Initial data for a head on collision of two Kerr-like black holes with close limit

Sergio Dain

Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1
14476 Golm
Germany

April 22, 2004

Abstract

We prove the existence of a family of initial data for the Einstein vacuum equation which can be interpreted as the data for two Kerr-like black holes in arbitrary location and with spin in arbitrary direction. This family of initial data has the following properties: (i) When the mass parameter of one of them is zero or when the distance between them goes to infinity, it reduces exactly to the Kerr initial data. (ii) When the distance between them is zero, we obtain exactly a Kerr initial data with mass and angular momentum equal to the sum of the mass and angular momentum parameters of each of them. The initial data depends smoothly on the distance, the mass and the angular momentum parameters.

1 Introduction

Perturbation analysis has been used successfully in the study of black-hole collisions (see [25] and references therein and also [2] for a recent approach which combines full numerical calculations with perturbation theory). In the case of a head-on collision a pure linear analysis provide a remarkable accurate description of the late state of the collision[25]. This calculation is based on the fact that the head-on initial data [24][11] can be written as a perturbation of the Schwarzschild initial data, the separation between the black-holes is the perturbation parameter. When the black holes have intrinsic angular momentum, the natural generalization of this idea is to consider perturbations around a Kerr background. The problem now is to construct proper initial data: a solution of the vacuum constraints equations that represents two black holes; such that,

1

when the separation between them is zero we recover the initial data of the final Kerr black hole. This property is usually called the 'close limit' of the initial data. The standard Bowen-York data[8] do not have a close limit to Kerr, since it does not include the Kerr initial data for any choice of the parameters. In [22] and [23] it has been computed numerically a class of axially symmetric initial data that has the desired close limit to the Kerr, and it has also been studied the linear evolution of them. However this initial data do not have the topology of two black holes, an extra undesirable singularity appear in the solution. The purpose of this article is to overcome this problem, generalize the construction for non-axially symmetric data and give a rigorous existence proof for this class of initial data.

The plan of the paper is the following. In section 2 we give a brief summary of the standard conformal technique for solving the constraints equations with many asymptotically flat ends. This class of initial data contain, in general, apparent horizons around the ends. The existence of apparent horizons lead us to interpret these data as representing initial data for black-holes collision. The evolution of them will presumably contain an event horizon, the final picture of the whole space time will be similar to the one shown in figure 60 of [19], which represents a collision and merging of two black holes. We describe in this section a set of existence theorems that are appropriate for our present purpose[15]. In section 3 we summarize properties of the Kerr initial data that have been proved in [14]. Together with the existence theorems of section 2 these are the essential tools for our construction. The new initial data is given in sections 4 and 5. The idea is very similar to the one given in [14], the important difference is the choice of the conformal metric and the conformal extrinsic curvature. In section 4 we construct remarkably simple initial data that represent a Schwarzschild and a Kerr black hole, with a close limit to a final Kerr black hole. In section 5 we generalize to two Kerr black holes at arbitrary location and with spin in arbitrary directions. The Kerr metric is characterized by the mass m and the angular momentum per mass a. The initial data given in section 5 depend smoothly on the parameters m_1, J_1^a, m_2, J_2^a and L, where m_1 and m_2 are the mass parameter of each of the black holes, J_1^a and J_2^a are the spin vectors of each of them and L is the distance between the black holes. We define $a_1^2 = m_1^2 |J_1|^2$ and $a_2^2 = m_2^2 |J_2|^2$, we always chose $m_1^2 > a_1^2$ and $m_2^2 > a_2^2$. The data satisfies the following two properties:

(i) *Far limit:* When m_1 is zero, we obtain the Kerr initial data with mass m_2 and angular momentum J_2. If we fix the location of the hole 1 and take the limit $L = \infty$, we obtain the Kerr initial data with mass m_1 and angular momentum J_1. The same is true if we exchange the holes 1 and 2, because the data is symmetric in the parameters 1 and 2.

(ii) *Close limit:* When $L = 0$ we obtain a Kerr initial data with mass $m_1 + m_2$ and angular momentum $J_K = J_1 + J_2$.

In section 6 possible generalizations are discussed. Finally, in appendix A, we state some known results about the momentum constraint in axial symmetry

in a coordinate independent way.

2 Solutions of the vacuum constraint equations with many asymptotically flat ends

The conformal approach to find solutions of the constraint equations with many asymptotically flat end points i_k is the following (cf. [12], [13] and the reference given there. The setting outlined here, where we have to solve (1), (2) on the compact manifold has been studied in [6], [16], [17], see also [5], [20] for an interesting application of this formalism). Let S be a compact manifold, denote by i_k a finite number of points in S, and define the manifold $\tilde{S}$ by $\tilde{S} = S \setminus \bigcup i_k$. We assume that h_{ab} is a positive definite metric on S, with covariant derivative D_a, and Ψ^{ab} is a trace-free symmetric tensor, which satisfies

$$D_a \Psi^{ab} = 0 \quad \text{on} \quad \tilde{S}. \tag{1}$$

Let θ a solution of

$$L_h \theta = -\frac{1}{8} \Psi_{ab} \Psi^{ab} \theta^{-7} \quad \text{on} \quad \tilde{S}, \tag{2}$$

where $L_h = D^a D_a - R/8$ and R is the Ricci scalar of the metric h_{ab}. Then the physical fields $(\tilde{h}, \tilde{\Psi})$ defined by $\tilde{h}_{ab} = \theta^4 h_{ab}$ and $\tilde{\Psi}^{ab} = \theta^{-10} \Psi^{ab}$ will satisfy the vacuum constraint equations on $\tilde{S}$. To ensure asymptotic flatness of the data at the points i_k we require at each point i_k

$$\Psi^{ab} = O(r^{-4}) \quad \text{as} \quad r \to 0, \tag{3}$$

$$\lim_{r \to 0} r\theta = c_k, \tag{4}$$

where the c_k are positive constants, x^i are normal coordinates centered at i_k and $r = (\sum_{i=1}^{3} (x^j)^2)^{1/2}$.

In order to get existence of solutions for equations (1) and (2) with the boundary conditions (3) and (4), we have to impose some restrictions on the conformal metric h_{ab}. We require that the Ricci scalar of h_{ab} is positive

$$R > 0. \tag{5}$$

It is physically reasonable to impose that h_{ab} is smooth on $\tilde{S}$. However, the differentiability of h_{ab} at the ends i_k is a very delicate issue, since it characterizes the fall-off behavior of the initial data near space like infinity. Strong assumptions (for example smoothness) can rule out physical important initial data (for example the Kerr initial data). A suitable condition for h_{ab}, which in particular include the Kerr initial data as we will see, is the following

$$h_{ab} \in W^{4,p}(S), \quad p \geq 3/2. \tag{6}$$

3

(see e.g. [1] for the definitions of the Sobolev, Lebesgue and Hölder spaces $W^{s,p}$, L^p and $C^{m,\alpha}$, $0 < \alpha < 1$). In particular this condition implies that h_{ab} is $C^{2,\alpha}(S)$. Existence of solutions for the constraint equations under the assumption (5) and (6) were proved in [15], in the following we briefly summarize these results. We will present simplified versions of the theorems, which are sufficient for our present purpose.

In order to find solutions of equation (2) with the boundary condition (4) we look first for functions θ_{i_k} which satisfies the linear equation

$$L_h \theta_{i_k} = 0, \text{ in } S \setminus \{i_k\}, \tag{7}$$

and at i

$$\lim_{r \to 0} r\theta_{i_k} = 1, \tag{8}$$

where $i_k \in S$ is an arbitrary point. Denote by $B_{i_k}(\epsilon)$ the open ball with center i_k and radius $\epsilon > 0$, where ϵ is chosen small enough such that B_{i_k} is a convex normal neighborhood of i_k. Choose a function $\chi_{i_k} \in C^\infty(S)$ which is non-negative and such that $\chi_{i_k} = 1$ in $B_{i_k}(\epsilon/2)$ and $\chi_{i_k} = 0$ in $S \setminus B_{i_k}(\epsilon)$. We have the following lemma.

Lemma 2.1 *Assume that h_{ab} satisfies (5) and (6). Let $i_k \in S$ an arbitrary point. Then, there exists a unique solution θ_{i_k} of equation (7) which satisfies condition (8) at i_k. Moreover, $\theta_{i_k} > 0$ in $S \setminus \{i_k\}$ and we can write $\theta_{i_k} = \chi_{i_k}/r + g_{i_k}$, with $g_{i_k} \in C^{1,\alpha}(S)$.*

For each of the end points i_k we have a corresponding function θ_{i_k}, we define θ_0 by

$$\theta_0 = \sum_{k=0}^{n} c_k \theta_{i_k}. \tag{9}$$

The initial data will have $n + 1$ asymptotic ends, the relevant cases for us are $n = 0, 1, 2$. It is important to note that θ_0^{-1} is in $C^\alpha(S)$, it is non-negative and vanishes only at the points i_k.

To obtain the solution θ, we write $\theta = \theta_0 + u$ and solve on S the following equation for u

$$L_h u = -\frac{1}{8} \theta_0^{-7} \Psi_{ab} \Psi^{ab} (1 + \theta_0^{-1} u)^{-7}. \tag{10}$$

We have the following existence result.

Theorem 2.2 *Assume the metric h_{ab} satisfies (5) and (6), and that $\theta_0^{-7} \Psi_{ab} \Psi^{ab} \in L^q(S)$, $q \geq 2$. Then, there exists a unique non-negative solution $u \in W^{2,q}(S)$ of equation (10). We have $u > 0$ unless $\Psi_{ab}\Psi^{ab} = 0$.*

We note that our assumptions on Ψ^{ab} impose rather mild restrictions, which are, in particular, compatible with the fall off requirement (3).

We turn now to the linear equation (1). A smooth solution on S can easily be obtained by known techniques (cf. [12]) However, in that case the initial data will have vanishing momentum and angular momentum. To obtain data without this restriction, we have to consider fields $\Psi^{ab} \in C^\infty(\tilde{S})$ which are singular at i_k. The existence of initial data with non trivial momentum and angular momentum has been studied in [7], where the role of the conformal symmetries is also discussed. In [15] we have generalized some of the results proved in [7] for smooth metrics to metrics in the class (6). We will present here a simplified version of these results. Consider the following tensor in $\mathbb{R}^3 \setminus \{0\}$

$$\Psi_J^{ab} = \frac{3}{r^3}(n^a \epsilon^{bcd} J_c n_d + n^b \epsilon^{acd} J_c n_d), \tag{11}$$

where J^a is a constant vector and $n^a = x^a/r$. This tensor is trace-free and divergence free with respect to the flat metric. The vector J^a will give the angular momentum of the initial data. This tensor is one of the explicit solutions of the momentum constraint for conformally flat metric studied in [8]. Consider the three sphere S^3. The tensor Ψ_J^{ab} can be extended to an smooth tensor on $S^3 - \{i_1\} - \{i_2\}$, where i_1 and i_2 are two different points of S^3.

We define $\bar{\Psi}_J^{ab}$ to be the trace free part with respect to h_{ab} of Ψ_J^{ab}; and $(\mathcal{L}_h w)^{ab}$ to be the conformal Killing operator $\mathcal{L}_h$, with respect to the metric h_{ab}, acting on a vector w^a. We have the following theorem.

Theorem 2.3 *Assume that the metric h_{ab} satisfies (6) and let S be S^3. Let i_1 and i_2 two different points in S^3 and $Q^{ab} \in W^{1,p}(S)$ an arbitrary symmetric, trace-free, tensor field. Then, there exist a unique vector field $w^a \in W^{2,q}(S)$, $1 < q < 3/2$, such that the tensor*

$$\Psi^{ab} = \bar{\Psi}_J^{ab} + Q^{ab} + (\mathcal{L}_h w)^{ab}, \tag{12}$$

satisfies the equation $D_a \Psi^{ab} = 0$ on $S^3 - \{i_1\} - \{i_2\}$.

The tensor Q^{ab} gives the regular part of the extrinsic curvature, which does not contribute to the angular momentum of the initial data. A more general version of this theorem, which in particular includes linear momentum, was proved in [15]. By superposition we obtain solutions with no trivial angular momentum with three or more asymptotic ends. It is important to recall that the tensor Ψ_J^{ab} is singular in two points, that is, we need at least two asymptotic ends. If we insist in having only one asymptotic end, then, the possible existence of conformal symmetries of the metric h_{ab} restricts the allowed value for the angular and linear momentum of the initial data. For example, if we chose h_{ab} to be h_{ab}^0, the standard metric of S^3, then the linear and angular momentum of the data have to vanish. For a complete discussion of this issue see [7] [15] and also [4].

The linear and angular momentum of the initial data at each end i_k depend only on the tensor Ψ^{ab} and not on the conformal factor θ, they are given by

$$P^a = -\frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{\partial B_{i_k}(\epsilon)} r^2 \Psi_{bc}(\delta^{ba} - 2n^b n^a)n^c dS_\epsilon, \tag{13}$$

$$J^a = -\frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{\partial B_{i_k}(\epsilon)} r \Psi_{cd} \epsilon^{abc} n^d n_b dS_\epsilon, \tag{14}$$

where dS_ϵ is the area element on the two-sphere $\partial B_{i_k}(\epsilon)$. If we compute these integrals for the particular tensor given by (12) we obtain $P^a = 0$ and the angular momentum is given by the constant vector J^a of (11).

The total mass of the initial data at each end i_k is given by

$$M_{i_k} = \hat{m}_{i_k} + 2c_{i_k} \sum_{k'=0,\, k'\neq k}^{n} c_{i_{k'}} \theta_{i_{k'}}(i_k) + 2u(i_k), \tag{15}$$

where

$$\hat{m}_{i_k} = 2c_{i_k} g_{i_k}(i_k). \tag{16}$$

The constant $\hat{m}_{i_k}$ is the mass of the initial data with only one end point i_k and $\Psi^{ab} = 0$.

Let us mention how some well known initial data are obtained in this setting. Consider S^3 (in the rest of the paper we will always work in this manifold) with the standard metric h_{ab}^0 and standard spherical coordinates (ψ, ϑ, ϕ). The south pole i_0 is given by $\psi = 0$ and the north pole i_π by $\psi = \pi$. The metric h_{ab}^0 has the form

$$h^0 \equiv d\psi^2 + \sin^2\psi(d\vartheta^2 + \sin^2\vartheta d\phi^2). \tag{17}$$

Denote by $(\psi_k, \vartheta_k, \phi_k)$ the coordinates of the point i_k. Using that h_{ab}^0 is conformally flat one easily obtains that the corresponding function θ_{i_k} is given by

$$\theta_{i_k} = \frac{1}{\sqrt{2}(1 - \cos\psi \cos\psi_k - \sin\psi \sin\psi_k \cos\gamma)^{1/2}}, \tag{18}$$

where

$$\cos\gamma = \cos\vartheta \cos\vartheta_k + \sin\vartheta \sin\vartheta_k \cos(\phi - \phi_k).$$

Note that in this case $\hat{m}_{i_k} = 0$.

Assume that $\Psi^{ab} = 0$, this implies $u = 0$ by equation (10). If we chose only one end i_0, we have $\theta_0 = \theta_{i_0}$ and we obtain the Minkowski initial data. If we chose two different ends i_0 and i_1 we obtain the Schwarzschild solution, the conformal factor is given by

$$\theta_s = \theta_{i_0} + m\theta_{i_1} \sin(\psi_1/2). \tag{19}$$

Where m is an arbitrary, positive, constant. We have chosen the constants $c_0 = 1$, $c_1 = m\sin(\psi_1/2)$, such that m is the total mass at any of the ends i_0 or i_1, as we can easily verify from equations (15) and (18).

If we have three different points i_0, i_1 and i_2, we obtain the Brill-Lindquist initial data[11]. The conformal factor is given by

$$\theta_{ss} = \theta_{i_0} + m_1\theta_{i_1} \sin(\psi_1/2) + m_2\theta_{i_2} \sin(\psi_2/2). \tag{20}$$

6

We have chosen the constants such that the mass at the at the end i_0 is $M_{i_0} = m_1 + m_2$. We chose this end point as the place were the observer is 'located'. The far limit of the data is when m_1 or m_2 is equal to zero, or when ψ_1 or ψ_2 is equal to zero (i.e.; when we put one of the holes at the 'infinity' i_0), in both cases this data reduce to Schwarzschild data (19) with mass m_1 and m_2 respectively. As a distance parameter L we can chose the Euclidean distance in $\mathbb{R}^3$ between the points i_1 and i_2. The close limit is given by $i_1 = i_2$ (and different from i_0), in this case we obtain Schwarzschild data with mass $m_0 + m_1$.

Using again that h^0_{ab} is conformally flat and the fact that (11) is divergence free with respect to the flat metric, we find that the following tensor

$$\Psi^{ab}_0 = \frac{3}{(\sin \psi)^3}(\hat{n}^a \epsilon^{bcd} J_c \hat{n}_d + \hat{n}^b \epsilon^{acd} J_c \hat{n}_d). \tag{21}$$

is trace-free and divergence free with respect to h^0_{ab}, where we have defined $\hat{n}_a = D_a \psi$. Ψ^{ab}_0 is smooth on $S^3 - \{i_0\} - \{i_\pi\}$ and is singular at the poles. In an analogous way we can construct a tensor Ψ^{ab}_1 which is singular at two arbitrary (but different) points i_0 and i_1. Let θ_0 given by

$$\theta_0 = \theta_{i_0} + m\theta_{i_1} \sin(\psi_1/2). \tag{22}$$

Let u_{by} the unique solution of (10) for h^0_{ab}, Ψ^{ab}_1 and θ_0 given by (22). The conformal factor is

$$\theta_{by} = \theta_0 + u_{by}. \tag{23}$$

This is the Bowen-York initial data for one black hole with spin (the other data discussed there can be obtained a similar form). The positive constant m is no longer the total mass of the data. The mass of the data at the end i_0 is given by $m + u_{by}(i_0)$. The generalization to more than two ends is straightforward: pick up another tensor Ψ^{ab}_2 which is singular at the ends i_0 and i_2, define $\Psi^{ab}_{12} = \Psi^{ab}_1 + \Psi^{ab}_2$; and consider θ_0 given by

$$\theta_0 = \theta_{i_0} + m_1 \theta_{i_1} \sin(\psi_1/2) + m_2 \theta_{i_2} \sin(\psi_2/2). \tag{24}$$

Let $\bar{u}_{by}$ the unique solution of (10) for h^0_{ab}, Ψ^{ab}_{12} and (24). The conformal factor is given by $\theta_{by} = \theta_0 + \bar{u}_{by}$. The mass at i_0 is $m_1 + m_2 + \bar{u}_{by}(i_0)$. This data neither have a far nor a close limit to the Kerr initial data.

3   The Kerr initial data

In [14] it has been proved that the Kerr initial data satisfies the hypothesis of the existence theorems of section 2, namely, that the conformal metric satisfies (5) and (6), and the conformal extrinsic curvature has a the form (12). In this section we briefly summarize this calculation.

Consider the Kerr metric in the Boyer-Lindquist coordinates $(t, \tilde{r}, \vartheta, \phi)$[21] [9], with mass m and angular momentum per mass a, such that $m^2 > a^2$. We define $\delta = \sqrt{m^2 - a^2}$.

Take any slice $t = const$. Denote by $\tilde{h}^k_{ab}$ the intrinsic three metric of the slice and by $\tilde{\Psi}^{ab}_k$ its extrinsic curvature. These slices are maximal, i. e. $\tilde{h}^k_{ab}\tilde{\Psi}^{ab}_k = 0$. There exist a coordinate transformation which maps the region outside the exterior horizon in to the three sphere S^3 such that $\tilde{h}^k_{ab}$ is smooth in $S^3 - \{i_0\} - \{i_\pi\}$. Moreover, there exist a conformal factor θ_k, which is singular at the poles $\{i_0\}$ and $\{i_\pi\}$

$$\lim_{\psi \to \pi}(\psi - \pi)\theta_k = \delta, \quad \lim_{\psi \to 0}\psi\theta_k = 1, \tag{25}$$

such that the rescaled metric $h^k_{ab} = \theta_k^{-4}\tilde{h}^k_{ab}$ is in the Sobolev space $W^{4,p}(S^3)$, $p < 3$. The conformal metric h^k_{ab} has the form

$$h^k_{ab} = h^0_{ab} + a^2 f v_a v_b, \tag{26}$$

where the smooth vector field v_a is given by $v_a \equiv \sin^2\psi\sin^2\vartheta(d\phi)_a$, and the function f, which contains the non-trivial part of the metric, is given explicity in [14]. The only property of this function that we will use is that it is analytic in the parameters m, a and it satisfies $f \in W^{4,p}(S)$. Since f is analytic in a, the Ricci scalar R is also analytic in a. For $a = 0$ we have that $R = 6$, the scalar curvature of h^0_{ab}. Thus, if a is sufficiently small, R will be a positive function on S^3. In the following we will assume the latter condition to be satisfied.

The metric (26) is axially symmetric, the norm of the Killing vector $\eta^a = (\partial/\partial\phi)^a$ is given by

$$\eta^k = \sin^2\psi\sin^2\vartheta(1 + a^2 f \sin^2\psi\sin^2\vartheta). \tag{27}$$

The conformal extrinsic curvature is defined by

$$\Psi^{ab}_k = \theta_k^{10}\tilde{\Psi}^{ab}_k. \tag{28}$$

The tensor Ψ^{ab}_k is smooth in $S^3 - \{i_0\} - \{i_\pi\}$ and at the poles it has the form

$$\Psi^{ab}_k = \Psi^{ab}_J + Q^{ab} \tag{29}$$

where Ψ^{ab}_J is given by (11) with $|J| = am$, Q^{ab} is trace free and $Q^{ab} \in W^{1,p}(S^3)$. If $a = 0$ then $\Psi^{ab}_k = 0$. Since the Kerr initial data satisfies the constraint, we have that

$$D_a\Psi^{ab}_k = 0, \tag{30}$$

where D_a is the connection of the metric h^k_{ab}. Using theorems 2.1 and 2.2, we can decompose the Kerr conformal factor like

$$\theta_k = \theta_{i_0} + \delta\theta_{i_\pi} + u_k, \tag{31}$$

where the function $u_k \in W^{2,q}(S^3)$.

8

4 Close limit initial data with Schwarzschild-like and Kerr-like asymptotic ends

We want to construct initial data that represent two black-holes, one of them a Kerr-like black hole and the other a Schwarzschild-like black hole. This data will have both a far and a close limit. The main simplification between this data and the one in the following section is that the momentum constraint is solved explicitly, due to the axial symmetry of the conformal metric, i.e.; we do not make use of theorem 2.3 in this case.

Let i_1 an arbitrary point in S^3 with coordinates $(\psi_1, \vartheta_1, \phi_1)$, this point will be the location of the Schwarzschild-like end. Let m_1 an arbitrary positive constant. Define the following metric

$$h_{ab}^{sk} = h_{ab}^0 + a^2(\mu f + \nu f^K)v_a v_b, \tag{32}$$

where $f^K = f(a, m + m_1)$ is the corresponding f function defined by equation (26) for a Kerr initial data with angular momentum per mass a and mass $m+m_1$. The functions μ and ν will depend smoothly on m_1, m and the distance L between the point i_1 and the point i_π. These functions have to be chosen in such a way that the metric (32) reduce to the conformal Kerr metric (m, a) when m_1 is zero or when L goes to infinity (far limit), and to the conformal Kerr metric $(m+m_1, a)$ when $L = 0$ (close limit). It is clear that the metric (32) satisfies (6) and, for small a, (5). To ensure that the metric is non-degenerate we impose

$$\mu, \nu \geq 0. \tag{33}$$

Define the dimensionless parameter $\epsilon = L^2/(mm_1)$. We impose the following conditions on μ and ν. In the far limit

$$\epsilon = \infty \Rightarrow \mu = 1, \quad \nu = 0; \tag{34}$$

and in the close limit

$$\epsilon = 0 \Rightarrow \mu = 0, \quad \nu = 1. \tag{35}$$

Of course, there is an enormous freedom in the choice of μ and ν which satisfies (33), (34) and (35). For example, we can take

$$\mu = \frac{\epsilon}{1 + \epsilon}, \quad \nu = \frac{1}{1 + \epsilon}. \tag{36}$$

The metric (32) is axially symmetric. The norm of the Killing vector is given by

$$\eta^{sk} = \sin^2 \psi \sin^2 \vartheta (1 + a^2 \sin^2 \psi \sin^2 \vartheta (\mu f + \nu f^K)). \tag{37}$$

9

In appendix A we prove that the following tensor satisfies the equation $D_a \Psi_{sk}^{ab} = 0$ with respect to the metric (32)

$$\Psi_{sk}^{ab} = \left(\frac{\eta^{sk}}{\eta^k}\right)^{3/2} \mu \Psi_k^{ab} + \left(\frac{\eta^{sk}}{\eta^K}\right)^{3/2} \nu \Psi_K^{ab}, \tag{38}$$

where η^K is the corresponding norm of the Killing vector for the Kerr metric $(m + m_1, a)$ and Ψ_K^{ab} is the corresponding conformal extrinsic curvature. In the far limit this tensor is equal to Ψ_k^{ab} and in the close limit is equal to Ψ_K^{ab}.

Consider the functions θ_{i_k} corresponding to the metric h_{ab}^{sk}, these functions exist by lemma 2.1. Define θ_0 by

$$\theta_0 = \theta_{i_0} + (\mu\delta + \nu\delta_K)\theta_{i_\pi} + \mu m_1 \sin(\psi_1/2)\theta_{i_1}, \tag{39}$$

where $\delta_K = \sqrt{(m + m_1)^2 - a^2}$. Let u_{sk} be the unique, non-negative, solution of equation (10), for the metric (32), extrinsic curvature (38) and θ_0 given by (39). The solution exists by theorem 2.2. Then the conformal factor is given by

$$\theta_{sk} = \theta_0 + u_{sk}. \tag{40}$$

Note that the conformal factor has also the proper close and far limit. This finish the construction. The data has a close limit to a Kerr initial data with mass $m + m_1$ and angular momentum $(m + m_1)a$. It also has far limit: when $m_1 = 0$ or $L = \infty$ (which implies $\psi_1 = 0$) we obtain exactly the Kerr initial data of mass m and angular momentum ma. When $m = a = 0$ we obtain the Schwarzschild initial data with mass m_1.

Summarizing, to construct the initial data define h_{ab}^{sk}, Ψ_{sk}^{ab} and θ_0 by equations (32), (38) and (39). Solve equation (10) for u_{sk} with respect to these tensors. Define θ_{sk} by (40). Then the physical initial data is given by $\tilde{h}_{ab}^{sk} = \theta_{sk}^4 h_{ab}^{sk}$ and $\tilde{\Psi}_{sk}^{ab} = \theta_{sk}^{-10}\Psi_{sk}^{ab}$.

5   Close limit initial data with two Kerr-like asymptotic ends

Take the Kerr initial data and make a rigid rotation such that the spin points in to the direction of an arbitrary vector J_1^a, and make a shift of the origin to the coordinate position of an arbitrary point i_1. Let the mass and the modulus of the angular momentum of this data be m_1 and $a_1 m_1$. We obtain a rescaled metric $h_{ab}^{k_1} = h_{ab}^0 + a_1^2 f_1 v_a^1 v_b^1$, where f_1 and v_a^1 are obtained from f and v_a by the rotation and the shift of the origin, they depend on the coordinates of the point i_1 and the vector J_1^a. In an analogous way we define the corresponding rescaled extrinsic curvature $\Psi_{k_1}^{ab}$. Take another vector J_2^a and another point i_2 and make the same construction. We define f_1^K to be the corresponding f_1 function of a Kerr initial data of mass $m_1 + m_2$, angular momentum $J_1^a + J_2^a$. Let $\Psi_{K_1}^{ab}$ be the corresponding conformal extrinsic curvature. The function f_2^K and the

10

tensor $\Psi^{ab}_{K_2}$ is defined in analogous way, with respect to the origin i_2. Define the following metric

$$h_{ab} = h^0_{ab} + \left(a_1^2 \mu f_1 + a_K^2 \nu_1 f_1^K\right) v_a^1 v_b^1 + \left(a_2^2 \mu f_2 + a_K^2 \nu_2 f_2^K\right) v_a^2 v_b^2. \qquad (41)$$

The smooth function μ, ν_1, ν_2 will be chosen following similar arguments as in the previous section, namely

$$\mu, \nu_1, \nu_2 \geq 0, \qquad (42)$$

we impose the far limit

$$\epsilon = \infty \Rightarrow \mu = 1, \quad \nu_1 = \nu_2 = 0; \qquad (43)$$

and the close limit

$$\epsilon = 0 \Rightarrow \mu = 0, \quad \nu_1 + \nu_2 = 1, \qquad (44)$$

where $\epsilon = L^2/(m_1 m_2)$. In order that this data reduce to the one constructed in the previous section when one of the hole is Schwarzschild, we impose

$$a_1 = 0 \Rightarrow \nu_1 = 0, \quad a_2 = 0 \Rightarrow \nu_2 = 0. \qquad (45)$$

As in the previous section, it is clear that the metric (41) satisfies (6) and (5), for small a_1 and a_2. As an explicit example for the functions μ, ν_1, ν_2 we can take μ defined by (36) and

$$\nu_1 = \frac{a_1^2}{(a_1^2 + a_2^2)(1 + \epsilon)}, \quad \nu_2 = \frac{a_2^2}{(a_1^2 + a_2^2)(1 + \epsilon)}. \qquad (46)$$

When the points i_1 and i_2 are chosen to be on the axis $\vartheta = 0$, the metric (41) is axially symmetric, we can apply, as in the previous section, the result of appendix A to solve for the momentum constraint. But in general, the metric (41) is not axially symmetric. To solve the momentum constraint in this case we use theorem 2.3. Define Ψ^{ab}_{kk} by

$$\Psi^{ab}_{kk} = \mu_1 \bar{\Psi}^{ab}_{k_1} + \nu_1 \bar{\Psi}^{ab}_{K_1} + \mu_2 \bar{\Psi}^{ab}_{k_2} + \nu_2 \bar{\Psi}^{ab}_{K_2} + (\mathcal{L}w)^{ab}, \qquad (47)$$

where the bar indicate that we take the trace free part with respect the metric (41), and w^a is the unique vector field such that Ψ^{ab}_{kk} is divergence free in $\tilde{S}$ with respect to the metric (41). Note that Ψ^{ab}_{kk} has a far limit to $\Psi^{ab}_{k_1}$ or $\Psi^{ab}_{k_2}$ and a close limit $(i_1 = i_2)$ to $\Psi^{ab}_{K_1} = \Psi^{ab}_{K_2}$.

The functions θ_{i_k} for the metric h^{kk}_{ab} exist by lemma 2.1, we define θ_0 by

$$\theta_0 = \theta_{i_0} + (\mu \delta_1 + \nu_1 \delta_K) \theta_{i_1} \sin(\psi_1/2) + (\mu \delta_2 + \nu_2 \delta_K) \theta_{i_2} \sin(\psi_2/2). \qquad (48)$$

Let u_{kk} be the unique solution of (10) with respect to (41), (49) and (48), which exist by theorem 2.2. The conformal factor is given by

$$\theta_{kk} = \theta_0 + u_{kk}. \qquad (49)$$

We see that this initial data satisfies the far and close limit with respect to the Kerr initial data.

11

6 Conclusions

We have constructed a family of initial data that represent two black-holes and have the desired far and close limit with respect to the Kerr initial data. The main point in the construction is the choice of the conformal metric and the conformal extrinsic curvature, given by (41) and (47). The conformal metric and extrinsic curvature are the 'free data', they determine uniquely a solution of the constraints, provided they satisfy certain conditions. These conditions are discussed in the existence theorems of section 2. In our case the data (41) and (47) satisfy these conditions since the Kerr initial data satisfy them, as it was discussed in section 3. The tensors (41) and (47) are, essentially, a superposition of three different Kerr initial data: the two 'initial' Kerr black holes and the 'final' Kerr black hole. In contrast to the close limit data constructed in [22], where only one Kerr initial data is used. In that case, when one forces the data to have the topology of two black holes, the equations produce an unwanted extra singularity.

The functions ν and μ, which determined the solution, are only required to satisfy the far and close limit conditions. In order to provide an explicit example, we have presented a particular choice of these functions, but other choices can be easily constructed. For example, in a previous work[14] we have made the simpler choice $\nu_1 = \nu_2 = 0$ and $\mu = 1$, which do not have the close limit property. It would be very interesting to know the influence of different choices in the final form of the gravitational waves emitted by the data.

The natural question is whether this construction can be generalized to include linear momentum. It is possible to add an extra term in the extrinsic curvature (12) which contains the linear momentum of each black hole. The existence theorem for the momentum constraint is exactly the same (see [15]). However, we will not have either Kerr or Schwarzschild in the far limit, since the Boyer-Lindquist slices are not boosted. This is exactly the same situation as for the boosted data given in [8]. That is, in order to produce initial data which have the far limit to Kerr and also linear momentum we have to study boosted maximal slices in Kerr. Even for Schwarzschild this is a non-trivial problem. On the other hand, we could impose only a close limit to Kerr. This can be, in principle, achieve with a proper choice of the functions μ and ν. We will study this generalization in a future work.

Acknowledgements

12

A   Appendix: The momentum constraint in axially symmetric initial data

We collect here some results regarding the momentum constraints in an axially symmetric background which are useful for constructing explicit solutions. What follows is, essentially, a re-writing of the results founded in [10] and [3] in a coordinate independent way.

Assume that we have a metric h_{ab} with a Killing vector η^a, which is hypersurface orthogonal. Define the norm η by $\eta = \eta^a \eta^b h_{ab}$.

Let Ψ^{ab} the tensor defined by[18]

$$\Psi^{ab} = \frac{2}{\eta} S^{(a} \eta^{b)}, \tag{50}$$

where S^a satisfies

$$\pounds_\eta S^a = 0, \quad S^a \eta_a = 0, \quad D_a S^a = 0, \tag{51}$$

$\pounds_\eta$ is the Lie derivative with respect η.

We use the Killing equation $D_{(a} \eta_{b)} = 0$, the fact that η^a is hypersurface orthogonal, (i.e.; it satisfies $D_a \eta_b = -\eta_{[a} D_{b]} \ln \eta$) and equations (51) to conclude that Ψ^{ab} is trace free and divergence free.

The metric h_{ab} has the following splitting

$$h_{ab} = e_{ab} + \frac{\eta_a \eta_b}{\eta}, \tag{52}$$

where e_{ab} is the intrinsic metric of the 2-surface orthogonal to η^a. Let assume that we have another metric h'_{ab} related to h_{ab} by

$$h'_{ab} = e_{ab} + \gamma \frac{\eta_a \eta_b}{\eta}, \quad \pounds_\eta \gamma = 0, \tag{53}$$

where γ is a positive function. The metric h'_{ab} has a Killing vector η^a, with $\eta'_a = \gamma \eta_a$, i.e.; $\gamma = \eta'/\eta$. One easily check that the vector $S'^a = S^a/\sqrt{\gamma}$ satisfies equations (51) with respect to the metric h'_{ab}. Then, the tensor Ψ'^{ab} defined by

$$\Psi'^{ab} = \frac{1}{\gamma^{3/2}} \Psi^{ab}, \tag{54}$$

is trace free and divergence free with respect to h'_{ab}.

The solution of equations (51) can be written in terms of a scalar potential ω

$$S^a = \frac{1}{\eta} \epsilon^{abc} \eta_b D_c \omega, \quad \pounds_\eta \omega = 0. \tag{55}$$

We have restricted ourselves to tensor of the form (50), since the Kerr extrinsic curvature has precisely this form. For a more general discussion, and also for an explicit computation of the potential ω for Kerr, see [3].

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] J. Baker, B. Brügmann, M. Campanelli, and C. O. Lousto. Gravitational waves from black hole collisions via an eclectic approach. *Classical and Quantum Gravity*, 17(20):L149–L156, 2000.

[3] J. Baker and R. Puzio. A new method for solving the initial value problem with application to multiple black holes. *Phys. Rev. D*, 59:044030, 1999.

[4] R. Beig. Generalized Bowen-York initial data. In S. Cotsakis and G. Gibbons, editors, *Mathematical and Quantum Aspects of Relativity and Cosmology*, volume 537, pages 55–69. Springer Lecture Note in Physics, 2000. Also in gr-qc/0005043.

[5] R. Beig and S. Husa. Initial data for general relativity with toroidal conformal symmetry. *Phys. Rev. D*, 50:7116–7118, 1994.

[6] R. Beig and N. O'Murchadha. Trapped surface in vacuum spacetimes. *Class. Quantum Grav.*, 11(2):419–430, 1994.

[7] R. Beig and N. O'Murchadha. The momentum constraints of general relativity and spatial conformal isometries. *Commun. Math. Phys.*, 176(3):723–738, 1996.

[8] J. M. Bowen and J. W. York, Jr. Time-asymmetric initial data for black holes and black-hole collisions. *Phys. Rev. D*, 21(8):2047–2055, 1980.

[9] R. H. Boyer and R. W. Lindquist. Maximal analytic extension of the Kerr metric. *J. Math. Phys.*, 8(2):265–281, 1967.

[10] S. Brandt and E. Seidel. The evolution of distorted rotating black holes III: Initial data. *Phys. Rev. D*, 54(2):1403–1416, 1996.

[11] D. Brill and R. W. Lindquist. Interaction energy in geometrostatics. *Phys.Rev.*, 131:471–476, 1963.

[12] Y. Choquet-Bruhat, J. Isenberg, and J. W. York, Jr. Einstein constraint on asymptotically euclidean manifolds. *gr-qc/9906095*, 1999.

[13] Y. Choquet-Bruhat and J. W. York, Jr. The Cauchy problem. In A. Held, editor, *General Relativity and Gravitation*, volume 1, pages 99–172. Plenum, New York, 1980.

[14] S. Dain. Initial data for two Kerr-like black holes. Submitted for publication, gr-qc/0012023, 2000.

[15] S. Dain and H. Friedrich. Asymptotically flat initial data with prescribed regularity. Submitted for publication, gr-qc/0102047, 2000.

[16] H. Friedrich. On static and radiative space-time. *Commun. Math. Phys.*, 119:51–73, 1988.

[17] H. Friedrich. Gravitational fields near space-like and null infinity. *J. Geom. Phys.*, 24:83–163, 1998.

[18] S. W. Hawking. The event horizont. In C. DeWitt and B. S. DeWitt, editors, *Black Holes*, pages 1–56. Gordon and Breach Science Publishers, New York, 1973.

[19] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time.* Cambridge University Press, Cambridge, 1973.

[20] S. Husa. Asymptotically flat initial data for gravitational wave spacetimes, conformal compactification and conformal symmetry. *gr-qc/9811005*, 1998. Contribution to the 8th Conference on Theoretical Physics, General Relativity and Gravitation in Bistrita-Nasaud, Romania.

[21] R. P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11:237–238, 1963.

[22] W. Krivan and R. Price. Initial data for superposed rotating black holes. *Phys. Rev. D*, 58:104003, 1998.

[23] W. Krivan and R. Price. Formation of a rotating black hole from a close-limit head-on collision. *Phys. Rev. Lett.*, 82(7):1358–1361, 1999.

[24] C. Misner. Wormhole initial conditions. *Phys.Rev.*, 118:1110, 1960.

[25] J. Pullin. The close limit of colliding black holes: an update. *Prog. Theor. Phys. Suppl.*, 136:107–120, 1999.