J}_\epsilon \ \mathrm{e}^{-\sqrt{\epsilon}S_\epsilon(g_{ij}^\perp,\tilde{\lambda})}. \tag{38}$$

Unfortunately, but not unexpectedly, there is no immediate way in either $d=3$ or $d=4$ of evaluating these integrals since they are not of Gaussian form. For the time being we are content with less, namely, understanding the role played by the conformal factor. For this purpose, let us concentrate on the leading divergence in $\tilde{\lambda}$ in the action (22),

$$\begin{aligned} S_D \ &= \ -k\int d^dx \ \mathrm{e}^{(d-1)\tilde{\lambda}}(\partial_0\tilde{\lambda})^2 = k\int d^dx \ \mathrm{e}^{(d-1)\tilde{\lambda}}\tilde{\lambda}(\partial_0 + \Gamma^\mu_{0\mu})\partial_0\tilde{\lambda} \\ &= \ -k\int d^dx \ \mathrm{e}^{(d-1)\tilde{\lambda}}\tilde{\lambda}\partial_0^\dagger\partial_0\tilde{\lambda} = -k\int d^dx \ \sqrt{\bar{g}}\mathrm{e}^{(d-1)\lambda}\tilde{\lambda}\partial_0^\dagger\partial_0\tilde{\lambda}, \end{aligned} \tag{39}$$

with the positive constant $k = (d-1)(d-2)/(16\pi G)$, neglecting all other terms (including boundary contributions) in the action.

What still stands in the way of our doing the $\tilde{\lambda}$-integration in (38) is the $\tilde{\lambda}$-dependence of the measure in the action and of the various Jacobians. Following the example of Distler and Kawai in two-dimensional Liouville gravity [27], we make the unrigorous, but well-motivated assumption that all measures with respect to the metric $g_{ij} = \mathrm{e}^{2\lambda}\bar{g}_{ij}$ can by suitable field redefinitions be turned into measures with respect to the constant-curvature metric $\bar{g}_{ij}$, where the Jacobian accompanying this variable change is assumed to be of the same functional form as the (exponentiated) original action[5]. Applying this philosophy to the truncated path integral, we can pull all functional determinants out of the $\tilde{\lambda}$-integral (since they are now defined with respect to the metric $\bar{g}_{ij}$), resulting in

$$\begin{aligned} Z_\epsilon = \int [\mathcal{D}g_{ij}^\perp]_\epsilon \sqrt{\det{}_{\mathrm{S}}((2+dC)(d-1)2\epsilon\bar{\partial}_0^\dagger\bar{\partial}_0) \ \det{}_{\mathrm{V}^{d-1}}(2\epsilon\bar{\nabla}_0^\dagger\bar{\nabla}_0) \ \det{}_{\mathrm{V}}(\bar{K}^\dagger \circ \bar{K})} \ \cdot \\ \int [\mathcal{D}\tilde{\lambda}]_\epsilon \ \mathrm{e}^{-\sqrt{\epsilon} \ k^{ren}\int d^dx\sqrt{\bar{g}} \ \tilde{\lambda}\bar{\partial}_0^\dagger\bar{\partial}_0\tilde{\lambda}+\dots}, \end{aligned} \tag{40}$$

[5] In principle other (higher-curvature) terms may be generated in the process, but they will not affect our cancellation argument for the conformal divergence.

where we have absorbed the effect of the new Jacobian in a renormalization of the gravitational coupling constant contained in k (and we are assuming that k^{ren} is still positive).[6]

Since $\tilde{\lambda}$ takes values on the entire real line, we can set $\mathcal{D}\tilde{\lambda} = \mathcal{D}\delta\tilde{\lambda}$ and perform the $\tilde{\lambda}$-integral to formally obtain $1/\sqrt{\det_{\rm S}(-\bar{\partial}_0^\dagger \bar{\partial}_0)}$. As can be seen, this term *is* cancelled by the scalar determinant in (40) *provided* that its prefactor is negative, that is, if $C < -\frac{2}{d}$. Obviously the determinants involved here are badly divergent and must in principle be regularized. However, since the two terms have the same functional form we expect the cancellation to go through independent of the regularization chosen. (What we have in mind as a typical non-perturbative regularization of the partition function (40) is a common frequency cutoff ω_{n_0} for the *entire* expression Z_ϵ (not just for the $\tilde{\lambda}$-integration). The regularized determinants then take the form $\det_{\rm S}^{(n_0)} \hat{\mathcal{O}} = \prod_{n \leq n_0} e_n$, where e_n is the eigenvalue of the n'th eigenfunction of the operator $\hat{\mathcal{O}}$, and where we are suppressing degeneracies of the eigenfunctions associated with the spatial directions.)

Thus we conclude that under the plausible assumption that the conformal part of the volume element $\sqrt{\det g}$ can be absorbed into a redefinition of the coupling constants and provided that the DeWitt metric is chosen with $C < -\frac{2}{d}$, the conformal divergence of the Euclidean wick-rotated gravitational path integral is cancelled *non-perturbatively* by a corresponding term in the measure, coming from a Faddeev-Popov determinant.

The condition $C < -\frac{2}{d}$ was found previously in the perturbative treatment of Mazur and Mottola [9], with a similar cancellation mechanism. We differ from their and other authors' treatments by obtaining the configuration space of "Euclidean gravity" through a non-perturbative Wick rotation of the gauge-fixed Lorentzian path integral $Z^{(-1)}$, expression (13). Wick rotation in proper-time gauge corresponds to a straightforward substitution $\epsilon \mapsto -\epsilon$ in our formulas. In other gauges, there is no immediate relation between the Euclidean and the Lorentzian sectors *beyond the perturbative regime around a fixed (typically flat) background metric.* In these cases, even if any non-perturbative results could be obtained for Euclidean signature, their implications for the physical, Lorentzian sector would be unclear.

By contrast, we have shown under what conditions a cancellation of the conformal divergence may take place in the full, non-perturbative theory of Lorentzian space-times. This lends further support to the finding of the quantum gravity model obtained from Lorentzian dynamical triangulations, whose effective measure (including entropy contributions) apparently leads to a non-perturbative cancellation of the "conformal sickness" of the action. The restriction $C < -\frac{2}{d}$ found in the continuum is not unnatural in the sense that this parameter region contains the only dynamically distinguished value $C = -2$ of the constant C (found after Legendre-

[6] There is a somewhat related path-integral treatment by Mazur, based on a conformal decomposition of Riemannian space-time metrics [28]. However, he concentrates on boundary rather than bulk terms in the effective action.

transforming the gravitational action in $d=3$ and 4).

Even with our assumption of the absorption of the scaling factors $e^{2\lambda}$, it is unlikely that one can make much progress in computing the continuum path integral in proper-time gauge explicitly, at least in four dimensions. Note that substantial simplifications occur in the case $d = 3$, where there are no further explicitly λ-dependent terms in the part of the action indicated by the dots in formula (40). In that case, it remains to evaluate

$$Z_\epsilon^{d=3} = \int [\mathcal{D}g_{ij}^\perp]_\epsilon \sqrt{\det_{V^{d-1}}(2\epsilon \bar{\nabla}_0^\dagger \bar{\nabla}_0)\, \det_V(\bar{K}^\dagger \circ \bar{K})}\; e^{-i\sqrt{-\epsilon}k^{ren}\bar{S}_\epsilon}, \tag{41}$$

with the action

$$\bar{S}_\epsilon^{d=3} = \int dt \Big(4\pi\chi + \int d^2x \sqrt{\det \bar{g}_{ij}} \Big(-2\Lambda - \frac{\epsilon}{4}(\partial_0 \bar{g}_{ij})(\bar{g}^{ik}\bar{g}^{jl} - \frac{1}{2}\bar{g}^{ij}\bar{g}^{kl})(\partial_0 \bar{g}_{kl})\Big)\Big). \tag{42}$$

Although this expression looks now tantalizingly simple, it is still non-polynomial in the remaining metric components. We will not attempt here to evaluate (41) further, but it would clearly be interesting to relate it to any one of the known exact results obtained in other approaches to three-dimensional quantum gravity (see, for example, [30]).

5 Perturbative evaluation of the 3d path integral

Although it was not our main motivation, one can take our formulation as the starting point for a perturbative expansion around a given classical solution. Depending on the solution one is interested in, the proper-time gauge is not necessarily the most convenient gauge choice *perturbatively*. Also, we do not expect to find anything new compared with previous calculations using other gauges. Nevertheless, the calculation we will perform illustrates the general procedure outlined in the main part of the paper and gives an explicit example of the cancellation mechanism at work.

We will study the path integral (38) by computing its perturbative expansion $Z^{(2)}$ around a particular classical solution. For the sake of simplicity, we choose the spatial slices to be flat two-tori (corresponding to a vanishing cosmological constant Λ), such that $M = [0,1] \times T^2$. There is a two-parameter set of classical solutions,

$$g_{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & V\frac{1}{\tau_2} & V\frac{\tau_1}{\tau_2} \\ 0 & V\frac{\tau_1}{\tau_2} & V\frac{\tau_1^2+\tau_2^2}{\tau_2} \end{pmatrix}, \tag{43}$$

in a proper-time coordinate system (t, x_1, x_2), where the x_i are periodic and rescaled to have period 1 [31, 30]. The metrics are parametrized by two modular parameters τ_α, and V denotes a constant spatial area. We will perform our perturbative calculation around any one of the "straight torus solutions" with $(\tau_1, \tau_2) = (0, \tau)$, $\tau > 0$, where the metric takes the diagonal form

$$g^0_{\mu\nu} = \text{diag}\ (\epsilon,\ V\frac{1}{\tau},\ V\tau). \tag{44}$$

Our starting point is the partition function

$$Z_\epsilon^{(2)} = \int [\mathcal{D}h_{ij}^\perp]_\epsilon \int [\mathcal{D}\delta\tilde{\lambda}]_\epsilon \; J_\epsilon \tilde{J}_\epsilon \; \mathrm{e}^{-\sqrt{\epsilon} S_\epsilon^{(2)}(h_{ij}^\perp, \delta\tilde{\lambda})}, \tag{45}$$

with the action given by

$$S_\epsilon^{(2)} = -\frac{\epsilon}{16\pi G} \int dt \int d^2x \sqrt{\det g_{ij}} \; \left(2\epsilon (\partial_0 \delta\tilde{\lambda})^2 - \frac{\epsilon}{4} (\partial_0 h_{ij}^\perp) g^{ik} g^{jl} (\partial_0 h_{kl}^\perp) \right), \tag{46}$$

and where both expressions are to be evaluated at the base metric (44). Since our end result will not depend on ϵ in a non-trivial way, we will from now on simply work with the Euclidean value $\epsilon = 1$ and drop the subscripts ϵ. We are using the notation $h_{ij} = \delta g_{ij}$ for the tangent vectors at $g^0_{\mu\nu} \in \mathcal{M}$, and decompose them according to (31), (32). We will parametrize the directions $h_{ij}^\perp$ perpendicular to the trace part explicitly as functions of the spatial vector fields ξ_i and infinitesimal modular parameters $\delta\tau_\alpha$, such that

$$\begin{aligned} h_{ij}^\perp &= \nabla_i \xi_j + \nabla_j \xi_i - g_{ij} \nabla_k \xi^k + \delta\tau_\alpha \langle \chi^{(\alpha)}, \Psi_{(\beta)} \rangle \; \delta^{\beta\gamma} \; \Psi_{(\gamma)ij} \\ &=: (\tilde{L}\xi)_{ij} + \delta\tau_\alpha \langle \chi^{(\alpha)}, \Psi_{(\beta)} \rangle \; \delta^{\beta\gamma} \; \Psi_{(\gamma)ij}, \qquad \text{with } \chi_{ij}^{(\alpha)} := \frac{\partial g_{ij}}{\partial \tau_\alpha}. \end{aligned} \tag{47}$$

This variable change is motivated by the standard decomposition of two-dimensional Riemannian metrics [29, 30]

$$g_{ij}(\vec{x}, t) = \mathrm{e}^{2\lambda(\vec{x},t)} f^*_{\vec{x},t} \; \tilde{g}_{ij}(\vec{x}, t), \tag{48}$$

followed by a shift $\lambda \mapsto \tilde{\lambda}$ of the conformal factor, c.f. (21). In the decomposition (48), $\tilde{g}_{ij}(\vec{x}, t)$ is one of a set of constant-curvature metrics, labelled by Teichmüller parameters τ_α, and f a spatial diffeomorphism, with generators ξ_i. Note that as a consequence of our gauge-fixing procedure, all quantities in (48) carry an additional proper-time dependence. To avoid misunderstandings, let us also point out that the diffeomorphisms f (which act in a standard way on the coordinates and g_{ij}'s, and map surfaces of constant t into themselves) are of course no longer invariances of the gauge-fixed action.

The $\Psi_{(\alpha)ij}$ form a basis for *ker* $\tilde{L}^\dagger$, that is, for the transverse traceless tensors, and we have chosen them to be orthonormal with respect to the scalar product $\langle , \rangle$ (which involves an integration over the spatial directions only). Explicitly, they are given by

$$\Psi_{(1)} = \sqrt{\frac{V}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi_{(2)} = \sqrt{\frac{V}{2}} \begin{pmatrix} \frac{1}{\tau} & 0 \\ 0 & -\tau \end{pmatrix}. \tag{49}$$

Associated with the variable change $h_{ij}^\perp \mapsto (\xi_i, \delta\tau_\alpha)$ is another Jacobian $\bar{J}$ which takes the form

$$\bar{J} = \sqrt{\det_{\mathrm{V}^2}(\tilde{L}^\dagger \tilde{L})} \; \det \langle \chi^{(\alpha)}, \Psi_{(\beta)} \rangle. \tag{50}$$

A straightforward calculation yields

$$\det_{\mathrm{V}^2}(\tilde{L}^\dagger\tilde{L}) = \det_{\mathrm{V}^2}(-2g^{ij}\nabla_i\nabla_j) \equiv \det_{\mathrm{V}^2}(-2\Box) \tag{51}$$

and

$$\det\langle\chi^{(\alpha)}, \Psi_{(\beta)}\rangle = -\frac{2V}{\tau^2}. \tag{52}$$

After the coordinate transformation on the tangent space of metrics, the partition function reads

$$Z^{(2)} = J\tilde{J}\bar{J} \int[\mathcal{D}\xi_i]\int[\mathcal{D}\delta\tau_\alpha]\int[\mathcal{D}\delta\tilde{\lambda}]\ \mathrm{e}^{-S^{(2)}(\xi,\delta\tilde{\lambda},\delta\tau)}, \tag{53}$$

with the action

$$S^{(2)} = -\frac{V}{16\pi G}\int dt\int d^2x\left(2(\partial_0\delta\tilde{\lambda})^2 + \frac{1}{2}g^{ij}\dot{\xi}_i\Box\dot{\xi}_j\right) + \frac{V}{16\pi G}\frac{1}{2\tau}\int dt\ (\partial_0\delta\vec{\tau})^2. \tag{54}$$

We have pulled out the determinants from under the functional integrations in (53), because they depend only on the fixed background metric g^0.

We can now perform the integrations over ξ_i, $\delta\tilde{\lambda}$ and $\delta\tau$, since the action is quadratic in these variables. Up to irrelevant (positive) constant terms, the two integrations yield

$$\xi_i\text{-integral:}\ \left(\det{}'_{\mathrm{V}^2}(-\partial_0^2)(-\Box)\right)^{-\frac{1}{2}} \tag{55}$$

$$\delta\tilde{\lambda}\text{-integral:}\ \left(\det{}'_{\mathrm{S}}(\partial_0^2)\right)^{-\frac{1}{2}} \tag{56}$$

$$\delta\tau_\alpha\text{-integral:}\ \left(\det{}'(-\partial_0^2)\right)^{-1}, \tag{57}$$

where by definition all zero-modes have been excluded, and the last determinant is that of a free two-dimensional particle with mass $m = \frac{1}{16\pi G}\frac{V}{\tau}$. The term coming from the $\delta\tilde{\lambda}$-integration is of course the ill-defined determinant associated with the conformal divergence.

A well-known subtlety arises in the evaluation of the determinant of $(\tilde{L}^\dagger\tilde{L})$ since the flat torus possesses Killing vectors, leading to zero-modes of this operator [32, 30, 24]. This can be taken care of by writing

$$\det_{\mathrm{V}^2}(\tilde{L}^\dagger\tilde{L}) = V_K^{-1}\det{}'_{\mathrm{V}^2}(\tilde{L}^\dagger\tilde{L}), \tag{58}$$

where V_K denotes the (infinite) volume of the diffeomorphism subgroup generated by the Killing vectors. Since we do not keep track of *positive* constant and infinite factors, we will simply drop this term. Also, it is not our aim here to investigate the possible physical significance of such zero-modes and we will remove them from now on wherever they occur.

Collecting all the determinants, the partition function is now given by

$$
\begin{aligned}
Z^{(2)} &\simeq J\tilde{J}\bar{J}'\frac{1}{\sqrt{\det'_{\mathrm{V}^2}(-\partial_0^2)\ \det'_{\mathrm{V}^2}(-\Box)\ \det'_{\mathrm{S}}(\partial_0^2)\ (\det'(-\partial_0^2))^2}} \\
&= \frac{\det\langle\chi,\Psi\rangle}{\det'(-\partial_0^2)}\ \frac{\sqrt{\det_{\mathrm{S}}4(2+3C)\ \det'_{\mathrm{V}}(F\circ L)^\dagger(F\circ L)}}{\sqrt{\det'_{\mathrm{V}^2}(-\partial_0^2)\ \det'_{\mathrm{S}}(\partial_0^2)}}.
\end{aligned}
\tag{59}
$$

The only non-trivial task remaining is the evaluation of the Faddeev-Popov determinant $\det_{\mathrm{V}}(F\circ L)^\dagger(F\circ L)$. At the metric $g^0\in\mathcal{M}$, we can compute this determinant explicitly, so there is no need to split off the time derivatives as we did in (30). As a warm-up, let us calculate the functional determinant $\det_{\mathrm{V}}(\nabla_0^\dagger\nabla_0)\equiv\det_{\mathrm{V}}(-\partial_0^2)$. In order not to have to deal with non-trivial boundary terms we demand that the eigenfunctions on $M=[0,1]\times T^2$ be periodic in the x_1- and x_2-direction, as well as in the time direction. A complete set of eigenfunctions is then given by

$$
\vec{\epsilon}^{\ (\kappa_1,\kappa_2,\omega,\mu)}=\vec{\alpha}^{(\mu)}\ \mathrm{e}^{i(\kappa_1x_1+\kappa_2x_2)}\ \mathrm{e}^{i\omega t},
\tag{60}
$$

with

$$
\vec{\alpha}^{(0)}=\sqrt{\frac{2}{V}}\begin{pmatrix}1\\0\\0\end{pmatrix},\ \vec{\alpha}^{(1)}=\sqrt{\frac{2}{\tau}}\begin{pmatrix}0\\1\\0\end{pmatrix},\ \vec{\alpha}^{(2)}=\sqrt{2\tau}\begin{pmatrix}0\\0\\1\end{pmatrix}.
\tag{61}
$$

The frequency ω takes discrete values $\omega=2\pi k$, $k=\pm1,\pm2,\pm3,\ldots$, and similarly the κ_i are given by $\kappa_i=2\pi k_i$, $k_i=0,\pm1,\pm2,\ldots$. The eigenvectors are orthonormal with respect to the scalar product (16), with discrete eigenvalues

$$
\nu^{(\kappa_1,\kappa_2,\omega,\mu)}=\omega^2.
\tag{62}
$$

The functional determinant is therefore the infinite product

$$
\det_{\mathrm{V}}(-\partial_0^2)=\prod_{\omega,\kappa_1,\kappa_2}\omega^6.
\tag{63}
$$

Next we determine the eigenfunctions $\vec{\eta}$ of the Faddeev-Popov operator in

$$
[(F\circ L)^\dagger(F\circ L)]_\mu{}^\nu\eta_\nu=\rho\eta_\mu.
\tag{64}
$$

Making an ansatz of the form

$$
\vec{\eta}=\vec{\sigma}(t)\mathrm{e}^{i\vec{\kappa}\cdot\vec{x}},
\tag{65}
$$

for the three-vectors, one obtains a coupled set of eigenvalue equations, namely,

$$
\begin{aligned}
\left(-4\partial_0^2+\frac{1}{V}(\tau\kappa_1^2+\frac{1}{\tau}\kappa_2^2)\right)\sigma_0-i\kappa_1\frac{\tau}{V}\partial_0\sigma_1-i\kappa_2\frac{1}{V\tau}\partial_0\sigma_2 &= \rho\sigma_0 &\quad (66)\\
-i\kappa_1\partial_0\sigma_0-\partial_0^2\sigma_1 &= \rho\sigma_1 &\quad (67)\\
-i\kappa_2\partial_0\sigma_0-\partial_0^2\sigma_2 &= \rho\sigma_2. &\quad (68)
\end{aligned}
$$

(We note in passing that it is clear from (67,68) that in the presence of boundaries at $t = 0, 1$ it would not be consistent to require a simultaneous vanishing of *all* components of $\vec{\sigma}$ at the boundaries.) One third of the eigenfunctions is easily found by setting $\sigma_0 = 0$. The eigenvectors are of the form

$$\vec{\eta}^{\,(\kappa_1,\kappa_2,\omega,0)} = \begin{pmatrix} 0 \\ \frac{1}{\tau} k_2 \\ -\tau k_1 \end{pmatrix} \mathrm{e}^{i\vec{\kappa}\cdot\vec{x}} \; e^{i\omega t}, \tag{69}$$

with eigenvalues

$$\rho^{(\kappa_1,\kappa_2,\omega,0)} = \omega^2. \tag{70}$$

For the remaining eigenfunctions one finds

$$\vec{\eta}^{\,(\kappa_1,\kappa_2,\omega,\pm)} = \begin{pmatrix} 1 \\ \frac{\omega}{\rho^{\pm}-\omega^2}\kappa_1 \\ \frac{\omega}{\rho^{\pm}-\omega^2}\kappa_2 \end{pmatrix} \mathrm{e}^{i\vec{\kappa}\cdot\vec{x}} \; e^{i\omega t}, \tag{71}$$

with the associated eigenvalues

$$\rho^{(k_1,k_2,\omega,\pm)} = \frac{1}{2}\Big(5\omega^2 + \frac{1}{V}(\tau\kappa_1^2 + \frac{1}{\tau}\kappa_2^2)\Big) \pm \frac{1}{2}\sqrt{\Big(5\omega^2 + \frac{1}{V}(\tau\kappa_1^2 + \frac{1}{\tau}\kappa_2^2)\Big)^2 - 16\omega^4}. \tag{72}$$

For fixed κ_1, κ_2 and ω, there are therefore three orthogonal eigenfunctions. The entire functional determinant is thus given by

$$\det{}'_{\mathrm{V}}(F\circ L)^\dagger(F\circ L) = \prod_{\omega\neq 0,\kappa_1,\kappa_2} \rho^{(\kappa_1,\kappa_2,\omega,0)}\rho^{(\kappa_1,\kappa_2,\omega,+)}\rho^{(\kappa_1,\kappa_2,\omega,-)} \equiv \prod_{\omega\neq 0,\kappa_1,\kappa_2} 4\omega^6. \tag{73}$$

This coincides (up to a constant factor) with the determinant of the kinetic term we calculated earlier, that is, $\det_{\mathrm{V}}\nabla_0^\dagger\nabla_0$. Substituting the results for the determinants back into (59), we observe again an exact cancellation of the infinite products *provided* that $C < -\frac{2}{3}$, in agreement with our earlier non-perturbative results. Since the regularized determinant (57) gives a term proportional to V/τ, our final result for the perturbative partition function around flat torus metrics of the type (44) is given by

$$Z^{(2)} = \frac{\det\langle\chi,\Psi\rangle}{\det'(-\partial_0^2)} \sim \frac{V^2}{\tau^3}. \tag{74}$$

6 Conclusions

Inspired by recent attempts of constructing a non-perturbative propagator for gravity by discrete methods, we have investigated the continuum gravitational propagator in proper-time gauge, concentrating on the role played by the conformal mode of the metric. Our starting point was the space of physical space-time metrics of Lorentzian signature. After performing a generalized Wick rotation, the partition function becomes real, but the Euclideanized action is seen to suffer from the usual "conformal sickness": as a result of conformal excitations it is unbounded below.

We then proceeded to determine the Faddeev-Popov determinants that arise during the coordinate changes on the space of metrics, the first from splitting off the gauge degrees of freedom associated with the diffeomorphisms of the base manifold, the second from isolating the part of the metric that has a negative-definite kinetic term. Although an explicit evaluation of the functional determinants and the non-perturbative path integral seem technically out of reach, we showed that under certain assumptions about the behaviour of the partition function under renormalization the conformal divergence in the action is cancelled by a corresponding Faddeev-Popov term in the measure. This conclusion also required that the signature of the DeWitt metric was chosen to be indefinite, i.e. the constant C in the DeWitt measure (used to define inner products on the tangent space of metrics) satisfied $C < -\frac{2}{d}$ which we argued was a natural condition. Our work can therefore be seen as a non-perturbative generalization of earlier findings by Mazur and Mottola [9] which – although acknowledged by the authorities [33] – are maybe not widely appreciated.

Our results reinforce the evidence coming from dynamically triangulated formulations of quantum gravity [16] that in a fully non-perturbative path integral the conformal mode is not a propagating degree of freedom and therefore the conformal divergence is simply absent, contrary to what one may have expected from just looking at the action or from considering reduced, cosmological models. Although geometric configurations with large and negative action exist, they are effectively suppressed by the non-trivial path-integral measure. This is an attractive proposition because it implies that the unboundedness of the action is no obstacle in principle to the construction of a gravitational path integral. Also, at a kinematical level (that is, before quantum gravity proper has been solved), it brings the covariant formulation of gravity into line with canonical treatments where the conformal degree of freedom is fixed by imposing the Hamiltonian constraint. Although in the path integral this constraint is not enforced explicitly, it seems that the measure nevertheless *does* know about it.

All our calculations were done in proper-time gauge, mimicking a similar procedure in the discrete Lorentzian gravity models of [10, 11]. Note that there is nothing wrong in principle with choosing a gauge in quantum gravity. Our choice of proper time does have an invariant geometric meaning, but this by no means entails that proper-time correlators assume a simple form in other gauges or that any interesting physical quantity will be easily expressible in proper-time gauge. This is simply an inevitable feature of gauge theories. Obviously we expect our result about the absence of the conformal sickness to be gauge-independent. In practice, an explicit check may not be straightforward, since our construction of a Wick rotation was closely tied to the proper-time gauge. There may be other gauge choices and other prescriptions of Wick-rotating, but we are currently not aware of any concrete proposals. As usual in quantum gravity, one is not exactly faced with an embarrassment of riches when trying to quantize the theory.

Clarifying the role of the conformal factor is only one step in analyzing the properties of non-perturbative path integrals for Lorentzian gravity. We do not expect that much progress can be made in a continuum formulation in evaluating these quantities explicitly and showing how the non-renormalizability of the perturbative approach is resolved non-perturbatively. For a solution of these more fundamental problems we must rely on the properly regulated discrete quantum gravity models that are currently under investigation.

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Appendix 1

In this appendix we show that every metric $g_{\mu\nu}$ in an infinitesimal neighbourhood of the constraint surface $\mathcal{C}$ can be uniquely decomposed into an element in $\mathcal{C}$ and an infinitesimal diffeomorphism, parametrized by a vector field ξ_μ. Writing the original metric as $g_{\mu\nu} = g^{\rm pt}_{\mu\nu} + h_{\mu\nu}$, we would like to understand under which conditions the tangent vector $h_{\mu\nu}$ can be uniquely decomposed as

$$h_{\mu\nu} = h^{\rm pt}_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{75}$$

where $h^{\rm pt}_{\mu\nu}$ is tangential to $\mathcal{C}$, i.e. of the form

$$h^{\rm pt}_{\mu\nu} = \begin{pmatrix} 0 & \vec{0} \\ \vec{0} & h_{ij} \end{pmatrix}. \tag{76}$$

This is tantamount to solving the set of equations

$$2\nabla_0 \xi_0 = h_{00} \tag{77}$$

$$\nabla_0 \xi_i + \nabla_i \xi_0 = h_{0i} \tag{78}$$

for $\vec{\xi}$, where the covariant derivatives refer to the base point metric $g^{\rm pt}_{\mu\nu}$. Since this metric is in proper-time gauge, its Christoffel symbols take the form

$$\begin{aligned} \Gamma^\mu_{00} &= \Gamma^0_{0\mu} = 0 \\ \Gamma^0_{lj} &= -\frac{1}{2}\, g_{lj,0} \\ \Gamma^i_{0j} &= \frac{1}{2}\, g^{ik} g_{jk,0} \\ \Gamma^i_{lj} &= \frac{1}{2}\, g^{ik} \left(g_{lk,j} + g_{jk,l} - g_{lj,k} \right). \end{aligned} \tag{79}$$

Using these explicit expressions we obtain

$$\xi_0(t,\mathbf{x}) = \frac{1}{2}\int_0^t dt'\ h_{00}(t',\mathbf{x}) \tag{80}$$

$$\partial_t \xi_i + a_{ij}\xi_j = b_i(t,\mathbf{x}), \tag{81}$$

where $b_i = h_{0i} - \partial_i \xi_0$, and $a_{ij} = -2\Gamma^i_{0j}$. The system of differential equations determining ξ_i has unique solutions if

1) b_i and a_{ij} are smooth and continuous in the interval of interest, namely, [0,t];

2) the a_{ij} satisfy a Lipschitz condition

$$\sum_{i=1}^{d-1} |a_{ij}| < k_j, \tag{82}$$

where the k_j are arbitrary constants, greater than zero.

These conditions are satisfied for all metrics $g^{\mathrm{pt}}_{\mu\nu}$, which we assume to be non-degenerate and sufficiently differentiable. The boundedness property (82) follows from the compactness of the base space ${}^{(d)}M = [0,t] \times {}^{(d-1)}\Sigma$.

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