Global properties of gravitational lens maps in a Lorentzian manifold setting

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Abstract

In a general-relativistic spacetime (Lorentzian manifold), gravitational lensing can be characterized by a lens map, in analogy to the lens map of the quasi-Newtonian approximation formalism. The lens map is defined on the celestial sphere of the observer (or on part of it) and it takes values in a two-dimensional manifold representing a two-parameter family of worldlines. In this article we use methods from differential topology to characterize global properties of the lens map. Among other things, we use the mapping degree (also known as Brouwer degree) of the lens map as a tool for characterizing the number of images in gravitational lensing situations. Finally, we illustrate the general results with gravitational lensing (a) by a static string, (b) by a spherically symmetric body, (c) in asymptotically simple and empty spacetimes, and (d) in weakly perturbed Robertson-Walker spacetimes.

1 Introduction

Gravitational lensing is usually studied in a quasi-Newtonian approximation formalism which is essentially based on the assumptions that the gravitational fields are weak and that the bending angles are small, see Schneider, Ehlers and Falco [1] for a comprehensive discussion. This formalism has proven to be very powerful for the calculation of special models. In addition it has also been used for proving general theorems on the qualitative features of gravitational lensing such as the possible number of images in a multiple imaging situation. As to the latter point, it is interesting to inquire whether the results can be reformulated in a Lorentzian manifold setting, i.e., to inquire to what extent the results depend on the approximations involved.

In the quasi-Newtonian approximation formalism one considers light rays in Euclidean 3-space that go from a fixed point (observer) to a point that is allowed to vary over a 2-dimensional plane (source plane). The rays are assumed to be straight lines with the only exception that they may have a sharp bend at a 2-dimensional plane (deflector plane) that is parallel to the source plane. (There is also a variant with several deflector planes to model deflectors which are not "thin".) For each concrete mass distribution, the deflecting angles are to be calculated with the help of Einstein's field equation, or rather of those remnants of Einstein's field equation that survive the approximations involved. Hence, at each point of the deflector plane the deflection angle is uniquely determined by the mass distribution. As a consequence, following light rays from the observer into the past always gives a unique "lens map" from the deflector plane to the source plane. There is "multiple imaging" whenever this lens map fails to be injective.

In this article we want to inquire whether an analogous lens map can be introduced in a spacetime setting, without using quasi-Newtonian approximations. According to the rules of general relativity, a spacetime is to be modeled by a Lorentzian manifold $(\mathcal{M}, g)$ and the light rays are to be modeled by the lightlike geodesics in $\mathcal{M}$. We shall assume that $(\mathcal{M}, g)$ is time-oriented, i.e., that the timelike and lightlike vectors can be distinguished into future-pointing and past-pointing in a globally consistent way. To define a general lens map, we have to fix a point $p \in \mathcal{M}$ as the event where the observation takes place and we have to look for an analogue of the deflector plane and for an analogue of the source plane. As to the deflector plane, there is an obvious candidate, namely the *celestial sphere* $\mathcal{S}_p$ at p. This can be defined as the the set of all one-dimensional lightlike subspaces of the tangent space $T_p\mathcal{M}$ or, equivalently, as the totality of all light rays issuing from p into the past. As to the source plane, however, there is no natural candidate. Following Frittelli, Newman and Ehlers [2, 3, 4], one might consider any timelike 3-dimensional submanifold $\mathcal{T}$ of the spacetime manifold as a substitute for the source plane. The idea is to view such a submanifold as ruled by worldlines of light sources. To make this more explicit, one could restrict to the case that $\mathcal{T}$ is a fiber bundle over a two-dimensional manifold $\mathcal{N}$, with fibers timelike and diffeomorphic to $\mathbb{R}$. Each fiber is to be interpreted as the worldline of a light source, and the set $\mathcal{N}$ may be identified with the set of all those worldlines. In this situation we wish to define a lens map $f_p : \mathcal{S}_p \longrightarrow \mathcal{N}$ by extending each light ray from p into the past until it meets $\mathcal{T}$ and then projecting onto $\mathcal{N}$. In general, this prescription does not give a well-defined map since neither existence nor uniqueness of the target value is guaranteed. As to existence, there might be some past-pointing lightlike geodesics from p that never reach $\mathcal{T}$. As to uniqueness, one and the same light ray might intersect $\mathcal{T}$ several times. The uniqueness problem could be circumvented by considering, on each past-pointing lightlike geodesic from p, only the first intersection with $\mathcal{T}$, thereby willfully excluding some light rays from the discussion. This comes up to ignoring every image that is hidden behind some other image of a light source with a worldline $\xi \in \mathcal{N}$. For the existence problem, however, there is no general solution. Unless one restricts to special situations, the lens map will be defined only on some subset $\mathcal{D}_p$ of $\mathcal{S}_p$ (which may even be empty). Also, one would like the lens map to be differentiable or at least continuous. This is guaranteed if one further restricts the domain $\mathcal{D}_p$ of the lens map by considering only light rays that meet $\mathcal{T}$ transversely.

Following this line of thought, we give a precise definition of lens maps in Section 2. We will be a little bit more general than outlined above insofar as the source surface need not be timelike; we also allow for the limiting case of a lightlike source surface. This has the advantage that we may choose the source surface "at infinity" in the case of an asymptotically simple and empty spacetime. In Section 3 we briefly discuss some general properties of the caustic of the lens map. In Section 4 we introduce the mapping degree (Brouwer degree) of the lens map as an important tool from differential topology. This will then give us some theorems on the possible number of images in gravitational lensing situations, in particular in the case that we have a "simple lensing neighborhood". The latter notion will be introduced and discussed in Section 5. We conclude with applying the general results to some examples in Section 6.

Our investigation will be purely geometrical in the sense that we discuss the influence of the spacetime geometry on the propagation of light rays but not the influence of the matter distribution on the spacetime geometry. In other words, we use only the geometrical background of general relativity but not Einstein's field equation. For this reason the "deflector", i.e., the matter distribution that is the cause of gravitational lensing, never explicitly appears in our investigation. However, information on whether the deflectors are transparent or non-transparent will implicitly enter into our considerations.

2 Definition of the lens map

As a preparation for precisely introducing the lens map in a spacetime setting, we first specify some terminology.

By a *manifold* we shall always mean what is more fully called a "real, finite-dimensional, Hausdorff, second countable (and thus paracompact) C^∞-manifold without boundary". Whenever we have a C^∞ vector field X on a manifold $\mathcal{M}$, we may consider two points in $\mathcal{M}$ as *equivalent* if they lie on the same integral curve of X. We shall denote the resultant quotient space, which may be identified with the set of all integral curves of X, by $\mathcal{M}/X$. We call X a *regular* vector field if $\mathcal{M}/X$ can be given the structure of a manifold in such a way that the natural projection $\pi_X : \mathcal{M} \longrightarrow \mathcal{M}/X$ becomes a C^∞-submersion. It is easy to construct examples of non-regular vector fields. E.g., if X has no zeros and is defined on $\mathbb{R}^n \setminus \{0\}$, then $\mathcal{M}/X$ cannot satisfy the Hausdorff property, so it cannot be a manifold according to our terminology. Palais [5] has proven a useful result which, in our terminology, can be phrased in the following way. If none of X's integral curves is closed or almost closed, and if $\mathcal{M}/X$ satisfies the Hausdorff property, then X is regular.

We are going to use the following terminology. A *Lorentzian manifold* is a manifold $\mathcal{M}$ together with a C^∞ metric tensor field g of Lorentzian signature $(+\cdots+-)$. A Lorentzian manifold is *time-orientable* if the set of all timelike vectors $\{Z \in T\mathcal{M} \,|\, g(Z,Z) < 0\}$ has exactly two connected components. Choosing one of those connected components as *future-pointing* defines a *time-orientation* for $(\mathcal{M}, g)$. A *spacetime* is a connected 4-dimensional time-orientable Lorentzian manifold together with a time-orientation.

We are now ready to define what we will call a "source surface" in a spacetime. This will provide us with the target space for lens maps.

Definition 1. $(\mathcal{T}, W)$ is called a *source surface* in a spacetime $(\mathcal{M}, g)$ if
(a) $\mathcal{T}$ is a 3-dimensional C^∞ submanifold of $\mathcal{M}$;
(b) W is a nowhere vanishing regular C^∞ vector field on $\mathcal{T}$ which is everywhere causal, $g(W,W) \le 0$, and future-pointing;
(c) $\pi_W : \mathcal{T} \longrightarrow \mathcal{N} = \mathcal{T}/W$ is a fiber bundle with fiber diffeomorphic to $\mathbb{R}$ and the quotient manifold $\mathcal{N} = \mathcal{T}/W$ is connected and orientable.

We want to interpret the integral curves of W as the worldlines of light sources. Thus, one should assume that they are not only causal but even timelike, $g(W,W) < 0$, since a light source should move at subluminal velocity. For technical reasons, however, we allow for the possibility that an integral curve of W is lightlike (everywhere or at some points), because such curves may appear as (C^1-)limits of timelike curves. This will give us the possibility to apply the resulting formalism to asymptotically simple and empty spacetimes in a convenient way, see Subsection 6.2 below. Actually, the causal character of W will have little influence upon the results we want to establish. What really matters is a transversality condition that enters into the definition of the lens map below.

Please note that, in the situation of Definition 1, the bundle $\pi_W : \mathcal{T} \longrightarrow \mathcal{N}$ is necessarily trivializable, i.e., $\mathcal{T} \simeq \mathcal{N} \times \mathbb{R}$. To prove this, let us assume that the flow of W is defined on all of $\mathbb{R} \times \mathcal{T}$, so it makes $\pi_W : \mathcal{T} \longrightarrow \mathcal{N}$ into a principal fiber bundle. (This is no restriction of generality since it can always be achieved by multiplying W with an appropriate function. This function can be determined in the following way. Owing to a famous theorem of Whitney [6], also see Hirsch [7], p.55, paracompactness guarantees that $\mathcal{T}$ can be embedded as a closed submanifold into $\mathbb{R}^n$ for some n. Pulling back the Euclidean metric gives a complete Riemannian metric h on $\mathcal{T}$ and the flow of the vector field $h(W,W)^{-1/2}W$ is defined on all of $\mathbb{R} \times \mathcal{T}$, cf. Abraham and Marsden [8], Proposition 2.1.21.) Then the result follows from the well known facts that any fiber bundle whose typical fiber is diffeomorphic to $\mathbb{R}^n$ admits a global section (see, e.g., Kobayashi and Nomizu [9], p.58), and that a principal fiber bundle is trivializable if and only if it admits a global section (see again [9], p.57).

Also, it is interesting to note the following. If $\mathcal{T}$ is any 3-dimensional submanifold of $\mathcal{M}$ that is foliated into timelike curves, then time orientability guarantees that these are the integral curves of a timelike vector field W. If we assume, in addition, that $\mathcal{T}$ contains no closed timelike curves, then it can be shown that $\pi_W : \mathcal{T} \longrightarrow \mathcal{N}$ is necessarily a fiber bundle with fiber diffeomorphic to $\mathbb{R}$, providing $\mathcal{N}$ satisfies the Hausdorff property, see Harris [10], Theorem 2. This shows that there is little room for relaxing the conditions of Definition 1.

Choosing a source surface in a spacetime will give us the target space $\mathcal{N} = \mathcal{T}/W$ for the lens map. To specify the domain of the lens map, we consider, at any point $p \in \mathcal{M}$, the set $\mathcal{S}_p$ of all lightlike directions at p, i.e., the set of all one-dimensional lightlike subspaces of $T_p\mathcal{M}$. We shall refer to $\mathcal{S}_p$ as to the *celestial sphere* at p. This is justified since, obviously, $\mathcal{S}_p$ is in natural one-to-one relation with the set of all light rays arriving at p. As it is more convenient to work with vectors rather than with directions, we shall usually represent $\mathcal{S}_p$ as a submanifold of $T_p\mathcal{M}$. To that end we fix a future-pointing timelike vector V_p in the tangent space $T_p\mathcal{M}$. The vector V_p may be interpreted as the 4-velocity of an observer at p. We now consider the set

$$\mathcal{S}_p = \left\{ Y_p \in T_p\mathcal{M} \,\middle|\, g(Y_p, Y_p) = 0 \text{ and } g(Y_p, V_p) = 1 \right\}. \tag{1}$$

It is an elementary fact that (1) defines an embedded submanifold of $T_p\mathcal{M}$ which is diffeomorphic to the standard 2-sphere S^2. As indicated by our notation, the set (1) can be identified with the celestial sphere at p, just by relating each vector to the direction spanned by it.

Representation (1) of the celestial sphere gives a convenient way of representing the light rays through p. We only have to assign to each $Y_p \in \mathcal{S}_p$ the lightlike geodesic $s \longmapsto \exp_p(sY_p)$ where $\exp_p : W_p \subseteq T_p\mathcal{M} \longrightarrow \mathcal{M}$ denotes the exponential map at the point p of the Levi-Civita connection of the metric g. Please note that this geodesic is past-pointing, because V_p was chosen future-pointing, and that it passes through p at the parameter value $s = 0$.

The lens map is defined in the following way. After fixing a source surface $(\mathcal{T}, W)$ and choosing a point $p \in \mathcal{M}$, we denote by $\mathcal{D}_p \subseteq \mathcal{S}_p$ the subset of all lightlike directions at p such that the geodesic to which this direction is tangent meets $\mathcal{T}$ (at least once) if sufficiently extended to the past, and if at the first intersection point q with $\mathcal{T}$ this geodesic is transverse to $\mathcal{T}$. By projecting q to $\mathcal{N} = \mathcal{T}/W$ we get the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N} = \mathcal{T}/W$, see Figure 1. If we use the representation (1) for $\mathcal{S}_p$, the definition of the lens map can be given in more formal terms in the following way.

Definition 2. Let $(\mathcal{T}, W)$ be a source surface in a spacetime $(\mathcal{M}, g)$. Then, for each $p \in \mathcal{M}$, the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N} = \mathcal{T}/W$ is defined in the following way. In the notation of

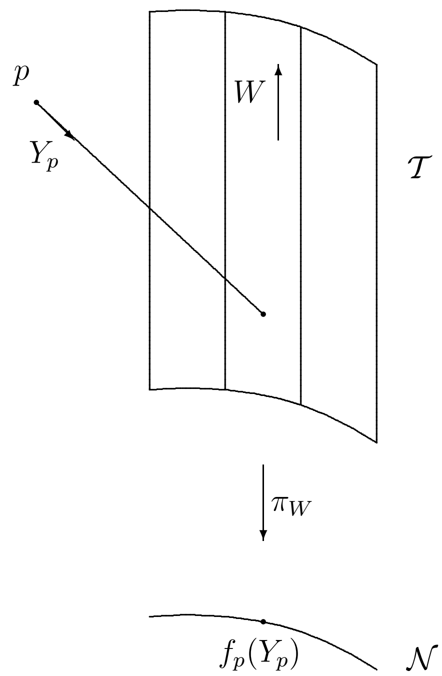

Figure 1: Illustration of the lens map

equation (1), let $\mathcal{D}_p$ be the set of all $Y_p \in \mathcal{S}_p$ such that there is a real number $w_p(Y_p) > 0$ with the properties
(a) sY_p is in the maximal domain of the exponential map for all $s \in [\,0\,, w_p(Y_p)]$;
(b) the curve $s \longmapsto \exp(sY_p)$ intersects $\mathcal{T}$ at the value $s = w_p(Y_p)$ transversely;
(c) $\exp_p(sY_p) \notin \mathcal{T}$ for all $s \in [\,0\,, w_p(Y_p)[$.
This defines a map $w_p : \mathcal{D}_p \longrightarrow \mathbb{R}$. The *lens map* at p is then, by definition, the map

$$f_p : \mathcal{D}_p \longrightarrow \mathcal{N} = \mathcal{T}/X\,, \quad f_p(Y_p) = \pi_W\big(\exp_p(w_p(Y_p)Y_p)\big)\,. \tag{2}$$

Here $\pi_W : \mathcal{T} \longrightarrow \mathcal{N}$ denotes the natural projection.

The transversality condition in part (b) of Definition 2 guarantees that the domain $\mathcal{D}_p$ of the lens map is an open subset of $\mathcal{S}_p$. The case $\mathcal{D}_p = \emptyset$ is, of course, not excluded. In particular, $\mathcal{D}_p = \emptyset$ whenever $p \in \mathcal{T}$, owing to part (c) of Definition 2.

Moreover, the transversality condition in part (b) of Definition 2, in combination with the implicit function theorem, makes sure that the map $w_p : \mathcal{D}_p \longrightarrow \mathbb{R}$ is a C^∞ map. As the exponential map of a C^∞ metric is again C^∞, and π_W is a C^∞ submersion by assumption, this proves the following.

Proposition 1. *The lens map is a C^∞ map.*

Please note that without the transversality condition the lens map need not even be continuous.

Although our Definition 2 made use of the representation (1), which refers to a timelike vector V_p, the lens map is, of course, independent of which future-pointing V_p has been chosen. We decided to index the lens map only with p although, strictly speaking, it depends on $\mathcal{T}$, on W, and on p. Our philosophy is to keep a source surface $(\mathcal{T}, W)$ fixed, and then to consider the lens map for all points $p \in \mathcal{M}$.

In view of gravitational lensing, the lens map admits the following interpretation. For $\xi \in \mathcal{N}$, each point $Y_p \in \mathcal{D}_p$ with $f_p(Y_p)$ corresponds to a past-pointing lightlike geodesic from p to the worldline ξ in $\mathcal{M}$, i.e., it corresponds to an image at the celestial sphere of p of the light source with worldline ξ. If f_p is not injective, we are in a multiple imaging situation. The converse need not be true as the lens map does not necessarily cover all images. There might be a past-pointing lightlike geodesic from p reaching ξ after having met $\mathcal{T}$ before, or being tangential to $\mathcal{T}$ on its arrival at ξ. In either case, the corresponding image is ignored by the lens map. The reader might be inclined to view this as a disadvantage. However, in Section 6 below we discuss some situations where the existence of such additional light rays can be excluded (e.g., asymptotically simple and empty spacetimes) and situations where it is desirable, on physical grounds, to disregard such additional light rays (e.g., weakly perturbed Robertson-Walker spacetimes with compact spatial sections).

It was already mentioned that the domain $\mathcal{D}_p$ of the lens map might be empty; this is, of course, the worst case that could happen. The best case is that the domain is all of the celestial sphere, $\mathcal{D}_p = \mathcal{S}_p$. We shall see in the following sections that many interesting results are true just in this case. However, there are several cases of interest where $\mathcal{D}_p$ is a proper subset of $\mathcal{S}_p$. If the domain of the lens map f_p is the whole celestial sphere, none of the light rays issuing from p into the past is blocked or trapped before it reaches $\mathcal{T}$. In view of applications to gravitational lensing, this excludes the possibility that these light rays meet a non-transparent deflector. In other words, it is a typical feature of gravitational lensing situations with non-transparent deflectors that $\mathcal{D}_p$ is not all of $\mathcal{S}_p$. Two simple examples, viz., a non-transparent string and a non-transparent spherical body, will be considered in Subsection 6.1 below.

3 Regular and critical values of the lens map

Please recall that, for a differentiable map $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ between two manifolds, $Y \in \mathcal{M}_1$ is called a *regular point* of F if the differential $T_Y F : T_Y\mathcal{M}_1 \longrightarrow T_{F(Y)}\mathcal{M}_2$ has maximal rank, otherwise Y is called a *critical point*. Moreover, $\xi \in \mathcal{M}_2$ is called a *regular value* of F if all $Y \in F^{-1}(\xi)$ are regular points, otherwise ξ is called a *critical value*. Please note that, according to this definition, any $\xi \in \mathcal{M}_2$ that is not in the image of F is regular. The well-known (*Morse-*)*Sard theorem* (see, e.g., Hirsch [7], p.69) says that the set of regular values of F is residual (i.e., it contains the intersection of countably many sets that are open and dense in $\mathcal{M}_2$) and thus dense in $\mathcal{M}_2$ and the critical values of F make up a set of measure zero in $\mathcal{M}_2$.

For the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N}$, we call the set

$$\mathrm{Caust}(f_p) = \big\{\, \xi \in \mathcal{N} \,\big|\, \xi \text{ is a critical value of } f_p \,\big\} \tag{3}$$

the *caustic* of f_p. The Sard theorem then implies the following result.

Proposition 2. *The caustic* $\mathrm{Caust}(f_p)$ *is a set of measure zero in* $\mathcal{N}$ *and its complement* $\mathcal{N} \setminus \mathrm{Caust}(f_p)$ *is residual and thus dense in* $\mathcal{N}$.

Please note that $\mathrm{Caust}(f_p)$ need not be closed in $\mathcal{N}$. Counter-examples can be constructed easily by starting with situations where the caustic is closed and then excising points from spacetime. For lens maps defined on the whole celestial sphere, however, we have the following result.

Proposition 3. *If $\mathcal{D}_p = \mathcal{S}_p$, the caustic* $\mathrm{Caust}(f_p)$ *is compact in $\mathcal{N}$.*

This is an obvious consequence of the fact that $\mathcal{S}_p$ is compact and that f_p and its first derivative are continuous.

As the domain and the target space of f_p have the same dimension, $Y_p \in \mathcal{D}_p$ is a regular point of f_p if and only if the differential $T_{Y_p} f_p : T_{Y_p}\mathcal{S}_p \longrightarrow T_{f_p(Y_p)}\mathcal{N}$ is an isomorphism. In this case f_p maps a neighborhood of Y_p diffeomorphically onto a neighborhood of $f_p(Y_p)$. The differential $T_{Y_p} f_p$ may be either orientation-preserving or orientation-reversing. To make this notion precise we have to choose an orientation for $\mathcal{S}_p$ and an orientation for $\mathcal{N}$. For the celestial sphere $\mathcal{S}_p$ it is natural to choose the orientation according to which the origin of the tangent space $T_p\mathcal{M}$ is to the inner side of $\mathcal{S}_p$. The target manifold $\mathcal{N}$ is orientable by assumption, but in general there is no natural choice for the orientation. Clearly, choosing an orientation for $\mathcal{N}$ fixes an orientation for $\mathcal{T}$, because the vector field W gives us an orientation for the fibers. We shall say that the orientation of $\mathcal{N}$ is *adapted* to some point $Y_p \in \mathcal{D}_p$ if the geodesic with initial vector Y_p meets $\mathcal{T}$ at the inner side. If $\mathcal{D}_p$ is connected, the orientation of $\mathcal{N}$ that is adapted to some $Y_p \in \mathcal{D}_p$ is automatically adapted to all other elements of $\mathcal{D}_p$. Using this terminology, we may now introduce the following definition.

Definition 3. A regular point $Y_p \in \mathcal{D}_p$ of the lens map f_p is said to have *even parity* (or *odd parity*, respectively) if $T_{Y_p} f_p$ is orientation-preserving (or orientation-reversing, respectively) with respect to the natural orientation on $\mathcal{S}_p$ and the orientation adapted to Y_p on $\mathcal{N}$. For a regular value $\xi \in \mathcal{N}$ of the lens map, we denote by $n_+(\xi)$ (or $n_-(\xi)$, respectively) the number of elements in $f_p^{-1}(\xi)$ with even parity (or odd parity, respectively).

Please note that $n_+(\xi)$ and $n_-(\xi)$ may be infinite, see the Schwarzschild example in Subsection 6.1 below. A criterion for $n_\pm(\xi)$ to be finite will be given in Proposition 8 below.

Definition 3 is relevant for gravitational lensing in the following sense. The assumption that Y_p is a regular point of f_p implies that an observer at p sees a neighborhood of $\xi = f_p(Y_p)$ in $\mathcal{N}$ as a neighborhood of Y_p at his or her celestial sphere. If we compare the case that Y_p has odd parity with the case that Y_p has even parity, then the appearance of the neighborhood in the first case is the mirror image of its appearance in the second case. This difference is observable for a light source that is surrounded by some irregularly shaped structure, e.g. a galaxy with curved jets or with lobes.

If ξ is a regular value of f_p, it is obvious that the points in $f_p^{-1}(\xi)$ are isolated, i.e., any Y_p in $f_p^{-1}(\xi)$ has a neighborhood in $\mathcal{D}_p$ that contains no other point in $f_p^{-1}(\xi)$. This follows immediately from the fact that f_p maps a neighborhood of Y_p diffeomorphically onto its image. In the next section we shall formulate additional assumptions such that the set $f_p^{-1}(\xi)$ is finite, i.e., such that the numbers $n_\pm(\xi)$ introduced in Definition 3 are finite. It is the main purpose of the next section to demonstrate that then the difference $n_+(\xi) - n_-(\xi)$ has some topological invariance properties. As a preparation for that we notice the following result which is an immediate consequence of the fact that the lens map is a local diffeomorphism near each regular point.

Proposition 4. *n_+ and n_- are constant on each connected component of $f_p(\mathcal{D}_p)\backslash\mathrm{Caust}(f_p)$.*

Hence, along any continuous curve in $f_p(\mathcal{D}_p)$ that does not meet the caustic of the lens map, the numbers n_+ and n_- remain constant, i.e., the observer at p sees the same number of images for all light sources on this curve. If a curve intersects the caustic, the number of images will jump. In the next section we shall prove that n_+ and n_- always jump by the same amount (under conditions making sure that these numbers are finite), i.e., the total number of images always jumps by an even number. This is well known in the quasi-Newtonian approximation formalism, see, e.g., Schneider, Ehlers and Falco [1], Section 6.

If $\mathrm{Caust}(f_p)$ is empty, transversality guarantees that $f_p(\mathcal{D}_p)$ is open in $\mathcal{N}$ and, thus a manifold. Proposition 4 implies that, in this case, f_p gives a C^∞ covering map from $\mathcal{D}_p$ onto $f_p(\mathcal{D}_p)$. As a C^∞ covering map onto a simply connected manifold must be a global diffeomorphism, this implies the following result.

Proposition 5. *Assume that* $\mathrm{Caust}(f_p)$ *is empty and that* $f_p(\mathcal{D}_p)$ *is simply connected. Then* f_p *gives a global diffeomorphism from* $\mathcal{D}_p$ *onto* $f_p(\mathcal{D}_p)$.

In other words, the formation of a caustic is necessary for multiple imaging provided that $f_p(\mathcal{D}_p)$ is simply connected. In Subsection 6.1 below we shall consider the spacetime of a non-transparent string. This will demonstrate that the conclusion of Proposition 5 is not true without the assumption of $f_p(\mathcal{D}_p)$ being simply connected.

In the rest of this subsection we want to relate the caustic of the lens map to the caustic of the past light cone of p. The past light cone of p can be defined as the image set in $\mathcal{M}$ of the map

$$F_p : (s, Y_p) \longmapsto \exp_p(sY_p) \tag{4}$$

considered on its maximal domain in $]\,0\,,\,\infty\,[\,\times \mathcal{S}_p$, and its caustic can be defined as the set of critical values of F_p. In other words, $q \in \mathcal{M}$ is in the caustic of the past light cone of p if and only if there is an $s_0 \in\,]\,0\,,\,\infty\,[$ and a $Y_p \in \mathcal{S}_p$ such that the differential $T_{(s_0,Y_p)}F_p$ has rank $k < 3$. In that case one says that the point $q = \exp_p(s_0 Y_p)$ is *conjugate* to p along the geodesic $s \longmapsto \exp_p(sY_p)$, and one calls the number $m = 3 - k$ the *multiplicity* of this conjugate point. As $F_p(\,\cdot\,, Y_p)$ is always an immersion, the multiplicity can take the values 1 and 2 only. (This formulation is equivalent to the definition of conjugate points and their multiplicities in terms of *Jacobi vector fields* which may be more familiar to the reader.) It is well known, but far from trivial, that along every lightlike geodesic conjugate points are isolated. Hence, in a compact parameter interval there are only finitely many points that are conjugate to a fixed point p. A proof can be found, e.g., in Beem, Ehrlich and Easley [11], Theorem 10.77.

After these preparations we are now ready to establish the following proposition. We use the notation introduced in Definition 2.

Proposition 6. *An element* $Y_p \in \mathcal{D}_p$ *is a regular point of the lens map if and only if the point* $\exp_p(w_p(Y_p)Y_p)$ *is not conjugate to* p *along the geodesic* $s \longmapsto \exp_p(sY_p)$. *A regular point* $Y_p \in \mathcal{D}_p$ *has even parity* (*or odd parity, respectively*) *if and only if the number of points conjugate to* p *along the geodesic* $[\,0\,, w_p(Y_p)] \longrightarrow \mathcal{M}\,,\ s \longmapsto \exp_p(sY_p)$ *is even* (*or odd, respectively*). *Here each conjugate point is to be counted with its multiplicity.*

Proof. In terms of the function (4), the lens map can be written in the form

$$f_p(Y_p) = \pi_W\big(F_p(w_p(Y_p), Y_p)\big)\ . \tag{5}$$

As $s \longmapsto F_p(s, Y_p)$ is an immersion transverse to $\mathcal{T}$ at $s = w_p(Y_p)$ and π_W is a submersion, the differential of f_p at Y_p has rank 2 if and only if the differential of F_p at $(w_p(Y_p), Y_p)$

has rank 3. This proves the first claim. For proving the second claim define, for each $s \in [0, w_p(Y_p)]$, a map

$$\Phi_s : T_{Y_p}\mathcal{S}_p \longrightarrow T_{f_p(Y_p)}\mathcal{N} \tag{6}$$

by applying to each vector in $T_{Y_p}\mathcal{S}_p$ the differential $T_{(s,Y_p)}F_p$, parallel-transporting the result along the geodesic $F_p(\,\cdot\,, Y_p)$ to the point $q = F_p\big(w_p(Y_p), Y_p\big)$ and then projecting down to $T_{f_p(Y_p)}\mathcal{N}$. In the last step one uses the fact that, by transversality, any vector in $T_q\mathcal{M}$ can be uniquely decomposed into a vector tangent to $\mathcal{T}$ and a vector tangent to the geodesic $F_p(\,\cdot\,, Y_p)$. For $s = 1$, this map Φ_s gives the differential of the lens map. We now choose a basis in $T_{Y_p}\mathcal{S}_p$ and a basis in $T_{f_p(Y_p)}\mathcal{N}$, thereby representing the map Φ_s as a (2×2)-matrix. We choose the first basis right-handed with respect to the natural orientation on $\mathcal{S}_p$ and the second basis right-handed with respect to the orientation on $\mathcal{N}$ that is adapted to Y_p. Then $\det(\Phi_0)$ is positive as the parallel transport gives an orientation-preserving isomorphism. The function $s \longmapsto \det(\Phi_s)$ has a single zero whenever $F_p(s, Y_p)$ is a conjugate point of multiplicity one and it has a double zero whenever $F_p(s, Y_p)$ is a conjugate point of multiplicity two. Hence, the sign of $\det(\Phi_1)$ can be determined by counting the conjugate points. □

This result implies that $\xi \in \mathcal{N}$ is a regular value of the lens map f_p whenever the worldline ξ does not pass through the caustic of the past light cone of p. The relation between parity and the number of conjugate points is geometrically rather evident because each conjugate point is associated with a "crossover" of infinitesimally neighboring light rays.

4 The mapping degree of the lens map

The mapping degree (also known as Brouwer degree) is one of the most powerful tools in differential topology. In this section we want to investigate what kind of information could be gained from the mapping degree of the lens map, providing it can be defined.

For the reader's convenience we briefly summarize definition and main properties of the mapping degree, following closely Choquet-Bruhat, Dewitt-Morette, and Dillard-Bleick [12], pp.477. For a more abstract approach, using homology theory, the reader may consult Dold [13], Spanier [14] or Bredon [15]. In this article we shall not use homology theory with the exception of the proof of Proposition 11.

The definition of the mapping degree is based on the following observation.

Proposition 7. *Let $F : \overline{\mathcal{D}} \subseteq \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a continuous map, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are oriented connected manifolds of the same dimension, $\mathcal{D}$ is an open subset of $\mathcal{M}_1$ with compact closure $\overline{\mathcal{D}}$ and $F|_{\mathcal{D}}$ is a C^∞ map. (Actually, C^1 would do.) Then for every $\xi \in \mathcal{M}_2 \setminus F(\partial\mathcal{D})$ which is a regular value of $F|_{\mathcal{D}}$, the set $F^{-1}(\xi)$ is finite.*

Proof. By contradiction, let us assume that there is a sequence $(y_i)_{i\in\mathbb{N}}$ with pairwise different elements in $F^{-1}(\xi)$. By compactness of $\overline{\mathcal{D}}$, we can choose an infinite subsequence that converges towards some point $y_\infty \in \overline{\mathcal{D}}$. By continuity of F, $F(y_\infty) = \xi$, so the hypotheses of the proposition imply that $y_\infty \notin \partial\mathcal{D}$. As a consequence, y_∞ is a regular point of $F|_{\mathcal{D}}$, so it must have an open neighborhood in $\mathcal{D}$ that does not contain any other element of $F^{-1}(\xi)$. This contradicts the fact that a subsequence of $(y_i)_{i\in\mathbb{N}}$ converges towards y_∞. □

If we have a map F that satisfies the hypotheses of Proposition 7, we can thus define, for every $\xi \in \mathcal{M}_2 \setminus F(\partial\mathcal{D})$ which is a regular value of $F|_{\mathcal{D}}$,

$$\deg(F, \xi) = \sum_{y \in F^{-1}(\xi)} \mathrm{sgn}(y) \, , \tag{7}$$

where $\mathrm{sgn}(y)$ is defined to be $+1$ if the differential $T_y F$ preserves orientation and -1 if $T_y F$ reverses orientation. If $F^{-1}(\xi)$ is the empty set, the right-hand side of (7) is set equal to zero. The number $\deg(F, \xi)$ is called the *mapping degree* of F at ξ. Roughly speaking, $\deg(F, \xi)$ tells how often the image of F covers the point ξ, counting each "layer" positive or negative depending on orientation.

The mapping degree has the following properties (for proofs see Choquet-Bruhat, Dewitt-Morette, and Dillard-Bleick [12], pp.477).

Property A: $\deg(F, \xi) = \deg(F, \xi')$ whenever ξ and ξ' are in the same connected component of $\mathcal{M}_2 \setminus F(\partial\mathcal{D})$.

Property B: $\deg(F, \xi) = \deg(F', \xi)$ whenever F and F' are homotopic, i.e., whenever there is a continuous map $\Phi : [0, 1] \times \overline{\mathcal{D}} \longrightarrow \mathcal{M}_2$, $(s, y) \longmapsto \Phi_s(y)$ with $\Phi_0 = F$ and $\Phi_1 = F'$ such that $\deg(\Phi_s, \xi)$ is defined for all $s \in [0, 1]$.

Property A can be used to extend the definition of $\deg(F, \xi)$ to the non-regular values $\xi \in \mathcal{M}_2 \setminus F(\partial\mathcal{D})$. Given the fact that, by the Sard theorem, the regular values are dense in $\mathcal{M}_2$, this can be done just by continuous extension.

Property B can be used to extend the definition of $\deg(F, \xi)$ to continuous maps $F : \overline{D} \longrightarrow \mathcal{M}_2$ which are not necessarily differentiable on $\mathcal{D}$. Given the fact that the C^∞ maps are dense in the continuous maps with respect to the C^0-topology, this can be done again just by continuous extension.

We now apply these general results to the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N}$. In the case $\mathcal{D}_p \neq \mathcal{S}_p$ it is necessary to extend the domain of the lens map onto a compact set to define the degree of the lens map. We introduce the following definition.

Definition 4. A map $\overline{f_p} : \overline{\mathcal{D}_p} \subseteq \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is called an *extension* of the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N}$ if
(a) $\mathcal{M}_1$ is an orientable manifold that contains $\mathcal{D}_p$ as an open submanifold;
(b) $\mathcal{M}_2$ is an orientable manifold that contains $\mathcal{N}$ as an open submanifold;
(c) the closure $\overline{\mathcal{D}_p}$ of $\mathcal{D}_p$ in $\mathcal{M}_1$ is compact;
(d) $\overline{f_p}$ is continuous and the restriction of $\overline{f_p}$ to $\mathcal{D}_p$ is equal to f_p.

If the lens map is defined on the whole celestial sphere, $\mathcal{D}_p = \mathcal{S}_p$, then the lens map is an extension of itself, $\overline{f_p} = f_p$, with $\mathcal{M}_1 = \mathcal{S}_p$ and $\mathcal{M}_2 = \mathcal{N}$. If $\mathcal{D}_p \neq \mathcal{S}_p$, one may try to continuously extend f_p onto the closure of $\mathcal{D}_p$ in $\mathcal{S}_p$, thereby getting an extension with $\mathcal{M}_1 = \mathcal{S}_p$ and $\mathcal{M}_2 = \mathcal{N}$. If this does not work, one may try to find some other extension. The string spacetime in Subsection 6.1 below will provide us with an example where an extension exists although f_p cannot be continuously extended from $\mathcal{D}_p$ onto its closure in $\mathcal{S}_p$. The spacetime around a spherically symmetric body with $R_o < 3m$ will provide us with an example where the lens map admits no extension at all, see Subsection 6.1 below.

Applying Proposition 7 to the case $F = \overline{f_p}$ immediately gives the following result.

Proposition 8. *If the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N}$ admits an extension $\overline{f_p} : \overline{\mathcal{D}_p} \subseteq \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, then for all regular values $\xi \in \mathcal{N} \setminus \overline{f_p}(\partial\mathcal{D}_p)$ the set $f_p^{-1}(\xi)$ is finite, so the numbers $n_+(\xi)$ and $n_-(\xi)$ introduced in Definition 3 are finite.*

If $\overline{f_p}$ is an extension of the lens map f_p, the number $\deg(\overline{f_p}, \xi)$ is a well defined integer for all $\xi \in \mathcal{N} \setminus \overline{f_p}(\partial \mathcal{D}_p)$, provided that we have chosen an orientation on $\mathcal{M}_1$ and on $\mathcal{M}_2$. The number $\deg(\overline{f_p}, \xi)$ changes sign if we change the orientation on $\mathcal{M}_1$ or on $\mathcal{M}_2$. This sign ambiguity can be removed if $\mathcal{D}_p$ is connected. Then we know from the preceding section that $\mathcal{N}$ admits an orientation that is adapted to all $Y_p \in \mathcal{D}_p$. As $\mathcal{N}$ is connected, this determines an orientation for $\mathcal{M}_2$. Moreover, the natural orientation on $\mathcal{S}_p$ induces an orientation on $\mathcal{D}_p$ which, for $\mathcal{D}_p$ connected, gives an orientation for $\mathcal{M}_1$.

In the rest of this paper we shall only be concerned with the situation that $\mathcal{D}_p$ is connected, and we shall always tacitly assume that the orientations have been chosen as indicated above, thereby fixing the sign of $\deg(f_p)$. Now comparison of (7) with Definition 3 shows that

$$\deg(\overline{f_p}, \xi) = n_+(\xi) - n_-(\xi) \tag{8}$$

for all regular values in $\mathcal{N} \setminus \overline{f_p}(\partial \mathcal{D}_p)$. Owing to Property A, this has the following consequence.

Proposition 9. *Assume that $\mathcal{D}_p$ is connected and that the lens map admits an extension $\overline{f_p} : \overline{\mathcal{D}_p} \subseteq \mathcal{M}_1 \longrightarrow \mathcal{M}_2$. Then $n_+(\xi) - n_-(\xi) = n_+(\xi') - n_-(\xi')$ for any two regular values ξ and ξ' which are in the same connected component of $\mathcal{N} \setminus \overline{f_p}(\partial \mathcal{D}_p)$. In particular, $n_+(\xi) + n_-(\xi)$ is odd if and only if $n_+(\xi') + n_-(\xi')$ is odd.*

We know already from Proposition 4 that the numbers n_+ and n_- remain constant along each continuous curve in $f_p(\mathcal{D}_p)$ that does not meet the caustic of f_p. Now let us consider a continuous curve $\alpha \ : \ ]-\varepsilon_0\, , \varepsilon_0\, [\longrightarrow f_p(\mathcal{D}_p)$ that meets the caustic at $\alpha(0)$ whereas $\alpha(\varepsilon)$ is a regular value of f_p for all $\varepsilon \neq 0$. Under the additional assumptions that $\mathcal{D}_p$ is connected, that f_p admits an extension, and that $\alpha(0) \notin \overline{f_p}(\partial \mathcal{D}_p)$, Proposition 9 tells us that $n_+\big(\alpha(\varepsilon)\big) - n_-\big(\alpha(\varepsilon)\big)$ remains constant when ε passes through zero. In other words, n_+ and n_- are allowed to jump only by the same amount. As a consequence, the total number of images $n_+ + n_-$ is allowed to jump only by an even number.

We now specialize to the case that the lens map is defined on the whole celestial sphere, $\mathcal{D}_p = \mathcal{S}_p$. Then the assumption of f_p admitting an extension is trivially satisfied, with $\overline{f_p} = f_p$, and the degree $\deg(f_p, \xi)$ is a well-defined integer for all $\xi \in \mathcal{N}$. Moreover, $\deg(f_p, \xi)$ is a constant with respect to ξ, owing to Property A. It is then usual to write simply $\deg(f_p)$ instead of $\deg(f_p, \xi)$. Using this notation, (8) simplifies to

$$\deg(f_p) = n_+(\xi) - n_-(\xi) \tag{9}$$

for all regular values ξ of f_p. Thus, the total number of images

$$n_+(\xi) + n_-(\xi) = \deg(f_p) + 2n_-(\xi) \tag{10}$$

is either even for all regular values ξ or odd for all regular values ξ, depending on whether $\deg(f_p)$ is even or odd.

In some gravitational lensing situations it might be possible to show that there is one light source $\xi \in \mathcal{N}$ for which $f_p^{-1}(\xi)$ consists of exactly one point, i.e., ξ is not multiply imaged. This situation is characterized by the following proposition.

Proposition 10. *Assume that $\mathcal{D}_p = \mathcal{S}_p$ and that there is a regular value ξ of f_p such that $f_p^{-1}(\xi)$ is a single point. Then $|\deg(f_p)| = 1$. In particular, f_p must be surjective and $\mathcal{N}$ must be diffeomorphic to the sphere S^2.*

Proof. The result $|\deg(f_p)| = 1$ can be read directly from (9), choosing the regular value ξ which has exactly one pre-image point under f_p. This implies that f_p must be surjective since a non-surjective map has degree zero. So $\mathcal{N}$ being the continuous image of the compact set $\mathcal{S}_p$ under the continuous map f_p must be compact. It is well known (see, e.g., Hirsch [7], p.130, Exercise 5) that for $n \geq 2$ the existence of a continuous map $F : S^n \longrightarrow \mathcal{M}_2$ with $\deg(F) = 1$ onto a compact oriented n-manifold $\mathcal{M}_2$ implies that $\mathcal{M}_2$ must be simply connected. As the lens map gives us such a map onto $\mathcal{N}$ (after changing the orientation of $\mathcal{N}$, if necessary), we have thus found that $\mathcal{N}$ must be simply connected. Owing to the well-known classification theorem of compact orientable two-dimensional manifolds (see, e.g., Hirsch [7], Chapter 9), this implies that $\mathcal{N}$ must be diffeomorphic to the sphere S^2. □

In the situation of Proposition 10 we have $n_+(\xi) + n_-(\xi) = 2n_-(\xi) \pm 1$, for all $\xi \in \mathcal{N} \setminus \mathrm{Caust}(f_p)$, i.e., the total number of images is odd for all light sources $\xi \in \mathcal{N} \simeq S^2$ that lie not on the caustic of f_p. The idea to use the mapping degree for proving an odd number theorem in this way was published apparently for the first time in the introduction of McKenzie [16]. In Proposition 10 one would, of course, like to drop the rather restrictive assumption that $f_p^{-1}(\xi)$ is a single point for some ξ. In the next section we consider a special situation where the result $|\deg(f_p)| = 1$ can be derived without this assumption.

5 Simple lensing neighborhoods

In this section we investigate a special class of spacetime regions that will be called "simple lensing neighborhoods". Although the assumption of having a simple lensing neighborhood is certainly rather special, we shall demonstrate in Section 6 below that sufficiently many examples of physical interest exist. We define simple lensing neighborhoods in the following way.

Definition 5. $(\mathcal{U}, \mathcal{T}, W)$ is called a *simple lensing neighborhood* in a spacetime $(\mathcal{M}, g)$ if
(a) $\mathcal{U}$ is an open connected subset of $\mathcal{M}$ and $\mathcal{T}$ is the boundary of $\mathcal{U}$ in $\mathcal{M}$;
(b) $(\, \mathcal{T} = \partial U,\, W\,)$ is a source surface in the sense of Definition 1;
(c) for all $p \in \mathcal{U}$, the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N} = \partial\mathcal{U}/W$ is defined on the whole celestial sphere, $\mathcal{D}_p = \mathcal{S}_p$;
(d) $\mathcal{U}$ does not contain an almost periodic lightlike geodesic.

Here the notion of being "almost periodic" is defined in the following way. Any immersed curve $\lambda : I \longrightarrow \mathcal{U}$, defined on a real interval I, induces a curve $\hat{\lambda} : I \longrightarrow P\mathcal{U}$ in the projective tangent bundle $P\mathcal{U}$ over $\mathcal{U}$ which is defined by $\hat{\lambda}(s) = \{\, c\dot{\lambda}(s) \,|\, c \in \mathbb{R}\,\}$. The curve λ is called *almost periodic* if there is a strictly monotonous sequence of parameter values $(s_i)_{i\in\mathbb{N}}$ such that the sequence $\big(\hat{\lambda}(s_i)\big)_{i\in\mathbb{N}}$ has an accumulation point in $P\mathcal{U}$. Please note that Condition (d) of Definition 5 is certainly true if the strong causality condition holds everywhere on $\mathcal{U}$, i.e., if there are no closed or almost closed causal curves in $\mathcal{U}$. Also, Condition (d) is certainly true if every future-inextendible lightlike geodesic in $\mathcal{U}$ has a future end-point in $\mathcal{M}$.

Condition (d) should be viewed as adding a fairly mild assumption on the future-behavior of lightlike geodesics to the fairly strong assumptions on their past-behavior that are contained in Condition (c). In particular, Condition (c) excludes the possibility that past-oriented lightlike geodesics are blocked or trapped inside $\mathcal{U}$, i.e., it excludes the case that

$\mathcal{U}$ contains non-transparent deflectors. Condition (c) requires, in addition, that the past-pointing lightlike geodesics are transverse to $\partial\mathcal{U}$ when leaving $\mathcal{U}$.

In the situation of a simple lensing neighborhood, we have for each $p \in \mathcal{U}$ a lens map that is defined on the whole celestial sphere, $f_p : \mathcal{S}_p \longrightarrow \mathcal{N} = \partial\mathcal{U}/W$. We have, thus, equation (9) at our disposal which relates the numbers $n_+(\xi)$ and $n_-(\xi)$, for any regular value $\xi \in \mathcal{N}$, to the mapping degree of f_p. (Please recall that, by Proposition 8, $n_+(\xi)$ and $n_-(\xi)$ are finite.) It is our main goal to prove that, in a simple lensing neighborhood, the mapping degree of the lens map equals ± 1, so $n(\xi) = n_+(\xi) + n_-(\xi)$ is odd for all regular values ξ. Also, we shall prove that a simple lensing neighborhood must be contractible and that its boundary must be diffeomorphic to $S^2 \times \mathbb{R}$. The latter result reflects the fact that the notion of simple lensing neighborhoods generalizes the notion of asymptotically simple and empty spacetimes, with $\partial\mathcal{U}$ corresponding to past lightlike infinity $\mathscr{I}^-$, as will be detailed in Subsection 6.2 below. When proving the desired properties of simple lensing neighborhoods we may therefore use several techniques that have been successfully applied to asymptotically simple and empty spacetimes before.

As a preparation we need the following lemma.

Lemma 1. *Let $(\mathcal{U}, \mathcal{T}, W)$ be a simple lensing neighborhood in a spacetime $(\mathcal{M}, g)$. Then there is a diffeomorphism Ψ from the sphere bundle $\mathcal{S} = \{ Y_p \in \mathcal{S}_p \,|\, p \in \mathcal{U} \}$ of lightlike directions over $\mathcal{U}$ onto the space $T\mathcal{N} \times \mathbb{R}^2$ such that the following diagramm commutes.*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\Psi} & T\mathcal{N} \times \mathbb{R}^2 \\ i_p \uparrow & & \downarrow \mathrm{pr} \\ \mathcal{S}_p & \xrightarrow{f_p} & \mathcal{N} \end{array} \tag{11}$$

Here i_p denotes the inclusion map and pr *is defined by dropping the second factor and projecting to the foot-point.*

Proof. We fix a trivialization for the bundle $\pi_W : \mathcal{T} \longrightarrow \mathcal{N}$ and identify $\mathcal{T}$ with $\mathcal{N} \times \mathbb{R}$. Then we consider the bundle $\mathcal{B} = \{ X_q \in \mathcal{B}_q \,|\, q \in \mathcal{T} \}$ over $\mathcal{T}$, where $\mathcal{B}_q \subset \mathcal{S}_q$ is, by definition, the subspace of all lightlike directions that are tangent to past-oriented lightlike geodesics that leave $\mathcal{U}$ transversely at q. Now we choose for each $q \in \mathcal{T}$ a vector $Q_q \in T_q\mathcal{M}$, smoothly depending on q, which is non-tangent to $\mathcal{T}$ and outward pointing. With the help of this vector field Q we may identify $\mathcal{B}$ and $T\mathcal{N} \times \mathbb{R}$ as bundles over $\mathcal{T} \simeq \mathcal{N} \times \mathbb{R}$ in the following way. Fix $\xi \in \mathcal{N}$, $X_\xi \in T_\xi\mathcal{N}$ and $s \in \mathbb{R}$ and view the tangent space $T_\xi\mathcal{N}$ as a natural subspace of $T_q(\mathcal{N} \times \mathbb{R})$, where $q = (\xi, s)$. Then the desired identification is given by associating the pair (X_ξ, s) with the direction spanned by $Z_q = X_\xi + Q_q - \alpha\, W(q)$, where the number α is uniquely determined by the requirement that Z_q should be lightlike and past-pointing. – Now we consider the map

$$\pi : \mathcal{S} \longrightarrow \mathcal{B} \simeq T\mathcal{N} \times \mathbb{R} \tag{12}$$

given by following each lightlike geodesic from a point $p \in \mathcal{U}$ into the past until it reaches $\mathcal{T}$, and assigning the tangent direction at the end-point to the tangent direction at the initial point. As a matter of fact, (12) gives a principal fiber bundle with structure group $\mathbb{R}$. To prove this, we first observe that the geodesic spray induces a vector field without zeros on $\mathcal{S}$. By multiplying this vector field with an appropriate function we get a vector field whose flow is defined on all of $\mathbb{R} \times \mathcal{S}$ (see the second paragraph after Definition 1 for how to find such a function). The flow of this rescaled vector field defines an $\mathbb{R}$-action on $\mathcal{S}$ such that (12) can be identified with the projection onto the space of orbits. Conditions (c) and (d) of Definition 5 guarantee that no orbit is closed or almost closed. Owing to a general result of Palais [5], this is sufficient to prove that this action makes (12) into a principal fiber bundle with structure group $\mathbb{R}$. However, any such bundle is trivializable, see, e.g., Kobayashi and Nomizu [9], p.57/58. Choosing a trivialization for (12) gives us the desired diffeomorphism Ψ from $\mathcal{S}$ to $\mathcal{B} \times \mathbb{R} \simeq T\mathcal{N} \times \mathbb{R}^2$. The commutativity of the diagram (11) follows directly from the definition of the lens map f_p. $\square$

With the help of this lemma we will now prove the following proposition which is at the center of this section.

Proposition 11. *Let $(\mathcal{U}, \mathcal{T}, W)$ be a simple lensing neighborhood in a spacetime $(\mathcal{M}, g)$. Then*
(a) $\mathcal{N} = \mathcal{T}/W$ *is diffeomorphic to the standard 2-sphere S^2;*
(b) $\mathcal{U}$ *is contractible;*
(c) *for all $p \in \mathcal{U}$, the lens map $f_p : \mathcal{S}_p \simeq S^2 \longrightarrow \mathcal{N} \simeq S^2$ has $|\deg(f_p)| = 1$; in particular, f_p is surjective.*

Proof. In the proof of part (a) and (b) we shall adapt techniques used by Newman and Clarke [17, 18] in their study of asymptotically simple and empty spacetimes. To that end it will be necessary to assume that the reader is familiar with homology theory. With the sphere bundle $\mathcal{S}$, introduced in Lemma 1, we may associate the *Gysin homology sequence*

$$\ldots \longrightarrow H_m(\mathcal{S}) \longrightarrow H_m(\mathcal{U}) \longrightarrow H_{m-3}(\mathcal{U}) \longrightarrow H_{m-1}(\mathcal{S}) \longrightarrow \ldots \tag{13}$$

where $H_m(\mathcal{X})$ denotes the m^{th} homology group of the space $\mathcal{X}$ with coefficients in a field $\mathbb{F}$. For any choice of $\mathbb{F}$, the Gysin sequence is an exact sequence of abelian groups, see, e.g., Spanier [14], p.260 or, for the analogous sequence of cohomology groups, Bredon [15], p.390. By Lemma 1, $\mathcal{S}$ and $\mathcal{N}$ have the same homotopy type, so $H_m(\mathcal{S})$ and $H_m(\mathcal{N})$ are isomorphic. Upon inserting this into (13), we use the fact that $H_m(\mathcal{U}) = \mathbf{1}$ (= trivial group consisting of the unit element only) for $m > 4$ and $H_m(\mathcal{N}) = \mathbf{1}$ for $m > 2$ because $\dim(\mathcal{U}) = 4$ and $\dim(\mathcal{N}) = 2$. Also, we know that $H_0(\mathcal{U}) = \mathbb{F}$ and $H_0(\mathcal{N}) = \mathbb{F}$ since $\mathcal{U}$ and $\mathcal{N}$ are connected. Then the exactness of the Gysin sequence implies that

$$H_m(\mathcal{U}) = \mathbf{1} \quad \text{for} \;\; m > 0 \tag{14}$$

and

$$H_1(\mathcal{N}) = \mathbf{1}\,, \quad H_2(\mathcal{N}) = \mathbb{F}\,. \tag{15}$$

From (15) we read that $\mathcal{N}$ is compact since otherwise $H_2(\mathcal{N}) = \mathbf{1}$. Moreover, we observe that $\mathcal{N}$ has the same homology groups and thus, in particular, the same Euler characteristic as the 2-sphere. It is well known that any two compact and orientable 2-manifolds are diffeomorphic if and only if they have the same Euler characteristic (or, equivalently, the same genus), see, e.g., Hirsch [7], Chapter 9. We have thus proven part (a) of the proposition. – To prove part (b) we consider the end of the exact *homotopy sequence* of the fiber bundle $\mathcal{S}$ over $\mathcal{U}$, see, e.g., Frankel [19], p.600,

$$\ldots \longrightarrow \pi_1(\mathcal{S}) \longrightarrow \pi_1(\mathcal{U}) \longrightarrow \mathbf{1}\,. \tag{16}$$

As $\mathcal{S}$ has the same homotopy type as $\mathcal{N} \simeq S^2$, we may replace $\pi_1(\mathcal{S})$ with $\pi_1(S^2) = \mathbf{1}$, so the exactness of (16) implies that $\pi_1(\mathcal{U}) = \mathbf{1}$, i.e., that $\mathcal{U}$ is simply connected. If, for some $m > 1$, the homotopy group $\pi_m(\mathcal{U})$ would be different from $\mathbf{1}$, the Hurewicz isomorphism theorem (see, e.g., Spanier [14], p.394 or Bredon [15], p.479, Corollary 10.10.) would give a contradiction to (14). Thus, $\pi_m(\mathcal{U}) = \mathbf{1}$ for all $m \in \mathbb{N}$, i.e., $\mathcal{U}$ is contractible. – We now prove part (c). Since $\mathcal{U}$ is contractible, the tangent bundle $T\mathcal{U}$ and thus the sphere bundle $\mathcal{S}$ over $\mathcal{U}$ admits a global trivialization, $\mathcal{S} \simeq \mathcal{U} \times S^2$. Fixing such a trivialization and choosing a contraction that collapses $\mathcal{U}$ onto some point $p \in \mathcal{U}$ gives a contraction $\tilde{i_p} : \mathcal{S} \longrightarrow \mathcal{S}_p$. Together with the inclusion map $i_p : \mathcal{S}_p \longrightarrow \mathcal{S}$ this gives us a homotopy equivalence between $\mathcal{S}_p$ and $\mathcal{S}$. (Please recall that a *homotopy equivalence* between two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ is a pair of continuous maps $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ and $\tilde{\varphi} : \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\varphi \circ \tilde{\varphi}$ can be continuously deformed into the identity on $\mathcal{Y}$ and $\tilde{\varphi} \circ \varphi$ can be continuously deformed into the identity on $\mathcal{X}$.) On the other hand, the projection pr from (11), together with the zero section $\tilde{\mathrm{pr}} : \mathcal{N} \longrightarrow T\mathcal{N} \times \mathbb{R}^2$ gives a homotopy equivalence between $T\mathcal{N} \times \mathbb{R}^2$ and $\mathcal{N}$. As a consequence, the diagram (11) tells us that the lens map $f_p = \mathrm{pr} \circ \Psi \circ i_p$ together with the map $\tilde{f}_p = \tilde{i}_p \circ \Psi^{-1} \circ \tilde{\mathrm{pr}}$ gives a homotopy equivalence between $\mathcal{S}_p \simeq S^2$ and $\mathcal{N} \simeq S^2$, so $f_p \circ \tilde{f}_p$ is homotopic to the identity. Since the mapping degree is a homotopic invariant (please recall Property B of the mapping degree from Section 4), this implies that $\deg(f_p \circ \tilde{f}_p) = 1$. Now the product theorem for the mapping degree (see, e.g., Choquet-Bruhat, Dewitt-Morette, and Dillard-Bleick [12], p.483) yields $\deg(f_p)\deg(\tilde{f}_p) = 1$. As the mapping degree is an integer, this can be true only if $\deg(f_p) = \deg(\tilde{f}_p) = \pm 1$. In particular, f_p must be surjective since otherwise $\deg(f_p) = 0$. □

In all simple examples to which this proposition applies the degree of f_p is, actually, equal to $+1$, and it is hard to see whether examples with $\deg(f_p) = -1$ do exist. The following consideration is quite instructive. If we start with a simple lensing neighborhood in a flat spacetime (or, more generally, in a conformally flat spacetime), then conjugate points cannot occur, so it is clear that the case $\deg(f_p) = -1$ is impossible. If we now perturb the metric in such a way that the simple-lensing-neighborhood property is maintained during the perturbation, then, by Property B of the degree, the equation $\deg(f_p) = +1$ is preserved. This demonstrates that the case $\deg(f_p) = -1$ cannot occur for weak gravitational fields (or for small perturbations of conformally flat spacetimes such as Robertson-Walker spacetimes).

Among other things, Proposition 11 gives a good physical motivation for studying degree-one maps from S^2 to S^2. In particular, it is an interesting problem to characterize the caustics of such maps. Please note that, by parts (a) and (c) of Proposition 11, $f_p(\mathcal{D}_p)$ is simply connected for all $p \in \mathcal{U}$. Hence, Proposition 5 applies which says that the formation of a caustic is necessary for multiple imaging.

Owing to (10), part (c) of Proposition 11 implies in particular that $n(\xi) = n_+(\xi) + n_-(\xi)$ is odd for all worldlines of light sources $\xi \in \mathcal{N}$ that do not pass through the caustic of the past light cone of p, i.e., if only light rays within $\mathcal{U}$ are taken into account the observer at p sees an odd number of images of such a worldline. It is now our goal to prove a similar 'odd number theorem' for a light source with worldline inside $\mathcal{U}$. As a preparation we establish the following lemma.

Lemma 2. *Let $(\mathcal{U}, \mathcal{T}, W)$ be a simple lensing neighborhood in a spacetime $(\mathcal{M}, g)$ and $p \in \mathcal{U}$. Let $J^-(p,\mathcal{U})$ denote, as usual, the* causal past *of p in $\mathcal{U}$, i.e., the set of all points in $\mathcal{M}$ that can be reached from p along a past-pointing causal curve in $\mathcal{U}$. Let $\partial_{\mathcal{U}} J^-(p,\mathcal{U})$ denote the boundary of $J^-(p,\mathcal{U})$ in $\mathcal{U}$. Then*

(a) *every point* $q \in \partial_{\mathcal{U}} J^-(p,\mathcal{U})$ *can be reached from* p *along a past-pointing lightlike geodesic in* $\mathcal{U}$*;*
(b) $\partial_{\mathcal{U}} J^-(p,\mathcal{U})$ *is relatively compact in* $\mathcal{M}$.

Proof. As usual, let $I^-(p,\mathcal{U})$ denote the chronological past of p in $\mathcal{U}$, i.e., the set of all points that can be reached from p along a past-pointing timelike curve in $\mathcal{U}$. To prove part (a), fix a point $q \in \partial_{\mathcal{U}} J^-(p,\mathcal{U})$. Choose a sequence $(p_i)_{i\in\mathbb{N}}$ of points in $\mathcal{U}$ that converge towards p in such a way that $p \in I^-(p_i,\mathcal{U})$ for all $i \in \mathbb{N}$. This implies that we can find for each $i \in \mathbb{N}$ a past-pointing timelike curve λ_i from p_i to q. Then the λ_i are past-inextendible in $\mathcal{U} \setminus \{q\}$. Owing to a standard lemma (see, e.g., Wald [20], Lemma 8.1.5) this implies that the λ_i have a causal limit curve λ through p that is past-inextendible in $\mathcal{U} \setminus \{q\}$. We want to show that λ is the desired lightlike geodesic. Assume that λ is not a lightlike geodesic. Then λ enters into the open set $I^-(p,\mathcal{U})$ (see Hawking and Ellis [21], Proposition 4.5.10), so λ_i enters into $I^-(p,\mathcal{U})$ for i sufficiently large. This, however, is impossible since all λ_i have past end-point on $\partial_{\mathcal{U}} J^-(p,\mathcal{U})$, so λ must be a lightlike geodesic. It remains to show that λ has past end-point at q. Assume that this is not true. Since λ is past-inextendible in $\mathcal{U} \setminus \{q\}$ this assumption implies that λ is past-inextendible in $\mathcal{U}$, so by condition (c) of Definition 5 λ has past end-point on $\partial\mathcal{U}$ and meets $\partial\mathcal{U}$ transversely. As a consequence, for i sufficiently large λ_i has to meet $\partial\mathcal{U}$ which gives a contradiction to the fact that all λ_i are within $\mathcal{U}$. – To prove part (b), we have to show that any sequence $(q_i)_{i\in\mathbb{N}}$ in $\partial_{\mathcal{U}} J^-(p,\mathcal{U})$ has an accumulation point in $\mathcal{M}$. So let us choose such a sequence. From part (a) we know that there is a past-pointing lightlike geodesic μ_i from p to q_i in $\mathcal{U}$ for all $i \in \mathbb{N}$. By compactness of $\mathcal{S}_p \simeq S^2$, the tangent directions to these geodesics at p have an accumulation point in $\mathcal{S}_p$. Let μ be the past-pointing lightlike geodesic from p which is determined by this direction. By condition (c) of Definition 5, this geodesic μ and each of the geodesics μ_i must have a past end-point on $\partial\mathcal{U}$ if maximally extended inside $\mathcal{U}$. We may choose an affine parametrization for each of those geodesics with the parameter ranging from the value 0 at p to the value 1 at $\partial\mathcal{U}$. Then our sequence $(q_i)_{i\in\mathbb{N}}$ in $\mathcal{U}$ determines a sequence $(s_i)_{i\in\mathbb{N}}$ in the interval $[0,1]$ by setting $q_i = \mu_i(s_i)$. By compactness of $[0,1]$, this sequence must have an accumulation point $s \in [0.1]$. This demonstrates that the q_i must have an accumulation point in $\mathcal{M}$, namely the point $\mu(s)$. □

We are now ready to prove the desired odd-number theorem for light sources with worldline in $\mathcal{U}$.

Proposition 12. *Let* $(\mathcal{U},\mathcal{T},W)$ *be a simple lensing neighborhood in a spacetime* $(\mathcal{M},g)$ *and assume that* $\mathcal{U}$ *does not contain a closed timelike curve. Fix a point* $p \in \mathcal{U}$ *and a timelike embedded* C^∞ *curve* γ *in* $\mathcal{U}$ *whose image is a closed topological subset of* $\mathcal{M}$. (*The latter condition excludes the case that* γ *has an end-point on* $\partial\mathcal{U}$.) *Then the following is true.*
(a) *If* γ *does not meet the point* p*, then there is a past-pointing lightlike geodesic from* p *to* γ *that lies completely within* $\mathcal{U}$ *and contains no conjugate points in its interior.* (*The end-point may be conjugate to the initial-point.*) *If this geodesic meets* γ *at the point* q*, say, then all points on* γ *that lie to the future of* q *cannot be reached from* p *along a past-pointing lightlike geodesic in* $\mathcal{U}$.
(b) *If* γ *meets neither the point* p *nor the caustic of the past light cone of* p*, then the number of past-pointing lightlike geodesics from* p *to* γ *that are completely contained in* $\mathcal{U}$ *is finite and odd.*

Proof. In the first step we construct a C^∞ vector field V on $\mathcal{M}$ that is timelike on $\mathcal{U}$, has γ as an integral curve, and coincides with W on $\mathcal{T} = \partial\mathcal{U}$. To that end we first choose any future-pointing timelike C^∞ vector field V_1 on $\mathcal{M}$. (Existence is guaranteed by our assumption of time-orientability.) Then we extend the vector field W to a C^∞ vector field V_2 onto some neighborhood $\mathcal{V}$ of $\mathcal{T}$. Since W is causal and future-pointing, V_2 may be chosen timelike and future-pointing on $\mathcal{V} \setminus \mathcal{T}$. (Here we make use of the fact that $\mathcal{T} = \partial\mathcal{U}$ is a closed subset of $\mathcal{M}$.) Finally we choose a timelike and future-pointing vector field V_3 on some neighborhood $\mathcal{W}$ of γ that is tangent to γ at all points of γ. (Here we make use of the fact that the image of γ is a closed subset of $\mathcal{M}$.) We choose the neighborhoods $\mathcal{V}$ and $\mathcal{W}$ disjoint which is possible since γ is completely contained in $\mathcal{U}$ and closed in $\mathcal{M}$. With the help of a partition of unity we may now combine the three vector fields V_1, V_2, V_3 into a vector field V with the desired properties.

In the second step we consider the quotient space $\mathcal{M}/V$. This space contains the open subset $\mathcal{U}/V$ whose boundary $\mathcal{T}/V = \mathcal{N}$ is, by Proposition 11, a manifold diffeomorphic to S^2. We want to show that $\mathcal{U}/V$ is a manifold (which, according to our terminology, in particular requires that $\mathcal{U}/V$ is a Hausdorff space). To that end we consider the map $j_p : \overline{\partial_{\mathcal{U}} J^-(p,\mathcal{U})} \longrightarrow \overline{\mathcal{U}}/V$ which assigns to each point $q \in \overline{\partial_{\mathcal{U}} J^-(p,\mathcal{U})}$ the integral curve of V passing through that point. (Overlining always means closure in $\mathcal{M}$.) Clearly, j_p is continuous with respect to the topology $\overline{\partial_{\mathcal{U}} J^-(p,\mathcal{U})}$ inherits as a subspace of $\mathcal{M}$ and the quotient topology on $\overline{\mathcal{U}}/V$. Moreover, $\overline{\partial_{\mathcal{U}} J^-(p,\mathcal{U})}$ intersects each integral curve of V at most once, and if it intersects one integral curve then it also intersects all neighbboring integral curves in $\overline{\mathcal{U}}$; this follows from Wald [20], Theorem 8.1.3. Hence, j_p is injective and its image is open in $\overline{\mathcal{U}}/V$. On the other hand, part (b) of Lemma 2 implies that the image of j_p is closed. Since the image of j_p is non-empty and connected, it must be all of $\overline{\mathcal{U}}/V$. (The domain of j_p and, thus, the image of j_p is non-empty because $\mathcal{U}$ does not contain a closed timelike curve. The domain and, thus, the image of j_p is connected since $\mathcal{U}$ is connected.) We have, thus, proven that j_p is a homeomorphism. This implies that the Hausdorff condition is satisfied on $\overline{\mathcal{U}}/V$ and, in particular, on $\mathcal{U}/V$. Since V is timelike and $\mathcal{U}$ contains no closed timelike curves, this makes sure that $\mathcal{U}/V$ is a manifold according to our terminology, see Harris [10], Theorem 2.

In the third step we use these results to prove part (a) of the proposition. Our result that j_p is a homeomorphism implies, in particular, that γ has an intersection with $\partial_{\mathcal{U}} J^-(p,\mathcal{U})$ at some point q. Now part (a) of Lemma 2 shows that there is a past-pointing lightlike geodesic from p to q in $\mathcal{U}$. This geodesic cannot contain conjugate points in its interior since otherwise a small variation would give a timelike curve from p to q, see Hawking and Ellis [21], Proposition 5.4.12, thereby contradicting $q \in \partial_{\mathcal{U}} J^-(p,\mathcal{U})$. The rest of part (a) is clear since all past-pointing lightlike geodesics in $\mathcal{U}$ that start at p are confined to $J^-(p,\mathcal{U})$.

In the last step we prove part (b). To that end we choose on the tangent space $T_p\mathcal{M}$ a Lorentz basis $(E_p^1, E_p^2, E_p^3, E_p^4)$ with E_p^4 future-pointing, and we identify each $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ with the past-pointing lightlike vector $Y_p = x^1 E_p^1 + x^2 E_p^2 + x^3 E_p^3 - |x| E_p^4$. With this identification, the lens map takes the form $f_p : S^2 \longrightarrow \mathcal{N} = \partial\mathcal{U}/V$, $x \longmapsto \pi_V\big(\exp_p(w_p(x)x)\big)$. We now define a continuous map $F : B \longrightarrow \mathcal{M}/V$ on the closed ball $B = \big\{ x \in \mathbb{R}^3 \,\big|\, |x| \leq 1 \,\big\}$ by setting $F(x) = \pi_V\big(\exp_p(w_p(\frac{x}{|x|})\, x)\big)$ for $x \neq 0$ and $F(0) = \pi_V(p)$. The restriction of F to the interior of $\mathcal{B}$ is a C^∞ map onto the manifold $\mathcal{U}/V$, with the exception of the origin where F is not differentiable. The latter problem can be circumvented by approximating F in the C^o-sense, on an arbitrarily small neighborhood of the origin, by a C^∞ map. Then the mapping degree $\deg(F)$ can be calculated (see, e.g., Choquet-Bruhat, Dewitt-Morette, and Dillard-Bleick [12], pp.477) with the help of the integral formula

$$\int_B F^*\omega = \deg(F) \int_{\mathcal{U}/V} \omega \tag{17}$$

where ω is any 3-form on $\mathcal{U}/V$ and the star denotes the pull-back of forms. For any 2-form ψ on $\mathcal{U}/V$, we may apply this formula to the form $\omega = d\psi$. With the help of the Stokes theorem we then find

$$\int_{S^2} F^*\psi = \deg(F) \int_{\mathcal{N}} \psi \, . \tag{18}$$

However, the restriction of F to $\partial B = S^2$ gives the lens map, so on the left-hand side of (18) we may replace $F^*\psi$ by $f_p^*\psi$. Then comparison with the integral formula for the degree of f_p shows that $\deg(F) = \deg(f_p)$ which, according to Proposition 11, is equal to ± 1. For every $\zeta \in \mathcal{U}/V$ that is a regular value of F, the result $\deg(F) = \pm 1$ implies that the number of elements in $F^{-1}(\zeta)$ is finite and odd. By assumption, the worldline $\gamma \in \mathcal{U}/V$ meets neither the point p nor the caustic of the past light cone of p. The first condition makes sure that our perturbation of F near the origin can be done without influencing the set $F^{-1}(\gamma)$; the second condition implies that γ is a regular value of F, please recall our discussion at the end of Section 3. This completes the proof. $\square$

If only light rays within $\mathcal{U}$ are taken into account, then Proposition 12 can be summarized by saying that, for light sources in a simple lensing neighborhood, the "youngest image" has always even parity and the total number of images is finite and odd.

In the quasi-Newtonian approximation formalism it is a standard result that a transparent gravitational lens produces an odd number of images, see Schneider, Ehlers and Falco [1], Section 5.4, for a detailed discussion. Proposition 12 may be viewed as a reformulation of this result in a Lorentzian geometry setting. It is quite likely that an alternative proof of Proposition 12 can be given by using the Morse theoretical results of Giannoni, Masiello and Piccione [22, 23]. Also, the reader should compare our results with the work of McKenzie [16] who used Morse theory for proving an odd-number theorem in certain globally hyperbolic spacetimes. Contrary to McKenzie's theorem, our Proposition 12 requires mathematical assumptions which can be physically interpreted rather easily.

6 Examples

6.1 Two simple examples with non-transparent deflectors

(a) Non-transparent string

As a simple example, we consider gravitational lensing in the spacetime $(\mathcal{M}, g)$ where $\mathcal{M} = \mathbb{R}^2 \times \big(\mathbb{R}^2 \setminus \{0\}\big)$ and

$$g = -dt^2 + dz^2 + dr^2 + k^2\, r^2\, d\varphi^2 \tag{19}$$

with some constant $0 < k < 1$. Here (t, z) denote Cartesian coordinates on $\mathbb{R}^2$ and (r, φ) denote polar coordinates on $\mathbb{R}^2 \setminus \{0\}$. This can be interpreted as the spacetime around a static non-transparent string, see Vilenkin [24], Hiscock [25] and Gott [26]. One should think of the string as being situated at the z-axis. Since the latter is not part of the spacetime, it is indeed justified to speak of a *non-transparent* string.

As $\partial/\partial t$ is a Killing vector field normalized to -1, the lightlike geodesics in $(\mathcal{M}, g)$ correspond to the geodesics of the space part. The latter is a metrical product of a real line with coordinate z and a cone with polar coordinates (r, φ). So the geodesics are straight lines if we cut the cone open along some radius $\varphi =$ const. and flatten it out in a plane. Owing to this simple form of the lightlike geodesics, the investigation of lens maps in this string spacetime is quite easy.

To work this out, choose some constant $R > 0$ and let $\mathcal{T}$ denote the hypercylinder $r = R$ in $\mathcal{M}$. Let W denote the restriction of the vector field $\partial/\partial t$ to $\mathcal{T}$. Then $(\mathcal{T}, W)$ is a source surface, with $\mathcal{N} = \mathcal{T}/W \simeq S^1 \times \mathbb{R}$. Henceforth we discuss the lens map f_p for any point $p \in \mathcal{M}$ at a radius $r < R$. There are no past-pointing lightlike geodesics from p that intersect $\mathcal{T}$ more than once or touch $\mathcal{T}$ tangentially, so the lens map f_p gives full information about all images at p of each light source $\xi \in \mathcal{N}$. The domain $\mathcal{D}_p$ of the lens map is given by excising a curve segment, namely a meridian including both end-points at the "poles", from the celestial sphere $\mathcal{S}_p$, so $\mathcal{D}_p \simeq \mathbb{R}^2$ is connected. The boundary of $\mathcal{D}_p$ in $\mathcal{S}_p$ corresponds to light rays that are blocked by the string before reaching $\mathcal{T}$. It is easy to see that the lens map cannot be continuously extended onto $\mathcal{S}_p$ (= closure of $\mathcal{D}_p$ in $\mathcal{S}_p$). Nonetheless, the lens map admits an extension in the sense of Definition 4. We may choose $\mathcal{M}_1 = S^2$ and $\mathcal{M}_2 = S^2$. Here $\mathcal{D}_p$ is embedded into the sphere in such a way that it covers a region $(\theta, \varphi) \in \,]0, \pi[\, \times \,]\,\varepsilon\,, 2\pi - \varepsilon[\,$, i.e., in comparison with the embedding into $\mathcal{S}_p$ the curve segment excised from the sphere has been "widened" a bit. The embedding of $\mathcal{N} \simeq S^1 \times \mathbb{R}$ into S^2 is made via Mercator projection.

As the string spacetime has vanishing curvature, the light cones in $\mathcal{M}$ have no caustics. Owing to our general results of Section 3, this implies that the caustic of the lens map is empty and that all images have even parity, so (8) gives $\deg(\overline{f_p}, \xi) = n_+(\xi) = n(\xi)$ for all $\xi \in \mathcal{N} \setminus \overline{f_p}(\partial \mathcal{D}_p)$.

The actual value of $n(\xi)$ depends on the parameter k that enters into the metric (19). If $i = 1/k$ is an integer, $\mathcal{N} \setminus \overline{f_p}(\partial \mathcal{D}_p)$ is connected and $n(\xi) = i$ everywhere on this set. If $i < 1/k < i + 1$ for some integer i, $\mathcal{N} \setminus \overline{f_p}(\partial \mathcal{D}_p)$ has two connected components, with $n(\xi) = i$ on one of them and $n(\xi) = i + 1$ on the other. Thus, the string produces multiple imaging and the number of images is (finite but) arbitrarily large if k is sufficiently small.

For all $k \in \,]0, 1[\,$, the lens map is surjective, $f_p(\mathcal{D}_p) = \mathcal{N} \simeq S^1 \times \mathbb{R}$. So this example shows that the assumption of $f_p(\mathcal{D}_p)$ being simply connected was essential in Proposition 5.

(b) Non-transparent spherical body

We consider the Schwarzschild metric

$$g = \left(1 - \tfrac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) - \left(1 - \tfrac{2m}{r}\right) dt^2 \tag{20}$$

on the manifold $\mathcal{M} = \,]R_o, \infty[\, \times S^2 \times \mathbb{R}$. In (20), r is the coordinate ranging over $]R_o, \infty[\,$, t is the coordinate ranging over $\mathbb{R}$, and θ and φ are spherical coordinates on S^2. This gives the static vacuum spacetime around a spherically symmetric body of mass m and radius R_o. Restricting the spacetime manifold to the region $r > R_o$ is a way of treating the central body as non-transparent. In the following we keep a value $R_o > 0$ fixed and we allow m to vary between $m = 0$ (flat space) and $m = R_o/2$ (black hole).

For discussing lens maps in this spacetime we fix a constant $R > 3R_o/2$. We denote by $\mathcal{T}$ the set of all points in $\mathcal{M}$ with coordinate $r = R$ and we denote by W the restriction of $\partial/\partial t$ to W. Then $(\mathcal{T}, W)$ is a source surface, with $\mathcal{N} = \mathcal{T}/W \simeq S^2$. It is our goal to discuss the properties of the lens map $f_p : \mathcal{D}_p \longrightarrow \mathcal{N}$ for a point $p \in \mathcal{M}$ with a radius coordinate $r < R$ in dependence of the mass parameter m. To that end we make use of well-known

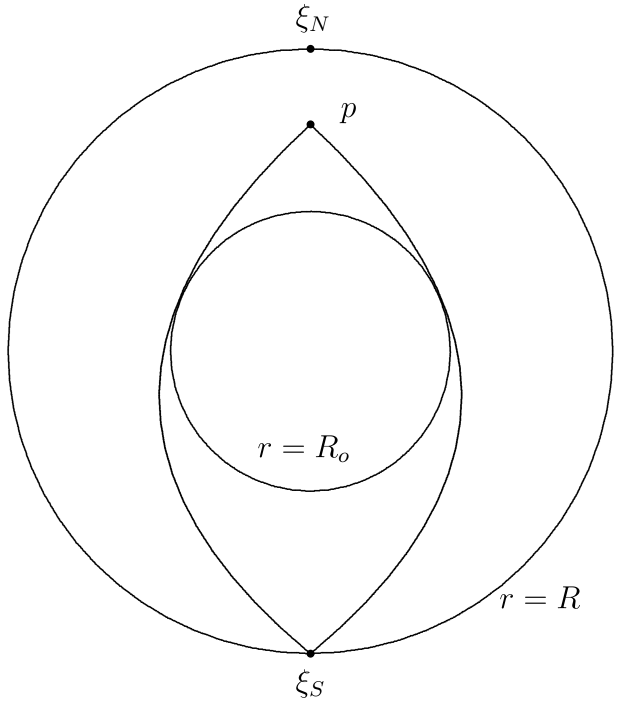

Figure 2: At $m = m_1$, the extended lens map $\overline{f_p}$ maps the boundary of $\mathcal{D}_p$ onto the south pole ξ_S.

properties of the lightlike geodesics in the Schwarzschild metric, see, e.g., Chandrasekhar [28], Section 20, for a comprehensive discussion. For determining the relevant features of the lens map it will be sufficient to concentrate on qualitative aspects of image positions. For quantitative aspects the reader may consult Virbhadra and Ellis [27].

We first observe that, for any $m \in [0, R_o/2]$, there is no past-pointing lightlike geodesic from p that intersects $\mathcal{T}$ more than once or touches $\mathcal{T}$ tangentially. This follows from the fact that in the region $r > 3m$ the radius coordinate has no local maximum along any light ray. So the lens map f_p gives full information about all images at p of light sources $\xi \in \mathcal{N}$.

For $m = 0$, the light rays are straight lines. The domain $\mathcal{D}_p$ of the lens map is given by excising a disc, including the boundary, from the celestial sphere $\mathcal{S}_p$, i.e., $\mathcal{D}_p \simeq \mathbb{R}^2$. The boundary of $\mathcal{D}_p$ corresponds to light rays grazing the surface of the central body, so f_p can be continuously extended onto the closure of $\mathcal{D}_p$ in $\mathcal{S}_p$, thereby giving an extension of f_p, in the sense of Definition 4, $\overline{f_p} : \overline{\mathcal{D}_p} \subseteq \mathcal{S}_p \longrightarrow \mathcal{N}$. In Figure 2, $\overline{f_p}(\partial\mathcal{D}_p)$ can be represented as a "circle of equal latitude" on the sphere $r = R$, with the image of f_p "to the north" of this circle. With increasing m, $\overline{f_p}(\partial\mathcal{D}_p)$ moves "south" until, at some value $m = m_1$, it has reached the "south pole" ξ_S. This is the situation depicted in Figure 2. From now on the lens map is surjective and ξ_S is seen as an *Einstein ring*, thereby indicating that a caustic has formed. Now $\overline{f_p}(\partial\mathcal{D}_p)$ moves north until, at some value $m = m_1'$, it has reached the "north pole" ξ_N. From now on ξ_N is seen as an Einstein ring, in addition to the regular image that exists from the beginning. With further increasing m, we find an infinite sequence of values $0 < m_1 < m_1' < m_2 < m_2' < \cdots < m_i < m_i' < \dots$ such that at $m = m_i$ a new Einstein ring of ξ_S and at m_i' a new Einstein ring of ξ_N comes into existence. For all intermediate values of m, $\overline{f_p}(\partial\mathcal{D}_p)$ divides $\mathcal{N}$ into two connected components. All points ξ in the southern component, with the exception of the south pole ξ_S, are regular values of the lens map. $f_p^{-1}(\xi)$ consists of exactly $2i$ points where i is the largest integer with $m_i < m$. There are i images of even parity, $n_+(\xi) = i$, and i images of odd parity, $n_-(\xi) = i$, hence $\deg(\overline{f_p}, \xi) = n_+(\xi) - n_-(\xi) = 0$. Similarly, all points ξ in the northern component, with the exception of the north pole ξ_N, are regular values of the lens map. $f_p^{-1}(\xi)$ consists of exactly $2i+1$ points, where i is the largest integer with $m_i' < m$. There are $i+1$ images of even parity, $n_+(\xi) = i+1$, and i images of odd parity, $n_-(\xi) = i$, hence $\deg(\overline{f_p}, \xi) = n_+(\xi) - n_-(\xi) = 1$. Both sequences $(m_i)_{i\in\mathbb{N}}$ and $(m_i')_{i\in\mathbb{N}}$ converge towards $m = R_o/3$. For $m \geq R_o/3$, the boundary of $\mathcal{D}_p$ corresponds to light rays that approach the sphere $r = 3m$ asymptotically in a neverending spiral motion, cf. Chandrasekhar [28], Figure 9 and Figure 10. The lens map no longer admits an extension in the sense of Definition 4, so we cannot assign a mapping degree to it. There are infinitely many concentric Einstein rings for both poles, and infinitely many isolated images for all other $\xi \in \mathcal{N}$, with both $n_+(\xi)$ and $n_-(\xi)$ being infinite. These features remain unchanged until the black-hole case $m = R_o/2$ is reached.

The fact that in this case the caustic of the lens map consists of just two points is rather exceptional. After a small perturbation of the spherical symmetry the caustic would show a completely different behavior. For regular $\xi \in \mathcal{N}$, however, the statements about $n_\pm(\xi)$ are stable against small perturbations.

Having studied Schwarzschild spacetimes around non-transparent bodies, the reader might ask what about transparent bodies, i.e., what about matching an interior solution to the exterior Schwarzschild solution at Radius R_o, with $R_o > 2m$, and allowing for light rays passing through the interior region. If $R_o > 3m$, and if there are no light rays trapped within the interior region, the resulting spacetime will be asymptotically simple and empty. Qualitative features of lens maps in this class of spacetimes are discussed in the following subsection. For a more explicit discussion of lens maps in the Schwarzschild spacetime of a transparent body, choosing a perfect fluid with constant density for the interior region, the reader is refered to Kling and Newman [29].

6.2 Asymptotically simple and empty spacetimes

Asymptotically simple and empty spacetimes are considered to be good models for the gravitational fields of transparent gravitating bodies that can be viewed as isolated from all other masses in the universe. The formal definition, which is essentially due to Penrose [30], cf., e.g. Hawking and Ellis [21], p. 222, reads as follows.

Definition 6. A spacetime $(\mathcal{M}, g,)$ is called *asymptotically simple* if there is a strongly causal spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ with the following properties.
(a) $\mathcal{M}$ is an open submanifold of $\tilde{\mathcal{M}}$ with a non-empty boundary $\partial\mathcal{M}$.
(b) There is a C^∞ function $\Omega : \tilde{\mathcal{M}} \longrightarrow \mathbb{R}$ such that $\mathcal{M} = \{\, p \in \tilde{\mathcal{M}} \mid \Omega(p) > 0 \,\}$, $\partial\mathcal{M} = \{\, p \in \tilde{\mathcal{M}} \mid \Omega(p) = 0 \,\}$, $d\Omega \neq 0$ everywhere on $\partial\mathcal{M}$ and $\tilde{g} = \Omega^2 g$ on $\mathcal{M}$.
(c) Every inextendible lightlike geodesic in $\mathcal{M}$ has past and future end-point on $\partial\mathcal{M}$.
$(\mathcal{M}, g)$ is called *asymptotically simple and empty* if, in addition,
(d) there is a neighborhood $\mathcal{V}$ of $\partial\mathcal{M}$ in $\tilde{\mathcal{M}}$ such that the Ricci tensor of g vanishes on $\mathcal{V} \cap \mathcal{M}$.

Condition (d) of Definition 6 is a way of saying that, sufficiently far away from the gravitating body under consideration, Einstein's vacuum field equation is satisfied. This assumption is reasonable for the spacetime around an isolated body producing gravitational lensing as long as cosmological aspects can be ignored.

The assumptions (a)–(d) of Definition 6 imply that $\partial\mathcal{M}$ is a $\tilde{g}$-lightlike hypersurface in $\tilde{\mathcal{M}}$ that has two connected components, usually denoted by $\mathscr{I}^+$ and $\mathscr{I}^-$ (cf., e.g., Hawking and Ellis, [21], p.222). Every inextendible lightlike geodesic in $\mathcal{M}$ has future end-point on $\mathscr{I}^+$ and past end-point on $\mathscr{I}^-$.

In the following we concentrate on $\mathscr{I}^-$ which is the relevant quantity in view of gravitational lensing. By construction, $\mathscr{I}^-$ is ruled by the integral curves of the $\tilde{g}$-gradient Z of Ω. (In coordinate notation, the vector field Z is defined by $Z^a = \tilde{g}^{ab}\,\partial_b\Omega$ on $\mathscr{I}^-$.) It is well known that Z is regular, with $\mathscr{I}^-/Z$ being diffeomorphic to S^2, and that the natural projection $\pi_Z : \mathscr{I}^- \longrightarrow \mathscr{I}^-/Z \simeq S^2$ makes $\mathscr{I}^-$ into a trivializable fiber bundle with typical fiber diffeomorphic to $\mathbb{R}$. For a full proof we refer to Newman and Clarke [17, 18]. (The argument given in Hawking and Ellis [21], Proposition 6.9.4, which is due to Geroch [31], is incomplete.) This result can be translated into our terminology in the following way.

Proposition 13. *In the case of an asymptotically simple and empty spacetime,* $(\mathscr{I}^-, Z)$ *is a source surface in the spacetime* $(\tilde{\mathcal{M}}, \tilde{g})$*, with* $\mathcal{N} = \mathscr{I}^-/Z$ *diffeomorphic to* S^2*.*

Each integral curve of Z can be written as the C^1-limit of a sequence $(\gamma_i)_{i\in\mathbb{N}}$ of timelike curves in $\mathcal{M}$. We may interpret the γ_i as a sequence of worldlines of light sources approaching infinity. From the viewpoint of the physical spacetime $(\mathcal{M}, g)$, it is thus justified to interpret the integral curves of Z as "light sources at infinity". With respect to the unphysical metric $\tilde{g}$, these worldlines are lightlike. With respect to the physical metric, however, they have no causal character at all, because the metric g is not defined on $\mathscr{I}^-$. It is, thus, a misinterpretation to say that the "light sources at infinity" move at the speed of light.

We shall now show that the formalism of "simple lensing neighborhoods" applies to the situation at hand. To that end, we observe that $\mathscr{I}^-$ is the boundary of $\mathcal{M}$ in the manifold $\tilde{M} \setminus \mathscr{I}^+$. This gives rise to the following result.

Proposition 14. *In the case of an asymptotically simple and empty spacetime,* $(\mathcal{M}, \mathscr{I}^-, Z)$ *is a simple lensing neighborhood in the spacetime* $(\tilde{\mathcal{M}} \setminus \mathscr{I}^+, \tilde{g}|_{\tilde{\mathcal{M}}\setminus\mathscr{I}^+})$*.*

Proof. Condition (a) of Definition 5 is obvious from Definition 6 and Condition (b) was just established. The proof of the remaining two conditions is based on the fact that on $\mathcal{M}$ the g-lightlike geodesics coincide with the $\tilde{g}$-lightlike geodesics (up to affine parametrization). Condition (d) of Definition 5 is satisfied since every lightlike geodesic in $\mathcal{M}$ has past end-point on $\mathscr{I}^-$ and future end-point on $\mathscr{I}^+$. Moreover, the arrival on $\mathscr{I}^\pm$ must be transverse since $\mathscr{I}^\pm$ is $\tilde{g}$-lightlike. This shows that Condition (c) of Definition 5 is satisfied as well. □

We can, thus, apply our results on simple lensing neighborhoods to asymptotically simple and empty spacetimes. As a first result, Proposition 11 tells us that every asymptotically simple and empty spacetime $\mathcal{M}$ must be contractible. This result is not new. It is well known that every asymptotically simple and empty spacetime is globally hyperbolic and, thus, homeomorphic to a product of a Cauchy surface $\mathcal{C}$ with the real line, $\mathcal{M} \simeq \mathcal{C} \times \mathbb{R}$, and that $\mathcal{C}$ is contractible. For a full proof we refer again to Newman and Clarke [17, 18]. The stronger result that $\mathcal{C}$ must be homeomorphic to $\mathbb{R}^3$ requires the assumption that the Poincaré conjecture is true (i.e., that every simply connected and compact 3-manifold is homeomorphic to S^3).

In addition, Proposition 11 gives us the following result.

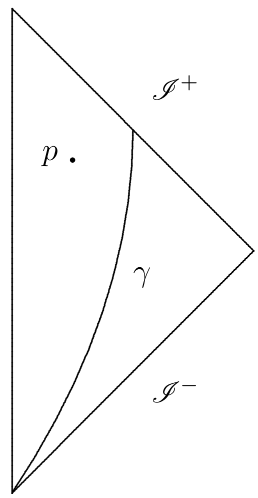

Figure 3: Illustration of Proposition 16

Proposition 15. *In the case of an asymptotically simple and empty spacetime, for all* $p \in \mathcal{M}$ *the lens map* $f_p : \mathcal{S}_p \longrightarrow \mathscr{I}^-/Z \simeq S^2$ *has* $|\deg(f_p)| = 1$.

The lens map f_p for "light sources at infinity" in an asymptotically simple and empty spacetime was already discussed in Perlick [32, 33]. In particular, a proof of the result $\deg(f_p) = 1$ was given in Theorem 6 of [32]. An equivalent statement, using a different terminology, can be found as Lemma 1 in Kozameh, Lamberti and Reula [34], together with a short proof. However, both these earlier proofs are incomplete. The proof in [32] is based on the idea to homotopically deform f_p into the identity, but it is not shown that the construction can be made in such a way that the dependence on the deformation parameter is, indeed, continuous. In [34], the authors write the future light cone (or, equivalently, the past light cone) of a point $p \in \mathcal{M}$ as the image of a map $\Phi :]0, \infty[\ \times S^2 \longrightarrow \mathcal{M}$, and they assign a *winding number* to each map $\Phi(s, \cdot)$. Since a winding number has to refer to a "center", the authors in [34] apparently take for granted that there is a timelike curve through p that has no further intersection with the light cone of p. The existence of such a curve, however, is an open question. With our Proposition 11 we have filled these gaps insofar as we have established the result $\deg(f_p) = \pm 1$. However, we have not shown whether, with our choice of orientations, the occurence of the minus sign can be ruled out.

Proposition 15 implies that every observer in p sees an odd number of images of each light source at infinity that does not pass through the caustic of the past light cone of p. (Here one has to refer to the $\tilde{g}$-cone which is an extension of the g-cone.) As an immediate consequence of Proposition 12, we find that a similar statement is true for light sources inside $\mathcal{M}$, see Figure 3.

Proposition 16. *Fix a point p and a timelike embedded C^∞ curve γ in an asymptotically simple and empty spacetime $(\mathcal{M}, g)$. Assume that the image of γ is a closed subset of $\tilde{\mathcal{M}} \setminus \mathscr{I}^+$ and that γ meets neither the point p nor the caustic of the past light cone of p. Then the number of past-pointing lightlike geodesics from p to γ in $\mathcal{M}$ is finite and odd.*

Let us conclude this subsection with a few remarks on spacetimes that are asymptotically simple but not empty. For any asymptotically simple spacetime it is easy to verify that $\partial\mathcal{M}$

has either one or two connected components, and that all lightlike geodesics in $\mathcal{M}$ have their past end-point in the same connected component of $\partial\mathcal{M}$. Let us denote this component by $\mathscr{I}^-$ henceforth. In order to apply our formalism of simple lensing neighborhoods the additional assumptions needed are that $\mathscr{I}^-$ is a fiber bundle with $\tilde{g}$-causal fibers diffeomorphic to $\mathbb{R}$ over an orientable basis manifold, and that all past-inextendible lightlike geodesics in $\mathcal{M}$ meet $\mathscr{I}^-$ transversely. If these assumptions are satisfied, our results on simple lensing neighborhoods apply. In particular, $\mathscr{I}^-$ must be diffeomorphic to $S^2 \times \mathbb{R}$ and $\mathcal{M}$ must be contractible.

As an interesting special case, we might modify Condition (d) of Definition 6 by requiring the Ricci tensor of g to be equal to $\Lambda\, g$ near $\partial\mathcal{M}$ with a positive or negative cosmological constant Λ. The resulting spacetimes are called *asymptotically deSitter* for $\Lambda > 0$ and *asymptotically anti-deSitter* for $\Lambda < 0$. It was verified already by Penrose [30] that then $\partial\mathcal{M}$ is $\tilde{g}$-spacelike for $\Lambda > 0$ and $\tilde{g}$-timelike for $\Lambda < 0$. Thus, the formalism of simple lensing neighborhoods is inappropriate for investigating asymptotically deSitter spacetimes, but it may be used for the investigation of asymptotically anti-deSitter spacetimes.

6.3 Weakly perturbed Robertson-Walker spacetimes

It is a characteristic feature of the lens map, as defined in this paper, that it is constructed by following each past-pointing lightlike geodesic up to its first intersection with the source surface only. Further intersections are ignored, i.e., some images are willfully excluded from the gravitational lensing discussion. In the preceding examples no such further intersections occured. We shall now discuss an example where they do occur but where it is physically well motivated to disregard them.

To that end we start out with a spacetime $(\mathcal{M}, g)$ with $\mathcal{M} = S^3 \times \mathbb{R}$ and

$$g = R(t)^2\big(- dt^2 + d\chi^2 + \sin^2\!\chi (d\theta^2 + \sin^2\!\theta\, d\phi^2)\big)\,. \tag{21}$$

Here $\chi \in [0, \pi]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ denote standard coordinates on S^3 (with the usual coordinate singularities), t denotes the projection from $\mathcal{M} = S^3 \times \mathbb{R}$ onto $\mathbb{R}$, and $R : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly positive but otherwise arbitrary C^∞ function. This is the general form of a Robertson-Walker spacetime with positive spatial curvature and natural topology which has no particle horizons. (Particle horizons are excluded by the assumption that the "conformal time" t runs over all of $\mathbb{R}$.)

Now fix a coordinate value $\chi_o \in \,]\,0\,, \pi/2[\,$ and let $\mathcal{U}$ denote the set of all points in $\mathcal{M}$ whose χ-coordinate is smaller than χ_o. Let W denote the restriction of the vector field $\partial/\partial t$ to the boundary ∂U. Then $(\mathcal{U}, \partial\mathcal{U}, W)$ is a simple lensing neighborhood. This is easily verified using the fact that the lightlike geodesics in $\mathcal{M}$ project to the geodesics of the standard metric on S^3. Our assumptions that t ranges over all of $\mathbb{R}$ and that $\chi_o < \pi/2$ are essential to make sure that, for all $p \in \mathcal{U}$, the lens map is defined on all of $\mathcal{S}_p$. In the case at hand, the lens map $f_p : \mathcal{S}_p \longrightarrow \partial\mathcal{U}/W$ is a global diffeomorphism for all points $p \in \mathcal{U}$. Actually, there are infinitely many past-pointing lightlike geodesics from any fixed $p \in \mathcal{U}$ to any fixed $\xi \in \partial\mathcal{U}/W$, but only one of them reaches ξ without having left $\mathcal{U}$. All the other ones make at least a half circle around the whole universe, so they will give rise to rather faint images as a consequence of absorption in the intergalactic medium. It is, thus, reasonable to assume that only the one image which enters into the lens map is actually visible. In this sense, disregarding all the other light rays is physically well motivated. Please note that all the infinitely many images of ξ are situated at just two points of the celestial sphere at p; the two brightest images cover all the other ones.

Now this example is boring in view of gravitational lensing because the lens map is a global diffeomorphism. However, we can switch to a more interesting situation by choosing a compact subset $\mathcal{K} \in S^3$ and modifying the metric on the set $\mathcal{K} \times \mathbb{R}$. In view of Einstein's field equation, this can be interpreted as introducing local mass concentrations that act as gravitational lens deflectors. If $\mathcal{K} \times \mathbb{R}$ is completely contained in $\mathcal{U}$, and if the modification of the metric is sufficiently small to make sure that, even after the modification, no light rays are past- or future-trapped inside $\mathcal{U}$, then $\mathcal{U}$ remains a simple lensing neighborhood. We have, thus, Proposition 11 at our disposal. Under the (very mild) additional assumption that, even after the perturbation, there are no closed timelike curves in $\mathcal{U}$, we may also use Proposition 12. This is a line of argument to the effect that, in a Robertson-Walker spacetime of the kind considered here, any transparent gravitational lens deflector produces an odd number of visible images. The assumption that there are no particle horizons was essential since otherwise the lens map would not be defined on the whole celestial sphere for all $p \in \mathcal{U}$.

A similar argument applies, of course, to Robertson-Walker spacetimes with non-compact spatial sections. Then we don't have to care about light rays traveling around the whole universe, so there are no additional images which are ignored by the lens map.

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