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Asymptotically Flat Initial Data with Prescribed Regularity at Infinity

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Abstract: We prove the existence of a large class of asymptotically flat initial data with non-vanishing mass and angular momentum for which the metric and the extrinsic curvature have asymptotic expansions at space-like infinity in terms of powers of a radial coordinate.

1. Introduction

An initial data set for the Einstein vacuum equations is given by a triple $(\tilde{S}, \tilde{h}_{ab}, \tilde{\Psi}_{ab})$, where $\tilde{S}$ is a connected 3-dimensional manifold, $\tilde{h}_{ab}$ a (positive definite) Riemannian metric, and $\tilde{\Psi}_{ab}$ a symmetric tensor field on $\tilde{S}$. The data will be called "asymptotically flat", if the complement of a compact set in $\tilde{S}$ can be mapped by a coordinate system $\tilde{x}^j$ diffeomorphically onto the complement of a closed ball in $\mathbb{R}^3$ such that we have in these coordinates

$$\tilde{h}_{ij} = \left(1 + \frac{2m}{\tilde{r}}\right)\delta_{ij} + O\left(\tilde{r}^{-2}\right), \tag{1}$$

$$\tilde{\Psi}_{ij} = O\left(\tilde{r}^{-2}\right), \tag{2}$$

as $\tilde{r} = (\sum_{j=1}^{3}(\tilde{x}^j)^2)^{1/2} \to \infty$. Here the constant m denotes the mass of the data, $a, b, c...$ denote abstract indices, $i, j, k...$, which take values 1, 2, 3, denote coordinates indices while δ_{ij} denotes the flat metric with respect to the given coordinate system $\tilde{x}^j$. Tensor indices will be moved with the metric h_{ab} and its inverse h^{ab}. We set $x_i = x^i$ and $\partial^i = \partial_i$. Our conditions guarantee that the mass, the momentum, and the angular momentum of the initial data set are well defined.

There exist weaker notions of asymptotic flatness (cf. [14]) but they are not useful for our present purpose. In this article we show the existence of a class of asymptotically flat initial data which have a more controlled asymptotic behavior than (1), (2) in the sense that they admit near space-like infinity asymptotic expansions of the form

$$\tilde{h}_{ij} \sim (1 + \frac{2m}{\tilde{r}})\delta_{ij} + \sum_{k\geq 2} \frac{\tilde{h}^k_{ij}}{\tilde{r}^k}, \tag{3}$$

$$\tilde{\Psi}_{ij} \sim \sum_{k\geq 2} \frac{\tilde{\Psi}^k_{ij}}{\tilde{r}^k}, \tag{4}$$

where $\tilde{h}^k_{ij}$ and $\tilde{\Psi}^k_{ij}$ are smooth functions on the unit 2-sphere (thought of as being pulled back to the spheres $\tilde{r} = const.$ under the map $\tilde{x}^j \to \tilde{x}^j/\tilde{r}$).

We are interested in such data for two reasons. The evolution of asymptotically flat initial data near space-like and null infinity has been studied in considerable detail in [23]. In particular the that article a certain "regularity condition" has been derived on the data near space-like infinity, which is expected to provide a criterion for the existence of a smooth asymptotic structure at null infinity. To simplify the lengthy calculations, the data considered in [23] have been assumed to be time-symmetric and to admit a smooth conformal compactification. With these assumptions the regularity condition is given by a surprisingly succinct expression. With the present work we want to provide data which will allow us to perform the analysis of [23] without the assumption of time symmetry but which are still "simple" enough to simplify the work of generalizing the regularity condition to the case of the non-trivial second fundamental form.

Thus we will insist in the present paper on the smooth conformal compactification of the metric but drop the time symmetry. A subsequent article will be devoted to the analysis of a class of more general data which will include in particular stationary asymptotically flat data.

The "regular finite initial value problem near space-like infinity", formulated and analyzed in [23], suggests how to calculate numerically entire asymptotically flat solutions to Einstein's vacuum field equations on finite grids. In the present article we provide data for such numerical calculations which should allow us to study interesting situations while keeping a certain simplicity in the handling of the initial data.

The difficulty of constructing data with the asymptotic behavior (3), (4) arises from the fact that the fields need to satisfy the constraint equations

$$\tilde{D}^b\tilde{\Psi}_{ab} - \tilde{D}_a\tilde{\Psi} = 0,$$
$$\tilde{R} + \tilde{\Psi}^2 - \tilde{\Psi}_{ab}\tilde{\Psi}^{ab} = 0,$$

on $\tilde{S}$, where $\tilde{D}_a$ is the covariant derivative, $\tilde{R}$ is the trace of the corresponding Ricci tensor, and $\tilde{\Psi} = \tilde{h}^{ab}\tilde{\Psi}_{ab}$. Part of the data, the "free data", can be given such that they are compatible with (3), (4). However, the remaining data are governed by elliptic equations and we have to show that (3), (4) are in fact a consequence of the equations and the way the free data have been prescribed.

To employ the standard techniques to provide solutions to the constraints, we assume

$$\tilde{\Psi} = 0, \tag{5}$$

such that the data correspond to a hypersurface which is maximal in the solution space-time.

We give an outline of our results. Because of the applications indicated above, we wish to control in detail the conformal structure of the data near space-like infinity. Therefore we shall analyze the data in terms of the conformal compactification (S, h_{ab}, Ψ_{ab}) of the "physical" asymptotically flat data. Here S denotes a smooth, connected, orientable, compact 3-manifold. It contains a point i such that we can write $\tilde{S} = S \setminus \{i\}$. The point i will represent, in a sense described in detail below, space-like infinity for the physical initial data.

By singling out more points in S and by treating the fields near these points in the same way as near i we could construct data with several asymptotically flat ends, since all the following arguments equally apply to such situations. However, for convenience we restrict ourselves to the case of a single asymptotically flat end.

We assume that h_{ab} is a positive definite metric on S with covariant derivative D_a and Ψ_{ab} is a symmetric tensor field which is smooth on $\tilde{S}$. In agreement with (5) we shall assume that Ψ_{ab} is trace free,

$$h^{ab}\,\Psi_{ab} = 0.$$

The fields above are related to the physical fields by rescaling

$$\tilde{h}_{ab} = \theta^4\,h_{ab}, \quad \tilde{\Psi}_{ab} = \theta^{-2}\,\Psi_{ab}, \tag{6}$$

with a conformal factor θ which is positive on $\tilde{S}$. For the physical fields to satisfy the vacuum constraints we need to assume that

$$D^a\Psi_{ab} = 0 \quad \text{on} \quad \tilde{S}, \tag{7}$$

$$(D_b D^b - \frac{1}{8}R)\theta = -\frac{1}{8}\Psi_{ab}\Psi^{ab}\theta^{-7} \quad \text{on} \quad \tilde{S}. \tag{8}$$

Equation (8) for the conformal factor θ is the Lichnerowicz equation, transferred to our context.

Let x^j be h-normal coordinates centered at i such that $h_{kl} = \delta_{kl}$ at i and set $r = (\sum_{i=1}^{3}(x^j)^2)^{1/2}$. To ensure asymptotic flatness of the data (6) we require

$$\Psi_{ab} = O(r^{-4}) \quad \text{as} \quad r \to 0, \tag{9}$$

$$\lim_{r\to 0} r\theta = 1. \tag{10}$$

In the coordinates $\tilde{x}^j = x^j/r^2$ the fields (6) will then satisfy (1), (2) (cf. [22,23] for this procedure).

Not all data as given by (6), which are derived from data h_{ab}, Ψ_{ab} as described above, will satisfy conditions (3), (4). We will have to impose extra conditions and we want to keep these conditions as simple as possible.

Since we assume the metric h_{ab} to be smooth on S, it will only depend on the behavior of θ near i whether condition (3) will be satisfied. Via Eq. (8) this behavior depends on Ψ_{ab}. What kind of condition do we have to impose on Ψ_{ab} in order to achieve (3)?

The following space of functions will play an important role in our discussion. Denote by B_a the open ball with center i and radius $r = a > 0$, where a is chosen small enough such that B_a is a convex normal neighborhood of i. A function $f \in C^\infty(\tilde{S})$ is said to be in $E^\infty(B_a)$ if on B_a we can write $f = f_1 + r f_2$ with $f_1, f_2 \in C^\infty(B_a)$ (cf. Definition 1). An answer to our question is given by the following theorem:

Theorem 1. *Let h_{ab} be a smooth metric on S with positive Ricci scalar R. Assume that Ψ_{ab} is smooth in $\tilde{S}$ and satisfies on B_a,*

$$r^8 \Psi_{ab}\Psi^{ab} \in E^\infty(B_a). \tag{11}$$

Then there exists on $\tilde{S}$ a unique solution θ of Eq. (8), which is positive, satisfies (10), and has in B_a the form

$$\theta = \frac{\hat{\theta}}{r}, \quad \hat{\theta} \in E^\infty(B_a), \quad \hat{\theta}(i) = 1. \tag{12}$$

In fact, we will get slightly more detailed information. We find that $\hat{\theta} = u_1 + r\,u_2$ on B_a with $u_2 \in E^\infty(B_a)$ and a function $u_1 \in C^\infty(B_a)$ which satisfies $u_1 = 1 + O(r^2)$ and

$$\left(D_b D^b - \frac{1}{8}R\right)\frac{u_1}{r} = \theta_R,$$

in $B_a \setminus \{i\}$, where θ_R is in $C^\infty(B_a)$ and vanishes at any order at i.

If θ has the form (12) then (3) will be satisfied due to our assumptions on h_{ab}.

Note the simplicity of condition (11). To allow for later generalizations, we shall discuss below the existence of the solution θ under weaker assumptions on the smoothness of the metric h_{ab} and the smoothness and asymptotic behavior of Ψ_{ab} (cf. Theorem 12). In fact, already the methods used in this article would allow us to deduce analogues of all our results under weaker differentiability assumptions; however, we are particularly interested in the C^∞ case because it will be convenient in our intended applications. If the metric is analytic on B_a it can be arranged that $\theta_R = 0$ and u_1 is analytic on B_a (and unique with this property, see [24] and the remark after Theorem 2). We finally note that the requirement $R > 0$, which ensures the solvability of the Lichnerowicz equation, could be reformulated in terms of a condition on the Yamabe number (cf. [29]).

It remains to be shown that condition (11) can be satisfied by tensor fields Ψ_{ab} which satisfy (7), (9). A special class of such solutions, namely those which extend smoothly to all of S, can easily be obtained by known techniques (cf. [16]). However, in that case the initial data will have vanishing momentum and angular momentum. To obtain data without this restriction, we have to consider fields $\Psi_{ab} \in C^\infty(\tilde{S})$ which are singular at i in the sense that they admit, in accordance with (2), (6), (10), at $i = \{r = 0\}$ asymptotic expansions of the form

$$\Psi_{ij} \sim \sum_{k \geq -4} \Psi^k_{ij} r^k \quad \text{with} \quad \Psi^k_{ij} \in C^\infty(S^2). \tag{13}$$

It turns out that condition (11) excludes data with non-vanishing linear momentum, which requires a non-vanishing leading order term of the form $O(r^{-4})$. In Sect. 3.4 we will show that such terms imply terms of the form $\log r$ in θ and thus do not admit expansion of the form (3). However, this does not necessarily indicate that condition (11) is overly restrictive. In the case where the metric h_{ab} is smooth it will be shown in Sect. 3.4 that a non-vanishing linear momentum always comes with logarithmic terms, irrespective of whether condition (11) is imposed or not.

There remains the question whether there exist fields Ψ_{ab} which satisfy (11) and have non-trivial angular momentum. The latter requires a term of the form $O(r^{-3})$ in (13). It turns out that condition (11) fixes this term to be of the form

$$\Psi^{AJ}_{ij} = \frac{A}{r^3}(3n_i n_j - \delta_{ij}) + \frac{3}{r^3}(n_j \epsilon_{kil} J^l n^k + n_i \epsilon_{ljk} J^k n^l), \tag{14}$$

where $n^i = x^i/r$ is the radial unit normal vector field near i and J^k, A are constants, the three constants J^k specifying the angular momentum of the data. The spherically symmetric tensor which appears here with the factor A agrees with the extrinsic curvature for a maximal (non-time symmetric) slice in the Schwarzschild solution (see for example [10]). Note that the tensor Ψ^{AJ}_{ij} satisfies condition (11) and the equation $\partial^i \Psi^{AJ}_{ij} = 0$ on $\tilde{S}$ for the flat metric. In the next theorem we prove an analogous result for general smooth metrics.

Theorem 2. *Let h_{ab} be a smooth metric in S. There exist trace-free tensor fields $\Psi_{ab} \in C^\infty(S \setminus \{i\})$ satisfying (13) with the following properties:*

(i) $\Psi_{ab} = \Psi^{AJ}_{ab} + \hat{\Psi}_{ab}$, *where Ψ^{AJ}_{ab} is given by (14) and $\hat{\Psi}_{ab} = O(r^{-2})$.*
(ii) $D^a \Psi_{ab} = 0$ *on $\tilde{S}$.*
(iii) $r^8 \Psi_{ab} \Psi^{ab}$ *satisfies condition (11).*

We prove a more detailed version of this theorem in Sect. (4.3). There it will be shown how to construct such solutions from free-data by using the York splitting technique ([35]). In Sect. 4.1 the case where h_{ab} is conformal to the Euclidean metric is studied in all generality.

2. Preliminaries

In this section we collect some known facts from functional analysis and the theory of linear elliptic partial differential equations.

Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}_0$ the set of non-negative integers. We use multi-indices $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n$ and set $|\beta| = \sum_{i=1}^n \beta_i$, $\beta! = \beta_1!\beta_2! \ldots \beta_n!$, $x^\beta = (x^1)^{\beta_1}(x^2)^{\beta_2} \ldots (x^n)^{\beta_n}$, $\partial^\beta u = \partial_{x^1}^{\beta_1} \partial_{x^2}^{\beta_2} \ldots \partial_{x^n}^{\beta_n} u$, $D^\beta u = D_1^{\beta_1} D_2^{\beta_2} \ldots D_n^{\beta_n} u$, and, for $\beta, \gamma \in \mathbb{N}_0^n$, $\beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n)$ and $\beta \leq \gamma$ if $\beta_i \leq \gamma_i$. We denote by Ω an open domain in $\mathbb{R}^3$ (resp. in S; quite often we will then choose $\Omega = B_a$).

We shall use the following functions spaces (see [1,25] for definitions, notations, and results): the set of m times continuously differentiable functions $C^m(\Omega)$, the Hölder space $C^{m,\alpha}(\Omega)$, where $0 < \alpha < 1$, the corresponding spaces $C^m(\bar{\Omega})$, $C^{m,\alpha}(\bar{\Omega})$, the space $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω, the Lebesgue space $L^p(\Omega)$, the Sobolev space $W^{m,p}(\Omega)$, and the local Sobolev space $W^{m,p}_{loc}(\Omega)$. For a compact manifold S we can also define analogous spaces $L^p(S)$, $C^{m,\alpha}(S)$, $W^{m,p}(S)$ (cf. [6]).

We shall need the following relations between these spaces.

Theorem 3 (Sobolev imbedding). *Let Ω be a $C^{0,1}$ domain in $\mathbb{R}^3$, let k, m, j be non-negative integers and $1 \leq p, q < \infty$. Then there exist the following imbeddings:*

(i) *If $mp < 3$, then*

$$W^{j+m,p}(\Omega) \subset W^{j,q}(\Omega), \quad p \leq q \leq 3p/(3 - mp).$$

(ii) *If* $(m-1)p < 3 < mp$, *then*

$$W^{j+m,p}(\Omega) \subset C^{j,\alpha}(\bar{\Omega}), \quad \alpha = m - 3/p.$$

Theorem 4. *Let* $u \in W^{1,1}(\Omega)$, *and suppose there exist positive constants* $\alpha \leq 1$ *and* K *such that*

$$\int_{B_R} |\partial u|\, d\mu \leq K R^{2+\alpha} \quad \textit{for all balls} \quad B_R \subset \Omega \quad \textit{of radius} \quad R > 0.$$

Then $u \in C^{\alpha}(\Omega)$.

Our existence proof for the non-linear equations relies on the following version of the compact imbedding for compact manifolds [6]:

Theorem 5 (Rellich–Kondrakov). *The following imbeddings are compact*

(i) $W^{m,q}(S) \subset L^p(S)$ *if* $1 \geq 1/p > 1/q - m/3 > 0$.
(ii) $W^{m,q}(S) \subset C^{\alpha}(S)$ *if* $m - \alpha > 3/q$, $0 \leq \alpha < 1$.

A further essential tool for the existence proof is the Schauder fixed point theorem [25]:

Theorem 6 (Schauder fixed point). *Let* B *be a closed convex set in a Banach space* V *and let* T *be a continuous mapping of* B *into itself such that the image* $T(B)$ *is precompact, i.e. has compact closure in* B. *Then* T *has a fixed point.*

We turn now to the theory of elliptic partial differential equations (see [12,14,25, 31]). Let $\mathbf{L}$ be a linear differential operator of order m on the compact manifold S which acts on tensor fields u. In the case where $u \sim u^{a_1 \dots a_{m_1}}$ is a contravariant tensor field of rank m_1, $\mathbf{L}$ has in local coordinates the form

$$\mathbf{L}u = \sum_{|\beta|=0}^{m} a^{j_1 \cdots j_{m_2}}{}_{i_1 \dots i_{m_1} \beta}\, D^{\beta}\, u^{i_1 \dots i_{m_1}} \equiv \sum_{|\beta|=0}^{m} a_{\beta}\, D^{\beta}\, u, \tag{15}$$

where the coefficients $a^{j_1 \cdots j_{m_2}}{}_{i_1 \dots i_{m_1} \beta} = a(x)^{j_1 \cdots j_{m_2}}{}_{i_1 \dots i_{m_1} \beta}$ are tensor fields of a certain smoothness, and D denotes the Levi–Civita connection with respect to the metric h. In the expression on the right-hand side we suppressed the indices belonging to the unknown and the target space. Assuming the same coordinates as above, we write for a given covector ξ_i at a point $x \in \Omega$ and multi-index β as usual $\xi^{\beta} = \xi_1^{\beta_1} \dots \xi_4^{\beta_4}$ and define a linear map $A(x, \xi) : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ by setting $(A(x, \xi)\, u)^{j_1 \cdots j_{m_2}} = \sum_{|\beta|=m} a(x)^{j_1 \cdots j_{m_2}}{}_{i_1 \dots i_{m_1}\, \beta}\, \xi^{\beta}\, u^{i_1 \dots i_{m_1}}$. The operator $\mathbf{L}$ is elliptic at x if for any $\xi \neq 0$ the map $A(x, \xi)$ is an isomorphism, $\mathbf{L}$ is elliptic on S if it is elliptic at all points of S.

We have the following L^p regularity result [2,3,14,31].

Theorem 7 (L^p regularity). *Let* $\mathbf{L}$ *be an elliptic operator of order* m *on* Ω *(resp.* S*) with coefficients* $a_{\beta} \in W^{s_{|\beta|}, p}(S)$, *where* $s_k > 3/p + k - m + 1$, *and* $p > 1$. *Let* s *be a natural number such that* $s_k \geq s - m \geq 0$. *Let* $u \in W^{m,p}_{\text{loc}}(\Omega)$ *(resp.* $W^{m,p}_{\text{loc}}(S)$*), with* $p > 1$, *be a solution of the elliptic equation* $\mathbf{L}u = f$.

(i) *If* $f \in W^{s-m,q}_{\text{loc}}(\Omega)$, $q \geq p$, *then* $u \in W^{s,q}_{\text{loc}}(\Omega)$.
(ii) *If* $f \in W^{s-m,q}(S)$, $q \geq p$, *then* $u \in W^{s,q}(S)$.

Furthermore, we have the Schauder interior elliptic regularity [3,19,25,31].

Theorem 8 (Schauder elliptic regularity). *Let* $\mathbf{L}$ *be an elliptic operator of order* m *on* Ω *with coefficients* $a_\beta \in C^{k,\alpha}(\bar{\Omega})$. *Let* $u \in W^{m,p}(\Omega)$, *with* $p > 1$, *be a solution of the elliptic equation* $\mathbf{L}u = f$, *with* $f \in C^{k,\alpha}(\bar{\Omega})$. *Then* $u \in C^{k+m,\alpha}(\Omega')$, *for all* $\Omega' \subset\subset \Omega$.

For linear elliptic equations we have the Fredholm alternative for elliptic operators on compact manifolds [12].

Theorem 9 (Fredholm alternative). *Let* $\mathbf{L}$ *be an elliptic operator of order* m *on* S *whose coefficients satisfy the hypothesis of Theorem 7. Let* s *be some natural number such that* $s_k \geq s - m \geq 0$ *and* $f \in L^p(S)$, $p > 1$. *Then the equation* $\mathbf{L}u = f$ *has a solution* $u \in W^{m,p}(S)$ *iff*

$$\int_S < v, f >_h d\mu = 0 \quad \textit{for all} \quad v \in \ker(\mathbf{L}^*).$$

Here $d\mu$ denotes the volume element determined by h and $\mathbf{L}^*$ the formal adjoint of $\mathbf{L}$, which for the operator (15) is given by

$$\mathbf{L}^* u = \sum_{|\beta|=0}^{m} (-1)^{|\beta|} D^\beta (a_\beta u). \tag{16}$$

Furthermore $<, >_h$ denotes the appropriate inner product induced by the metric h_{ab}. In our case, where u and f will be vector fields f^a and u^a, we have $< u, f >_h = f^a u_a$.

Let

$$Lu = \partial_i (a^{ij} \partial_j u + b^i u) + c^i \partial_i u + du, \tag{17}$$

be a linear elliptic operator of second order with principal part in divergence form on Ω which acts on scalar functions. An operator of the form (17) may be written in the form (15) provided its principal coefficients a^{ij} are differentiable.

We shall assume that L is strictly elliptic in Ω; that is, there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^n. \tag{18}$$

We also assume that L has bounded coefficients; that is for some constants Λ and $\nu \geq 0$ we have for all $x \in \Omega$,

$$\sum |a^{ij}|^2 \leq \Lambda^2, \quad \lambda^{-2} \sum \left(|b^i|^2 + |c^i|^2 \right) + \lambda^{-1} |d| \leq \nu^2. \tag{19}$$

In order to formulate the maximum principle, we have to impose that the coefficient of u satisfy the non-positivity condition

$$\int_\Omega (dv - b^i \partial_i v)\, dx \leq 0 \quad \forall v \geq 0,\ v \in C_0^1(\Omega). \tag{20}$$

We have the following versions of the maximum principle [25].

Theorem 10 (Weak Maximum Principle). *Assume that* L *given by (17) satisfies conditions (18), (19) and (20). Let* $u \in W^{1,2}(\Omega)$ *satisfy* $Lu \geq 0$ (≤ 0) *in* Ω. *Then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ \quad \left(\inf_\Omega u \geq \inf_{\partial\Omega} u^- \right),$$

where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \min\{u(x), 0\}$.

Theorem 11 (Strong Maximum Principle). *Assume that L given by (17) satisfies conditions (18), (19) and (20). Let $u \in W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ in Ω. Then, if for some ball $B \subset\subset \Omega$ we have*

$$\sup_B u = \sup_\Omega u \geq 0,$$

the function u must be constant in Ω.

Because u is assumed to be only in $W^{1,2}$ the inequality $Lu \geq 0$ has to be understood in the weak sense (see [25] for details).

3. The Hamiltonian Constraint

In this section we will prove Theorem 1.

3.1. Existence. The existence of solutions to the Lichnerowicz equation has been studied under various assumptions (cf. [15,16,28] and the reference given there). The setting outlined above, where we have to solve (8), (10) on the compact manifold S, has been studied in [8,22,23].

In general the "physical" metric provided by an asymptotically flat initial data set will not admit a smooth conformal compactification at space-like infinity. Explicit examples for such situations can be obtained by studying space-like slices of stationary solutions like the Kerr solution. To allow for later generalizations of the present work which would admit also stationary solutions we shall prove the existence result of Theorem 1 for metrics h_{ab} which are not necessarily smooth. In the proof we will employ Sobolev spaces $W^{m,p}(S)$ and the corresponding imbeddings and elliptic estimates (in particular, there will be no need for us to employ weighted Sobolev spaces with weights involving the distance to the point i). With these spaces and standard L^p elliptic theory we will also be able to handle the mild r^{-1}-type singularity at i which occurs on the right-hand side of Eq. (8).

The conformal Laplacian or Yamabe operator

$$L_h = h^{ab} D_a D_b - \frac{1}{8} R,$$

which appears on the left-hand side of (8), is a linear elliptic operator of second order whose coefficients depend on the derivatives of the metric h up to second order. The smoothness to be required of the metric h is determined by the following considerations. In the existence proof we need:

(i) The existence of normal coordinates. This suggests that we assume $h \in C^{1,1}(S)$.
(ii) The maximum principle, Theorems 10 and 11. The required boundedness of the Ricci scalar R imposes restrictions on the second derivative of h.
(iii) The elliptic L^p estimate, Theorem 7. This requires that $h \in W^{3,p}(S)$ for $p > 3/2$.

Since the right-hand side of Eq. (8) is in L^2 the assumption that $h \in W^{3,p}(S)$, $p > 3$, would be sufficient to handle Eq. (8). However, when we will discuss the momentum constraint in Sect. 4.2, we will wish to be able to handle cases where $p < 3/2$. In these cases the conditions of Theorem 7 suggest that we assume that $h \in W^{4,p}(S)$. In order to simplify our hypothesis we shall assume in the following that

$$h_{ab} \in W^{4,p}(S), \quad p > 3/2. \tag{21}$$

The imbedding theorems then imply that $h_{ab} \in C^{2,\alpha}(S)$, $0 < \alpha < 1$, whence $R \in C^{\alpha}(S)$. We note that (21) is not the weakest possible assumption but it will be sufficient for our future applications.

Lemma 1. *Assume that h satisfies (21) and $R > 0$. Then:*

(i) $L_h : W^{2,q}(S) \to L^q(S)$, $q > 1$, *defines an isomorphism.*
(ii) *If $u \in W^{1,2}(S)$ and $L_h u \leq 0$, then $u \geq 0$; if, moreover, $L_h u \neq 0 \in L^q(S)$, then $u > 0$.*

Proof. (i) To show injectivity, assume that $L_h u = 0$. By elliptic regularity u is smooth enough such that we can multiply this equation with u and integrate by parts to obtain

$$\int_S \left(D^a u D_a u + \frac{1}{8} R u^2 \right) d\mu_h = 0.$$

Since $R > 0$ it follows that $u = 0$. Surjectivity follows then by Theorem 9 since $L_h = L_h^*$. Boundedness of L_h is immediately implied by the assumptions while the inequality

$$||u||_{W^{2,p}(S)} \leq C ||L_h u||_{L^p(S)},$$

which follows from the elliptic estimates underlying Theorem 7 and the injectivity of L_h (see e.g [12] for this well known result), implies the boundedness of L_h^{-1}.

(ii) If we have $u \leq 0$ in some region of S, it follows that $\sup_S(-u) \geq 0$. Then there is a region in S in which we can apply the maximum principle to the function $-u$ to conclude that u must be a non-positive constant whence $L_h u = -Ru/8 \geq 0$ in that region. In the case where $L_h u < 0$ we would arrive at a contradiction. In the case where $L_h u \leq 0$ we conclude that $u = 0$ in the given region and a repetition of the argument gives the desired result. □

To construct an approximate solution we choose normal coordinates x^j centered at i such that (after a suitable choice of $a > 0$) we have in the open ball B_a in these coordinates

$$h_{ij} = \delta_{ij} + \hat{h}_{ij}, \quad h^{ij} = \delta^{ij} + \hat{h}^{ij} \tag{22}$$

with

$$\hat{h}_{ij} = O(r^2), \quad \hat{h}^{ij} = O(r^2), \quad x^i \hat{h}_{ij} = 0, \quad x_i \hat{h}^{ij} = 0.$$

Notice that $\hat{h}_{ij}$, $\hat{h}^{ij}$, defined by the equations above, are not necessarily related to each other by the usual process of raising indices.

Denoting by Δ the flat Laplacian with respect to the coordinates x^j, we write on B_a

$$L_h = \Delta + \hat{L}_h,$$

with

$$\hat{L}_h = \hat{h}^{ij}\partial_i\partial_j + b^i\partial_i - \frac{1}{8}R. \tag{23}$$

We note that

$$b^i = O(r).$$

Choose a function $\chi_a \in C^\infty(S)$ which is non-negative and such that $\chi_a = 1$ in $B_{a/2}$ and $\chi_a = 0$ in $S \setminus B_a$. Denote by δ_i the Dirac delta distribution with source at i.

Lemma 2. *Assume that h satisfies (21) and $R > 0$. Then, there exists a unique solution θ_0 of the equation $L_h\theta_0 = -4\pi\delta_i$. Moreover $\theta_0 > 0$ in $\tilde{S}$ and we can write $\theta_0 = \chi_a/r + g$ with $g \in C^\alpha(S)$, $0 < \alpha < 1$.*

Proof. Observing that $1/r$ defines a fundamental solution to the flat Laplacian, we obtain

$$\Delta\left(\frac{\chi_a}{r}\right) = -4\pi\delta_i + \hat{\chi}, \tag{24}$$

where $\hat{\chi}$ is a smooth function on S with support in $B_a \setminus B_{a/2}$. The ansatz $\theta_0 = \chi_a/r + g$ translates the original equation into an equation for g,

$$L_h g = -\hat{L}_h\left(\frac{\chi_a}{r}\right) - \hat{\chi}.$$

A direct calculation shows that $\hat{L}_h(\chi_a r^{-1}) \in L^q(S)$, $q < 3$. By Lemma 1 there exists a unique solution $g \in W^{2,q}(S)$ to this equation which by the imbedding theorem is in $C^\alpha(S)$.

To show that θ_0 is strictly positive, we observe that it is positive near i (because r^{-1} is positive and g is bounded) and apply the strong maximum principle to $-\theta_0$. □

We use the conformal covariance of the equation to strengthen the result on the differentiability of the function g. Consider a conformal factor

$$\omega_0 = e^{f_0} \text{ with } f_0 \in C^\infty(S) \text{ such that } f_0 = \frac{1}{2}x^j x^k L_{jk}(i) \text{ on } \quad B_a, \tag{25}$$

where we use the normal coordinates x^k and the value of the tensor

$$L_{ab} \equiv R_{ab} - \frac{1}{4}Rh_{ab}, \tag{26}$$

at i. Then the Ricci tensor of the metric

$$h'_{ab} = \omega_0^4 h_{ab} \tag{27}$$

vanishes at the point i and, since we are in three dimensions, the Riemann tensor vanishes there too. Hence the connection and metric coefficients satisfy in the coordinates x^k

$$\Gamma_i'{}^j{}_k = O(r^2), \quad h'_{ij} = \delta_{ij} + O(r^3). \tag{28}$$

Corollary 1. *The function g found in Lemma 2 is in $C^{1,\alpha}(S)$, $0 < \alpha < 1$.*

Proof. With $\omega_0 = e^{f_0}$ and h'_{ab} as above, we note that

$$L_{h'}(\theta'_0) = \omega_0^{-5} L_h(\theta_0), \tag{29}$$

where

$$\theta'_0 = \omega_0^{-1}\theta_0. \tag{30}$$

We apply now the argument of the proof of Lemma 2 to the function θ'_0. Since we have by Eq. (28) that $\hat{L}_{h'}(\chi_a r^{-1}) \in L^\infty(S)$, it follows that $\theta'_0 = \chi_a/r + g'$, where $g' \in C^{1,\alpha}(S)$. We use Eq. (30), the fact that $\omega_0 = 1 + O(r^2)$, and Lemma 4 to obtain the desired result. □

We note that the function

$$\theta_0^{-1} = \frac{r}{\chi_a + rg}, \tag{31}$$

is in $C^\alpha(S)$, it is non-negative and vanishes only at i. To obtain θ, we write $\theta = \theta_0 + u$ and solve on S the following equation for u:

$$L_h u = -\frac{1}{8}\theta_0^{-7}\Psi_{ab}\Psi^{ab}(1 + \theta_0^{-1}u)^{-7}. \tag{32}$$

Theorem 12. *Assume that $h_{ab} \in W^{4,p}(S)$ with $p > 3/2$, that $R > 0$ on S, and that $\theta_0^{-7}\Psi_{ab}\Psi^{ab} \in L^q(S)$, $q \geq 2$. Then there exists a unique non-negative solution $u \in W^{2,q}(S)$ of Eq. (32). We have $u > 0$ on S unless $\Psi_{ab}\Psi^{ab} = 0 \in L^q(S)$.*

We note that our assumptions on Ψ_{ab} impose rather mild restrictions, which are, in particular, compatible with the fall off requirement (9). By the imbedding Theorem 3 we will have $u \in C^\alpha(S)$, $\alpha = 2 - 3/q$, for $q > 3$; and $u \in C^{1,\alpha}(S)$, for $q > 3$.

Proof. The proof is similar to that given in [8], with the difference that we impose weaker smoothness requirements. Making use of Lemma 1, we define a non-linear operator $T : B \to C^0(S)$, with a subset B of $C^0(S)$ which will be specified below, by setting

$$T(u) = L_h^{-1} f(x, u),$$

where

$$f(x, u) = -\frac{1}{8}\,\theta_0^{-7}\,\Psi_{ab}\Psi^{ab}\,g(x, u) \tag{33}$$

with

$$g(x, u) = \left(1 + \theta_0^{-1}u\right)^{-7}.$$

In the following we will suppress the dependence of f and g on x. Let $\psi \in W^{2,q}(S) \subset C^\alpha(S)$ be the function satisfying $\psi = T(0)$ and set $B = \{u \in C^0(S) : 0 \leq u \leq \psi\}$, which is clearly a closed, convex subset of the Banach space $C^0(S)$.

We want to use the Schauder theorem to show the existence of a point $u \in B$ satisfying $u = T(u)$. This will be the solution to our equation.

We show that T is continuous. Observing the properties of θ_0^{-1} noted above, we see that g defines a continuous map $g : B \to L^2$. Using the Cauchy–Schwarz inequality, we get

$$\big\| |f(u_1) - f(u_2)||_{L^2} \leq \Big\| \frac{1}{8}\theta_0^{-7}\Psi_{ab}\Psi^{ab} \Big\|_{L^2} ||g(u_1) - g(u_2)\big\|_{L^2} .$$

By Theorems 1, 3 we know that the map $L^2(S) \to W^{2,2}(S) \to C^0(S)$, where the first arrow denotes the map L_h^{-1} and the second arrow the natural injection, is continuous. Together these observations give the desired result.

We show that T maps B into itself. If $u \geq 0$ we have $f \leq 0$ whence, by Lemma 1, $T(u) = L_h^{-1}(f(u)) \geq 0$. If $u_1 \geq u_2$ it follows that $f(u_1) \geq f(u_2)$ whence, again by Lemma 1, $T(u_2) - T(u_1) = L_h^{-1}(f(u_2) - f(u_1)) \geq 0$. We conclude from this that for $u \in B$ we have $0 \leq T(u) \leq T(0) = \psi$.

Finally, $T(B)$ is precompact because $W^{2,2}(S)$ is compactly embedded in $C^0(S)$ by Theorem 5. Thus the hypotheses of Theorem 6 are satisfied and there exists a fixed point u of T in B. By its construction we have $u \in L^q(S)$ whence, by elliptic regularity, $u \in W^{2,q}(S)$.

To show its uniqueness, assume that u_1 and u_2 are solutions to (32). Observing the special structure of g and the identity

$$a^{-7} - c^{-7} = (c-a)\sum_{j=0}^{6} a^{j-7}c^{-1-j},$$

which holds for positive numbers a and c, we find that we can write $L_h(u_1 - u_2) = c\,(u_1 - u_2)$ with some function $c \geq 0$. The maximum principle thus allows us to conclude that $u_1 = u_2$. The last statement about the positivity of u follows from Lemma 1, (ii). □

3.2. Asymptotic expansions near i of solutions to the Lichnerowicz equation. The aim of this section is to introduce the function spaces $E^{m,\alpha}$, to point out the simple consequences listed in Lemma 6, and to prove Theorem 13. These are the tools needed to prove Theorem 1.

Let $m \in \mathbb{N}_0$, and denote by $\mathcal{P}_m$ the space of homogeneous polynomials of degree m in the variables x^j. The elements of $\mathcal{P}_m$ are of the form $\sum_{|\beta|=m} C_\beta\, x^\beta$ with constant coefficients C_β. Note that r^2 is in $\mathcal{P}_2$ but r is not in $\mathcal{P}_1$. We denote by $\mathcal{H}_m$ the set of homogeneous harmonic polynomials of degree m, i.e. the set of $p \in \mathcal{P}_m$ such that $\Delta p = 0$. For $s \in \mathbb{Z}$, we define the vector space $r^s\mathcal{P}_m$ as the set of functions of the form $r^s p$ with $p \in \mathcal{P}_m$.

Lemma 3. *Assume $s \in \mathbb{Z}$. The Laplacian defines a bijective linear map*

$$\Delta : r^s\mathcal{P}_m \to r^{s-2}\mathcal{P}_m,$$

in either of the following cases:

(i) $s > 0$,
(ii) $s < 0$, $|s|$ *is odd and* $m + s \geq 0$.

Note that the assumptions on m and s imply that the function $\Delta(r^s\, p_m) \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ defines a function in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ which represents $\Delta(r^s\, p_m)$ in the distributional sense.

Proof. Since Δ maps $\mathcal{P}_m$ into $\mathcal{P}_{m-2}$, we find from

$$\Delta(r^s p) = r^{s-2}\left(s(s+1+2m)p + r^2 \Delta p\right), \tag{34}$$

that $\Delta(r^s p) \in r^{s-2}\mathcal{P}_m$.

We show now that the map is bijective for certain values of s and m. Because $r^s\mathcal{P}_m$ and $r^{s-2}\mathcal{P}_m$ have the same finite dimension, we need only show that the kernel is trivial for some s and m. The vector space $\mathcal{P}_m$ can be written as a direct sum

$$\mathcal{P}_m = \mathcal{H}_m \oplus r^2\mathcal{H}_{m-2} \oplus r^4\mathcal{H}_{m-4} \cdots , \tag{35}$$

(cf. [20]). If $\Delta(r^s p) = 0$, we get from (35) that

$$0 = \Delta(r^s p) = \sum_{0\le k\le m/2} \Delta(r^{s+2k} h_{m-2k}),$$

with $h_{m-2k} \in H_{m-2k}$. Applying (34), we obtain

$$0 = \sum_{0\le k\le m/2} r^{2k}(s+2k)(s+1+2(m-k))h_{m-2k},$$

which allows us to conclude by (35) that

$$(s+2k)\,(s+1+2(m-k))\,h_{m-2k} = 0.$$

Since by our assumptions $(s+2k)(s+1+2(m-k)) \neq 0$, it follows that the polynomials h_{m-2k} vanish, whence $r^s p = 0$. □

We will need the following technical lemma regarding Hölder functions:

Lemma 4. *Suppose $m \in \mathbb{N}$, $0 < \alpha < 1$, $f \in C^{m,\alpha}(B_a)$, and T_m denotes the Taylor polynomial of order m of f. Then $f_R \equiv f - T_m$ is in $C^{m,\alpha}(B_a)$ and satisfies, if $|\beta| \le m$,*

$$\partial^\beta f_R = O\left(r^{m-|\beta|+\alpha}\right) \quad as \quad r \to 0.$$

Moreover, let s be an integer such that $s \le 1$ and $m + s - 1 \ge 0$. Then f_R satisfies:

(i) $r^{s-2} f_R \in W^{m+s-1,p}(B_\epsilon)$, *for* $p < 3/(1-\alpha)$, $0 < \epsilon < a$.
(ii) *If* $m + s - 1 \ge 1$ *then* $r^{s-2} f_R \in C^{m+s-2,\alpha}(B_\epsilon)$.
(iii) $r f_R \in C^{m,\alpha}(B_\epsilon)$.

Proof. The relation

$$|\partial^\beta f_R| \le C|x|^{m-|\beta|+\alpha}, \quad x \in \bar{B}_\epsilon, \tag{36}$$

is a consequence of Lemma 14.

(i) We have

$$\partial^\beta(r^{s-2} f_R) = \sum_{\beta'+\gamma'=\beta} C_{\beta'}\partial^{\beta'} f_R\, \partial^{\gamma'}(r^{s-2}),$$

with certain constants $C_{\beta'}$, and the derivatives of r^{s-2} are bounded for $x \in \bar{B}_\epsilon$ by

$$|\partial^{\gamma'} r^{s-2}| \leq C r^{s-2-|\gamma'|}.$$

Observing (36), we obtain

$$|\partial^\beta (r^{s-2} f_R)| \leq C r^{m-|\beta|+s-2+\alpha},$$

for $x \in \bar{B}_\epsilon$, whence, by our hypothesis $m+s-1 \geq 0$,

$$|\partial^\beta (r^{s-2} f_R)| \leq C r^{-1+\alpha}, \tag{37}$$

for $|\beta| \leq m+s-1$. Using that $r^{-1+\alpha}$ is in $L^p(B_\epsilon)$ for $p < 3/(1-\alpha)$, we conclude that $\partial^\beta (r^{s-2} f) \in L^p(B_\epsilon)$ whence $r^{s-2} f_R \in W^{m+s-1,p}(B_\epsilon)$.

(ii) From the relation above and Theorem 3 we conclude for $m+s-1 \geq 1$ that $r^{s-2} f_R \in C^{m+s-2,\alpha'}(B_\epsilon)$ with $\alpha' = 1 - 3/p < \alpha$. To show that α' can in fact be chosen equal to α, we use the sharp result of Theorem 4. Set $g = \partial^{\beta'}(r^{s-2} f_R)$ for some $\beta' \leq m+s-2$. Let z be an arbitrary point of B_ϵ and $B_R(z)$ be a ball with center z and radius R such that $B_R \subset B_a$. Using the inequality (37), we obtain

$$\int_{B_R(z)} |\partial g| d\mu \leq C \int_{B_R(z)} r^{\alpha-1} d\mu. \leq C \int_{B_R(0)} r^{\alpha-1} d\mu = C' R^{2+\alpha}$$

(cf. [25] p. 159 for the second estimate). Applying now Lemma 4, we conclude that $g \in C^\alpha(B_\epsilon)$, whence $r^{s-2} f_R \in C^{m+s-2,\alpha}(B_\epsilon)$.

(iii) We have

$$\partial^\beta (r f_R) = r \partial^\beta f_R + f_1,$$

where, with certain constants $C_{\beta,\beta'}$,

$$f_1 = \sum_{\beta \neq \beta' + \gamma' \leq \beta} C_{\beta,\beta'} \partial^{\beta'} f_R \partial^{\gamma'} r.$$

Note that $r \partial^\beta f \in C^\alpha(B_a)$, since r is Lipschitz continuous.

Using the bound

$$|\partial^{\gamma'} r| \leq C r^{-|\gamma'|+1},$$

and the bound (36) we obtain

$$|\partial f_1| \leq C r^\alpha.$$

Thus $f_1 \in W^{1,p}$ for all p, whence, by Theorem 3, $f_1 \in C^\alpha$ for $\alpha < 1$. □

The following function spaces will be important for us.

Definition 1. *For $m \in \mathbb{N}_0$ and $0 < \alpha < 1$, we define the space $E^{m,\alpha}(B_a)$ as the set $E^{m,\alpha}(B_a) = \{f = f_1 + r f_2 \,:\, f_1, f_2 \in C^{m,\alpha}(B_a)\}$. Furthermore we set $E^\infty(B_a) = \{f = f_1 + r f_2 \,:\, f_1, f_2 \in C^\infty(B_a)\}$.*

Note that the decompositions above are not unique. If $f = f_1 + rf_2$, $f_1, f_2 \in C^{m,\alpha}(B_a)$, then also $f = f_1 + rf_R + r(f_2 - f_R)$ with $f_1 + rf_R$, $f_2 - f_R \in C^{m,\alpha}(B_a)$ if f_R is given as in Lemma 4. Obviously, $E^\infty(B_a) \subset E^{m,\alpha}(B_a)$ for all $m \in \mathbb{N}_0$. The converse is not quite immediate.

Lemma 5. *If $f \in E^{m,\alpha}(B_a)$ for all $m \in \mathbb{N}_0$, then $f \in E^\infty(B_a)$.*

Proof. Assume that $f \in E^{m,\alpha}(B_a)$ for all m. Take an arbitrary m and write $f = f_1 + rf_2$ with $f_1, f_2 \in C^{m,\alpha}(B_a)$. To obtain a unique representation, we write f_1 and f_2 as the sum of their Taylor polynomials of order m and their remainders,

$$f_1 = \sum_{j=0}^{m} p_j^1 + f_R^1, \quad f_2 = \sum_{j=0}^{m} p_j^2 + f_R^2, \tag{38}$$

with $p_j^1, p_j^2 \in \mathcal{P}_j$ and $f_R^1, f_R^2 = O(r^{m+\alpha})$. From this we get the representation

$$f = \left(\sum_{j=0}^{m} p_j^1 + r \sum_{j=0}^{m-1} p_j^2 \right) + f_R, \tag{39}$$

where $f_R \equiv f_R^1 + r(f_R^2 + p_m^2) \in C^{m,\alpha}(B_a)$ and $f_R = O(r^{m+\alpha})$ by Lemma 4. This decomposition is unique: if we had $f = 0$, the fast fall-off of f_R at the origin would imply that the term in brackets, whence also each of the polynomials and f_R, must vanish.

Since m was arbitrary, we conclude that the function f determines a unique sequence of polynomials p_j^2, $j \in \mathbb{N}_0$ as above. By Borel's theorem (cf. [18]) there exists a function $v_2 \in C^\infty(B_a)$ (not unique) such that

$$v_2 - \sum_{j=0}^{m} p_j^2 = O(r^{m+1}), \;\; m \in \mathbb{N}_0. \tag{40}$$

We show that the function $v_1 \equiv f - rv_2$ is $C^{m-1}(B_a)$ for arbitrary m, i.e. $v_1 \in C^\infty(B_a)$. Using (39), we obtain

$$v_1 = \left(\sum_{j=0}^{m} p_j^1 + f_R \right) + r \left(\sum_{j=0}^{m-1} p_j^2 - v_2 \right). \tag{41}$$

The first term is in $C^{m,\alpha}(B_a)$ by the observations above, the second term is in $C^{m,\alpha}(B_a)$ by (40) and Lemma 4. □

While we cannot directly apply elliptic regularity results to these spaces, they are nevertheless appropriate for our purposes. This follows from the following observation, which will be extended to more general elliptic equations and more general smoothness assumptions in Theorem 13 and in Appendix B.

If u is a solution to the Poisson equation

$$\Delta u = \frac{f}{r},$$

with $f \in E^{m,\alpha}(B_a)$, $m \geq 1$, then $u \in E^{m+1,\alpha}(B_a)$. This can be seen as follows. If we write $f = f_1 + r\, f_2 \in E^{m,\alpha}(B_a)$ in the form

$$\frac{f}{r} = \frac{T_m}{r} + f_R,$$

where T_m is the Taylor polynomial of order m of f_1, the remainder f_R is seen to be in $C^{m-1,\alpha}(B_a)$ by Lemma 4.

By Lemma 3, there exists a polynomial $\hat{T}_m$ such that

$$\Delta(r\hat{T}_m) = \frac{T_m}{r}.$$

Then $u_R \equiv u - r\hat{T}_m$ satisfies $\Delta u_R = f_R$ and Theorem 8 implies that $u_R \in C^{m+1,\alpha}(B_a)$, whence $u \in E^{m+1,\alpha}(B_a)$.

To generalize these arguments to equations with non-constant coefficients and to non-linear equations we note the following observation.

Lemma 6. *For $f, g \in E^{m,\alpha}(B_a)$ we have*

(i) $f + g \in E^{m,\alpha}(B_a)$.
(ii) $fg \in E^{m,\alpha}(B_a)$.
(iii) *If $f \neq 0$ in B_a, then $1/f \in E^{m,\alpha}(B_a)$.*

Analogous results hold for functions in $E^{\infty}(B_a)$.

Proof. The first two assertions are obvious, for (iii) we need only consider a small ball B_ϵ centered at the origin because r is smooth elsewhere. If $f = f_1 + rf_2$, $f_1, f_2 \in C^{m,\alpha}(B_a)$, we have $1/f = v_1 + rv_2$ with

$$v_1 = \frac{f_1}{(f_1)^2 - r^2(f_2)^2}, \quad v_2 = \frac{-f_2}{(f_1)^2 - r^2(f_2)^2}.$$

These functions are in $C^{m,\alpha}(B_\epsilon)$ for sufficiently small $\epsilon > 0$ because our assumptions imply that $f_1(0) \neq 0$. The $E^{\infty}(B_a)$ case is similar. □

We consider now a general linear elliptic differential operator L of second order

$$L = a^{ij}\partial_i\partial_j + b^i\partial_i + c. \quad (42)$$

It will be assumed in this section that

$$a^{ij},\ b^i,\ c \in C^{\infty}(\bar{B}_a). \quad (43)$$

We express the operator in normal geodesic coordinates x^i with respect to a^{ij}, centered at the origin of B_a, such that

$$a^{ij}(x) = \delta^{ij} + \hat{a}^{ij},$$

with

$$\hat{a}^{ij} = O(r^2), \quad (44)$$

and

$$x_j\hat{a}^{ij} = 0, \quad x \in B_a. \quad (45)$$

For the differential operator $\hat{L}$, given by

$$\hat{L}u = \hat{a}^{ij}(x)\partial_i \partial_j u + b^i(x)\partial_i u + c(x)u,$$

we find

Lemma 7. *Suppose $p \in \mathcal{P}_m$. Then the function U defined by $\hat{L}(r^s p) = r^{s-2}U$ is C^∞ and satisfies $U = O(r^{m+1})$. If in addition $b^i = O(r)$ (as in the case of the Yamabe operator L_h), then $U = O(r^{m+2})$.*

Proof. A direct calculation, observing (45), gives

$$U = \hat{a}^{ij}(s\delta_{ij} p + r^2 \partial_j \partial_i p) + b^i(s x_i p + r^2 \partial_i p) + cr^2 p,$$

which guarantees our result. □

In the following we shall use the splitting $L = \Delta + \hat{L}$, where Δ is the flat Laplacian in the normal coordinates x^i.

Theorem 13. *Let $u \in W^{2,p}_{\text{loc}}(B_a)$ be a solution of*

$$Lu = r^{s-2} f,$$

where L is given by (42) with (43) and $s \in \mathbb{Z}$, $p > 1$.

(i) *Assume $s = 1$ and $f \in E^{m,\alpha}(B_a)$. Then $u \in E^{1,\alpha'}(B_a)$, $0 < \alpha' < \alpha$, if $m = 0$ and $u \in E^{m+1,\alpha}(B_a)$ if $m \geq 1$. If $f \in E^\infty(B_a)$, then $u \in E^\infty(B_a)$.*
(ii) *If $s < 0$, $|s|$ is odd, $f \in C^{m,\alpha}(B_a)$ with $m + s - 1 \geq 0$, and $f = O(r^{s_0})$, with $s_0 + s \geq 0$; then u has the form*

$$u = r^s \sum_{k=s_0}^{m} u_k + u_R,$$

where $u_k \in \mathcal{P}_m$, $Lu_R = O(r^{m+\alpha+s-2})$, and $u_R \in C^{1,\alpha'}(B_a)$, $0 < \alpha' < \alpha$, if $m + s - 1 = 0$, $u_R \in C^{m+s,\alpha}(B_a)$ if $m + s - 1 \geq 1$.

If $f \in C^\infty(B_a)$, then u can be written in the form $u = r^s v_1 + v_2$ with $v_1, v_2 \in C^\infty(B_a)$, $v_1 = O(r^{s_0})$, and

$$L(r^s v_1) = r^{s-2} f + \theta_R, \quad L(v_2) = -\theta_R,$$

where $\theta_R \in C^\infty(B_a)$ and all its derivatives vanish at the origin.

Proof. In both cases we write $f = T_m + f_R$ with a polynomial T_m of order m,

$$T_m = \sum_{k=m_0}^{m} t_k, \quad t_k \in \mathcal{P}_m.$$

Case (i): $m_0 = 0$ and $f = f_1 + r f_2$, where f_1, f_2 are in $C^{m,\alpha}(B_a)$. We define T_m to be the Taylor polynomial of order m of f_1.
Case (ii): $m_0 = s_0$ and we define T_m to be the Taylor polynomial of order m of f.

We show that u has in both cases the form

$$u = r^s \sum_{k=m_0}^{m} u_k + u_R, \tag{46}$$

with $u_k \in \mathcal{P}_k$ and $u_R \in C^{m+s,\alpha}(B_a)$. For this purpose Lemma 3 will be used to determine the polynomials u_k in terms of t_k by a recurrence relation. The differentiability of u_R follows then from Lemma 4, elliptic regularity, and the elliptic equation satisfied by u_R.

The recurrence relation is defined by

$$\Delta(r^s u_{m_0}) = r^{s-2} t_{m_0}, \quad \Delta(r^s u_k) = r^{s-2}(t_k - U_k^{(k)}), \quad k = m_0 + 1, \dots, m, \tag{47}$$

where, given $u_{m_0}, \dots, u_{k-1}$, we define $U_k^{(k)}$ as follows. By Lemma (7) the functions

$$U^{(k)} = r^{-s+2}\, \hat{L}\left(r^s \sum_{j=m_0}^{k-1} u_j \right), \tag{48}$$

which will be defined successively for $k = m_0 + 1, \dots, m + 1$, are C^∞ and $U^{(k)} = O(r^{m_0+1})$. Thus we can write by Lemma (14)

$$U^{(k)} = \sum_{j=m_0+1}^{m} U_j^{(k)} + U_R^{(k)},$$

where $U_R^{(k)} = O(r^{m+\alpha})$ and $U_j^{(k)} \in \mathcal{P}_j$ denotes the homogeneous polynomial of order j in the Taylor expansion of $U^{(k)}$.

By Lemma (3) the recurrence relation (47) is well defined for the cases (i) and (ii). Note that

$$U_j^{(k')} = U_j^{(k)}, \;\; m_0 + 1 \le j \le k \quad \text{if} \quad k < k' \le m + 1, \tag{49}$$

because we have by Lemma 7,

$$U^{(k')} - U^{(k)} = r^{-s+2} \hat{L}\left(r^s \sum_{j=k}^{k'-1} u_j \right) = O(r^{k+1}).$$

With the definitions above and the identity (49), which allows us to replace $U_k^{(k)}$ by $U_k^{(m+1)}$, the original equation for u implies for the function u_R defined by (46) the equation

$$L u_R = r^{s-2} \left(U_R^{(m+1)} + f_R \right). \tag{50}$$

Case (i): We use Lemma (4), (i) to conclude that the right-hand side of Eq. (50) is in $L^p(B_\epsilon)$ if $m = 0$ and in $C^{m-1,\alpha}(B_\epsilon)$ if $m \ge 1$. Now Theorems 7 and 8 imply that $u_R \in C^{1,\alpha'}(B_a)$, $\alpha' < \alpha$, if $m = 0$ and $u_R \in C^{m+1,\alpha}(B_a)$ if $m \ge 1$. For the E^∞ case we use Lemma 5.

Case (ii): By our procedure we have

$$Lu_R = O(r^{m+\alpha+s-2}).$$

If $m+s-1=0$ we use part (i) of Lemma 4 to conclude that the right-hand side of Eq. (50) is in $L^p(B_a)$ and Theorems 7, 3 to conclude that $u_R \in C^{1,\alpha'}(B_a)$. If $m+s-1 \geq 1$ we use part (ii) of Lemma 4 to conclude that the right-hand side of Eq. (50) is in $C^{m+s-2,\alpha}(B_a)$. Elliptic regularity then implies that $u_R \in C^{m+s,\alpha}(B_a)$. The C^∞ case follows by an analogous argument as in the proof of Lemma 5, since the polynomials $u_{m_0}, \ldots, u_m$ obtained for an integer m' with $m' > m$ coincide with those obtained for m, i.e. the procedure provides a unique sequence of polynomials u_k, $k = m_0, \ldots$. □

More general expansions, which include logarithmic terms, have been studied (in a somewhat different setting) in [30], where results similar to those given in 13 have been derived. Definition 1 is tailored to the case in which no logarithmic terms appear and leads to a considerable simplification of the proofs as well as to a more concise statement of the results as compared with those given in [30].

Corollary 2. *Assume that the hypotheses on u of Theorem 13 are satisfied. Let θ_0 be a distributional solution of $L\,\theta_0 = -4\pi\delta_i$ in B_a. Then we can write*

$$\theta_0 = r^{-1}u_1 + u_2, \tag{51}$$

with $u_1, u_2 \in C^\infty(B_a)$, $u_1(0) = 1$, $L(r^{-1}u_1) = -4\pi\delta_i + \theta_R$, where $\theta_R \in C^\infty(B_a)$ and all its derivatives vanish at i. In the particular case of the Yamabe operator L_h with respect to a smooth metric h we have $u_1 = 1 + O(r^2)$.

Proof. Using that $\Delta(r^{-1}) = -4\pi\delta_i$ in B_a, we obtain for $u = \theta_0 - 1/r$,

$$Lu = -\hat{L}(r^{-1}). \tag{52}$$

By Lemma 7 we have $\hat{L}(r^{-1}) = r^{-3}U$, with $U \in C^\infty(B_a)$ and $U = O(r)$. Our assertion now follows from Theorem 13. For the last assertion we use that in the case of L_h we have $U = O(r^2)$. □

We note that the functions u_1, u_2 are in fact analytic and $\theta_R \equiv 0$ if the coefficients of L are analytic in B_a (cf. [24]).

3.3. Proof of Theorem 1. There exists a unique solution $\theta = \theta_0 + u$ of Eq. (8) with θ_0 as Lemma 2 and u as in Theorem 12. Since the operator L_h satisfies the hypothesis of Corollary 2 we can write on B_a,

$$\theta_0 = \frac{u_1}{r} + w, \tag{53}$$

where u_1, w are smooth functions and $L_h(r^{-1}u_1) = \theta_R$ on $B_a \setminus i$, with θ_R as described in Corollary 2.

Given the solution $u = u(x)$, we can read Eq. (32) in B_a as an equation for u,

$$L_h u = \frac{f(x)}{r}, \tag{54}$$

with f considered as a given function of x

$$f(x) = -\frac{r^8 \Psi_{ab}\Psi^{ab}}{8(r\theta_0 + ru)^7}.$$

By hypothesis $r^8\Psi_{ab}\Psi^{ab} \in E^\infty(B_a)$, by Corollary 2 we have $r\,\theta_0 \in E^\infty(B_a)$, and by Theorem 12 the solution u is in $C^\alpha(B_a) \subset E^{0,\alpha}(B_a)$. By Lemma 6 we thus have $f \in E^{0,\alpha}(B_a)$. By Theorem 13 Eq. (54) implies that $u \in E^{1,\alpha'}(B_a)$, $0 < \alpha' < \alpha$ which implies in turn that $f \in E^{1,\alpha'}(B_a)$. Repeating the argument, we show inductively that u, whence f is in $E^{m,\alpha'}(B_a)$ for all $m \geq 0$. Lemma 5 now implies that $u \in E^\infty(B_a)$. □

3.4. Solution of the Hamiltonian constraint with logarithmic terms. The example

$$\Delta(\log r h_m) = r^{-2} h_m (2m+1), \tag{55}$$

$h_m \in \mathcal{H}_m$, shows that logarithmic terms can occur in solutions to the Poisson equation even if the source has only terms of the form $r^s p$ with $p \in \mathcal{P}_m$. This happens in the cases where the Laplacian does not define a bijection between $r^{s+2}\mathcal{P}_m$ and $r^s\mathcal{P}_m$, cases which are excluded in Lemma 3. We shall use this to show that logarithmic terms can occur in the solution to the Lichnerowicz equation if the condition $r^8\Psi_{ab}\Psi^{ab} \in E^\infty(B_a)$ is not satisfied. Our example will be concerned with initial data with non-vanishing linear momentum.

We assume that in a small ball B_a centered at i the tensor Ψ^{ab} has the form

$$\Psi^{ab} = \Psi^{ab}_P + \Psi^{ab}_R, \tag{56}$$

where Ψ^{ab}_P is given in normal coordinates by (76) and $\Psi^{ab}_R = O(r^{-3})$ is a tensor field such that Ψ^{ab} satisfies Eq. (7). The existence of such tensors, which satisfy also

$$\theta_0^{-7}\Psi_{ab}\Psi^{ab} \in L^q(S), \quad q \geq 2, \tag{57}$$

and, by Lemma 13,

$$r^8\Psi_{ab}\Psi^{ab} = \psi + r\psi^R \quad \text{in} \quad B_a, \tag{58}$$

will be shown in Sect. 4.3. Here the function ψ^R is in $C^\alpha(B_a)$ and ψ is given explicitly by

$$\psi \equiv r^8 \Psi_{P\,ij}\Psi^{ij}_P = c + r^{-2}h_2, \tag{59}$$

with $c = \frac{15}{16}P^2$, $P^2 = P^i P_i$, and

$$h_2 = \frac{3}{8}r^2\left(3(P^i n_i)^2 - P^2\right), \tag{60}$$

where, in accordance with the calculations in Sect. 4.3, the latin indices in the expressions above are moved with the flat metric. We note that $h_2 \in \mathcal{H}_2$ and ψ is not continuous. The tensor Ψ^{ab} satisfies condition (9) and the three constants P^i given by (76) represent the momentum of the initial data. Since Ψ^{ab}_P is trace-free and divergence-free with respect to the flat metric, we could, of course, choose h_{ab} to be the flat metric and $\Psi^{ab}_R = 0$. This would provide one of the conformally flat initial data sets discussed in [11]. We are interested in a more general situation.

Lemma 8. *Let h_{ab} be a smooth metric, and let Ψ^{ab} be given by (56) such that conditions (57) and (58) hold. Then, there exists a unique, positive, solution to the Hamiltonian constraint (8). In B_a it has the form*

$$\theta = \frac{w_1}{r} + \frac{1}{2}m + \frac{1}{32}\, r\, \left((9\,(P^i n_i)^2 - 33\,P^2\right) + \frac{7}{16}\, m\, r^2 \left(\frac{5}{4}\, P^2 + \frac{3}{5}\,(3\,(P^i n_i)^2 - P^2)\, \log r\right) + u_R, \quad (61)$$

where the constant m is the total mass of the initial data, w_1 is a smooth function with $w_1 = 1 + O(r^2)$, and $u_R \in C^{2,\alpha}(B_a)$ with $u_R(0) = 0$.

Since w_1 is smooth and u_R is in $C^{2,\alpha}(B_a)$, there cannot occur a cancellation of logarithmic terms. For non-trivial data, for which $m \neq 0$, the logarithmic term will always appear. In the case where h_{ab} is flat and $\Psi_R^{ab} = 0$ an expansion similar to (61) has been calculated in [26].

Proof. The existence and uniqueness of the solution has been shown in Sect. 3.1. To derive (61) we shall try to calculate each term of the expansion and to control the remainder as we did in the proof of Theorem 13. However, Lemma 3 will not suffice here, we will have to use Eq. (55).

By Corollary 2 we have

$$\theta_0 = \frac{w_1}{r} + w,$$

with $w, w_1 \in C^\infty(B_a)$ and $w_1 = 1 + O(r^2)$. By Theorem 12 the unique solution u of Eq. (54) is in $C^\alpha(B_a)$. Equation (54) has the form

$$L_h u = (\frac{\psi}{r} + \psi^R)\, f(x,u) \quad \text{with} \quad f(x,u) = -\frac{1}{(w_1 + r(u+w))^7}.$$

By $m \equiv 2\,(u(0) + w(0))$ is given the mass of the initial data. Since $u \in C^\alpha(B_a)$, we find

$$f = -1 + \frac{7}{2}\, m\, r + f_R, \quad f_R = O(r^{1+\alpha}). \quad (62)$$

If we set

$$u_1 = -\frac{c}{2} r + \frac{1}{4r} h_2 + \frac{7}{2}\, m\, \left(\frac{c}{6}\, r^2 + \frac{1}{5}\, \log r\, h_2\right), \quad (63)$$

we find from (34), (55) that

$$\Delta u_1 = \frac{\psi}{r}\left(-1 + \frac{7}{2}\, m\, r\right), \quad (64)$$

and that $v = u - u_1$ satisfies

$$L_h v = \psi^R \left(-1 + \frac{7}{2}\, m\, r\right) - \hat{L}_h u_1 + \frac{\psi f_R}{r}. \quad (65)$$

We shall show that this equation implies that the function v is in $C^{2,\alpha}(S)$.

Since $\psi^R \in C^\alpha(B_a)$, the first term on the right-hand side of (65) is in $C^\alpha(B_a)$. By a direct calculation (using that the coefficient b^i of $\hat{L}_h$ is $O(r)$) we find that $\hat{L}_h u_1 \in W^{2,p}(B_a)$, $p < 3$, which implies by the Sobolev imbedding theorem that $\hat{L}_h u_1 \in C^\alpha(B_a)$. The third term on the right-hand side of (65) is more delicate, because it depends on the solution v. From Theorem 12 and Eq. (63) we find that $v \in C^\alpha(B_a) \cap W^{2,p}(B_a)$, where, due to (63), we need to assume $p < 3$. However, since $f_R = O(r^{1+\alpha})$ and ψ is bounded, the function $\psi f_R/r$ is bounded and the right-hand side of (65) is in $L^\infty(B_a)$. Thus Theorem 7 implies that $v \in W^{2,p}(B_a)$ for $p > 3$, whence $v \in C^{1,\alpha}(B_a)$ by the Sobolev imbedding theorem. It follows that we can differentiate f_R, considered as a function of x, to find that $\partial_i f_R = O(r)$. It follows that $\psi f_R/r$ is in $W^{1,p}(B_a)$, $p > 3$, and thus in $C^\alpha(B_a)$. Since the right-hand side of (65) is in $C^\alpha(B_a)$, it follows that $v \in C^{2,\alpha}(B_a)$. □

4. The Momentum Constraint

4.1. The momentum constraint on Euclidean space. In the following we shall give an explicit construction of the smooth solutions to the equation $\partial_a \Psi^{ab} = 0$ on the 3-dimensional Euclidean space $\mathbb{E}^3$ (in suitable coordinates $\mathbb{R}^3$ endowed with the flat standard metric) or open subsets of it. Another method to obtain such solutions has been described in [7], multipole expansions of such tensors have been studied in [9].

Let i be a point of $\mathbb{E}^3$ and x^k a Cartesian coordinate system with origin i such that in these coordinates the metric of $\mathbb{E}^3$, denoted by δ_{ab}, is given by the standard form δ_{kl}. We denote by n^a the vector field on $\mathbb{E}^3 \setminus \{i\}$ which is given in these coordinates by $x^k/|x|$.

Denote by m_a and its complex conjugate $\bar{m}_a$ complex vector fields, defined on $\mathbb{E}^3$ outside a lower dimensional subset and independent of $r = |x|$, such that

$$m_a m^a = \bar{m}_a \bar{m}^a = n_a m^a = n_a \bar{m}^a = 0, \quad m_a \bar{m}^a = 1. \tag{66}$$

There remains the freedom to perform rotations $m_a \to e^{if} m_a$ with functions f which are independent of r.

The metric has the expansion

$$\delta_{ab} = n_a n_b + \bar{m}_a m_b + m_a \bar{m}_b,$$

while an arbitrary symmetric, trace-free tensor Ψ_{ab} can be expended in the form

$$r^3 \Psi_{ab} = \xi(3 n_a n_b - \delta_{ab}) + \sqrt{2}\eta_1 n_{(a}\bar{m}_{b)} + \sqrt{2}\bar{\eta}_1 n_{(a} m_{b)} + \bar{\mu}_2 m_a m_b + \mu_2 \bar{m}_a \bar{m}_b, \tag{67}$$

with

$$\xi = \frac{1}{2} r^3 \Psi_{ab} n^a n^b, \quad \eta_1 = \sqrt{2} r^3 \Psi_{ab} n^a m^b, \quad \mu_2 = r^3 \Psi_{ab} m^a m^b.$$

Since Ψ_{ab} is real, the function ξ is real while η_1, μ_2 are complex functions of spin weight 1 and 2 respectively.

Using in the equation

$$\partial_a \Psi^{ab} = 0, \tag{68}$$

the expansion (67) and contracting suitably with n^a and m^a, we obtain the following representation of (68):

$$4r\partial_r\xi + \bar{\eth}\eta_1 + \eth\bar{\eta}_1 = 0, \tag{69}$$

$$r\partial_r\eta_1 + \bar{\eth}\mu_2 - \eth\xi = 0. \tag{70}$$

Here ∂_r denotes the radial derivative and $\eth$ the edth operator of the unit two-sphere (cf. [32] for definition and properties). By our assumptions the differential operator $\eth$ commutes with ∂_r.

Let ${}_sY_{lm}$ denote the spin weighted spherical harmonics, which coincide with the standard spherical harmonics Y_{lm} for $s = 0$. The ${}_sY_{lm}$ are eigenfunctions of the operator $\bar{\eth}\eth$ for each spin weight s. More generally, we have

$$\bar{\eth}^p\eth^p({}_sY_{lm}) = (-1)^p\frac{(l-s)!}{(l-s-p)!}\frac{(l+s+p)!}{(l+s)!}{}_sY_{lm}. \tag{71}$$

If μ_s denotes a smooth function on the two-sphere of integral spin weight s, there exists a function μ of spin weight zero such that $\eta_s = \eth^s\eta$. We set $\eta^R = \mathrm{Re}(\eta)$ and $\eta^I = i\,\mathrm{Im}(\eta)$, such that $\eta = \eta^R + \eta^I$, and define

$$\eta_s^R = \eth^s\eta^R, \quad \eta_s^I = \eth^s\eta^I,$$

such that $\eta_s = \eta_s^R + \eta_s^I$. We have

$$\overline{\eth^s\eta^R} = \bar{\eth}^s\eta^R, \quad \overline{\eth^s\eta^I} = -\bar{\eth}^s\eta^I.$$

Using these decompositions now for η_1 and μ_2, we obtain Eq. (69) in the form

$$2r\partial_r\xi = -\bar{\eth}\eth\eta^R. \tag{72}$$

Applying $\bar{\eth}$ to both sides of Eq. (70) and decomposing into real and imaginary part yields

$$r\partial_r\bar{\eth}\eth\eta^I = -\bar{\eth}^2\eth^2\mu^I, \tag{73}$$

$$2r\partial_r(r\partial_r\xi) + \bar{\eth}\eth\xi = \bar{\eth}^2\eth^2\mu^R. \tag{74}$$

Since the right-hand side of (72) has an expansion in spherical harmonics with $l \geq 1$ and the right hand sides of (73), (74) have expansions with $l \geq 2$, we can determine the expansion coefficients of the unknowns for $l = 0, 1$. They can be given in the form

$$\xi = A + rQ + \frac{1}{r}P, \quad \eta^I = iJ + \text{const.}, \quad \eta^R = rQ - \frac{1}{r}P + \text{const.},$$

with

$$P = \frac{3}{2}P^a n_a, \quad Q = \frac{3}{2}Q^a n_a, \quad J = 3J^a n_a, \tag{75}$$

where A, P^a, Q^a, J^a are arbitrary constants. Using (67), we obtain the corresponding tensors in the form (cf. ([11])

$$\Psi_P^{ab} = \frac{3}{2r^4}\left(-P^a n^b - P^b n^a - (\delta^{ab} - 5n^a n^b)\, P^c n_c\right), \tag{76}$$

$$\Psi_J^{ab} = \frac{3}{r^3}\left(n^a \epsilon^{bcd} J_c n_d + n^b \epsilon^{acd} J_c n_d\right), \tag{77}$$

$$\Psi_A^{ab} = \frac{A}{r^3}\left(3n^a n^b - \delta^{ab}\right), \tag{78}$$

$$\Psi_Q^{ab} = \frac{3}{2r^2}\left(Q^a n^b + Q^b n^a - (\delta^{ab} - n^a n^b)\, Q^c n_c\right). \tag{79}$$

We assume now that ξ and η^I have expansions in terms of spherical harmonics with $l \geq 2$. Then there exists a smooth function λ_2 of spin weight 2 such that

$$\xi = \bar{\eth}^2 \lambda_2^R, \quad \eta_1^I = \bar{\eth}\lambda_2^I.$$

Using these expressions in Eqs. (72)–(74) and observing that for smooth spin weighted functions μ_s with $s > 0$ we can have $\bar{\eth}\mu_s = 0$ only if $\mu_s = 0$, we obtain

$$\eth\eta^R = -2r\partial_r \bar{\eth}\lambda_2^R, \quad \eth^2\mu^I = -r\partial_r\lambda_2^I,$$
$$\eth^2\mu^R = 2r\partial_r\left(r\partial_r\lambda_2^R\right) - 2\lambda_2^R + \eth\bar{\eth}\lambda_2^R.$$

We are thus in a position to describe the general form of the coefficients in the expression (67)

$$\xi = \bar{\eth}^2\lambda_2^R + A + r\,Q + \frac{1}{r}\,P, \tag{80}$$

$$\eta_1 = -2\,r\,\partial_r\,\bar{\eth}\lambda_2^R + \bar{\eth}\lambda_2^I + r\,\eth Q - \frac{1}{r}\,\eth P + i\,\eth J, \tag{81}$$

$$\mu_2 = 2\,r\,\partial_r(r\,\partial_r\,\lambda_2^R) - 2\,\lambda_2^R + \eth\bar{\eth}\lambda_2^R - r\,\partial_r\,\lambda_2^I. \tag{82}$$

Theorem 14. *Let λ be an arbitrary complex C^∞ function in $B_a \setminus \{i\} \subset \mathbb{E}^3$ with $0 < a \leq \infty$, and set $\lambda_2 = \eth^2\lambda$. Then the tensor field*

$$\Psi^{ab} = \Psi_P^{ab} + \Psi_J^{ab} + \Psi_A^{ab} + \Psi_Q^{ab} + \Psi_\lambda^{ab}, \tag{83}$$

where the first four terms on the right-hand side are given by (76)–(78) while Ψ_λ^{ab} is is obtained by using in (67) only the part of the coefficients (80)–(82) which depends on λ_2, satisfies the equation $D^a\Psi_{ab} = 0$ in $B_a \setminus \{i\}$. Conversely, any smooth solution in $B_a \setminus \{i\}$ of this equation is of the form (83).

Obviously, the smoothness requirement on λ can be relaxed since $\Psi_\lambda^{ab} \in C^1(B_a \setminus \{i\})$ if $\lambda \in C^5(B_a \setminus \{i\})$. Notice, that no fall-off behaviour has been imposed on λ at i and that it can show all kinds of bad behaviour as $r \to 0$.

Since we are free to choose the radius a, we also obtain an expression for the general smooth solution on $\mathbb{E}^3 \setminus \{i\}$. By suitable choices of λ we can construct solutions Ψ_λ^{ab} which are smooth on $\mathbb{E}^3$ or which are smooth with compact support.

Given a subset S of $\mathbb{R}^3$ which is compact with boundary, we can use the representation (83) to construct hyperboloidal initial data ([21]) on S with a metric h which is Euclidean on all of S or on a subset U of S. In the latter case we would require Ψ_λ^{ab} to vanish on $S \backslash U$. In the case where the trace-free part of the second fundamental form implied by h on ∂S vanishes and the support of Ψ^{ab} has empty intersection with ∂S the smoothness of the corresponding hyperboloidal initial data near the boundary follows from the discussion in ([5]). Appropriate requirements on h and Ψ^{ab} near ∂S which ensure the smoothness of the hyperboloidal data under more general assumptions can be found in ([4]).

There exists a 10-dimensional space of conformal Killing vector fields on $\mathbb{E}^3$. In the cartesian coordinates x^i a generic conformal Killing vector field ξ_0^a has components

$$\xi_0^i = k^j \left(2\, x_j\, x^i - \delta_j{}^i\, x_l\, x^l\right) + \epsilon^i{}_{jk}\, S^j\, x^k + a x^i + q^i, \tag{84}$$

where k^i, S^i, q^i are arbitrary constant vectors and a an arbitrary number. In terms of the "physical coordinates" $y^i = x^i/|x|^2$, with respect to which i represents infinity, we see that k^i, S^i, q^i, and a generate translations, rotations, "special conformal transformations", and dilatations respectively.

For $0 < \epsilon < a$ we set $S_\epsilon = \{|x| = \epsilon\}$ and denote by dS_ϵ the surface element on it. For the tensor field Ψ^{ab} of (83) we obtain

$$\frac{1}{8\pi} \int_{S_\epsilon} \Psi^{ab}\, n_a\, \xi_{0b}\, dS_\epsilon = \left(P^a k_a + J^a S_a + A\, a + Q^a q_a\right). \tag{85}$$

We note that the integral is independent of ϵ and, more importantly, independent of the choice of λ. Thus the function λ neither contributes to the momentum

$$P^a = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{S_\epsilon} r^2\, \Psi_{bc}\, n^b\, (2n^c n^a - \delta^{ca})\, dS_\epsilon, \tag{86}$$

nor to the angular momentum

$$J^a = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{S_\epsilon} r\, \Psi_{bc}\, n^b \epsilon^{cad}\, n_d\, dS_\epsilon, \tag{87}$$

of the data.

If we use the coordinates x^i to identify $\mathbb{E}^3$ with $\mathbb{R}^3$ and map the unit sphere $S^3 \subset \mathbb{R}^4$ by stereographic projection through the south pole onto $\mathbb{R}^3$, the point i, i.e. $x^i = 0$, will correspond to the north pole, which will be denoted in the following again by i. The south pole, denoted in the following by i', will represent infinity with respect to the coordinates x^i and the origin with respect to the coordinates y^i. We use x^i and y^i as coordinates on $S^3 \setminus \{i'\}$ and $S^3 \setminus \{i\}$ respectively. If h^0 is the standard metric on S^3, we have in the coordinates x^i

$$h^0_{kl} = \theta^{-2} \delta_{kl}, \quad \theta = \left(\frac{1 + r^2}{2}\right)^{1/2}. \tag{88}$$

We assume that the function λ is smooth in $\mathbb{E}^3 \setminus \{i\}$ and set $\tilde{\Psi}^{ab} = \theta^{10}\, \Psi^{ab}$ with Ψ_{ab} as in (83). Then, we have by general rescaling laws (cf. (111), (112)),

$$D_a \tilde{\Psi}^{ab} = 0 \text{ in } S^3 \setminus \{i, i'\}, \tag{89}$$

where D_a denotes the connection corresponding to h^0. The smoothness of $\tilde{\Psi}^{ab}$ near i can be read off from Eqs. (67) and (80)–(82). In order to study its smoothness near i'

we perform the inversion to obtain the tensor in the coordinates y^i. It turns out that we obtain the same expressions as before if we make the replacements

$$n^i \to -n^i, \quad m^i \to m^i,$$
$$r \to 1/r, \quad \xi \to \xi, \quad \eta_1 \to -\eta_1, \quad \mu \to \mu,$$
$$P^a \to Q^a, \quad J^a \to -J^a, \quad A \to A, \quad Q^a \to P^a.$$

Thus the tensors (76)–(78), the first two of which are the only ones which contribute to the momentum and angular momentum, are singular in i' as well as in i. Observing again the conformal covariance, we can state the following result.

Corollary 3. *The general smooth solution of the equation $D_a \tilde{\Psi}^{ab} = 0$ on $S^3 \setminus \{i, i'\}$ with respect to the metric $\omega^4 h^0$, where h^0 denotes the standard metric on the unit 3-sphere and $\omega \in C^\infty(S^3)$, $\omega > 0$, is given by $\tilde{\Psi}^{ab} = (\omega^{-1}\theta)^{10}\,\Psi^{ab}$ with Ψ^{ab} as in (83) and $\lambda \in C^\infty(\mathbb{E}^3 \setminus \{i\})$. If we require the solution to be bounded near i' (in particular, if we construct solutions with only one asymptotically flat end), the quantities P^a, J^a, A, Q^a must vanish.*

We can now provide tensor fields which satisfy condition (11) and thus prove a special case of Theorem 2.

Theorem 15. *Denote by Ψ^{ab} a tensor field of the type (83). If $r\lambda \in E^\infty(B_a)$ and $P^a = 0$, then $r^8 \Psi_{ab}\Psi^{ab} \in E^\infty(B_a)$.*

Only the part of λ which in an expansion in terms of spherical harmonics is of order $l \geq 2$ contributes to Ψ_{ab}. We note that the condition $r\lambda \in E^\infty(B_a)$ entails that this part is of order r. The singular parts of Ψ_{ab} are of the form (77)–(78).

For the proof we need certain properties of the $\eth$ operator. If m^a is suitably adapted to standard spherical coordinates, the $\eth$ operator, acting on functions of spin weight s, acquires the form

$$\eth\eta_s = -(\sin\theta)^s \left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\left((\sin\theta)^{-s}\eta_s\right). \tag{90}$$

The operator $\bar{\eth}\eth$ acting on functions with $s = 0$ is the Laplace operator on the unit sphere, i.e. we have the identity

$$\bar{\eth}\eth f = r^2\Delta f - x^i x^j \partial_i\partial_j f - 2x^i\partial_i f, \tag{91}$$

where Δ denotes the Laplacian on $\mathbb{R}^3$. The commutator is given by

$$(\bar{\eth}\eth - \eth\bar{\eth})\eta_s = 2s\eta_s. \tag{92}$$

From this formula we obtain for $q \in \mathbb{N}$ by induction the relations

$$\left(\bar{\eth}\eth^q - \eth^q\bar{\eth}\right)\eta_s = (2qs + (q-1)q)\eth^{q-1}\eta_s, \tag{93}$$

$$\left(\eth\bar{\eth}^q - \bar{\eth}^q\eth\right)\eta_s = (-2qs + (q-1)q)\bar{\eth}^{q-1}\eta_s. \tag{94}$$

In particular, it follows that the functions η and μ, which satisfy $\eth\eta = \eta_1$ and $\eth^2\mu = \mu_2$, are given by

$$\eta = -2r\partial_r\left(\bar{\eth}\eth\lambda^R + 2\lambda^R\right) + \bar{\eth}\eth\lambda^I + 2\lambda^I + rQ - \frac{P}{r} + iJ, \tag{95}$$

$$\mu = 2r\partial_r\left(r\partial_r\lambda^R\right) + \bar{\eth}\eth\lambda^R - 2r\partial_r\lambda^I. \tag{96}$$

Lemma 9. *Let $f, g \in E^\infty$ be two complex functions (spin weight zero). Then, the functions $\bar{\eth}^q f \eth^q g$, $q \in \mathbb{N}$, of spin weight zero are also in E^∞.*

Remarkably, the statement is wrong if we replace E^∞ by C^∞: the calculation in polar coordinates, using (90), gives

$$\eth x^1\,\bar{\eth}x^3 = -x^1\,x^3 + i\,r\,x^2.$$

Proof. For $q = 1$ the proof follows from two identities. The first one is a simple consequence of the Leibniz rule

$$\eth f\bar{\eth}g + \bar{\eth}f\eth g = \bar{\eth}\eth(fg) - f\bar{\eth}\eth g - g\bar{\eth}\eth f. \tag{97}$$

Since $\bar{\eth}\eth$ maps by (91) smooth functions into smooth functions, it follows that $(\eth f\bar{\eth}g + \bar{\eth}f\eth g) \in E^\infty$ if $f, g \in E^\infty$ (here we can replace E^∞ by C^∞).

The other identity reads

$$\eth f\bar{\eth}g - \bar{\eth}f\eth g = 2\,i\,r\,\epsilon_l{}^{jk}\,x^l\,\partial_j f\,\partial_k g. \tag{98}$$

It is obtained by expressing (90) in the Cartesian coordinates x^l. Important for us is the appearance of the factor r. A particular case of this relation has been derived in [27]. It follows that $(\eth f\bar{\eth}g - \bar{\eth}f\eth g) \in E^\infty$ if $f, g \in E^\infty$. Taking the difference of (97) and (98) gives the desired result.

To obtain the result for arbitrary q, we proceed by induction. The Leibniz rule gives

$$\eth^{q+1}f\bar{\eth}^{q+1}g = \bar{\eth}\eth(\eth^q f\bar{\eth}^q g) - \bar{\eth}\eth^q f\eth\bar{\eth}^q g - \eth^q f\eth\bar{\eth}^{q+1}g - \bar{\eth}^q g\eth\bar{\eth}\eth^q f.$$

The induction hypothesis for q and (91) imply that the first term on the right-hand side is in E^∞. The factors appearing in the following terms can be written by (93) and (94) in the form:

$$\bar{\eth}\eth^q f = \eth^{q-1}(\eth\bar{\eth}f + q(q-1)f), \quad \eth\bar{\eth}^q g = \bar{\eth}^{q-1}(\eth\bar{\eth}g + q(q-1)g),$$
$$\eth\bar{\eth}^{q+1}g = \bar{\eth}^q(\bar{\eth}\eth g + q(q-1)g), \qquad \eth\bar{\eth}\eth^q f = \eth^q(\eth\bar{\eth}f + q(q-1)f).$$

Since the functions in parentheses are, by (91), in E^∞, the induction hypothesis implies that each of the products is in E^∞. □

Proof of Theorem 15. In terms of the coefficients (80)–(82) we have

$$r^8\Psi_{ab}\Psi^{ab} = r^2\left(2\mu_2\bar{\mu}_2 + 2\eta_1\bar{\eta}_1 + 3\xi^2\right). \tag{99}$$

Since $r\lambda$ is in $E^\infty(B_a)$, Eqs. (95) and (96) imply that $r\xi, r\eta, r\mu \in E^\infty$. The conclusion now follows from Lemma 9. □

4.2. The general case: Existence. The existence of solutions to the momentum constraint for asymptotically flat initial data has been proved in weighted Sobolev spaces (cf. [12,15–17] and the references given there) and in weighted Hölder spaces [13]. The existence of initial data with non-trivial momentum and angular momentum and the role of conformal symmetries have been analysed in some detail in [9]. In this section we will prove existence of solutions to the momentum constraint with non-trivial momentum and angular momentum following the approach of [9]. We generalize some of the results shown in [9] to metrics in the class (21). The results of the previous section will be important for the analysis of the general case.

We use the York splitting methods to reduce the problem of solving the momentum constraint to solving a linear elliptic system of equations. Let the conformal metric h on the initial hypersurface S be given. We use it to define the overdetermined elliptic conformal Killing operator $\mathcal{L}_h$, which maps vector fields v^a onto symmetric h-trace-free tensor fields according to

$$(\mathcal{L}_h v)^{ab} = D^a v^b + D^b v^a - \frac{2}{3} h^{ab} D_c v^c, \tag{100}$$

and the underdetermined elliptic divergence operator δ_h which maps symmetric h-trace-free tensor fields Φ^{ab} onto vector fields according to

$$(\delta_h \Phi)^a = D_b \Phi^{ba}. \tag{101}$$

Let Φ^{ab} be a symmetric h-trace-free tensor field and set

$$\Psi^{ab} = \Phi^{ab} - (\mathcal{L}_h v)^{ab}. \tag{102}$$

Then Ψ^{ab} will satisfy the equation $D_a \Psi^{ab} = 0$ if the vector field v^a satisfies

$$\mathbf{L}_h v^a = D_b \Phi^{ab}, \tag{103}$$

where the operator $\mathbf{L}_h$ is given by

$$\mathbf{L}_h v^a = D_b(\mathcal{L}_h v)^{ab} = D_b D^b v^a + \frac{1}{3} D^a D_b v^b + R^a{}_b v^b. \tag{104}$$

Since $\Psi^{ab}(\mathcal{L}_h v)_{ab} = 2\, D_a(\Psi^{ab} v_b) - 2\,(\delta_h \Psi)^a v_a$ for arbitrary vector fields v^a and symmetric h-trace-free tensor fields Ψ^{ab}, $\mathcal{L}_h$ has formal adjoint $\mathcal{L}_h^* = -2\,\delta_h$. Thus

$$\mathbf{L}_h = -\frac{1}{2}\, \mathcal{L}_h^* \circ \mathcal{L}_h, \tag{105}$$

and the operator is seen to be elliptic.

Provided the given data are sufficiently smooth, we can use Theorem 9 to show the existence of solutions to (103).

Lemma 10 (Regular case). *Assume that the metric satisfies (21) and Φ^{ab} is an h-trace-free symmetric tensor field in $W^{1,p}(S)$, $p > 1$. Then there exists a unique vector field $v^a \in W^{2,p}(S)$ such that the tensor field $\Psi^{ab} = \Phi^{ab} - (\mathcal{L}_h v)^{ab}$ satisfies the equation $D_a \Psi^{ab} = 0$ in S.*

Proof. The well known argument that the condition of Theorem 9 will be satisfied extends to our case. Assume that the vector field ξ^a is in the kernel of L_h, i.e.

$$\mathbf{L}_h \xi^a = 0 \text{ in } S. \tag{106}$$

Since the metric satisfies (21), elliptic regularity gives

$$\xi^a \in C^{2,\alpha}(S). \tag{107}$$

This smoothness suffices to conclude from (105), (106) that $0 = -2\,(\xi, \mathbf{L}_h \xi)_{L^2} = (\mathcal{L}_h \xi, \mathcal{L}_h \xi)_{L^2}$, whence

$$(\mathcal{L}_h \xi)_{ab} = 0. \tag{108}$$

This implies for an arbitrary symmetric, h-trace-free tensor field $\Phi^{ab} \in W^{1,p}(S)$, $p > 1$, the relation $0 = (\mathcal{L}_h \xi, \Phi)_{L^2} = -2\,(\xi, \delta_h \Phi)_{L^2}$, which shows that the Fredholm condition will be satisfied for any choice of Φ^{ab} in (103). □

We call the case above the "regular case" because the solution still satisfies the condition $\Psi^{ab} \in W^{1,p}(S)$. While this allows us to have solutions diverging like $O(r^{-1})$ at given points, it excludes solutions with non-vanishing momentum or angular momentum.

We note that by (108) the kernel of $\mathbf{L}_h$ consists of conformal Killing fields. Let ξ^a be such a vector field. Using (107) and Lemma 14 we find that we can write in normal coordinates centered at the point i of S

$$\xi^k = \xi_0^k + O(r^{2+\alpha}), \tag{109}$$

where ξ_0^k is the "flat" conformal Killing field (84) with coefficients given by

$$k_a = \frac{1}{6} D_a D_b \xi^b(i), \quad S^a = \epsilon^a{}_{bc} D^b \xi^c(i), \quad q^a = \xi^a(i), \quad a = \frac{1}{3} D_a \xi^a(i). \tag{110}$$

Since S is connected, the integrability conditions for conformal Killing fields (cf. [34]) entail that these ten "conformal Killing data at i" determine the field ξ^a uniquely on S.

With a conformal rescaling of the metric with a smooth, positive, conformal factor ω

$$h_{ab} \to h'_{ab} = \omega^4 h_{ab}, \tag{111}$$

which implies a corresponding change of the connection $D_a \to D'_a$, we associate the rescalings

$$\Psi^{ab} \to \Psi'^{ab} = \omega^{-10} \Psi^{ab}, \quad \xi^a \to \xi'^a = \xi^a, \tag{112}$$

for h-trace free, symmetric tensor fields Ψ^{ab} and Killing fields ξ^a. Then the conformal Killing operator and the divergence operator satisfy

$$(\mathcal{L}_{h'} v)^{ab} = \omega^{-4} (\mathcal{L}_h v)^{ab}, \quad D'_a \Psi'^{ab} = \omega^{-10} D_a \Psi^{ab}. \tag{113}$$

If we write $\omega = e^f$, the conformal Killing data transform as

$$k'^a = k^a + 2a D^a f(i) + \epsilon^{abc} S_b D_c f(i) + q_c D^a D^c f(i), \tag{114}$$

$$S'^a = S^a + 2\epsilon^{abc} q_b D_c f(i), \tag{115}$$

$$a' = a + q^a D_a f(i), \tag{116}$$

$$q'^a = q^a. \tag{117}$$

A vector field ξ on S will be called an "asymptotically conformal Killing field" at the point i if it satisfies in normal coordinates x^k centered at i Eq. (109) with (84). While any conformal Killing field is an asymptotically conformal Killing field, the converse need not be true. Let ξ^a be an asymptotically conformal Killing field and consider the integral

$$I_\xi = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \Psi^{ab} \, n_a \, \xi_b \, dS_\epsilon, \tag{118}$$

where in the coordinates x^k the unit normal n^k to the sphere ∂B_ϵ around i approaches $x^k/|x|$ for small ϵ. As shown in the previous section, the integral vanishes if the tensor Ψ^{ab} is of order r^{-1} at i. If, however, Ψ^{ab} is of order r^{-n}, $n = 2, 3, 4$, at i, the integral gives non-trivial results and can be understood in particular as a linear form on the momentum and angular momentum of the data.

We recall two important properties of the integral I_ξ. The first is the fact that the integral I_ξ is invariant under the rescalings (111), (112). The second property is concerned with the presence of conformal symmetries. Let Φ^{ab} be an arbitrary h-trace-free tensor field and v^a an arbitrary vector field. Using Gauss' theorem, we obtain

$$2 \int_{\partial B_\epsilon} \Phi^{ab} \, n_a \, v_b \, dS_\epsilon = -2 \int_{S - B_\epsilon} v_a \, D_b \Phi^{ab} \, d\mu - \int_{S - B_\epsilon} (\mathcal{L}_h v)_{ab} \Phi^{ab} \, d\mu, \tag{119}$$

with the orientation of n^k as above. If we apply this equation to (118), the first term on the right-hand side will vanish if $D_a \Psi^{ab} = 0$ in $S \setminus \{i\}$, while the second term on the right-hand side need not vanish if ξ^a is only an asymptotically conformal Killing vector field. However, if ξ is a conformal Killing field, we get $I_\xi = 0$. Thus the presence of Killing fields entails restrictions on the values allowed for the momentum and angular momentum of the data. For this reason the presence of conformal symmetries complicates the existence proof. Note that we are dealing with vector fields ξ^a which satisfy the conformal Killing equation (108) everywhere in S; in general, a small local perturbation of the metric will destroy this conformal symmetry.

Observing the conformal covariance (113) of the divergence equation, we perform a rescaling of the form (25), (27) such that the metric h' has in h'-normal coordinates x^k in B_a centered at i local expression h'_{kl} with

$$h'_{kl} - \delta_{kl} = O(r^3), \quad \partial_j h'_{kl} = O(r^2). \tag{120}$$

In these coordinates let $\Psi^{ik}_{\text{flat}} \in C^\infty(B_a \setminus \{i\})$ be a trace free symmetric and divergence free tensor with respect to the flat metric δ_{kl},

$$\delta_{ik} \, \Psi^{ik}_{\text{flat}} = 0, \qquad \partial_i \Psi^{ik}_{\text{flat}} = 0 \text{ in } B_a \setminus \{i\}, \tag{121}$$

with

$$\Psi^{ik}_{\text{flat}} = O(r^{-4}), \quad \partial_j \Psi^{ik}_{\text{flat}} = O(r^{-5}) \text{ as } r \to 0. \tag{122}$$

All these tensors have been described in Theorem (14). Note that the conditions (122) are essentially conditions on the function λ which characterizes the part Ψ^{ab}_λ of (83). Denote by $\Phi^{ab}_{\text{sing}} \in C^\infty(S \setminus \{i\})$ the h'-trace free tensor which is given on $B_a \setminus \{i\}$ by

$$\Phi^{ab}_{\text{sing}} = \chi \left(\Psi^{ab}_{\text{flat}} - \frac{1}{3} h'_{cd} \, \Psi^{cd}_{\text{flat}} \, h'^{ab} \right), \tag{123}$$

and vanishes elsewhere. Here χ denotes a smooth function of compact support in B_a equal to 1 on $B_{a/2}$. By our assumptions we have then

$$D'_a \Phi^{ab}_{\rm sing} = O(r^{-2}) \text{ as } \to 0. \tag{124}$$

Theorem 16 (Singular case). *Assume that the metric h satisfies (21), ω_0 denotes the conformal factor (25), $h' = \omega_0^4\, h$, $\Phi^{ab}_{\rm sing}$ is the tensor field defined above, and $\Phi^{ab}_{\rm reg}$ is a symmetric h-trace free tensor field in $W^{1,p}(S)$, $p > 1$.*

i) *If the metric h admits no conformal Killing fields on S, then there exists a unique vector field $v^a \in W^{2,q}(S)$, with $q = p$ if $p < 3/2$ and $1 < q < 3/2$ if $p \geq 3/2$, such that the tensor field*

$$\Psi^{ab} = \omega_0^{10} \left(\Phi^{ab}_{\rm sing} + \Phi^{ab}_{\rm reg} + (\mathcal{L}_{h'} v)^{ab} \right), \tag{125}$$

satisfies the equation $D_a \Psi^{ab} = 0$ in $S \setminus \{i\}$.

ii) *If the metric h admits conformal Killing fields ξ^a on S, a vector field v^a as specified above exists if and only if the constants k^a, S^a, a, q^a (partly) characterizing the tensor field $\Phi^{ab}_{\rm sing}$ (cf. (83), (76)–(79)), satisfy the equation*

$$P^a\, k_a + J^a\, S_a + A\, a + \left(P^c\, L_c{}^a(i) + Q^a\right)\, q_a = 0, \tag{126}$$

for any conformal Killing field ξ^a of h, where the constants k_a, S_a, a, q_a characterizing ξ^a are given by (110).

In both cases the momentum and angular momentum (cf. (86), (87)) of Ψ^{ab} agree with those of the tensor $\Phi^{ab}_{\rm sing}$. These quantities can thus be prescribed freely in case (i).

Proof. Because of (124) we can consider $D'_a(\Phi^{ab}_{\rm sing} + \Phi^{ab}_{\rm reg})$ as a function in $L^p(S)$, $1 < p < 3/2$. In case (i) the kernel of the operator $\mathcal{L}_{h'}$ appearing in the equation

$$D'_a \left(\Phi^{ab}_{\rm sing} + \Phi^{ab}_{\rm reg} + (\mathcal{L}_{h'} v)^{ab} \right) = 0,$$

is trivial and we can apply Theorem 9 to show that the equation above determines a unique vector field v^a with the properties specified above. After the rescaling we have $D_a \Psi^{ab} = 0$ by (113).

In case (ii) the kernel of $\mathcal{L}_{h'}$ is generated by the conformal Killing fields $\xi'^a = \xi^a$ of h'. If we express (119) in terms of the metric h', the tensor field $\Phi^{ab}_{\rm sing} + \Phi^{ab}_{\rm reg}$, and the vector field ξ'^a, take the limit $\epsilon \to 0$ and use Eq. (85), we find that the Fredholm condition of Theorem 9 is satisfied if and only if for every conformal Killing field ξ'^a of h' we have

$$P^a\, k'_a + J^a\, S'_a + A\, a' + Q^a\, q'_a = 0,$$

where the constants k'_a, S'_a, a', q'_a are given by Eq. (110), expressed in terms of ξ'^a and h'. By (114)–(117) this condition is identical with (126).

As shown in the previous section, we can choose $\Phi^{ab}_{\rm sing}$ such that the corresponding momentum and angular momentum integrals take preassigned values, which can be chosen freely in case (i) and need to satisfy (126) in case (ii). These values will agree with those obtained for $\omega_0^{-10} \Psi^{ab}$ due to the regularity properties of v^a. After the rescaling the values of the momentum and the angular momentum remain unchanged because $\omega_0 = 1 + O(r^2)$. □

We note that it is the presence of the 10-dimensional space of conformal Killing fields on the standard 3-sphere which led to the observation made in Corollary (3). The latter generalizes as follows.

Corollary 4. *If S is an arbitrary compact manifold, h satisfies (21), and we allow for $p \geq 2$ asymptotic ends i_k, $1 \leq k \leq p$, we can choose the sets of constants $(P^a_k, J^a_k, A_k, Q^a_k)$ arbitrarily in the ends i_k, $1 \leq k \leq p-1$. Which constants can be chosen at the end i_p depends on the conformal Killing fields admitted by h.*

Proof. This follows from the observation that in the case of p ends Eq. (126) generalizes to an equation of the form

$$\sum_{l=1}^{p} P^a_l \, k^l_a + J^a_l \, S^l_a + A_l \, a^l + \left(P^c_l \, L_c{}^a(i_l) + Q^a_l\right) q^l_a = 0,$$

where the constants bear for given l the same meaning with respect to the point i_l as the constants in (126) with respect to i. □

The case of spaces conformal to the unit 3-sphere (S^3, h_0) is very exceptional. A result of Obata [33], discussed in [9] in the context of the constraint equations, says that unless the manifold (S, h) is conformal to (S^3, h_0) there exists a smooth conformal factor such that in the rescaled metric h' every conformal Killing field is in fact a Killing field. Thus the dimension of the space of conformal Killing fields cannot exceed 6. In fact, it has been shown in [9] that in that case h' can admit at most four independent Killing fields and only one of them can be a rotation. In this situation Eq. (126), written in terms of the metric h', reduces by (110) to

$$J^a \, S'_a + \left(Q^a + P^b \, L'_a{}^b(i)\right) q_a = 0,$$

since $D'_a \xi^a = 0$ for a Killing field. The constants P^a and A can be prescribed arbitrarily. If there does exist a rotation among the Killing fields, the equation above implies

$$J^a \, S'_a = 0, \quad \left(Q^a + P^b \, L'_a{}^b(i)\right) q_a = 0.$$

4.3. Asymptotic expansions near i of solutions to the momentum constraint. In this section we shall prove an analogue of Theorem 13 for the operator $\mathbf{L}_h$ defined in (104). It will be used to analyse the behaviour of the solutions to the momentum constraint considered in Theorem (16) near i and to show the existence of a general class of solutions which satisfy condition (11). Our result rests on the close relation between the operator $\mathbf{L}_h$ and the Laplace operator.

We begin with a discussion on $\mathbb{R}^3$ and write $x_i = x^i$, $\partial^i = \partial_i$. The flat space analogue of $\mathbf{L}_h$ on $\mathbb{R}^3$ is given by

$$\mathbf{L}_0 v^k = \Delta v^k + \frac{1}{3} \partial^k \partial_l \, v^l, \tag{127}$$

where Δ denotes the flat space Laplacian and v^k a vector field on some neighbourhood of the origin in $\mathbb{R}^3$.

The following spaces of vector fields whose components are homogeneous polynomials of degree m and smooth functions respectively will be important for us.

Definition 2. *Let* $m \in \mathbb{N}$, $m \geq 1$. *We define the real vector spaces* $\mathcal{Q}_m$, $\mathcal{Q}_\infty(B_a)$ *by*

$$\mathcal{Q}_m = \{v \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) |\, v^i \in \mathcal{P}_m, \quad v^i x_i = r^2 v \text{ with } v \in \mathcal{P}_{m-1}\},$$
$$\mathcal{Q}_\infty(B_a) = \{v \in C^\infty(B_a, \mathbb{R}^3) |\, v^i x_i = r^2 v \text{ with } v \in C^\infty(B_a)\}.$$

The following lemma, an analogue of Lemma 3, rests on the conditions imposed on the vector fields above.

Lemma 11. *Suppose* $s \in \mathbb{Z}$. *Then the operator* $\mathbf{L}_0$ *defines a linear, bijective mapping of vector spaces*

$$\mathbf{L}_0 : r^s \mathcal{Q}_m \to r^{s-2} \mathcal{Q}_m,$$

in the following cases:

(i) $s > 0$,
(ii) $s < 0$, $|s|$ *is odd and* $m + s \geq 0$.

Note that the assumptions on m and s imply that the vector field $\mathbf{L}_0(r^s p^i) \in C^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{R}^3)$ defines a vector field in $L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ which represents $\mathbf{L}_0(r^s p^i)$ in the distributional sense.

Proof. For s as above and $p^i \in \mathcal{Q}_m$ there exists some $p_{m-1} \in \mathcal{P}_{m-1}$ with

$$\partial_k(r^s p^k) = r^s q_{m-1} \text{ with } q_{m-1} = s\, p_{m-1} + \partial_k p^k \in \mathcal{P}_{m-1}. \tag{128}$$

With Eq. (34) it follows that

$$\mathbf{L}_0(r^s p^i) = r^{s-2} \hat{p}^i \text{ with}$$
$$\hat{p}^i = s\,(s + 1 + 2m)\, p^i + r^2 \Delta p^i + \frac{1}{3}\,(s\, x^i\, q_{m-1} + r^2\, \partial^i q_{m-1}) \in \mathcal{P}_m.$$

Moreover, $\hat{p}^i \in \mathcal{Q}_m$ because $\hat{p}^i x_i = r^2\, \hat{p}_{m-1}$ with

$$\hat{p}_{m-1} = s\,(s + 1 + 2m)\, p_{m-1} + x_i\, \Delta p^i + \frac{1}{3}(s\, q_{m-1} + x_i\, \partial^i q_{m-1}) \in \mathcal{P}_{m-1}.$$

To show that the kernel of the map is trivial, assume that $\mathbf{L}_0(r^s p^i) = 0 \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. Taking a (distributional) derivative we obtain

$$0 = \partial_i\, \mathbf{L}_0(r^s p^i) = \frac{4}{3}\, \Delta(\partial_i(r^s\, p^i)) = \frac{4}{3}\, \Delta(r^s\, q_{m-1}). \tag{129}$$

When $s > 0$ or $|s|$ odd and $m - 1 + s \geq 0$ we use Lemma 3 to conclude that $q_{m-1} = 0$. We insert this in the equation $\mathbf{L}_0(r^s p^i) = 0$ to obtain $\Delta(r^s p^i) = 0$ and conclude again by Lemma 3 that $p^i = 0$.

There remains the case $s + m = 0$ with $|s|$ odd. Expanding q_{m-1} in Eq. (129) in harmonic polynomials (cf. (35)), we get

$$0 = \Delta(r^{-m} q_{m-1}) = \sum_{0 \leq k \leq (m-1)/2} \Delta(r^{2k-m} h_{m-1-2k}),$$

whence, by (34),

$$\sum_{0\le k\le (m-1)/2} r^{2k-m-2}\,(m-2\,k)\,(m-2\,k-1)\,h_{m-1-2k}=0.$$

Since this sum is direct each summand must vanish separately. Since m is odd, the only factor $(m-2\,k-1)$ which vanishes occurs when $2k=m-1$, from which we conclude that $q_{m-1}=r^{m-1}h_0$ with a constant h_0. Since Eq. (129), which reads now $h_0\,\Delta(r^{-1})=0$, holds in the distributional sense, it follows that $h_0=0$. □

Unless noted otherwise we shall assume in the following that the metric h is of class C^∞ and that it is chosen in its conformal class such that its Ricci tensor vanishes at i (cf. (25), (27)). By x^i will always be denoted a system of h-normal coordinates centered at i and all our calculations will be done in these coordinates. Thus we have

$$h_{kl}=\delta_{kl}+O(r^3),\quad \partial_j h_{kl}=O(r^2).$$

We write the operator $\mathbf{L}_h$ in the form

$$\mathbf{L}_h=\mathbf{L}_0+\hat{\mathbf{L}}_h,$$

where, with the notation of (22),

$$(\hat{\mathbf{L}}_h v)_i=\hat h^{jk}\partial_j\partial_k v_i+\frac{1}{3}\hat h^{jk}\partial_i\partial_k v_j+B^{jk}{}_i\,\partial_j\,v_k+A^j{}_i\,v_j, \tag{130}$$

with

$$B^{kj}{}_i=-2\,h^{jl}\,\Gamma_l{}^k{}_i-\frac{4}{3}\,h^{lf}\,\Gamma_l{}^j{}_f\,h^k{}_i+\frac{1}{3}\,\partial_i\,h^{jk},$$

and $A^j{}_i$ is a function of the metric coefficients and their first and second derivatives. The fields $A^j{}_i$, $B^{kj}{}_i$ are smooth and satisfy

$$A^j{}_i=O(r),\quad B^{kj}{}_i=O(r^2), \tag{131}$$

and, because $x_k\,x^i\,\Gamma_i{}^k{}_j=0$ at the point with normal coordinates x^k,

$$x_k\,x^i\,B^{kj}{}_i=-\frac{4}{3}\,r^2\,h^{lf}\,\Gamma_l{}^j{}_f. \tag{132}$$

Similarly, we write the operator $\mathcal{L}_h$ in the form

$$\mathcal{L}_h=\mathcal{L}_0+\hat{\mathcal{L}}_h.$$

Lemma 12. *Suppose $p^i\in\mathcal{Q}_m$. Then $\hat{\mathbf{L}}_h(r^s p^i)=r^{s-2}U^i$ with some $U^i\in\mathcal{Q}_\infty(B_a)$ which satisfies $U^i=O(r^{m+3})$.*

Proof. Using (130), we calculate $r^{-s+2}\,\hat{\mathbf{L}}_h(r^s p^i)$ and find

$$U_i=\hat h^{kj}\left(s\delta_{kj}\,p^i+r^2\partial_k\partial_j\,p_i\right)+\frac{1}{3}\hat h^{jk}\left(sx_i\,\partial_k\,p_j+r^2\partial_k\partial_i\,p_j+\delta_{ki}\,p_j\right)$$
$$+\,B^{kj}{}_i\left(sx_k\,p_j+r^2\partial_k\,p_i\right)+r^2A^j{}_i\,p_j. \tag{133}$$

Thus U^i is smooth. Using (131) we obtain that $U_i=O(r^{m+3})$, using (132) we find $x^iU_i=r^2f$ with some smooth function f. □

We are in a position now to prove for $m = \infty$ the analogue of point (ii) of Theorem 13.

Theorem 17. *Assume that h is smooth, $s \in \mathbb{Z}$, $s < 0$, $|s|$ odd, $F^i \in C^\infty(B_a)$, and $J^i \in \mathcal{Q}_\infty(B_a)$ with $J^i = O(r^{s_0})$ for some $s_0 \geq |s|$.*

Then, if $v^i \in W^{2,p}_{\mathrm{loc}}(B_a)$ solves

$$(\mathbf{L}_h v)^i = r^{s-2} J^i + F^i,$$

it can be written in the form

$$v^i = r^s v_1^i + v_2^i, \tag{134}$$

with $v_1^i \in \mathcal{Q}_\infty(B_a)$, $v_1^i = O(r^{s_0})$, $v_2^i \in C^\infty(B_a)$.

Proof. The proof is similar to that of Theorem 13. For given $m \in \mathbb{N}$ we can write by our assumptions $J^i = T_m^i + J_R^i$, where $J_R^i = O(r^{m+1})$ and T_m^i denotes the Taylor polynomial of J^i of order m. Because $J^i \in \mathcal{Q}_\infty(B_a)$, its Taylor polynomial can be written in the form

$$T_m^i = \sum_{k=s_0}^{m} t_k^i \quad \text{with} \quad t_k^i \in \mathcal{Q}_k.$$

We define now a function v_R^i (depending on m) by

$$v^i = r^s \sum_{k=s_0}^{m} v_k^i + v_R^i.$$

The quantities $(v_k^i) \in \mathcal{Q}_k$ are determined by the recurrence relation

$$\mathbf{L}_0(r^s v_{s_0}^i) = r^{s-2} t_{s_0}^i, \quad \mathbf{L}_0(r^s v_k^i) = r^{s-2}(t_k^i - U_k^{(k)i}),$$

where, for given k, the quantity $U_k^{(k)i} \in \mathcal{Q}_k$ is obtained as follows. The function

$$U^{(k)i} \equiv r^{-s+2}\, \hat{\mathbf{L}} \left(r^s \sum_{j=s_0}^{k-1} v_k^i \right),$$

has by Lemma 12 an expansion

$$U^{(k)i} = \sum_{j=s_0+2}^{m} U_j^{(k)i} + U_R^{(k)i} \quad \text{with} \quad U_j^{(k)i} \in \mathcal{Q}_j,$$

from which we read off $U_k^{(k)i}$. By Lemma 11 the recurrence relation is well defined.

With these definitions, the remainder v_R^i satisfies the equation

$$\mathbf{L} v_R^i = r^{s-2}\left(U_R^{(m+1)i} + J_R^i \right) + F^i.$$

By Lemma 4 the right-hand side of this equation is in $C^{m+s-2,\alpha}(B_\epsilon)$. By elliptic regularity we have $v_R^i \in C^{m+s,\alpha}(B_a)$. Since m was arbitrary, the conclusion follows now by an argument similar to the one used in the proof of Lemma 5. $\square$

Theorem 17 will allow us to prove that the solutions of the momentum constraint obtained in Sect. 4.2 have an expansion of the form (13), if we impose near i certain conditions on the data which can be prescribed freely.

In definition (123) of the field $\Phi^{ab}_{\rm sing}$, which enters Theorem (16), we assume that $\Psi^{ab}_{\rm flat}$ is of the form (83) with $\lambda \equiv 0$, i.e. it is given by the tensor fields (76)–(79). In order to write $\Psi^{ab}_{\rm flat}$ in a convenient form, we introduce vector fields which are given in normal coordinates by

$$v^i_P = -\frac{1}{4} P^k \partial_k \partial^i r^{-1} = r^{-5} p^i_P, \qquad p^i_P = \frac{1}{4}(r^2 P^i - 3 x^i P^k x_k) \in \mathcal{Q}_2, \tag{135}$$

$$v^i_J = \epsilon^{ijk} J_j \partial_k r^{-1} = r^{-3} p^i_J, \qquad p^i_J = -\epsilon^{ijk} J_j x_k \in \mathcal{Q}_1, \tag{136}$$

$$v^i_A = \frac{1}{2} A\, \partial^i r^{-1} = r^{-3} p^i_A, \qquad p^i_A = -\frac{1}{2} A\, x^i \in \mathcal{Q}_1, \tag{137}$$

$$v^i_Q = -2\, Q^i r^{-1} + \frac{1}{4} Q^k \partial_k \partial^i r = r^{-3} p^i_Q, \quad p^i_Q = -\frac{7}{4} r^2 Q^i - \frac{1}{4} x^i Q^k x_k \in \mathcal{Q}_2, \tag{138}$$

where P^i, J^i, A, Q^i are chosen such that the vector fields satisfy

$$(\mathcal{L}_0 v_P)^{ab} = \Psi^{ab}_P, \quad (\mathcal{L}_0 v_J)^{ab} = \Psi^{ab}_J, \quad (\mathcal{L}_0 v_Q)^{ab} = \Psi^{ab}_Q, \quad (\mathcal{L}_0 v_A)^{ab} = \Psi^{ab}_A, \tag{139}$$

with Ψ^{ab}_P, Ψ^{ab}_J, Ψ^{ab}_A, Ψ^{ab}_Q as given by (76)–(79). We have on $B_a \setminus \{i\}$,

$$(\mathbf{L}_0 v_P)^a = 0, \quad (\mathbf{L}_0 v_J)^a = 0, \quad (\mathbf{L}_0 v_A)^a = 0, \quad (\mathbf{L}_0 v_Q)^a = 0, \tag{140}$$

and can thus write on $B_{a/2} \setminus \{i\}$,

$$\Phi^{ij}_{\rm sing} = \big(\mathcal{L}_0(v_P + v_J + v_A + v_Q)\big)^{ij} - \frac{1}{3} h^{ij} h_{kl} \big(\mathcal{L}_0(v_P + v_J + v_A + v_Q)\big)^{kl}. \tag{141}$$

Of the field $\Phi^{ab}_{\rm reg} \in W^{1,p}(S)$, $p > 1$, entering Theorem 16 we assume that it can be written near i in the form

$$\Phi^{ab}_{\rm reg} = r^s\, \Phi^{ab}_{\rm 1reg} + \Phi^{ab}_{\rm 2reg}, \tag{142}$$

where $s \leq -1$ is some integer which will be fixed later on, $\Phi^{ab}_{\rm 1reg}$, $\Phi^{ab}_{\rm 2reg}$ are smooth in B_a and such that $\Phi^{ij}_{\rm 1reg} = O(r^{-s-1})$, and $x_i\, x_j\, \Phi^{ij}_{\rm 1reg} = r^2\, \Phi$ with some $\Phi \in C^\infty(B_a)$. Then

$$J^i_{\rm reg} \equiv D_j\, \Phi^{ij}_{\rm reg} = r^{s-2} \hat{J}^i + D_j\, \Phi^{ij}_{\rm 2reg}, \tag{143}$$

with $\hat{J}^i = r^2 D_j\, \Phi^{ij}_{\rm 1reg} + s\, x_j\, \Phi^{ij}_{\rm 1reg} \in \mathcal{Q}_\infty$ and $\hat{J}^i = O(r^{-s})$.

Using Theorem 17, we obtain for the solutions of Theorem 16 (where we can set by our present assumptions $\omega_0 \equiv 1$, $h \equiv h'$) the following result.

Corollary 5. *With the tensor fields* $\Phi^{ab}_{\rm sing}$, $\Phi^{ab}_{\rm reg}$ *given by (141), (142) respectively, let the vector field* v^a *be such that*

$$\Psi^{ab} = \Phi^{ab}_{\rm sing} + \Phi^{ab}_{\rm reg} + (\mathcal{L}v)^{ab}, \tag{144}$$

satisfies $D_a\,\Psi^{ab} = 0$ *in* $B_a \setminus \{i\}$.

(i) *If* $P^a = 0$ *in (141) and* $s = -3$ *in (142), the vector field* v^a *can be written in the form*

$$v^i = r^{-3}\,v^i_1 + v^i_2, \quad \textit{with} \quad v^i_1 \in \mathcal{Q}_\infty(B_a), \quad v^i_1 = O(r^3), \quad v^i_2 \in C^\infty(B_a).$$

(ii) *If* $J^a = 0$, $A = 0$, $Q^a = 0$ *in (141) and* $s = -5$ *in (142), the vector field* v^a *can be written in the form*

$$v^i = r^{-5}\,v^i_1 + v^i_2, \quad \textit{with} \quad v^i_1 \in \mathcal{Q}_\infty(B_a), \quad v^i_1 = O(r^5), \quad v^i_2 \in C^\infty(B_a).$$

Proof. In both cases the vector field v^a satisfies $\mathbf{L}_h v^a = -J^a_{\rm sing} - J^a_{\rm reg}$ with $J^a_{\rm reg}$ given by (143) and $J^a_{\rm sing} = D_b\,\Phi^{ab}_{\rm sing}$. By Eq. (140) we have in case (i) $J^a_{\rm sing} = (\mathbf{L}_h(v_J + v_A + v_Q))^a = (\hat{\mathbf{L}}_h(v_J + v_A + v_Q))^a$, and in case (ii) $J^a_{\rm sing} = (\mathbf{L}_h\,v_P))^a = (\hat{\mathbf{L}}_h\,v_P))^a$ on $B_{a/2} \setminus \{i\}$. The results now follow from Eqs. (135)–(138), Lemma 12, and Theorem 17. □

We are in a position now to describe the behaviour of the scalar field $\Psi_{ab}\,\Psi^{ab}$ near i.

Lemma 13. *The tensor field (144) satisfies*
in case (i) $r^8\,\Psi_{ab}\,\Psi^{ab} \in E^\infty(B_a)$,
in case (ii) $r^8\,\Psi_{ab}\,\Psi^{ab} = \psi + r\,\psi^R$ *where* $\psi^R \in C^\alpha(B_a)$ *and* $\psi = \frac{15}{16}\,P_i\,P^i + r^{-2}\,h_2$ *with harmonic polynomial* $h_2 = \frac{3}{8}\,r^2\,(3\,(P^i\,n_i)^2 - P_i\,P^i)$.

Proof. For $w^i \in \mathcal{Q}_\infty(B_a)$ with $x_i\,w^i = r^2\,\hat{w}$, $\hat{w} \in C^\infty(B_a)$ we have

$$(\mathcal{L}_h(r^s\,w))^{ij} = r^s\,((\mathcal{L}_h w)^{ij} - \frac{2}{3}\,s\,h^{ij}\,\hat{w}) + r^{s-2}\,2\,s\,x^{(i}\,w^{j)}. \tag{145}$$

We set $s = -3$ and $w^i = p^i_J + p^i_A + p^i_Q$ in case (i) and $s = -5$ and $w^i = p^i_P$ in case (ii) and we write $x_i\,v^i_1 = r^2\,\hat{v}_1$ with $\hat{v}_1 \in C^\infty(B_a)$. Observing the equation above we get on $B_{a/2} \setminus \{i\}$ a representation

$$\Psi^{ij} = r^s\,H^{ij} + r^{s-2}\,K^{ij} + L^{ij}$$

with fields

$$H^{ij} = (\mathcal{L}_0\,w)^{ij} - \frac{2}{3}\,h^{ij}\,h_{kl}\,(\mathcal{L}_0\,w)^{kl} - \frac{2}{3}\,s\,\hat{w}\,\left(\delta^{ij} + 2\,h^{ij}\,(1 - \frac{1}{3}\,h_{kl}\,\delta^{kl})\right)$$
$$+ (\mathcal{L}_h\,v_1)^{kl} - \frac{2}{3}\,s\,h^{ij}\,\hat{v}_1 + \Phi^{ij}_{1\rm reg},$$
$$K^{ij} = 2\,s\,(x^{(i}\,w^{j)} + x^{(i}\,v^{j)}_1), \quad L^{ij} = \Phi^{ij}_{2\rm reg} + (\mathcal{L}_h\,v_2)^{ij},$$

which are in $C^\infty(B_{a/2})$. Since a direct calculation gives $K_{ij}\, K^{ij} = r^2 K$ with $K \in C^\infty(B_{a/2})$, we get

$$\Psi_{ij}\, \Psi^{ij} = r^{2s-2}\,(2\, H_{ij}\, K^{ij} - K) + r^{2s}\, H_{ij}\, H^{ij} + r^{s-2}\, 2\, K_{ij}\, L^{ij} + r^s\, 2\, H_{ij} L^{ij} + L_{ij}\, L^{ij},$$

from which we can immediately read off the desired result in case (i). In case (ii) it is obtained from our assumptions by a detailed calculation of $r^{2s-2}\,(2\, H_{ij}\, K^{ij} - K) + r^{2s}\, H_{ij}\, H^{ij}$. □

Combining the results above and observing the conformal invariance of the equations involved, we obtain the following detailed version of Theorem 2. We use here the notation of Theorem 16.

Theorem 18. *Assume that the metric h is smooth and Ψ^{ab} is the solution of the momentum constraint determined in Theorem 16. If*

(i) $\Phi^{ab}_{\text{sing}} = \Psi^{ab}_J + \Psi^{ab}_A + \Psi^{ab}_Q - \frac{2}{3}\, h^{ab}\, h_{cd}\,(\Psi^{cd}_J + \Psi^{cd}_A + \Psi^{cd}_Q)$ *in* $B_{a/2}$,

(ii) $\Phi^{ab}_{\text{reg}} = r^{-3}\, \Phi^{ab}_{1\text{reg}} + \Phi^{ab}_{2\text{reg}}$ *with* $\Phi^{ab}_{1\text{reg}}, \Phi^{ab}_{2\text{reg}} \in C^\infty(B_a)$ *such that* $\Phi^{ab}_{1\text{reg}} = O(r^2)$, *and* $x_a\, x_a\, \Phi^{ab}_{1\text{reg}} = r^2\, \Phi$ *with some* $\Phi \in C^\infty(B_a)$,

then Ψ^{ab} satisfies condition (11).

A. On Hölder Functions

In this section we want to prove an estimate concerning Hölder continuous functions.

Let B be an open ball in $\mathbb{R}^n$, $n \geq 1$, centered at the origin. Suppose $f \in C^k(U)$ for some $k \geq 0$ and m is a non-negative integer with $m \leq k$. Then we can write

$$f = \sum_{|\beta|<m} \frac{1}{\beta!}\, \partial^\beta f(0)\, x^\beta + m \int_0^1 (1-t)^{m-1} \sum_{|\beta|=m} \frac{1}{\beta!}\, \partial^\beta f(t\,x)\, x^\beta\, d\,t$$
$$= \sum_{|\beta|\leq m} \frac{1}{\beta!}\, \partial^\beta f(0)\, x^\beta + m \int_0^1 (1-t)^{m-1} \sum_{|\beta|=m} \frac{1}{\beta!}\, (\partial^\beta f(t\,x) - \partial^\beta f(0))\, x^\beta\, d\,t,$$

where the first line is a standard form of Taylor's formula and the second line a slight modification thereof. We denote by $T_m(f)$ the Taylor polynomial of order m and by $R_m(f)$ the modified remainder, i.e. the first and the second term of the second line respectively.

Lemma 14. *Suppose $f \in C^{m,\alpha}(U)$. Then $f - T_m(f) \in C^{m,\alpha}(U)$ and we have for $\beta \in \mathbb{N}^n_0$, $|\beta| \leq m$,*

$$|\partial^\beta (f - T_m(f))(x)| \leq |x|^{m+\alpha-|\beta|} \sum_{|\gamma|=m-|\beta|} \frac{1}{\gamma!}\, c_{\gamma+\beta} \quad \text{on} \quad U, \tag{146}$$

where the constants c_δ denote the Hölder coefficients satisfying $|\partial^\delta f(x) - \partial^\delta f(0)| \leq c_\delta\, |x|^\alpha$ in U for $\delta \in \mathbb{N}^n_0$, $|\delta| = m$.

Proof. Applying the modified Taylor formula to f and then to its derivatives, we get

$$\partial^{\beta}(f - T_m(f)) = T_{m-|\beta|}(\partial^{\beta} f) + R_{m-|\beta|}(\partial^{\beta} f) - \partial^{\beta} T_m(f).$$

We show that

$$T_{m-|\beta|}(\partial^{\beta} f) - \partial^{\beta} T_m(f) = 0. \tag{147}$$

To prove this equation we use induction on n. For $n = 1$ the result follows by a direct calculation. To perform the induction step we assume $n \geq 2$ and show that the statement for $n - 1$ implies that for n.

We write $x = (x', x^n)$ for $x \in \mathbb{R}^n$ and $\beta = (\beta', \beta_n)$ for $\beta \in \mathbb{N}_0^n$, etc. Then we find the equalities

$$\begin{aligned}
\partial^{\beta} T_m(f) &= \partial^{\beta'} \partial^{\beta_n} \left(\sum_{\gamma_n=0}^{m} T_{m-\gamma_n}(\partial^{\gamma_n} f) \, \frac{1}{\gamma_n!} (x^n)^{\gamma_n} \right) \\
&= \sum_{\gamma_n=\beta_n}^{m-|\beta|+\beta_n} T_{m-|\beta'|-\gamma_n} \left(\partial^{\beta'} \partial^{\gamma_n} f \right) \frac{1}{(\gamma_n - \beta_n)!} (x^n)^{\gamma_n - \beta_n} \\
&= \sum_{\gamma_n=0}^{m-|\beta|} T_{m-|\beta|-\gamma_n} \left(\partial^{\beta'} \partial^{\gamma_n + \beta_n} f \right) \frac{1}{\gamma_n!} (x^n)^{\gamma_n} = T_{m-|\beta|}(\partial^{\beta} f).
\end{aligned}$$

Here the first line is a simple rewriting where we denote by $T_{m-\gamma_n}(\partial^{\gamma_n} f)$ the Taylor polynomial of order $m - \gamma_n$ of the function $\partial^{\gamma_n} f(x', 0)$ of $n - 1$ variables. In the second line the derivatives are taken and the induction hypothesis is used. The third line is obtained by redefining the index γ_n and using a similar rewriting as in the first line.

With (147) the estimate (146) follows immediately by estimating the integral defining $R_{m-|\beta|}(\partial^{\beta} f)$. □

B. An Additional Result

In this section we prove a certain extension of Theorem 13.

Theorem 19. *Let u be a distribution satisfying $Lu = f$, where $f \in E^{m,\alpha}(B_a)$, and the coefficient of the elliptic operator L are in $C^{m,\alpha}(B_a)$. Then*

$$u = r^3 \sum_{k=0}^{m} u_k + u_R \in E^{m+2,\alpha}(B_a), \tag{148}$$

with $u_k \in \mathcal{P}_k$ and $u_R \in C^{m+2,\alpha}(B_a)$.

Proof. We follow the proof of Theorem 13 using Schauder instead of L^p estimates. Since $f = f_1 + r f_2 \in E^{m,\alpha}(B_a)$ we have

$$f = r\, T_m + f_R \quad \text{with} \quad T_m = \sum_{k=0}^{m} t_k,$$

where T_m is the Taylor polynomial of order m of f_2 and $t_k \in \mathcal{P}_k$.

Consider the recurrence relation

$$\Delta(r^3 u_0) = r t_0, \quad \Delta(r^3 u_k) = r \left(t_k - U_k^{(k)} \right), \quad 1 \leq k \leq m,$$

which is obtained by defining U_k, $k = 1, \ldots, m$, by

$$\hat{L} \left(r^3 \sum_{j=0}^{k-1} u_j \right) = r U^{(k)},$$

and defining $U_j^{(k)}$ and $U_j^{(m+1)}$ as in the proof of Theorem 13. The equation for u and (148) then imply for u_R Eq. (50) with $s = 3$. Since by our assumptions and Lemma 4 the right-hand side of this equation is in $C^{m,\alpha}(B_a)$, the interior Schauder estimates of Theorem 8 imply that $u_R \in C^{m+2}(B_a)$. $\square$

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