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Gauge theories of spacetime symmetries

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Gauge theories of conformal spacetime symmetries are presented which merge features of Yang-Mills theory and general relativity in a new way. The models are local but nonpolynomial in the gauge fields, with a nonpolynomial structure that can be elegantly written in terms of a metric (or vielbein) composed of the gauge fields. General relativity itself emerges from the construction as a gauge theory of spacetime translations. The role of the models within a general classification of consistent interactions of gauge fields is discussed as well.

I. INTRODUCTION AND CONCLUSION

In this work new gauge theories of conformal spacetime symmetries are constructed which merge features of Yang-Mills theories and general relativity in an interesting way. This concerns both the Lagrangians and the gauge transformations of these models. The Lagrangians are local but nonpolynomial in the gauge fields, as general relativistic Lagrangians are local but nonpolynomial in the gravitational (metric or vielbein) fields. In fact, they are formally very similar to general relativistic Lagrangians, except that the metric and vielbein are polynomials in the conformal gauge fields, cf. eqs. (25), (38). Moreover, general relativity itself emerges from the construction as a gauge theory of spacetime translations (see section VI).

The (infinitesimal) conformal gauge transformations contain a Yang-Mills type transformation and a general coordinate transformation, with the remarkable property that both parts are tied to each other by the fact that they involve the *same* gauge parameter fields, cf. eq. (23). This unites the symmetry principles of Yang-Mills theory and general relativity in an interesting way and reflects that the models are gauge theories of spacetime symmetries in a very direct sense. The latter also manifests itself in the explicit dependence of the Lagrangians and the gauge transformations on conformal spacetime Killing vector fields. This, among other things, distinguishes the models presented here from gauge theories of conformal symmetries constructed in the past, such as supergravity models [1–13], or, more recently, models presented in Refs. [14,15].

At this point a comment seems to be in order. Particular models constructed in this work admit field redefinitions (of fields that occur in the action, and of gauge parameter fields) which completely remove the explicit dependence of the Lagrangian and gauge transformations on conformal Killing vector fields, and cast the models in more conventional form. In particular, the standard formulation of general relativity arises in this way through field redefinitions which trade metric or vielbein variables for gauge fields of translations. It is possible, and quite likely, that the (nonsupersymmetric version of) models constructed in [1–13] can be reproduced analogously. However, it seems to be impossible to eliminate the dependence on conformal Killing vector fields in a generic model constructed here.

The models are not only interesting for their own sake, but also in the context of a systematic classification of consistent interactions of gauge fields in general, which is quite a challenging problem and partly motivated this work. Such a classification was started in [16,17] using the BRST cohomological approach to consistent deformations of gauge theories [18]. The starting point of that investigation was the free Maxwell Lagrangian $L^{(0)} = -(1/4)\sum_A F^A_{\mu\nu}F^{\mu\nu A}$ for a set of vector gauge fields A^A_μ in flat spacetime. In the deformation approach one asks whether the action and its gauge symmetries can be nontrivially deformed, using an expansion in deformation parameters.

In [16,17] complete results were derived for Poincaré invariant deformations of the free Maxwell Lagrangian to first order in the deformation parameters. The result is that the most general first order deformation which is invariant under the standard Poincaré transformations contains at most four types of nontrivial interaction vertices: (i) polynomials in the field strengths and their first or higher order derivatives; (ii) Chern-Simons vertices of the form $A \wedge F \wedge \ldots \wedge F$ (present only in odd spacetime dimensions); (iii) cubic interaction vertices $f_{ABC}A^A_\mu A^B_\nu F^{\mu\nu C}$ where $f_{ABC} = f_{[ABC]}$ are antisymmetric constant coefficients; (iv) vertices of the form $A_\mu j^\mu$ where j^μ is a gauge invariant Noether current of the free theory[1]. First order deformations which are not required to be Poincaré invariant were also investigated. The results are similar, apart from a few (partly unsettled) details (cf. comments at the end of section 13.2 in [17]).

Self-interacting theories for vector gauge fields with interaction vertices (i), (ii) or (iii) are very well known. Those of type (i) occur, for instance, in the Euler-Heisenberg Lagrangian [19] or the Born-Infeld theory [20]. Lately, vertices (i) which are not Lorentz invariant attracted attention in the context of so-called noncommutative $U(1)$ gauge theory because the interactions in that model can be written as an infinite sum of such vertices by means of a field redefinition ("Seiberg-Witten map") [21] (field redefinitions of this type are automatically taken care of by the BRST cohomological approach: two deformations related by such a field redefinition are equivalent in that approach). From the deformation point of view, vertices (i) and (ii) are somewhat less interesting because they are gauge invariant [in case (ii) modulo a total derivative] under the original gauge transformations of Maxwell theory.

In contrast, vertices (iii) and (iv) are not gauge invariant under the gauge transformations of the free model; rather, they are invariant only on-shell (in the free model) modulo a total derivative and therefore they give rise to nontrivial deformations of the gauge transformations. This makes them particularly interesting. Interaction vertices (iii) are of course well known: they are encountered in Yang-Mills theories [22,23] and lead to a non-Abelian deformation of the commutator algebra of gauge transformations. But what about vertices (iv)? Such vertices are familiar from the coupling of vector gauge fields to matter fields, such as the coupling of the electromagnetic gauge field A_μ to a fermion current $j^\mu = \bar{\psi}\gamma^\mu\psi$, but what do we know about vertices involving gauge invariant currents made up of the gauge fields themselves?

As a matter of fact, it depends on the spacetime dimension whether or not Poincaré invariant vertices (iv) are present at all. In three dimensions such vertices exist and occur in 3-dimensional Freedman-Townsend models [24,25]. In contrast, they do not exist in four dimensions because Maxwell theory in four dimensions has no symmetry that gives rise to a Noether current needed for a Poincaré invariant vertex (iv) (this follows from the results of [26]). It is likely, though not proved, that this result in four dimensions extends to higher dimensions.

However, it must be kept in mind that this result on vertices (iv) in four dimensions concerns only Poincaré invariant interactions. The new gauge theories constructed here contain vertices (iv) that are *not* invariant under the standard Poincaré transformations because they involve gauge invariant Noether currents of spacetime symmetries themselves. Such vertices exist in all spacetime dimensions because there is a gauge invariant form of the Noether currents of the Poincaré symmetries [27,28]. The corresponding deformations of the gauge transformations incorporate Poincaré symmetries in the deformed gauge transformations. This promotes global Poincaré symmetries to local ones, yielding gauge theories of Poincaré symmetries. In four-dimensional spacetime, the construction can be extended to the remaining conformal transformations because dilatations and special conformal transformations also give rise to vertices (iv).[2] For this reason I shall focus on models in four-dimensional spacetime; however, all formulas are also valid in all other dimensions when restricted to gauge theories of Poincaré symmetries, see section VI.

The organization of the paper is the following. Section II treats a relatively simple example with only one gauge field and one vertex (iv) involving a Noether current of a conformal symmetry in four-dimensional spacetime. This results in a prototype model with just one conformal gauge symmetry. In section III the prototype model is rewritten by casting its gauge transformations in a more suitable form and introducing a gauge field dependent "metric". This paves the road for the generalization of the prototype model in section IV where four-dimensional gauge theories of the full conformal algebra or any of its subalgebras are constructed. These models involve not only first order interaction vertices (iv) but in addition also Yang-Mills type interaction vertices (iii) because in general the involved conformal symmetries do not commute. Then, in section V, the construction is further extended by including other fields (matter fields and gauge fields). Section VI explains the relation to general relativity.

II. PROTOTYPE MODEL

Let us first examine deformations of the Maxwell action for only one gauge field A_μ,

$$S^{(0)} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the standard Abelian field strength and indices μ are raised with the Minkowski

[1] Note the difference from vertices (iii): the latter are also of the form $A^A_\mu j^\mu_A$, but the currents $j^\mu_A = f_{ABC}A^B_\nu F^{\mu\nu C}$ are not gauge invariant.

[2] There are infinitely many additional vertices (iv) that are not Poincaré invariant because free Maxwell theory has infinitely many inequivalent Noether currents [29–32]. They are not related to spacetime symmetries. I did not investigate whether or not they also give rise to interesting gauge theories.

metric $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ $[F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}]$. Action (1) is invariant under the gauge transformations

$$\delta_\lambda^{(0)} A_\mu = \partial_\mu \lambda \tag{2}$$

and under global conformal transformations

$$\delta_\xi A_\mu = \xi^\nu F_{\nu\mu} \tag{3}$$

where ξ^μ is a conformal Killing vector field (no matter which one) of flat four-dimensional spacetime,[3]

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2}\eta_{\mu\nu}\partial_\rho \xi^\rho \quad (\xi_\mu = \eta_{\mu\nu}\xi^\nu). \tag{4}$$

(3) is the gauge covariant form [27,28] of a conformal transformation and gives rise to the gauge invariant Noether current

$$j^\mu = \xi^\nu T_\nu{}^\mu, \quad T_\nu{}^\mu = -\frac{1}{4}\delta_\nu^\mu F_{\rho\sigma}F^{\rho\sigma} + F_{\nu\rho}F^{\mu\rho}. \tag{5}$$

A first order deformation $S^{(1)}$ of action (1) that is of type (iv) and the corresponding first order deformation $\delta_\lambda^{(1)}$ of the gauge transformations (2) are

$$S^{(1)} = \int d^4x\, A_\mu j^\mu, \quad \delta_\lambda^{(1)} A_\mu = \lambda \xi^\nu F_{\nu\mu}. \tag{6}$$

Indeed, it can be readily checked that $S^{(1)}$ and $\delta_\lambda^{(1)}$ fulfill the first order invariance condition

$$\delta_\lambda^{(0)} S^{(1)} + \delta_\lambda^{(1)} S^{(0)} = 0.$$

One may now proceed to higher orders. This amounts to looking for higher order terms $S^{(k)}$ and $\delta_\lambda^{(k)}$ satisfying

$$\sum_{i=0}^{k} \delta_\lambda^{(i)} S^{(k-i)} = 0, \quad k = 2, 3, \ldots$$

It turns out that the deformation exists to all orders but that one obtains infinitely many terms giving rise to a nonpolynomial structure. This calls for a more efficient construction of the complete deformation. Let me briefly sketch two strategies, without going into details. The first one is a detour to a first order formulation: one casts the original free Lagrangian in first order form $(1/4)G^{\mu\nu}(G_{\mu\nu} - 2F_{\mu\nu})$ where $G_{\mu\nu} = -G_{\nu\mu}$ are auxiliary fields, deforms this first order model analogously to (6), and finally eliminates the auxiliary fields. Another strategy is the use of a technique applied in [33,34]: in view of (3), one defines a modified field strength $\hat{F}_{\mu\nu}$ implicitly through the relations $\hat{F}_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ and $D_\mu A_\nu = \partial_\mu A_\nu - A_\mu \xi^\rho \hat{F}_{\rho\nu}$, solves these relations for $\hat{F}_{\mu\nu}$ and finally constructs the action and gauge transformations in terms of $\hat{F}_{\mu\nu}$ and A_μ. Both strategies work and yield the same action and gauge transformations:

$$L = -\frac{1}{4}(1 + \xi^\rho A_\rho)\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}, \tag{7}$$

$$\delta_\lambda A_\mu = \partial_\mu \lambda + \lambda\, \xi^\nu \hat{F}_{\nu\mu} \tag{8}$$

with $\hat{F}_{\mu\nu}$ given by

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{A_\mu \xi^\rho F_{\rho\nu} - A_\nu \xi^\rho F_{\rho\mu}}{1 + \xi^\sigma A_\sigma}\ . \tag{9}$$

$\hat{F}_{\mu\nu}$ can be interpreted as the field strength for the gauge transformations (8) because its gauge transformation does not contain derivatives of λ: indeed, a straightforward, though somewhat lengthy, computation gives

$$\delta_\lambda \hat{F}_{\mu\nu} = \frac{\lambda}{1 + \xi^\sigma A_\sigma}\, \mathcal{L}_\xi \hat{F}_{\mu\nu} \tag{10}$$

where $\mathcal{L}_\xi$ is the standard Lie derivative along ξ^μ,

$$\mathcal{L}_\xi \hat{F}_{\mu\nu} = \xi^\rho \partial_\rho \hat{F}_{\mu\nu} + \partial_\mu \xi^\rho \hat{F}_{\rho\nu} + \partial_\nu \xi^\rho \hat{F}_{\mu\rho}\ .$$

Using (10), as well as (4), it is easy to verify that the Lagrangian (7) transforms under the gauge transformations (8) into a total derivative,

$$\delta_\lambda L = -\frac{1}{4}\partial_\mu(\lambda\, \xi^\mu \hat{F}_{\rho\nu}\hat{F}^{\rho\nu}). \tag{11}$$

Furthermore, owing to (10), the algebra of the gauge transformations (8) is obviously Abelian, i.e., two gauge transformations with different parameter fields, denoted by λ and λ', respectively, commute:

$$[\delta_\lambda, \delta_{\lambda'}] = 0. \tag{12}$$

I remark that, for notational convenience, I have suppressed the gauge coupling constant (= deformation parameter) in the formulas given above; it can be easily introduced in the usual way by substituting rescaled fields κA_μ and $\kappa\lambda$ for A_μ and λ, respectively, and then dividing the Lagrangian by κ^2. Expanding the resulting action and gauge transformations in κ, one obtains $S = S^{(0)} + \kappa S^{(1)} + O(\kappa^2)$ and $\delta_\lambda = \delta_\lambda^{(0)} + \kappa\delta_\lambda^{(1)} + O(\kappa^2)$ with $S^{(1)}$ and $\delta_\lambda^{(1)}$ as in (6). This shows that (7) and (8) complete the first order deformation (6) to all orders. Note that the completion contains infinitely many terms and is nonpolynomial but local in the gauge fields, as promised.

[3]The construction is not restricted to flat spacetime but applies analogously to any fixed background metric $\hat{g}_{\mu\nu}$ with at least one conformal Killing vector field ξ^μ. Then (4) turns into $\mathcal{L}_\xi \hat{g}_{\mu\nu} = (1/2)\hat{g}_{\mu\nu}\hat{D}_\rho \xi^\rho$ and subsequent formulas change accordingly.

III. REFORMULATION OF THE PROTOTYPE MODEL

In the remainder of this work I shall first rewrite and then generalize the prototype model with the Lagrangian (7) and the gauge transformations (8). A surprising feature of the Lagrangian (7) is that its nonpolynomial structure can be written in terms of the "metric"

$$g_{\mu\nu} = \eta_{\mu\nu} + \xi_\mu A_\nu + \xi_\nu A_\mu + \xi_\rho \xi^\rho A_\mu A_\nu \ , \tag{13}$$

where, again, $\xi_\mu = \eta_{\mu\nu}\xi^\nu$. The inverse and determinant of this metric are

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{\xi^\mu A^\nu + \xi^\nu A^\mu}{1+\xi^\sigma A_\sigma} + \frac{A_\rho A^\rho \xi^\mu \xi^\nu}{(1+\xi^\sigma A_\sigma)^2} \ ,$$
$$\det(g_{\mu\nu}) = -(1+\xi^\mu A_\mu)^2,$$

where $A^\mu = \eta^{\mu\nu}A_\nu$. Using these formulas one readily verifies that the Lagrangian (7) can be written as

$$L = -\frac{1}{4}\sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{14}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\sqrt{g} = |\det(g_{\mu\nu})|^{1/2} = 1 + \xi^\mu A_\mu$ (assuming $1 + \xi^\mu A_\mu > 0$). Furthermore, it can be easily checked that the gauge transformations (8) can be rewritten as

$$\delta_\omega A_\mu = \partial_\mu \omega + \omega \xi^\nu \partial_\nu A_\mu + \partial_\mu(\omega\xi^\nu) A_\nu \tag{15}$$

where ω is constructed of λ, ξ^μ and A_μ according to

$$\omega = \frac{\lambda}{1+\xi^\mu A_\mu} \ . \tag{16}$$

(15) is exactly the same transformation as (8), but written in terms of ω instead of λ. Since λ was completely arbitrary, ω is also completely arbitrary, and can thus be used as gauge parameter field in place of λ. Note that (15) is polynomial in the gauge fields, in contrast to (8).

To understand the gauge invariance of the model, and to generalize it subsequently, the following observation is crucial: under the gauge transformations (15) of the gauge fields, the metric (13) transforms according to

$$\delta_\omega g_{\mu\nu} = \mathcal{L}_\varepsilon g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \omega \partial_\rho \xi^\rho, \tag{17}$$

where $\mathcal{L}_\varepsilon g_{\mu\nu}$ is the Lie derivative of $g_{\mu\nu}$ along $\varepsilon^\mu = \omega\xi^\mu$:

$$\mathcal{L}_\varepsilon g_{\mu\nu} = \varepsilon^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \varepsilon^\rho g_{\rho\nu} + \partial_\nu \varepsilon^\rho g_{\mu\rho},$$
$$\varepsilon^\mu = \omega\xi^\mu. \tag{18}$$

In order to verify equation (17), one has to use the conformal Killing vector equations (4). Equations (15) and (17) make it now easy to understand the gauge invariance of the action with Lagrangian (14). Note that the last two terms on the right hand side of (15) are nothing but the Lie derivative $\mathcal{L}_\varepsilon A_\mu$ of A_μ along ε^μ:

$$\delta_\omega A_\mu = \partial_\mu \omega + \mathcal{L}_\varepsilon A_\mu.$$

Hence the gauge transformation of A_μ is the sum of a standard Abelian gauge transformation with parameter ω and a general coordinate transformation with parameters ε^μ [of course, these two transformations are related because of $\varepsilon^\mu = \omega\xi^\mu$]. As a consequence, the gauge transformation of $F_{\mu\nu}$ is given just by the Lie derivative along ε^μ, $\delta_\omega F_{\mu\nu} = \mathcal{L}_\varepsilon F_{\mu\nu}$. (17) has the form of a general coordinate transformation of $g_{\mu\nu}$ with parameters ε^μ plus a Weyl transformation with parameter $-(1/2)\omega\partial_\rho\xi^\rho$. As the Lagrangian is invariant under Weyl transformations of $g_{\mu\nu}$ (we are still discussing the four-dimensional case), it transforms under gauge transformations δ_ω just like a scalar density under general coordinate transformations with parameters ε^μ: $\delta_\omega L = \partial_\mu(\varepsilon^\mu L)$. This is exactly equation (11), owing to $\varepsilon^\mu = \omega\xi^\mu = \lambda\xi^\mu/(1+\xi^\nu A_\nu)$ and $L/(1+\xi^\nu A_\nu) = -(1/4)\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}$. A final remark on the prototype model is that the gauge transformations no longer commute when expressed in terms of ω rather than in terms of λ:

$$[\delta_\omega, \delta_{\omega'}] = \delta_{\omega''}, \quad \omega'' = \omega'\xi^\mu\partial_\mu\omega - \omega\xi^\mu\partial_\mu\omega'. \tag{19}$$

The reason for this is that the redefinition (16) of the gauge parameter field involves the gauge field A_μ.

IV. GENERALIZATION

The prototype model found above will now be generalized by gauging more than only one conformal symmetry in four-dimensional flat spacetime. Let $\mathcal{G}$ be the Lie algebra of the full conformal group or any of its subalgebras. Let us pick a basis of $\mathcal{G}$ and label its elements by an index A [since the conformal group in four dimensions is 15-dimensional, we have $A = 1, \ldots, N$ with $1 \le N \le 15$]. The corresponding set of conformal Killing vector fields is denoted by $\{\xi^\mu_A\}$. Since $\mathcal{G}$ is a Lie algebra, one can choose the ξ's such that

$$\xi^\nu_A \partial_\nu \xi^\mu_B - \xi^\nu_B \partial_\nu \xi^\mu_A = f_{BA}{}^C \xi^\mu_C \tag{20}$$

where $f_{AB}{}^C$ are the structure constants of $\mathcal{G}$ in the chosen basis. I associate one gauge field A^A_μ and one gauge parameter field ω^A with each element of $\mathcal{G}$ and introduce the following generalization of the gauge transformations (15):

$$\delta_\omega A^A_\mu = D_\mu \omega^A + \omega^B \xi^\nu_B \partial_\nu A^A_\mu + \partial_\mu(\omega^B \xi^\nu_B) A^A_\nu \tag{21}$$

where

$$D_\mu \omega^A = \partial_\mu \omega^A + A^B_\mu f_{BC}{}^A \omega^C. \tag{22}$$

The part $D_\mu\omega^A$ of $\delta_\omega A^A_\mu$ is familiar from Yang-Mills theory; the remaining part is the Lie derivative of A^A_μ along a vector field ε^μ containing the gauge parameter fields ω^A,

$$\delta_\omega A_\mu^A = D_\mu \omega^A + \mathcal{L}_\varepsilon A_\mu^A, \quad \varepsilon^\mu = \omega^B \xi_B^\mu. \tag{23}$$

The commutator of two gauge transformations is

$$[\delta_\omega, \delta_{\omega'}] = \delta_{\omega''} \ ,$$
$$\omega''^A = \omega^B \omega'^C f_{BC}{}^A + \omega'^B \xi_B^\mu \partial_\mu \omega^A - \omega^B \xi_B^\mu \partial_\mu \omega'^A. \tag{24}$$

The crucial step for constructing an action which is invariant under these gauge transformations is the following generalization of the prototype metric (13):

$$g_{\mu\nu} = \eta_{\mu\nu} + \xi_{A\mu} A_\nu^A + \xi_{A\nu} A_\mu^A + \xi_{A\rho} \xi_B^\rho A_\mu^A A_\nu^B, \tag{25}$$

with $\xi_{A\mu} = \eta_{\mu\nu} \xi_A^\nu$. This metric behaves under gauge transformations (21) similarly as the prototype metric (13) under gauge transformations (15):

$$\delta_\omega g_{\mu\nu} = \mathcal{L}_\varepsilon g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \omega^A \partial_\rho \xi_A^\rho, \tag{26}$$

with ε^μ as in (23). To verify (26), one has to use (4) (which holds for each ξ_A^μ) and (20). Note that (21) is the sum of a Yang-Mills gauge transformation with parameter fields ω^B and a general coordinate transformation with parameter fields $\varepsilon^\mu = \omega^B \xi_B^\mu$, while (26) has the form of a general coordinate transformation with parameters ε^μ plus a Weyl transformation with parameter $-(1/2)\omega^A \partial_\rho \xi_A^\rho$. This immediately implies that the following Lagrangian is invariant modulo a total derivative under gauge transformations (21):

$$L = -\frac{1}{4} \sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}^A F_{\rho\sigma}^B d_{AB}, \tag{27}$$

where d_{AB} is a symmetric $\mathcal{G}$-invariant tensor,

$$d_{AB} = d_{BA}, \quad f_{CA}{}^D d_{DB} + f_{CB}{}^D d_{AD} = 0, \tag{28}$$

and the $F_{\mu\nu}^A$ are field strengths familiar from Yang-Mills theory:

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f_{BC}{}^A A_\mu^B A_\nu^C. \tag{29}$$

Owing to (28), the Lagrangian (27) is invariant under Yang-Mills transformations of the $F_{\mu\nu}^A$. Furthermore it is invariant under Weyl transformations of $g_{\mu\nu}$. Hence, it transforms under gauge transformations (21) just like a scalar density under general coordinate transformations with parameters $\varepsilon^\mu = \omega^A \xi_A^\mu$:

$$\delta_\omega L = \partial_\mu (\omega^A \xi_A^\mu L). \tag{30}$$

Again, the Lagrangian is local but nonpolynomial in the gauge fields because it contains the inverse metric $g^{\mu\nu}$. The latter is

$$g^{\mu\nu} = \eta^{\mu\nu} - \xi_A^\mu \hat{A}^{A\nu} - \xi_A^\nu \hat{A}^{A\mu} + \xi_A^\mu \xi_B^\nu \hat{A}_\rho^A \hat{A}^{B\rho}, \tag{31}$$

where $\hat{A}^{A\mu} = \eta^{\mu\nu} \hat{A}_\nu^A$, with

$$\hat{A}_\mu^A = A_\mu^B E_B{}^A, \quad E_B{}^C (\delta_C^A + \xi_C^\mu A_\mu^A) = \delta_B^A. \tag{32}$$

The second equation in (32) expresses that the $E_B{}^A$ are the entries of a matrix E which inverts the matrix $1 + M$ where M is the matrix with entries $\xi_B^\mu A_\mu^A$. E can thus be written as an infinite (geometric) series of matrix products of M:

$$E = \sum_{k=0}^\infty (-M)^k, \quad M_B{}^A \equiv \xi_B^\mu A_\mu^A. \tag{33}$$

A gauge coupling constant κ can be introduced as before by means of the substitutions $A_\mu^A \to \kappa A_\mu^A$, $\omega^A \to \kappa \omega^A$, $L \to L/\kappa^2$. Equivalently, one may use $f_{AB}{}^C \to \kappa f_{AB}{}^C$, $\xi_A^\mu \to \kappa \xi_A^\mu$. Of course, the zeroth order Lagrangian is positive definite only for appropriate choices of $\mathcal{G}$. For instance, one may choose a $\mathcal{G}$ that is Abelian or compact; then there is a basis of $\mathcal{G}$ such that $d_{AB} = \delta_{AB}$. The simplest case is a one-dimensional $\mathcal{G}$ and reproduces the prototype model. Choices such as $\mathcal{G} = so(2,4)$ (full conformal algebra) or $\mathcal{G} = so(1,3)$ (Lorentz algebra) do not give a positive definite zeroth order Lagrangian because these algebras are not compact (one cannot achieve $d_{AB} = \delta_{AB}$).

V. INCLUSION OF MATTER FIELDS AND FURTHER GAUGE FIELDS

Using the metric (25), it is straightforward to extend the models of the previous section so as to include further fields. First I discuss the case of just one (real) scalar field ϕ and introduce the gauge transformation

$$\delta_\omega \phi = \omega^A \xi_A^\mu \partial_\mu \phi + \frac{1}{4} \phi \omega^A \partial_\mu \xi_A^\mu. \tag{34}$$

A contribution to the Lagrangian which is gauge invariant modulo a total derivative is

$$L_\phi = \frac{1}{2} \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} \sqrt{g} R \phi^2 \tag{35}$$

with $g_{\mu\nu}$ and $g^{\mu\nu}$ as before in (25) and (31), and R the Riemannian curvature scalar built from $g_{\mu\nu}$,

$$R = g^{\mu\nu} R_{\mu\rho\nu}{}^\rho,$$
$$R_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}{}^\sigma + \Gamma_{\mu\lambda}{}^\sigma \Gamma_{\nu\rho}{}^\lambda - (\mu \leftrightarrow \nu),$$
$$\Gamma_{\mu\nu}{}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

Using (26), one easily derives the gauge variation of R:

$$\delta_\omega R = \varepsilon^\mu \partial_\mu R + \frac{1}{2} R \omega^A \partial_\mu \xi_A^\mu$$
$$- \frac{3}{2} g^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma_{\mu\nu}{}^\rho \partial_\rho)(\omega^A \partial_\sigma \xi_A^\sigma). \tag{36}$$

This makes it is easy to verify the gauge invariance of (35): L_ϕ transforms as a scalar density under standard general coordinate transformations of $g_{\mu\nu}$ and ϕ; therefore the first term in (34) and the first term in (26) make a contribution $\partial_\mu(\varepsilon^\mu L_\phi)$ to $\delta_\omega L_\phi$; the second terms in (34) and (26) contribute a total derivative to $\delta_\omega L_\phi$ because L_ϕ is invariant modulo a total derivative under Weyl transformations of $g_{\mu\nu}$ and ϕ with weights of ratio -2 (in four dimensions). The complete transformation reads

$$\delta_\omega L_\phi = \partial_\mu\Big[\omega^A\xi^\mu_A L_\phi + \frac{1}{8}\sqrt{g}g^{\mu\nu}\phi^2\partial_\nu(\omega^A\partial_\rho\xi^\rho_A)\Big]. \quad (37)$$

To include fermions, I introduce the "vierbein"

$$e_\mu{}^\nu = \delta^\nu_\mu + \xi^\nu_A A^A_\mu. \quad (38)$$

The term vierbein is used because $e_\mu{}^\nu$ is related to the "metric" (25) through

$$g_{\mu\nu} = \eta_{\rho\sigma}e_\mu{}^\rho e_\nu{}^\sigma. \quad (39)$$

Furthermore the vierbein transforms under the gauge transformations (21) according to

$$\begin{aligned}\delta_\omega e_\mu{}^\nu &= \varepsilon^\rho\partial_\rho e_\mu{}^\nu + \partial_\mu\varepsilon^\rho e_\rho{}^\nu\\ &\quad + C_\rho{}^\nu e_\mu{}^\rho - \frac{1}{4}e_\mu{}^\nu\omega^A\partial_\rho\xi^\rho_A\end{aligned} \quad (40)$$

with ε^μ as in (23) and

$$C_\mu{}^\nu = -\frac{1}{2}\omega^A(\partial_\mu\xi^\nu_A - \eta^{\nu\sigma}\eta_{\mu\rho}\partial_\sigma\xi^\rho_A). \quad (41)$$

Note that (40) has indeed the familiar form of the transformation of vierbein fields in general relativity: the lower index of $e_\mu{}^\nu$ transforms as a "world index" (it sees only the general coordinate transformation with parameters ε^μ) while the upper index transforms as a "Lorentz index" (it sees only "Lorentz transformations with parameters $C_\mu{}^\nu$" – the Lorentz character is due to $C^{\mu\nu} = -C^{\nu\mu}$ where $C^{\mu\nu} = \eta^{\mu\rho}C_\rho{}^\nu$). In addition (40) contains a Weyl transformation with parameter $-(1/2)\omega^A\partial_\rho\xi^\rho_A$. I now define a "spin connection" $\omega_\mu{}^{\nu\rho}$:

$$\begin{aligned}\omega_\mu{}^{\nu\rho} &= E_\sigma{}^\nu E_\lambda{}^\rho\eta^{\sigma\kappa}\eta^{\lambda\tau}\omega_{\mu\kappa\tau},\\ \omega_{\mu\nu\rho} &= \omega_{[\mu\nu]\rho} - \omega_{[\nu\rho]\mu} + \omega_{[\rho\mu]\nu},\\ \omega_{[\mu\nu]\rho} &= \frac{1}{2}e_\rho{}^\sigma\eta_{\sigma\lambda}(\partial_\mu e_\nu{}^\lambda - \partial_\nu e_\mu{}^\lambda)\end{aligned} \quad (42)$$

where $E_\mu{}^\nu$ is the inverse vierbein ($E_\mu{}^\nu e_\nu{}^\rho = \delta^\rho_\mu$),

$$E_\mu{}^\nu = \delta^\nu_\mu - \hat{A}^A_\mu\xi^\nu_A \quad (43)$$

with $\hat{A}^A_\mu$ as in (32). Since $\omega_\mu{}^{\nu\rho}$ is constructed of $e_\mu{}^\nu$ in exactly the same manner as one constructs the spin connection of the vierbein in general relativity, one infers from (40) that $\omega_\mu{}^{\nu\rho}$ transforms under the gauge transformations (21) according to

$$\begin{aligned}\delta_\omega\omega_\mu{}^{\nu\rho} &= \partial_\mu C^{\nu\rho} - \omega_\mu{}^{\sigma\nu}C_\sigma{}^\rho + \omega_\mu{}^{\sigma\rho}C_\sigma{}^\nu\\ &\quad + \varepsilon^\sigma\partial_\sigma\omega_\mu{}^{\nu\rho} + \partial_\mu\varepsilon^\sigma\omega_\sigma{}^{\nu\rho}\\ &\quad + \frac{1}{4}(e_\mu{}^\rho E_\sigma{}^\nu - e_\mu{}^\nu E_\sigma{}^\rho)\eta^{\sigma\lambda}\partial_\lambda(\omega^A\partial_\tau\xi^\tau_A)\end{aligned} \quad (44)$$

where $C^{\nu\rho} = \eta^{\nu\sigma}C_\sigma{}^\rho$ with $C_\sigma{}^\rho$ as in (41). I denote a fermion field by ψ (without displaying its spinor indices), and introduce the gauge transformations

$$\delta_\omega\psi = \omega^A\xi^\mu_A\partial_\mu\psi - \frac{1}{2}C^{\mu\nu}\sigma_{\mu\nu}\psi + \frac{3}{8}\psi\omega^A\partial_\mu\xi^\mu_A \quad (45)$$

where $4\sigma_{\mu\nu}$ is the commutator of γ-matrices, using the conventions

$$\begin{gathered}\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu},\\ \sigma_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad \gamma_\mu = \eta_{\mu\nu}\gamma^\nu.\end{gathered}$$

A contribution to the Lagrangian which is invariant modulo a total derivative under the gauge transformations (21) and (45) is

$$L_\psi = \mathrm{i}\sqrt{g}\,\bar\psi\gamma^\nu E_\nu{}^\mu(\partial_\mu\psi + \frac{1}{2}\omega_\mu{}^{\sigma\rho}\sigma_{\sigma\rho}\psi). \quad (46)$$

L_ψ transforms under the gauge transformations like a scalar density under general coordinate transformations with parameters $\varepsilon^\mu = \omega^A\xi^\mu_A$ because the "Lorentz" and "Weyl" parts of the gauge transformation of the fermion, vierbein and spin connection cancel each other completely,

$$\delta_\omega L_\psi = \partial_\mu(\omega^A\xi^\mu_A L_\psi). \quad (47)$$

The inclusion of standard Yang-Mills gauge fields A^I_μ is even simpler: the contribution to the Lagrangian is just the standard Yang-Mills Lagrangian in the metric (25),

$$L_{\mathrm{YM}} = -\frac{1}{4}\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}d_{IJ}, \quad (48)$$

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + f_{JK}{}^I A^J_\mu A^K_\nu \quad (49)$$

where $f_{JK}{}^I$ and d_{IJ} are the structure constants and an invariant symmetric tensor of some Lie algebra $\mathcal{G}_{\mathrm{YM}}$. Note that the difference from (27) is that now the field strengths $F^I_{\mu\nu}$ involve the gauge fields of $\mathcal{G}_{\mathrm{YM}}$ while the metric $g_{\mu\nu}$ is composed of the gauge fields of $\mathcal{G}$. The conformal gauge transformations of A^I_μ are just the standard Lie derivatives along $\varepsilon^\mu = \omega^A\xi^\mu_A$,

$$\delta_\omega A^I_\mu = \omega^B\xi^\nu_B\partial_\nu A^I_\mu + \partial_\mu(\omega^B\xi^\nu_B)A^I_\nu. \quad (50)$$

Since L_{YM} is invariant under Weyl transformations of $g_{\mu\nu}$, it transforms under the conformal gauge transformations (21) and (50) like a scalar density under general coordinate transformations with parameters ε^μ,

$$\delta_\omega L_{\mathrm{YM}} = \partial_\mu(\omega^A\xi^\mu_A L_{\mathrm{YM}}). \quad (51)$$

In addition L_{YM} is invariant under the usual Yang-Mills gauge transformations $\delta_\alpha A^I_\mu = \partial_\mu\alpha^I + A^J_\mu f_{JK}{}^I\alpha^K$ for arbitrary gauge parameter fields α^I.

It is straightforward to construct further interaction terms, such as $\sqrt{g}\phi^4$ or Yukawa-interactions $\sqrt{g}\phi\bar\psi\psi$, and to extend the construction to scalar fields or fermions transforming nontrivially under $\mathcal{G}_{\rm YM}$. In fact, it is even possible to construct models where the "matter fields" transform under $\mathcal{G}$ according to a nontrivial representation. I shall only discuss the case of scalar fields transforming under a nontrivial representation of $\mathcal{G}$; the extension to fermions is straightforward. Of course, the notion "scalar fields" should be used cautiously when these fields sit in a nontrivial representation of $\mathcal{G}$ as they may or may not transform nontrivially under Lorentz transformations (depending on the choice of $\mathcal{G}$ and its representation). I denote these "scalar fields" by ϕ^i. The corresponding representation matrices of $\mathcal{G}$ are denoted by T_A and chosen such that they represent $\mathcal{G}$ with the same structure constants $f_{AB}{}^C$ as in (20), i.e.,

$$T^i_{Ak}T^k_{Bj}-T^i_{Bk}T^k_{Aj}=f_{AB}{}^C T^i_{Cj}. \tag{52}$$

Further properties of the representation will not matter to the construction. In place of (34), the gauge transformations now read

$$\delta_\omega\phi^i=-\omega^A T^i_{Aj}\phi^j+\omega^A\xi^\mu_A\partial_\mu\phi^i+\frac{1}{4}\phi^i\omega^A\partial_\mu\xi^\mu_A. \tag{53}$$

Accordingly, one introduces covariant derivatives

$$D_\mu\phi^i=\partial_\mu\phi^i+A^A_\mu T^i_{Aj}\phi^j. \tag{54}$$

These covariant derivatives transform under gauge transformations (21), (53) according to

$$\begin{aligned}\delta_\omega D_\mu\phi^i=&-\omega^A T^i_{Aj}D_\mu\phi^j+\mathcal{L}_\varepsilon D_\mu\phi^i\\&+\frac{1}{4}(D_\mu\phi^i)\omega^A\partial_\nu\xi^\nu_A+\frac{1}{4}\phi^i\partial_\mu(\omega^A\partial_\nu\xi^\nu_A)\end{aligned}$$

where $\mathcal{L}_\varepsilon D_\mu\phi^i=\varepsilon^\nu\partial_\nu D_\mu\phi^i+\partial_\mu\varepsilon^\nu D_\nu\phi^i$ with ε^μ as in (23). The generalization of the Lagrangian (35) is simply

$$\tilde{L}_\phi=\sqrt{g}\Big[\frac{1}{2}g^{\mu\nu}D_\mu\phi^i D_\nu\phi^j-\frac{1}{12}R\phi^i\phi^j\Big]d_{ij} \tag{55}$$

where d_{ij} is a $\mathcal{G}$-invariant symmetric tensor,

$$d_{ij}=d_{ji},\quad d_{kj}T^k_{Ai}+d_{ik}T^k_{Aj}=0. \tag{56}$$

Using (56) and arguments analogous to those that led to (37), one infers that

$$\delta_\omega\tilde{L}_\phi=\partial_\mu\Big[\omega^A\xi^\mu_A L_\phi+\frac{1}{8}\sqrt{g}g^{\mu\nu}\phi^i\phi^j d_{ij}\partial_\nu(\omega^A\partial_\rho\xi^\rho_A)\Big].$$

VI. RELATION TO GENERAL RELATIVITY

So far we have worked in four-dimensional spacetime. Actually the whole construction goes through without any change in an arbitrary dimension if we restrict it to isometries of the flat metric rather than considering all conformal symmetries. In other words, all formulas given above hold in arbitrary dimension if we impose

$$\partial_\mu\xi^\mu_A=0. \tag{57}$$

When (57) holds, the gauge transformations δ_ω are local Poincaré transformations. This raises the question of whether there is a relation to general relativity. The answer to this question is affirmative and easily obtained from the following observation: when (57) holds, the "Einstein-Hilbert action" constructed from the metric (25),

$$S_{\rm EH}=-\frac{1}{2}\int d^nx\,\sqrt{g}R, \tag{58}$$

is invariant under gauge transformations (21) because equation (26) reduces to a general coordinate transformation of $g_{\mu\nu}$ with parameters $\varepsilon^\mu=\omega^A\xi^\mu_A$. Now, consider the special case of an action given just by (58) (without any additional terms), and assume that $\{A^A_\mu\}$ contains (at least) the gauge fields of all spacetime translations. Then we may interpret (25) as a field redefinition which just substitutes new fields $g_{\mu\nu}$ for certain combinations of the original field variables. Since the action depends on the gauge fields only via the new fields $g_{\mu\nu}$, it reproduces the standard theory of pure gravitation as described by general relativity.

In fact, the argument is even more transparent when one works with the vielbein (38) rather than with the metric (25) [according to (39), the metric can be written in terms of the vielbein, and thus action (58) can also be written in terms of the vielbein, as usual]. That is, we may label the translations by an index ν and choose the corresponding Killing vector fields as $\xi^\mu_\nu=\delta^\mu_\nu$. Accordingly, the gauge fields of translations are denoted by A^ν_μ. (38) may then be interpreted as a field redefinition that substitutes $e_\mu{}^\nu$ for A^ν_μ. This field redefinition is clearly local and invertible (at least locally), as (38) can obviously be solved for A^ν_μ in terms of $e_\mu{}^\nu$ and the gauge fields of Lorentz transformations.

The same argument applies when we add to the integrand of (58) the first term of the matter Lagrangian (35) (the second term is not needed since we consider only gauged Poincaré transformations here), the fermion Lagrangian (46) or the Yang-Mills type Lagrangian (48). Since these contributions also depend on the gauge fields A^A_μ only via the $e_\mu{}^\nu$, the same field redefinition implies the equivalence to general relativity coupled to matter fields in the standard way.

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