An application of the DR–duality theory for compact groups to endomorphism categories of C*–algebras with nontrivial center

Hellmut Baumgärtel

Mathematical Institute, University of Potsdam,
Am Neuen Palais 10, Postfach 601 553,
D–14415 Potsdam, Germany.
baumg@rz.uni-potsdam.de

Fernando Lledó*

Max–Planck–Institut für Gravitationsphysik,
Albert–Einstein–Institut, Am Mühlenberg 1,
D–14476 Golm, Germany.
lledo@aei-potsdam.mpg.de

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Dedicated to Sergio Doplicher and John E. Roberts on the occasion of their 60th birthdays.

Abstract

The main result of the Doplicher/Roberts (DR–) duality theory for compact groups, applied to the special case of endomorphism categories with permutation and conjugation structure of a fixed C*–algebra $\mathcal{A}$ with trivial center, says that such a category can be characterized as the category of all canonical endomorphisms of $\mathcal{A}$ w.r.t. an (essentially uniquely determined) Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$, where $\mathcal{G}$ is a compact automorphism group of $\mathcal{F}$ and $\mathcal{A}'\cap\mathcal{F}=\mathbb{C}1\!\!1$.

In [4] C*–Hilbert systems $\{\mathcal{F},\mathcal{G}\}$ are considered where the fixed point algebra $\mathcal{A}$ has nontrivial center $\mathcal{Z}$ and where $\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}$ is satisfied. The corresponding category of all canonical endomorphisms of $\mathcal{A}$ contains characteristic mutually isomorphic subcategories of the DR–type which are connected with the choice of distinguished $\mathcal{G}$–invariant algebraic Hilbert spaces within the corresponding $\mathcal{G}$–invariant Hilbert $\mathcal{Z}$–modules.

We present in this paper the solution of the corresponding inverse problem. More precisely, assuming that the given endomorphism category $\mathcal{T}$ of a C*–algebra $\mathcal{A}$ with center $\mathcal{Z}$ contains a certain subcategory of the DR–type, a Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$ is constructed such that $\mathcal{T}$ is isomorphic to the category of all canonical endomorphisms of $\mathcal{A}$ w.r.t. $\{\mathcal{F},\mathcal{G}\}$ and $\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}$. Furthermore, there is a natural equivalence relation between admissible subcategories and it is shown that two admissible subcategories yield $\mathcal{A}$–module isomorphic Hilbert extensions iff they are equivalent. The essential step of the solution is the application of the standard DR–theory to the assigned subcategory.

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1 Introduction

One of the origins of the DR–duality theory for compact groups (cf. [10]) is the analysis of superselection structures formulated in the context of Algebraic Quantum Field Theory, where

Present address: Institute of Pure and Applied Mathematics, RWTH–Aachen, Templergraben 55, D–52056 Aachen, Germany. E-Mail: lledo@iram.rwth-aachen.de endomorphism categories $\mathcal{T}$ of a fixed C–algebra $\mathcal{A}$ with trivial center (the so–called algebra of quasilocal observables) appear. These categories $\mathcal{T}$ are equipped with a conjugation and a permutation structure. Application of the general DR–duality theory yields to the result that $\mathcal{T}$ can be characterized by a compact symmetry group $\mathcal{G}$ via the construction of a suitable Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$, where $\mathcal{F}$ is called the field algebra (see [9, 11, 1]). The condition $\mathcal{A}'\cap\mathcal{F} = \mathbb{C}\mathbb{1}$ is crucial for this construction.

This result suggests the natural question for conditions of endomorphism categories $\mathcal{T}$ of C*–algebras $\mathcal{A}$ with nontrivial center $\mathcal{Z}$ such that there is still a description of $\mathcal{T}$ by a compact group. Already the paper [10, Sections 2,3] starts with nontrivial center. The point there is that the "flip property" for the permutator $\epsilon(\rho,\sigma)$ and the intertwiners $A \in (\rho,\rho')$, $B \in (\sigma,\sigma')$, i.e.

$$\epsilon(\rho',\sigma')A \times B = B \times A\,\epsilon(\rho,\sigma),$$

is assumed to be valid for all intertwiners. Only later on the condition $(\iota,\iota) = \mathbb{C}\mathbb{1}$ is added to arrive at the famous DR–theorem. Also in the context of more general categories (that do not assume the existence of a permutator) some results for nontrivial (ι,ι) are stated (cf. [13, Section 2]).

In [4] we present several results on general Hilbert C*–systems, where $\mathcal{A}'\cap\mathcal{F} = \mathcal{Z} \supset \mathbb{C}\mathbb{1}$ is assumed. For example, properties of the corresponding category of canonical endomorphisms are proved and the breakdown of the Galois correspondence between the symmetry group $\mathcal{G}$ and the stabilizer of its fixed point algebra $\mathcal{A}$ is stated. Further a new concept of "irreducibility" is proposed. In [2] further properties of "irreducible endomorphisms" are mentioned and for special automorphism categories the solution of the "inverse problem" (given the category, construct an assigned Hilbert extension) is briefly described. The paper [5] contains a description of the status of the problem for automorphism categories as a counterpart of the special case where $\mathcal{Z} = \mathbb{C}\mathbb{1}$. In the previous two references the corresponding group $\mathcal{G}$ is abelian.

The present paper describes the solution of the inverse problem in the general case. First, sufficient conditions for the endomorphism category $\mathcal{T}$ of $\mathcal{A}$ are stated such that there exists a Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$ and $\mathcal{T}$ turns out to be the category of all canonical endomorphisms of $\mathcal{A}$. The crucial assumption is the existence of a certain subcategory $\mathcal{T}_{\mathcal{C}}$ of $\mathcal{T}$ of the DR–type. The goal of this note is to show that the DR–properties of $\mathcal{T}_{\mathcal{C}}$ are sufficient to characterize the category $\mathcal{T}$ as the category of all canonical endomorphisms of $\mathcal{A}$ w.r.t. a suitable Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$, where $\mathcal{G}$ is the characteristic compact DR–group for $\mathcal{T}_{\mathcal{C}}$, interpreted as an automorphism group of the Hilbert extension. The conditions are also necessary. This is an easy implication of the results for Hilbert C*–systems, obtained in [4]. Second, a uniqueness result is stated: Two admissible subcategories of $\mathcal{T}$ yield $\mathcal{A}$–module isomorphic Hilbert extensions iff they are equivalent (in a precise sense formulated in the following section).

The present paper is structured in 4 sections: In the following section we present the postulates for the category $\mathcal{T}$ that assure the unique (up to $\mathcal{A}$–module isomorphy) extension of the C*–algebra $\mathcal{A}$ (cf. Theorem 2.4, which is our main result). In Subsection 3.1 we introduce the new notion of irreducible endomorphism in $\mathcal{T}$ and show some of its consequences. Finally, we give in the following two subsections the proof of Theorem 2.4.

For convience of the reader we finish this section recalling some useful definitions concerning Hilbert extensions of a C*–algebra $\mathcal{A}$: A C*–system $\{\mathcal{F},\mathcal{G}\}$, where $\mathcal{G} \subset \operatorname{aut}\mathcal{F}$ is compact w.r.t. the topology of pointwise norm convergence, is called Hilbert if $\operatorname{spec}\mathcal{G} = \hat{\mathcal{G}}$ (the dual of $\mathcal{G}$) and if each spectral subspace $\mathcal{F}_D$, $D \in \hat{\mathcal{G}}$, contains an algebraic Hilbert space $\mathcal{H}_D$ such that $\mathcal{G}$ acts invariantly on $\mathcal{H}_D$ and the unitary representation $\mathcal{G}\restriction\mathcal{H}_D$ is a element of the equivalence class D. A Hilbert space $\mathcal{H} \subset \mathcal{F}$ is called algebraic if the scalar product $\langle\cdot,\cdot\rangle$ of $\mathcal{H}$ is given by $\langle A,B\rangle\mathbb{1} := A^*B$, $A,B \in \mathcal{H}$. A Hilbert system $\{\mathcal{F},\mathcal{G}\}$ is called a Hilbert extension of a C*–algebra $\mathcal{A}$ iff $\mathcal{A} \subset \mathcal{F}$ is the fixed point algebra of $\mathcal{G}$. Two Hilbert extensions $\{\mathcal{F}_1,\mathcal{G}\}$, $\{\mathcal{F}_2,\mathcal{G}\}$ are called $\mathcal{A}$–module isomorphic if the exists an isomorphism $J\colon \mathcal{F}_1 \to \mathcal{F}_2$, such that $J(A) = A$, A 2 A , that intertwines between the corresponding group actions. Finally, the stabilizer stab A for a unital C*{subalgebra A F is defined by stab A := fg 2 aut F jg(A) = A for all A 2 A g: (The terminology Hilbert system can be traced back to [8], where in the case just mentioned the spectrum of G is called the Hilbert spectrum .)

2 Assumptions on the endomorphism category and the main result

In the present section we will collect the assumptions on the category T of suitable endomorphisms of a unital C*{algebra A that guarantees the main theorem stated at the end of this section. For standard notions within category theory we refer to [14].

Let T be a tensor C*{category of unital endomorphisms of A [10]. We denote the objects by ; ; ;::: 2 Ob T . The arrows between objects ; are given as usual by the intertwiner spaces (;) := fX 2 A j X (A) = (A)X ; A 2 A g and we put A B := A (B) for A 2 (; ^{0}); B 2 (; ^{0}), so that A B 2 (; $^{0\,0}$). By we denote the identical endomorphism and (;) = Z is the center of A . Note that (;) is a left (Z){ and a right (Z){module, i.e. (Z)(;) (Z) = (;) (Z) (;). The conditions on T are given by:

P.1.1 T is closed w.r.t. direct sums , i.e. if ; 2 Ob T , then there are isometries V; W 2 A with V W = 0; V V + W W = 1l such that ( ) := V ( )V + W ( )W 2 Ob T : In this case V 2 (;); W 2 (;).

P.1.2 T is closed w.r.t. subobjects < , i.e. if 2 Ob T and is a unital endomorphism of A such that there is an isometry V 2 (;), then 2 Ob T . In this case ( ) = V ( )V :

P.1.3 T is closed w.r.t. complementary subobjects, i.e. if 2 Ob T and < , then there is a subobject ^{0}< such that = ^{0}.

P.2 T contains a C*{subcategory T_C with Ob T_C = Ob T ; where the arrows $(;)_C$ (;) satisfy the following properties:

P.2.1 $(;)_C$ $(;)_C$ $(;)_C$;

P.2.2 $(;)_C$ $(;)_C$;

P.2.3 $(;)_C$ $(;)_C$,

P.2.4 $(;)_C$ $(;)_C$,

P.2.5 $(;)_C$ = C1l;

P.2.6 Any finite set of linearly independent elements $F_1; F_2; \ldots; F_n$ 2 $(;)_C$ (a complex Banach space), is linearly independent modulo (Z) in (;), i.e. if $\sum_{j=1}^{n} {}_j F_j$ = 0; $_j$ 2 C, implies $_j$ = 0, then also $\sum_{j=1}^{n} F_j$ (Z_j) = 0 implies (Z_j) = 0. Moreover, $(;)_C$ generates (;), i.e. (;) = $(;)_C$ (Z) = (Z)$(;)_C$ (Z).

P.2.7 T_C is closed w.r.t. direct sums, subobjects and complementary subobjects, i.e. now the required projections and isometries must ly in the corresponding intertwiner spaces $(;)_C$.

P.3 There is a permutation structure[1], on T_C, i.e. there is a mapping f ; g ! (;) 2 $(;)_C$, where (;) is a unitary satisfying:

P.3.1 (;) (;) = 1l;

[1] In [10] categories as T_C satisfying the properties below are called symmetric.

P.3.2 $(;) = (;) = 1\!\!1;$

P.3.3 $(;) = (;) \quad ((;));$

P.3.4 $({}^0; {}^0)A \quad B = B \quad A \quad (;)$ for all $A \in (; {}^0)_C ; B \in (; {}^0)_C :$

P.4 There is a conjugation structure on T_C, i.e. to each $\in \mathrm{Ob}\, T_C$ there corresponds a conjugate $\overline{} \in \mathrm{Ob}\, T$ and intertwiners $R \in (; \overline{})_C ; S \in (; \overline{})_C$ such that

P.4.1 $S \quad (R \quad) = 1\!\!1; \quad R \quad \overline{}(S \quad) = 1\!\!1;$

P.4.2 $S = (\overline{};)R :$

2.1 Remark (i) The preceding axioms imply that the subcategory T_C satisfies the postulates of the DR–theory developed in [10], so that we can apply standard results from this theory.

(ii) Note that the definition of subobject in (P.1.2) is not a straightforward generalization to nontrivial center situation of the one given in [10]. Namely, if E is a selfadjoint central projection $E \in Z$ with $0 < E < 1\!\!1$, then there is no isometry $W \in A$ such that $E = W W$, because in that case we simply have

$$1\!\!1 = W \quad W \; W \quad W = W \quad E\, W = E\, W \quad W = E :$$

(iii) From (P.4) it already follows that T has conjugates (cf. [13, Section 2]). Further (P.2.7) also shows that T is closed under direct sums, subobjects and complementary subobjects. (P.2.1)–(P.2.3) imply $(; {}^0)_C \quad (; {}^0)_C \quad (; {}^{0\,0})_C$ and from (P.2.4) it follows immediately that the equality $(;)_C = (;)_C$ holds.

(iv) (P.1.1) implies an "a priori property" of A, namely there are two isometries $V ; W \in A ; V \; W = 0; V V + W W = 1\!\!1$. (P.1.3) implies that if $E \in (;)$ is a projection such that there is an isometry V with $V V = E$, then there is also an isometry W with $W W = 1\!\!1 \quad E :$

2.2 Remark Let $\mathrm{Ob}\, T \ni \; \mapsto V \in (;)$ be a choice of unitaries that satisfy

$$V \quad = V \quad V : \tag{1}$$

Note that (1) implies $V = 1\!\!1$, because $V = V = V \quad V = V V$. This choice allows to define from the subcategory T_C of T another subcategory T_C^0 of T satisfying the same properies as T_C. Indeed, put

$$(;)_C^0 := V \; (;)_C V \quad (;) \tag{2}$$

and the corresponding permutation structure ${}^0(;)$ for T_C^0 is given by

$$ {}^0(;) := V \quad V \quad (;) \quad (V \; V \;) : \tag{3}$$

It is easy to check that ^{0} satisfies the properties in (P.3). The corresponding conjugates R^0 are defined by

$$R^0 := V_{-} \; R \quad \text{and} \quad S^0 := {}^0(; \overline{})R^0 : \tag{4}$$

Then it is straightforward to verify the postulates (P.2)–(P.4) for the new subcategory.

The preceding remark suggests to define an equivalence relation between different subcategories of T:

2.3 Definition The subcategories $\mathcal{T}_C$ and $\mathcal{T}_C^0$, satisfying the postulates (P.2)–(P.4), are called equivalent, if there is an assignment

$$\mathrm{Ob}\,\mathcal{T} \ni \quad \to V \in (\;;\;); \text{ with } V \text{ unitary and } V \quad = V \quad V \;;$$

such that Eqs. (2),(3) and (4) hold.

2.4 Theorem (i) Let $\mathcal{T}$ satisfy the postulates (P.1)–(P.4) before. Then there exists a Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ of $\mathcal{A}$ with $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ such that $\mathcal{T}$ is isomorphic to the category of all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$.

(ii) Further let $\mathcal{T}_C, \mathcal{T}_C^0$ be two subcategories of $\mathcal{T}$ satisfying the postulates (P.2)–(P.4). Then the corresponding Hilbert extensions are $\mathcal{A}$–module isomorphic iff $\mathcal{T}_C$ and $\mathcal{T}_C^0$ are equivalent.

3 Proof of Theorem 2.4

In the present section we will give a (constructive) proof of the previous theorem. For this purpose we will use well–known results already stated in [10, 1, 4] as well as in [13, Sections 2,3].

First we study properties of $\mathcal{T}$ implied by the existence of the subcategory $\mathcal{T}_C$, in particular we introduce the notion of irreducibility in the context of $\mathcal{T}$ and prove the decomposition theorem.

3.1 Irreducibility and decomposition theorem

Since the C*–algebra $\mathcal{A}$ has a nontrivial center $\mathcal{Z}$ it is immediate that one needs to extend the notion of irreducible objects to the category $\mathcal{T}$ (cf. e.g. [4, Section 5]). We propose

3.1 Definition $\in \mathrm{Ob}\,\mathcal{T}$ is called irreducible if $(\;;\;) = \mathcal{Z}$. We denote the set of all irreducible objects of $\mathcal{T}$ by $\mathrm{Irr}\,\mathcal{T}$ and by $\mathrm{Irr}_0\,\mathcal{T}$ a complete system of irreducible and mutually disjoint objects of $\mathcal{T}$.

Note that irreducibility of in the sense of Definition 3.1 and of in the usual sense as an object in $\mathcal{T}_C$ coincide, because $\in \mathrm{Irr}\,\mathcal{T}$ iff $(\;;\;)_C = \mathbb{C}\mathbb{1}$. We state some further consequences of this definition.

3.2 Lemma (I) If $\;;\; \in \mathrm{Irr}\,\mathcal{T}$, then either is unitarily equivalent to or they are disjoint (i.e. $(\;;\;) = \{0\}$).

(II) The following properties are equivalent:

(i) is irreducible,

(ii) $\overline{}$ is irreducible,

(iii) $(\;;\;) = (\mathcal{Z})$,

(iv) $(\;;\overline{}\;) = R\,\mathcal{Z}$, where R is a conjugate according to property (P.4).

(III) If is irreducible, then $(\;;\;)_C$ is an algebraic Hilbert space in $\mathcal{A}$ for each $\in \mathrm{Ob}\,\mathcal{T}$ and $(\;;\;) = (\;;\;)_C\,\mathcal{Z}$ is a right–$\mathcal{Z}$–Hilbert module with the scalar product $\langle X, Y\rangle := X\;\; Y$:

(IV) is irreducible iff there is no subobject of .

Proof: (I) Let $(\; ; \;)$ $\{0\}$, so that the inclusion $(\; ; \;)_C$ $\{0\}$ is also proper. Then it is straightforward to construct a unitary $U \in (\; ; \;)_C$.

(II) The proof uses the vector space isomorphisms between intertwiner spaces (see, for example [13, Lemma 2.1]) together with the link between conjugates and permutation given by assumption (P.4.2). Namely, the latter implies that $(\; ; \;) = \; (Z) \; i \; (\bar{\ } ; \bar{\ }) = \bar{\ }(Z)$ for all $\in \mathrm{Ob}\,T$, while the vector space isomorphisms yield $(\; ; \;) = Z \; i \; (\bar{\ } ; \bar{\ }) = \bar{\ }(Z)$:

(III) It follows immediately from (P.2.3)–(P.2.5).

(IV) If is irreducible, then it is straightforward to see that any isometry $W \in (\; ; \;)$ is actually a unitary, because $W W = : E \in (\; ; \;)$, hence $E = 1\!\!1$ by Remark 2.1 (ii). To show the reverse implication assume that is not irreducible. Now the estimate

$$A \quad (R \; R)^2 \quad (A); \quad 0 \quad A \in (\; ; \;)_C ;$$

where denotes the corresponding left inverse (see for example [13, Lemma 2.7]), implies that the C*–algebra $(\; ; \;)_C$ $C1\!\!1$ is finite–dimensional, hence must have proper subobjects (see e.g. [6, Lemma 11.1.27 and 11.1.29]). The crucial fact is that $(A) \in C1\!\!1$ for $A \in (\; ; \;)_C$. ■

The previous lemma implies in particular that the restriction of irreducible objects to Z are automorphisms of Z.

3.3 Corollary For every $\in \mathrm{Irr}\,T$ one has that $:= \restriction Z \in \mathrm{aut}\,Z$. Then, according to Gelfand's theorem, there exist corresponding homeomorphisms of $\mathrm{spec}\,Z$, denoted by $f \in C(\mathrm{spec}\,Z)$ which are given by $(Z)(\;) = Z(f^{\;1}(\;))$; $Z \in Z$; $\in \mathrm{spec}\,Z$.

3.4 Remark Corollary 3.3 shows that for an irreducible the second possibility considered in [4, Remark 5.5] of a proper inclusion Z (Z) is actually not realized.

We define the dimension of any object $\in \mathrm{Ob}\,T$ in the usual way by $d(\;)1\!\!1 := R \; R$; which satisfies the standard properties of multiplicativity, additivity etc. (recall that $R \in (\; ; \bar{\ } \;)$ and $d(\;) > 0$). Now using the DR–theory for T_C (cf. [10, Sections 2,3]) one arrives at the crucial decomposition statement for objects.

3.5 Proposition Let $\in \mathrm{Ob}\,T$. Then

(I) $d(\;) \in \mathbb{N}$;

(II) $= \bigoplus_{j=1}^{r} \bigoplus_{l=1}^{m(\; j; \;)} \; {}_{jl}$, with ${}_{jl} := {}_j \in \mathrm{Irr}_0\,T$; $l = 1;2;\ldots m(\; j; \;)$ and $d(\;) = \sum_{j=1}^{r} m(\; j; \;)d(\; j)$, where $m(\; ; \;) := \dim(\; ; \;)_C$.

(III) If $;$ $\in \mathrm{Ob}\,T$, then is unitarily equivalent to $i \; m(\; ; \;) = m(\; ; \;)$ for all .

3.6 Remark Statement (II) in Proposition 3.5 means explicitly

$$(\;) = \sum_{;j} W_{\;j} \; (\;) W_{\;j} ;$$

where $W_{\;j} \in (\; j; \;)_C$, $W_{\;j} W_{\;j} = 1\!\!1$, $W_{\;j} W_{\;0 j^0} = 0$, for $(\; ;j) \neq (\; ^0;j^0)$ and $\sum_{;j} W_{\;j} W_{\;j} = 1\!\!1$. Now $\{W_{\;j}\}_j$ is an orthonormal basis of the Hilbert module $(\; ; \;)$ and this implies that every orthonormal basis of the Hilbert module $(\; ; \;)$ can be used in the decomposition formula for . Note that the Hilbert modules $(\; ; \;)$ for $\in \mathrm{Irr}_0\,T$ are mutually orthogonal in A.

3.2 Construction of the Hilbert extension $\{\mathcal{F},\mathcal{G}\}$

In this subsection we will prove part (i) of Theorem 2.4 by constructing the Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$ that satisfies $\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}$.

We can proceed following the strategy already presented in [1, Sections 3-6]. To each $\rho\in\mathrm{Irr}_0\,\mathcal{T}$ we assign a Hilbert space $\mathcal{H}_\rho$ with $\dim\mathcal{H}_\rho=d(\rho)$ and, using orthonormal bases $\{\Psi_{\rho j}\}_j$ of $\mathcal{H}_\rho$, we define the $\mathcal{A}$–left module

$$\mathcal{F}_0 := \Big\{\sum_{\rho,j} A_{\rho j}\Psi_{\rho j} \;\Big|\; A_{\rho j}\in\mathcal{A};\ \text{finite sum}\Big\},$$

where the $\{\Psi_{\rho j}\}_{\rho j}$ form an $\mathcal{A}$–module basis of $\mathcal{F}_0$. $\mathcal{F}_0$ is independent of the special choice of the bases $\{\Psi_{\rho j}\}_j$ of $\mathcal{H}_\rho$ and putting $\Psi_{\rho j}A:=\rho(A)\Psi_{\rho j}$, $\mathcal{F}_0$ turns out to be a bimodule.

Further we define Hilbert spaces (recall that $\rho\prec\sigma$ means ρ is a subobject of σ).

$$\mathcal{H}_\sigma := \bigoplus_{\rho\prec\sigma}(\rho,\sigma)_{\mathcal{C}}\,\mathcal{H}_\rho \quad\text{and}\quad \mathcal{H}_\sigma\subset\mathcal{F}_0;\quad \sigma\in\mathrm{Ob}\,\mathcal{T}\,; \tag{5}$$

as well as the right–$\mathcal{Z}$–Hilbert modules

$$\boldsymbol{\mathcal{H}}_\rho := \mathcal{H}_\rho\mathcal{Z}=\mathcal{Z}\mathcal{H}_\rho \quad\text{and}\quad \boldsymbol{\mathcal{H}}_\sigma := \bigoplus_{\rho\prec\sigma}(\rho,\sigma)\boldsymbol{\mathcal{H}}_\rho=\mathcal{H}_\sigma\mathcal{Z}\,;$$

with the corresponding $\mathcal{Z}$–scalar product

$$\langle X,Y\rangle := \sum_{\rho,j}\rho^{-1}(X_{\rho j}^*Y_{\rho j});\ \text{where}$$

$$X=\sum_{\rho,j}X_{\rho j}\Psi_{\rho j};\ X_{\rho j}\in(\rho,\sigma);\ Y=\sum_{\rho,j}Y_{\rho j}\Psi_{\rho j};\ Y_{\rho j}\in(\rho,\sigma).$$

The preceding comments show that we have established the following functor $\mathbf{F}$ between the categories $\mathcal{T}$ (resp. $\mathcal{T}_{\mathcal{C}}$) and the corresponding category of Hilbert $\mathcal{Z}$–modules (resp. Hilbert spaces); (cf. e.g. [4, Section 4] and [1, Corollary 3.3]).

3.7 Lemma The functor $\mathbf{F}$ given by

$$\mathrm{Ob}\,\mathcal{T}\ni\sigma\mapsto\boldsymbol{\mathcal{H}}_\sigma\subset\mathcal{F}_0 \quad\text{and}\quad (\sigma,\tau)\ni A\mapsto\mathbf{F}(A)\in\mathcal{L}_{\mathcal{Z}}(\boldsymbol{\mathcal{H}}_\sigma\to\boldsymbol{\mathcal{H}}_\tau)\,;$$

where $\mathbf{F}(A)X:=AX$, $X\in\boldsymbol{\mathcal{H}}_\sigma$, defines an isomorphism between the corresponding categories and $\mathbf{F}(A^*)$ is the module adjoint w.r.t. $\langle\cdot,\cdot\rangle$. Similarly, one can apply $\mathbf{F}$ to $\mathcal{T}_{\mathcal{C}}$ in order to obtain the associated subcategory of algebraic Hilbert spaces $\mathcal{H}_\sigma$ and arrows $\mathbf{F}((\sigma,\tau)_{\mathcal{C}})\subset\mathcal{L}(\mathcal{H}_\sigma\to\mathcal{H}_\tau)$.

Proof: Similar as in [4, p. 791 f]. ∎

Now we can apply the results in [1] to the subcategory $\mathbf{F}(\mathcal{T}_{\mathcal{C}})$, in order to enrich gradually the structure of $\mathcal{F}_0$:

3.8 Lemma There exists a product structure on $\mathcal{F}_0$ with the properties

$$\mathrm{span}\,\{\Phi\Psi\mid\Phi\in\mathcal{H}_\sigma,\ \Psi\in\mathcal{H}_\tau\}=\mathcal{H}_{\sigma\tau}\,;$$

$$\epsilon(\sigma,\tau)\Phi\Psi=\Psi\Phi\,;\quad \Phi\in\mathcal{H}_\sigma\,;\ \Psi\in\mathcal{H}_\tau\,;$$

$$\langle XY,X'Y'\rangle=\langle X,X'\rangle\,\langle Y,Y'\rangle\,;\quad X,X'\in\boldsymbol{\mathcal{H}}_\sigma\,;\ Y,Y'\in\boldsymbol{\mathcal{H}}_\tau\,.$$

Note that for orthonormal bases $\{ _j\}_j, \{ _k\}_k$ of $H \; ; H$, respectively, we obtain from Lemma 3.8 that

$$(\; ; \;) = \sum_{j,k} \quad _k \; _j \; _k \; _j \, .$$

As in [1, Section 5] we introduce the notion of a conjugated basis $\overline{\ }_j$ of $H_{\overline{\ }}$ w.r.t. an orthonormal basis $_j$ of H such that $R = \sum_j \; _j \; \overline{\ }_j \; _j$. This is necessary in order to put a compatible *–structure on F_0.

3.9 Lemma Let $\overline{\ }_{,j}$ be a conjugated basis corresponding to the basis $_{,j}$; $\in \mathrm{Irr}_0 T$, and define $_j := R \; \overline{\ }_j$; $j = 1, 2, \ldots, d( )$. Then F_0 turns into a *–algebra. The Hilbert spaces H and the corresponding modules H are algebraic, i.e.

$$\langle X, Y \rangle = X \; Y; \quad X, Y \in H \; .$$

The objects $\in \mathrm{Ob}\, T$ are identified as canonical endomorphisms

$$(A) = \sum_{j=1}^{d( )} \; _j A \; _j \, .$$

In F_0 one has natural projections onto the –component of the decomposition:

$$(\sum_{,j} \; A \; _j \; _j) := \sum_{j=1}^{d( )} A \; _j \; _j ; \quad \in \mathrm{Irr}_0 T \, .$$

To put a C*–norm $\| \; \|$ we argue as in [1, Section 6]. Its construction is essentially based on the following A–scalar product on F_0

$$\langle F, G \rangle := \sum_{,j} \frac{1}{d( )} A \; _j B \; _j ; \text{ for } F := \sum_{,j} A \; _j \; _j ; G := \sum_{,j} B \; _j \; _j ;$$

3.10 Lemma The scalar product $\langle \; , \; \rangle$ satisfies $\langle F, G \rangle = F \; G$ and is selfadjoint w.r.t. $\langle \; , \; \rangle$. The projections and the scalar product have continuous extensions to $F := \mathrm{clo}_{\| \|} F_0$ and $F = \mathrm{span}\{A H \}$.

Finally, the symmetry group w.r.t. $\langle \; , \; \rangle$ is defined by the subgroup of all automorphisms $g \in \mathrm{aut} F$ satisfying $\langle gF_1, gF_2 \rangle = \langle F_1, F_2 \rangle$. It leads to

3.11 Lemma The symmetry group coincides with the stabilizer stab A of A and the modules H are invariant w.r.t. stab A.

Proof: Use [4, Lemma 7.1] (cf. also with the arguments given in [1, Section 6]). ∎

This suggests to consider the subgroup $G \subset$ stab A consisting of all elements of stab A leaving even the Hilbert spaces H invariant. Then it turns out that the pair $\{F, G\}$ satisfies the properties needed to prove Theorem 2.4, i.e. $\{F, G\}$ is a Hilbert extension of A. The following result concludes the proof of part (i) of Theorem 2.4.

3.12 Lemma G is compact and the spectrum spec$G$ on F coincides with the dual $\hat{G}$. For $\in \mathrm{Irr}\, T$ the Hilbert spaces H are irreducible w.r.t. G; i.e. there is a bijection $\mathrm{Irr}_0 T \ni \; \leftrightarrow \; D \in \hat{G}$. Moreover A coincides with the fixed point algebra of the action of G in F and $A' \cap F = Z$.

3.13 Remark (i) From [4, Section 7] it follows that stab A is in general not compact.

(ii) The characterization of stab A given in [4, Theorem 7.11] in terms of functions contained in $C(\mathrm{spec}Z \to G)$ is in general not correct, although in some special cases like the one–dimensional torus $G := T$ it is true that stab $A = C(\mathrm{spec}Z \to T)$ (cf. [3]). It is though possible to give a similar characterization of stab A in terms of functions contained in $C(\mathrm{spec}Z \to \mathrm{Mat}(C))$.

3.3 Uniqueness result

Now we prove part (ii) of Theorem 2.4. First assume that the subcategories T_C and T_C^0 are equivalent. We consider the Hilbert extension F assigned to T_C. The corresponding invariant Hilbert spaces are given by (5). Now we change these Hilbert spaces by

$$H \to V H =: H^0 :$$

Using the function F of Lemma 3.7 so that $L_G(H \to H) := F((\ ;\ )_C) = (\ ;\ )_C$ we obtain

$$L_G(V H \to V H) = V L_G(H \to H) V = (\ ;\ )_C^0 : \tag{6}$$

Further, w.r.t. the "new Hilbert spaces" we obtain the 'primed' permutators and conjugates of the second subcategory. This means, it is sufficient to prove that if the subcategory T_C is given, then two Hilbert extensions, assigned to $(T; T_C)$ according to the first part of the theorem, are always A–module isomorphic. Now let $F_1; F_2$ be two Hilbert extensions assigned to T_C. For $\in \mathrm{Irr}_0 T$ let $f\ ^1_j g_j, f\ ^2_j g_k$ be orthonormal bases of the Hilbert spaces $H^1; H^2$, respectively. Then

$$ {}^r_j\ {}^r_k = \sum_{;l} K\ ^l_{jk}\ {}^r_l; \quad K\ ^l_{jk} \in (\ ;\ )_C; \ r = 1; 2:$$

Therefore the definition

$$J\Big(\sum_{;j} A_j\ ^1_j\Big) := \sum_{;j} A_j\ ^2_j$$

is easily seen to extend to an A–module isomorphism from F_1 onto F_2 (see [6, p. 203 ff.]).

Second, we assume that the Hilbert extensions $F_1; F_2$ assigned to $T_C^1; T_C^2$, respectively, are A–module isomorphic. The G–invariant Hilbert spaces are given by (5). Now let J be an A–module isomorphism $J : F_1 \to F_2$ so that

$$J(H^1) = \sum_{<} (\ ;\ )_C^1 J(H^1)$$

and again the $J(H^1)$ form a system of G–invariant Hilbert spaces in F_2. Further we have the system H^2 in F_2. That is, to each we obtain two G–invariant Hilbert spaces H^2 and $J(H^1)$ that are contained in the Hilbert module H^2. Let $f\ _{;j} g_j, f\ _{;j} g_j$ be orthonormal bases of $J(H^1)$, H^2, respectively. Then obviously $V := \sum_j\ _{;j}\ _{;j}$ is a unitary with $V \in (\ ;\ )$ and $H^2 = V J(H^1)$. Further, for $X \in H^1$, $Y \in H^1$ (hence $X Y \in H^1$) we have

$$V J(X) V J(Y) = V (V) J(XY) = V J(XY);$$

and this implies $V = V V$. Finally, we argue as in (6) to obtain

$$V (\ ;\ )_C^1 V = (\ ;\ )_C^2 :$$

and the latter equation implies Eqs. (2)–(4).

4 Conclusions

In the present paper we present the solution of the problem of finding the unique (up to $\mathcal{A}$–module isomorphy) Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ of a unital C*–algebra $\mathcal{A}$ with nontrivial center $\mathcal{Z}$, given a suitable endomorphism category $\mathcal{T}$ of $\mathcal{A}$, which we characterize in Section 2. The extension satisfies $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ and the essential step for its construction is the specification of a subcategory $\mathcal{T}_C$ of $\mathcal{T}$, which is of the well–known DR–type. From the point of view of the DR–theory the appearence of the subcategory $\mathcal{T}_C \subset \mathcal{T}$ is quite natural, since the group appearing in the extension is still compact. There are several directions in which the present results could be generalized. First, we hope that the inclusion situation $\mathcal{T}_C \subset \mathcal{T}$ may also be relevant for braided tensor categories, since in this context there are 2–dimensional physically relevant models where a nontrivial center appears (see e.g. [7, 12]). Second, one could try to find extensions, where the condition on the relative commutant $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ (which is crucial for our approach) is not satisfied anymore. In this context non unitarily equivalent irreducible endomorphisms will no longer be disjoint and one needs probably to replace the free modules $\mathcal{H}$ that appeared in our approach by more general C*–Hilbert modules. Finally, we hope that the present results as well as those in [4] will motivate a more systematic study of the representation theory of (say compact) groups over Hilbert C*–modules.

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