

Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime

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Abstract. Some years ago, Radzikowski has found a characterization of Hadamard states for scalar quantum fields on a four-dimensional globally hyperbolic spacetime in terms of a specific form of the wavefront set of their two-point functions (termed ‘wavefront set spectrum condition’), thereby initiating a major progress in the understanding of Hadamard states and the further development of quantum field theory in curved spacetime. In the present work, we extend this important result on the equivalence of the wavefront set spectrum condition with the Hadamard condition from scalar fields to vector fields (sections in a vector bundle) which are subject to a wave-equation and are quantized so as to fulfill the covariant canonical commutation relations, or which obey a Dirac equation and are quantized according to the covariant anti-commutation relations, in any globally hyperbolic spacetime having dimension three or higher.

In proving this result, a gap which is present in the published proof for the scalar field case will be removed. Moreover we determine the short-distance scaling limits of Hadamard states for vector-bundle valued fields, finding them to coincide with the corresponding flat-space, massless vacuum states.

1 Introduction

In quantum field theory on curved spacetime, Hadamard states have acquired a prominent status; they are now recognized as defining the class of physical states for quantum fields obeying linear wave equations on any globally hyperbolic spacetime. The original motivation for introducing Hadamard states was the observation that they allow a

arXiv:math-ph/0008029 v2 23 Feb 2001

definition of the expectation value of the energy-momentum tensor with reasonable properties [41, 16, 43], thus Hadamard states may be viewed as a subclass of the states with finite energy density. This rests basically on the fact that the two-point functions of Hadamard states all have — by the very definition of Hadamard states — the same singular part which is determined by the spacetime metric and the wave equation obeyed by the quantum field (via the ‘Hadamard recursion relations’) and which mimics the singular behaviour of the vacuum state’s two-point function for linear quantum fields in flat spacetime.

A major progress in the study of Hadamard states was initiated by the observation that, for the free scalar field, the Hadamard condition on the two-point function of a quantum field state can be characterized in terms of a particular, antisymmetric form of the wavefront set of the two-point function [32]. This particular form of the two-function’s wavefront set is reminiscent of the form of the support of the Fourier-transformed two-point function of a quantum field in the vacuum state on Minkowski spacetime and hence has been called “wavefront set spectrum condition” in [32] and “microlocal spectrum condition” in [5]. A generalization to n -point functions has been suggested in [5]. In the present work, we will say that a state ω fulfills the microlocal spectrum condition if the wavefront set of its two-point function ω_2 assumes the same specific, anti-symmetric form known for Hadamard states of a free scalar field. Expressed in formulae, this means that the relations (5.9,5.10) in Sec. 5 hold.

The equivalent translation of the property of a two-point function to be of Hadamard form into specific properties of its wavefront set made it possible to apply the powerful methods of microlocal analysis (see e.g. the monographs [22, 23, 37]) in the study of Hadamard states. We mention here the following results that consequently arose:

- (a) It has been shown that the Hadamard form of states of the free scalar field is incompatible with a wide class of spacetime backgrounds which are initially globally hyperbolic and then develop closed timelike curves [24].
- (b) “Worldline energy inequalities” have been established for Hadamard states [13]. Such energy inequalities signify lower bounds for the expectation value of the energy density integrated along timelike curves for a suitable class of physical states (for instance, Hadamard states). (We refer to [13] and the review [14] for further discussion and references.)
- (c) A covariant definition of Wick-polynomials of the free scalar field has been given, and generalizations of the flat space spectrum condition to curved spacetime by a “microlocal spectrum condition” [5].
- (d) A local, covariant perturbative construction of $P(\phi)_4$ theories on curved spacetime has been developed along the lines of the approach by Epstein and Glaser [4].

In a recent work [34] we have shown that each ground state or KMS-state (thermal equilibrium state) of any vector-valued quantum field obeying a hyperbolic linear wave-equation on a stationary, globally hyperbolic spacetime fulfills the microlocal spectrum condition. The present paper may be viewed as accompanying our work [34]. We shall present a characterization of the Hadamard condition for vector fields obeying a wave equation or Dirac equation on any globally hyperbolic spacetime in terms of a specific form of the

wavefront set of the corresponding two-point functions — in other words, we generalize the results of [32] on the equivalence of Hadamard condition and microlocal spectrum condition from the case of scalar fields to that of vector fields and Dirac fields. Moreover, we shall consider not only 4-dimensional spacetime, but all spacetime dimensions ≥ 3 .

Since two-point functions of Hadamard states differ by a C^∞ -kernel, it is easy to show that the results of [38, 39] generalize from the scalar field case to the effect that all quasifree Hadamard states of a vector-valued linear quantum field (fulfilling canonical commutation or anti-commutation relations) induce locally unitarily equivalent representations of the field algebra. This may provide a starting point for generalizing the results of [4] on the Epstein-Glaser approach to perturbative construction of interacting quantum fields in curved spacetime from scalar fields to vector fields which may have more direct physical relevance.

We should like to point out that, in the case of the Dirac field on globally hyperbolic spacetimes, results similar to ours have already been obtained in a couple of other works. The first of these is the PhD thesis by Köhler [27] who shows that, in four spacetime dimensions, the Hadamard form of the two-point function of quasifree states for the Dirac field can be characterized by the microlocal spectrum condition. This result is essentially the same as our Thm. 5.8 for the said case. In some more recent works, Kratzert [29] and Hollands [21] consider the Dirac field on n -dimensional globally hyperbolic spacetimes. They also present results on the equivalence of Hadamard form and microlocal spectrum condition. Moreover, both authors investigate also the polarization set of the two-point functions of Hadamard states. The polarization set is a generalization of the wavefront set for vector-bundle distributions introduced by Dencker [7]. In components of a local frame for a vector-bundle, a vector-bundle distribution u is locally represented as an element (u_1, \dots, u_r) of $\oplus^r \mathcal{D}'(\mathbb{R}^n)$ where r is the dimension of the fibres and n is the dimension of the base-manifold (see Sec. 2.3). Then the elements in the polarization set of u are vectors $(x, \xi; v) \in (T^*\mathbb{R}^n \setminus \{0\}) \oplus \mathbb{C}^r$ where the vectors v describe, roughly speaking, which of u 's components has the “most singular” behaviour in the microlocal sense, and (x, ξ) describes the directions of worst decay in Fourier-space of those “most singular” components, like in the wavefront set. The projection of the polarization set of u onto its $(T^*\mathbb{R}^n \setminus \{0\})$ -part yields the wavefront set of u , defined as the union of the wavefront sets of all its components u_1, \dots, u_r which are scalar distributions. In their works, Kratzert and Hollands determine, among other things, the polarization set of the two-point functions for Hadamard states of the Dirac field and they show that Dencker’s connection, which describes the propagation of singularities of the polarization set, coincides in this case with the lifted spin-connection. Thereby they arrive at a characterization of Hadamard states of the Dirac field in terms of a specific form of the polarization sets of the corresponding two-point functions. This characterization is somewhat more detailed than ours in terms of the wavefront set since the polarization set contains more information than the wavefront set. However, as is already seen from the works [27, 29, 21], the microlocal spectrum condition in terms of the wavefront set completely characterizes the Hadamard condition as long as the principal part of the wave operator whose wave-equation is obeyed by the quantum field is scalar. We will exclusively consider this case, as Hadamard forms for more general wave operators have, to our knowledge, never been considered.

The contents of this paper are as follows: In Chapter 2 we summarize the definition

and basic properties of the wavefront set for scalar and vector-bundle distributions on manifolds. This material is included mainly to establish our notation, and to render the paper, for the convenience of the reader, as self-contained as reasonably possible. An auxiliary result relating the wavefront set of a distribution to the wavefront set of its short-distance scaling limit is also given.

Chapter 3 contains the definition of wave-operators and Dirac-operators on vector-bundles over globally hyperbolic spacetimes of any dimension $m \geq 3$. (Since we consider only Majorana-spinors, there are further restrictions on m in the Dirac-operator case.) Much of the material in that chapter is patterned along the references [10, 11, 26, 19, 40]. We also quote the ‘propagation of singularities theorem’ for distributional solutions of wave-operators, needed later, from [12, 7].

In Chapter 4 we give a discussion of quantum fields obeying canonical commutation relations (CCR) or canonical anti-commutation relations (CAR). We also explain how CCR- or CAR-quantum fields are associated with wave-operators or Dirac-operators, respectively.

In the fifth chapter we begin with the definition of Hadamard states for vector-valued linear quantum fields obeying a wave equation or a Dirac equation in a globally hyperbolic spacetime of dimension ≥ 3 . Our definition mimics the approach by Kay and Wald [25] for the scalar case, so we are really defining “globally Hadamard states” whose full definition is a bit involved.

Then we state in Sec. 5.2 the result on the ‘propagation of Hadamard form’ in the generality needed for the present purposes and sketch the proof, which is an entirely straightforward adaptation of the proof in [18] (as clarified in [25]) for the scalar field case.

In a further step, Sec. 5.3, we determine the short distance scaling limits of Hadamard states which are found to coincide with the two-point functions of the flat-space vacua for multi-component free fields satisfying massless Klein-Gordon or Dirac equations.

Finally, we present our main result as Thm. 5.8 in Sec. 5.4, asserting that Hadamard states of a vector-valued quantum field satisfying a wave-equation and CCR, or a Dirac equation and CAR, can be characterized by the specific form of the wavefront set of their two-point functions exactly as in the scalar field case. Prior to proving that result, we will point out that the original proof of the statement for scalar fields in [32] contains a gap, and we shall provide the means to complete the argument with the help of the result on the propagation of Hadamard form. (That gap affects also the proofs of the equivalence of Hadamard form and microlocal spectrum condition for Dirac fields in the works [27, 29, 21] since their authors rely on Radzikowski’s argument.)

Several technical issues related to Hadamard forms have been put into the Appendix. Among them are the precise forms of Hadamard recursion relations for wave-operators on vector-valued fields as well as the relation of Riesz-type distributions to Hadamard forms. A considerable part of that material has been taken from the monograph [19], which we would like to advertise as a most valuable source regarding the mathematics of Hadamard forms.

2 On the Wavefront Set

2.1 Wavefront Sets of Scalar Distributions

Let $n \in \mathbb{N}$ and $v \in \mathcal{D}'(\mathbb{R}^n)$. One calls $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ a *regular directed point* for v if there are $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi(x) \neq 0$, and a conical open neighbourhood Γ of k in $\mathbb{R}^n \setminus \{0\}$ [i.e. Γ is an open neighbourhood of k , and $k \in \Gamma \Leftrightarrow \mu k \in \Gamma \forall \mu > 0$], such that

$$\sup_{\tilde{k} \in \Gamma} (1 + |\tilde{k}|)^N |\widehat{\chi v}(\tilde{k})| \leq C_N < \infty$$

holds for all $N \in \mathbb{N}$, where $\widehat{\chi v}$ denotes the Fourier transform of the distribution $\chi \cdot v$.

Definition 2.1. $\text{WF}(v)$, the *wavefront set* of $v \in \mathcal{D}'(\mathbb{R}^n)$, is defined as the complement in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ of the set of all regular directed points for v .

Thus, $\text{WF}(v)$ consists of pairs (x, k) of points x in configuration space, and k in Fourier space, so that the Fourier transform of $\chi \cdot v$ isn't rapidly decaying along the direction k for large $|k|$, no matter how closely χ is concentrated around x .

If $\phi : U \rightarrow U'$ is a diffeomorphism between open subsets of \mathbb{R}^n , and $v \in \mathcal{D}'(U)$, then it holds that $\text{WF}(\phi^*v) = {}^tD\phi^{-1}\text{WF}(v)$ where ${}^tD\phi^{-1}$ denotes the transpose of the inverse tangent map (or differential) of ϕ , with ${}^tD\phi^{-1}(x, k) = (\phi(x), {}^tD\phi^{-1} \cdot k)$ for all $(x, k) \in \text{WF}(v)$ and $\phi^*v(f) = v(f \circ \phi)$, $f \in \mathcal{D}(U')$. This transformation behaviour of the wavefront set allows it to define the wavefront set $\text{WF}(v)$ of a scalar distribution $v \in \mathcal{D}'(X)$ on any n -dimensional manifold X [as usual, we take manifolds to be Hausdorff, connected, 2nd countable, C^∞ and without boundary] by using coordinates: Let $\kappa : U \rightarrow \mathbb{R}^n$ be a coordinate system around a point $q \in X$. Then the dual tangent map is an isomorphism ${}^tD\kappa : T_q^*X \rightarrow \mathbb{R}^n$. We will use the notational convention $(q, \xi) \in T^*X \Leftrightarrow \xi \in T_q^*X$. Then let $(q, \xi) \in T^*X \setminus \{0\}$ and $(x, k) := {}^tD\kappa^{-1}(q, \xi) = (\kappa(q), {}^tD\kappa^{-1} \cdot \xi)$, so that (x, k) is in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

Definition 2.2. We define $\text{WF}(v)$ by saying that $(q, \xi) \in \text{WF}(v)$ iff $(x, k) \in \text{WF}(\kappa^*v)$ where κ^*v is the chart expression of v .

Owing to the transformation properties of the wavefront set under local diffeomorphisms one can see that this definition is independent of the choice of the chart κ , and moreover, $\text{WF}(v)$ is a subset of $T^*X \setminus \{0\}$, the cotangent bundle with the zero section removed.

It is straightforward to deduce from the definition that

$$\text{WF}\left(\sum_j v_j\right) \subset \bigcup_j \text{WF}(v_j) \tag{2.1}$$

for any collection of finitely many $v_1, \dots, v_m \in \mathcal{D}'(X)$, and

$$\text{WF}(Av) \subset \text{WF}(v), \quad v \in \mathcal{D}'(X), \tag{2.2}$$

for any partial differential operator A with smooth coefficients. (This generalizes to pseudodifferential operators A .) It is also worth noting that $\text{WF}(v)$ is a closed conic

subset of $T^*X \setminus \{0\}$ where conic means $(q, \xi) \in \text{WF}(v) \Leftrightarrow (q, \mu\xi) \in \text{WF}(v) \forall \mu > 0$. Another important property is the following: Denote by p_{M^*} the base projection of T^*X , i.e. $p_{M^*} : (q, \xi) \mapsto q$. Then for all $v \in \mathcal{D}'(X)$ there holds

$$p_{X^*}\text{WF}(v) = \text{sing supp } v \quad (2.3)$$

where $\text{sing supp } v$ is the *singular support* of v .

Definition 2.3. For $v \in \mathcal{D}'(X)$, $\text{sing supp } v$ is defined as the complement of all points $q \in X$ for which there is an open neighbourhood U and a smooth n -form $\alpha_U \in \Omega^n(U)$ so that

$$v(h) = \int_U h \cdot \alpha_U \quad \text{for all } h \in \mathcal{D}(U).$$

In other words, v is given by an integral over a smooth n -form exactly if $\text{WF}(v)$ is empty.

2.2 Vector Bundles and Morphisms

Let \mathfrak{X} be a C^∞ vector bundle over a base manifold N ($\dim N = n$) with typical fibre \mathbb{C}^r or \mathbb{R}^r and bundle projection π_N . The space of smooth sections of \mathfrak{X} will be denoted by $C^\infty(\mathfrak{X})$ and $C_0^\infty(\mathfrak{X})$ denotes the subspace of smooth sections with compact support. These spaces can be equipped with locally convex topologies similar to those of the test-function spaces $\mathcal{E}(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$, see e.g. [8, 9]. We denote by $(C^\infty(\mathfrak{X}))'$ and $(C_0^\infty(\mathfrak{X}))'$ the respective spaces of continuous linear functionals, and by $C_0^\infty(\mathfrak{X}_U)$ the space of all smooth sections in \mathfrak{X} with compact support in the open subset U of N .

It will be useful to introduce the following terminology. Let X be any smooth manifold. Then we say that ρ is a local diffeomorphism of X if there are two open subsets $U_1 = \text{dom } \rho$ and $U_2 = \text{Ran } \rho$ of X so that $\rho : U_1 \rightarrow U_2$ is a diffeomorphism. If $U_1 = U_2 = X$, then ρ is a diffeomorphism of X . Now let ρ be a (local) diffeomorphism of the base manifold N . Then we say that R is a (local) *bundle map of \mathfrak{X} covering ρ* if R is a smooth map from $\pi_N^{-1}(\text{dom } \rho)$ to $\pi_N^{-1}(\text{Ran } \rho)$ with $\pi_N \circ R = \rho$ and mapping the fibre over each $q \in \text{dom } \rho$ linearly into the fibre over $\rho(q)$. If this map is also one-to-one and if R is also a local diffeomorphism, then R will be called a (local) *morphism of \mathfrak{X} covering ρ* .

Each (local) bundle map R of \mathfrak{X} covering a (local) diffeomorphism ρ of N induces a (local) action on $C_0^\infty(\mathfrak{X})$, that is, a continuous linear map $R^* : C_0^\infty(\mathfrak{X}_{\text{dom } \rho}) \rightarrow C_0^\infty(\mathfrak{X}_{\text{Ran } \rho})$ given by

$$R^*f := R \circ f \circ \rho^{-1}, \quad f \in C_0^\infty(\mathfrak{X}_{\text{dom } \rho}). \quad (2.4)$$

We finally note that the terminology introduced above applies equally well to the case where ρ is a local diffeomorphism between base manifolds of *different* vector bundles.

2.3 Wavefront Set of Vectorbundle Distributions

Let \mathfrak{X} again be a C^∞ vector bundle as before. Then let $U \subset N$ be an open subset and let (e_1, \dots, e_r) be a local trivialization, or local frame, of \mathfrak{X} over U . That means

the e_j , $j = 1, \dots, r$ are sections in $C^\infty(\mathfrak{X}_U)$ so that, for each $q \in U$, $(e_1(q), \dots, e_r(q))$ forms a linear basis of the fibre $\pi_N^{-1}(\{q\})$. Such a local trivialization induces a one-to-one correspondence between $C_0^\infty(\mathfrak{X}_U)$ and $\oplus^r \mathcal{D}(U)$ by assigning to each $f \in C_0^\infty(\mathfrak{X}_U)$ the $(f^1, \dots, f^r) \in \oplus^r \mathcal{D}(U)$ with ¹

$$f^a e_a = f.$$

This, in turn, induces a one-to-one correspondence between $(C_0^\infty(\mathfrak{X}_U))'$ and $\oplus^r \mathcal{D}'(U)$, via mapping $u \in (C_0^\infty(\mathfrak{X}_U))'$ to the $(u_1, \dots, u_r) \in \oplus^r \mathcal{D}'(U)$ given by

$$u_a(h) = u(h \cdot e_a), \quad h \in \mathcal{D}(U).$$

With this notation, one defines for $u \in (C_0^\infty(\mathfrak{X}_U))'$ the wavefront set as

$$\text{WF}(u) := \bigcup_{a=1}^r \text{WF}(u_a),$$

i.e. the wavefront set of u is defined as the union of the wavefront sets of the scalar component-distributions in any local trivialization over U . Using (2.1) and (2.2) it is straightforward to see that this definition is independent of the choice of local trivializations. Therefore, one is led to the following

Definition 2.4. Let $u \in (C_0^\infty(\mathfrak{X}))'$, $(q, \xi) \in \text{T}^*N \setminus \{0\}$. Then (q, ξ) is defined to be in $\text{WF}(u)$ if, for any neighbourhood U of q over which \mathfrak{X} trivializes, (q, ξ) is in $\text{WF}(u_U)$ where u_U is the restriction of u to $C_0^\infty(\mathfrak{X}_U)$.

The properties of $\text{WF}(u)$ are similar to those in the case of scalar distributions; obviously (2.1) and (2.2) generalize to the vectorbundle case. Also, $\text{WF}(u)$ is a closed conic subset of $\text{T}^*N \setminus \{0\}$, and it holds that

$$p_{N^*} \text{WF}(u) \subset \text{sing supp } u, \quad u \in (C_0^\infty(\mathfrak{X}))', \quad (2.5)$$

where $p_{N^*} : \text{T}^*N \rightarrow N$ is the cotangent bundle projection, and the counterpart of Def. 2.3 relevant here is:

Definition 2.5. For $u \in (C_0^\infty(\mathfrak{X}))'$, $\text{sing supp } u$ is defined as the complement of all points in N for which there are an open neighbourhood U , a smooth section $\nu \in C^\infty(\mathfrak{X}^*)$ in the dual bundle \mathfrak{X}^* to \mathfrak{X} , and a smooth n -form $\alpha_U \in \Omega^n(U)$ so that

$$u(f) = \int_U \nu(f) \cdot \alpha_U, \quad f \in C_0^\infty(\mathfrak{X}_U).$$

As in the scalar case, it is very useful to know the behaviour of the wavefront set under (local) morphisms of \mathfrak{X} . The following Lemma provides this information. The proof can be given by simply adapting the arguments well-known for the scalar case.

¹summation over repeated indices is implied

Lemma 2.6. *Let U_1 and U_2 be open subsets of N , and let $R : \mathfrak{X}_{U_1} \rightarrow \mathfrak{X}_{U_2}$ be a vector bundle map covering a diffeomorphism $\rho : U_1 \rightarrow U_2$. Let $u \in (C_0^\infty(\mathfrak{X}_{U_1}))'$. Then it holds that*

$$\text{WF}(R^*u) \subset {}^tD\rho\text{WF}(u) = \{(\rho^{-1}(x), {}^tD\rho \cdot \xi) : (x, \xi) \in \text{WF}(u)\}, \quad (2.6)$$

where ${}^tD\rho$ denotes the transpose (or dual) of the tangent map of ρ . If R is even a bundle morphism, then the inclusion (2.6) becomes an equality.

Note that the above Lemma applies equally well to the case of bundle morphisms covering diffeomorphisms between base spaces of *different* vector bundles.

2.4 Scaling Limits

In the present subsection we consider scaling limits of vector-bundle distributions.

Let \mathfrak{X} be a vector bundle with base N as before. Let $q \in N$ and let $\kappa : U \rightarrow O \subset \mathbb{R}^n$ be a coordinate chart around q . We assume that the chart range O is convex and that $\kappa(q) = 0$. Then we define the following semi-group $(\check{\delta}_\lambda)_{1 > \lambda > 0}$ of local diffeomorphisms of O :

$$\check{\delta}_\lambda(y) := \lambda \cdot y, \quad y \in O, \quad 1 > \lambda > 0.$$

This induces a semi-group $(\delta_\lambda)_{1 > \lambda > 0}$ of local diffeomorphisms of U according to

$$\delta_\lambda = \kappa^{-1} \circ \check{\delta}_\lambda \circ \kappa, \quad 1 > \lambda > 0.$$

(Note that $(\delta_\lambda)_{1 > \lambda > 0}$ depends on κ , which is not reflected by our notation.)

Now let $(D_\lambda)_{1 > \lambda > 0}$ be a family of local morphisms of \mathfrak{X} so that D_λ covers δ_λ for each $1 > \lambda > 0$.

Definition 2.7. Let $u \in (C_0^\infty(\mathfrak{X}_U))'$. If the limit

$$u^{(0)}(f) := \lim_{\lambda \rightarrow 0} u(D_\lambda^* f)$$

exists for all $f \in C_0^\infty(\mathfrak{X}_U)$ and does not vanish for all f , then $u^{(0)}$ will be called the *scaling limit distribution* with respect to $(D_\lambda)_{1 > \lambda > 0}$ at q .

Clearly, the scaling limit distribution is then a member of $(C_0^\infty(\mathfrak{X}_U))'$. The following result will later be of interest.

Proposition 2.8. *Let $u^{(0)}$ be the scaling limit distribution of a $u \in (C_0^\infty(\mathfrak{X}_U))'$ at q with respect to some $(D_\lambda)_{1 > \lambda > 0}$ such that*

$$\max_{1 \leq a, b \leq r} |D_b^a(\lambda)| \leq c\lambda^{-\nu}, \quad 0 < \lambda < \lambda_0, \quad (2.7)$$

holds for the components $D_b^a(\lambda)$ of D_λ in any local trivialization of \mathfrak{X} near q with suitable constants $c, \nu > 0$.

Then

$$(q, \xi) \in \text{WF}(u^{(0)}) \Rightarrow (q, \xi) \in \text{WF}(u). \quad (2.8)$$

Proof. We will establish the relation

$$(q, \xi) \notin \text{WF}(u) \Rightarrow (q, \xi) \notin \text{WF}(u^{(0)}) \quad (2.9)$$

which is equivalent to (2.8). Using the chart, we identify T_q^*N with \mathbb{R}^n , and we identify the components $(u_1^{(0)}, \dots, u_r^{(0)})$ of $u^{(0)}$ and (u_1, \dots, u_r) of u with respect to a local trivialization of \mathfrak{X} near q with elements of $\mathcal{D}'(O)$ where O is the chart range. (Note that it is no restriction to assume that \mathfrak{X} trivialises on the chart domain since only the behaviour of u in any arbitrarily small neighbourhood of q is relevant here.) With the indicated identifications provided by the chart, the required relation (2.9) reads

$$(0, \xi) \notin \text{WF}(u_a) \text{ for all } 1 \leq a \leq r \quad \Rightarrow \quad (0, \xi) \notin \text{WF}(u_a^{(0)}) \text{ for all } 1 \leq a \leq r$$

and

$$u(D_\lambda^* f) = D_b^a(\lambda) u_a(f^b \circ \delta_\lambda^{-1}) = D_b^a(\lambda) u_a^{[\lambda]}(f^b)$$

in components of the local trivialization, where we have introduced

$$u_a^{[\lambda]}(f) := u_a(f \circ \delta_\lambda^{-1}), \quad f \in \mathcal{D}(O).$$

Now let $(0, \xi) \notin \text{WF}(u_a)$ for all $1 \leq a \leq r$. This means that there is a function $\chi \in C_0^\infty(O)$ with $\chi(0) = 1$ and an open conic neighbourhood Γ of ξ (in $\mathbb{R}^n \setminus \{0\}$) so that, for all $m > 0$,

$$\sup_{k \in \Gamma} |\widehat{\chi u_a}(k)| (1 + |k|^m) \leq C_m \quad (2.10)$$

holds for all $1 \leq a \leq r$ with suitable $C_m > 0$.

Now choose some $\chi_0 \in C_0^\infty(O)$ with $\chi_0(0) = 1$ and $\text{supp}(\chi_0) \subset \text{supp}(\chi)$. Then the $\widehat{\chi_0 u_a^{(0)}}(k)$ are analytic functions of k , hence bounded on compact sets. This implies that it suffices to show that there are an open conic neighbourhood Γ_0 of ξ and some $m_0 > 0$ so that for all $m > m_0$

$$\sup_{k \in \Gamma_0} \left| \widehat{\chi_0 u_a^{(0)}}(k) \right| |k|^m < C'_m \quad (2.11)$$

holds for all $1 \leq a \leq r$ with suitable $C'_m > 0$, in order to conclude that $(0, \xi) \notin \text{WF}(u_a^{(0)})$ for all $1 \leq a \leq r$.

To prove that (2.11) holds, we observe that

$$((\chi u_a)^{[\lambda]})^\wedge(k) = \widehat{\chi u_a}(\lambda^{-1}k), \quad 1 > \lambda > 0, \quad k \in \mathbb{R}^n.$$

Furthermore, since (2.10) holds and since the cone Γ is scale-invariant, we see that

$$\sup_{1 > \lambda > 0, k \in \Gamma} |\widehat{\chi u_a}(\lambda^{-1}k)| |\lambda^{-1}k|^m \leq C_m$$

for all $1 \leq a \leq r$. Thus, if $m \geq \nu$, we obtain from assumption (2.7), for all $1 \leq a \leq r$,

$$\begin{aligned} \sup_{1 > \lambda > 0, k \in \Gamma} |D_a^b(\lambda) ((\chi u_b)^{[\lambda]})^\wedge(k)| |k|^m &\leq \sup_{1 > \lambda > 0, k \in \Gamma} cr^2 \lambda^{-\nu} \max_{1 \leq b \leq r} |\widehat{\chi u_b}(\lambda^{-1}k)| |k|^m \\ &\leq cr^2 \sup_{1 > \lambda > 0, k \in \Gamma} \max_{1 \leq b \leq r} |\widehat{\chi u_b}(\lambda^{-1}k)| |\lambda^{-1}k|^m \\ &\leq cr^2 C_m. \end{aligned}$$

Observing that $\chi_0 u_a^{[\lambda]} = \chi_0(\chi u_a)^{[\lambda]}$ for $1 > \lambda > 0$ and using also that

$$(\chi_0 u_a^{(0)})^\wedge(k) = \lim_{\lambda \rightarrow 0} D_a^b(\lambda) (\chi_0 u_b^{[\lambda]})^\wedge(k)$$

holds for all $k \in \mathbb{R}^n$, $1 \leq a \leq r$, the desired bound (2.11) is implied by the last estimate. \square

3 Wave-operators and Dirac-operators

3.1 Wave-operators on Vector-bundles over Curved Spacetimes

We shall investigate the situation of general vector-valued fields propagating over a curved spacetime. Thus, the basic object of our considerations is a vector bundle \mathfrak{V} with typical fibre \mathbb{C}^r , base manifold M ($\dim M = m$) and base projection π_M . The base manifold is to have the structure of a spacetime, so it will be assumed that M is endowed with a Lorentzian metric g having signature $(+, -, \dots, -)$. Thus (M, g) is a Lorentzian spacetime-manifold. Within the scope of the present work, we will impose further regularity conditions on the causal structure of (M, g) . First, we assume that (M, g) is time-orientable, i.e. that there exists a global, timelike vectorfield on M . A further assumption which we make is that (M, g) be globally hyperbolic. This means that there exists a Cauchy-surface in (M, g) , which by definition is a C^0 -hypersurface in M which is intersected exactly once by each inextendible g -causal curve in M . It can be shown that (M, g) is globally hyperbolic if and only if there exists an $m - 1$ -dimensional manifold Σ and a diffeomorphism $\Psi : \mathbb{R} \times \Sigma \rightarrow M$ so that, for each $t \in \mathbb{R}$, $\Sigma_t = \Psi(\{t\} \times \Sigma)$ is a Cauchy-surface in (M, g) . This means that a globally hyperbolic spacetime can be smoothly foliated by a C^∞ -family $\{\Sigma_t\}_{t \in \mathbb{R}}$ of Cauchy-surfaces.

The causal structure of globally hyperbolic spacetimes is, in a sense, “best behaved”. In particular, it has the consequence that if v is a non-zero lightlike vector in $T_q M$ for any $q \in M$, then the maximal geodesic γ which it defines (i.e. $\gamma : I \subset \mathbb{R} \rightarrow M$ is a solution of the geodesic equation with $\gamma(0) = q$ and $\frac{d}{dt}\gamma(t)|_{t=0} = v$, and any other such curve that has the same properties cannot properly extend γ) is endpointless (inextendible), and thus there is for each Cauchy-surface C in (M, g) exactly one parameter value t so that $\gamma(t) \in C$.

Let us also collect the notation for the causal future/past sets. If $p \in M$, then one denotes by $J^\pm(p)$ the subset of all points q in M which lie on any future/past directed causal curve [continuous, piecewise smooth] starting at p . For $G \subset M$, $J^\pm(G)$ is defined as $\bigcup\{J^\pm(p) : p \in G\}$. For any subset Σ of M one defines its future/past domain of

dependence, $D^\pm(\Sigma)$, as the set of all points p in M such that each past/future-inextendible causal curve starting at p intersects Σ at least once. Then $D(\Sigma)$ denotes $\overline{D^+(\Sigma) \cup D^-(\Sigma)}$. A set $G' \subset M$ is called *causally separated* from G if $G' \cap (\overline{J^+(G)} \cup \overline{J^-(G)}) = \emptyset$. Note that the relation of causal separation is symmetric in G and G' . The reader is referred to [20, 42] for a more detailed discussion of causal structure.

After this brief reminder concerning some basic properties of Lorentzian spacetimes, we turn now to wave-operators. A linear partial differential operator

$$P : C_0^\infty(\mathfrak{Y}) \rightarrow C_0^\infty(\mathfrak{Y})$$

will be said to have *metric principal part* if, upon choosing a local trivialization of \mathfrak{Y} over $U \subset M$ in which sections $f \in C_0^\infty(\mathfrak{Y}_U)$ take the component representation (f^1, \dots, f^r) , and a chart $(x^\mu)_{\mu=1}^m$, one has the following coordinate representation for P :²

$$(Pf)^a(x) = g^{\mu\nu}(x)\partial_\mu\partial_\nu f^a(x) + A^{\nu a}_b(x)\partial_\nu f^b(x) + B^a_b(x)f^b(x).$$

Here, ∂_μ denotes the coordinate derivative $\frac{\partial}{\partial x^\mu}$, and $A^{\nu a}_b$ and B^a_b are suitable collections of smooth, complex-valued functions. Observe that thus the principal part of P diagonalizes in all local trivializations (it is “scalar”).

We will further suppose that there is a morphism Γ of \mathfrak{Y} covering the identity map of M which acts as an involution ($\Gamma \circ \Gamma = \text{id}_{\mathfrak{Y}}$) and operates anti-isomorphically on the fibres. Therefore, Γ acts like a complex conjugation in each fibre space, and the Γ -invariant part \mathfrak{Y}° of \mathfrak{Y} is a vector bundle with typical fibre isomorphic to \mathbb{R}^r . If P has metric principal part and is in addition Γ -invariant, i.e.

$$\Gamma \circ P \circ \Gamma = P,$$

then P will be called a *wave operator*. [Note that we have written here Γ where we should have written Γ^* , however this appears justified since Γ covers the identity, so we adopt this convention since it results in a simpler notation.]

It is furthermore worth noting that, given any wave operator, there is a uniquely determined covariant derivative, or linear connection, $\nabla^{(P)}$ on \mathfrak{Y} , characterized by the property

$$2 \cdot \nabla_{\text{grad}\varphi}^{(P)} f = P(\varphi f) - \varphi P(f) - (\square\varphi)f \tag{3.1}$$

for all $\varphi \in C_0^\infty(M, \mathbb{R})$ and all $f \in C_0^\infty(\mathfrak{Y}^\circ)$ [19, Chp. 6]. Here, \square denotes the d’Alembertian operator associated with g on the scalar functions. Then there exists also a bundle map V of \mathfrak{Y}° covering the identity on the base manifold M such that

$$Pf = g^{\mu\nu}\nabla_\mu^{(P)}\nabla_\nu^{(P)}f + Vf, \quad f \in C_0^\infty(\mathfrak{Y}^\circ).$$

(Here we have followed our convention to denote the induced action of the bundle map covering id_M simply by V instead of V^* .)

²Greek indices are raised and lowered with $g^\mu_\nu(x)$, latin indices with δ^a_b .

3.2 Propagation of Singularities

We consider a wave operator P for a vector bundle \mathfrak{V} over a spacetime manifold (M, g) (for the present subsection, we need not require that (M, g) be globally hyperbolic). Let $w \in (C_0^\infty(\mathfrak{V}) \otimes C_0^\infty(\mathfrak{V}))'$. Then we call w a *bisolution for the wave operator P up to C^∞* , or, for short, *bisolution mod C^∞* , if there are two smooth sections $\varphi, \psi \in C^\infty(\mathfrak{V}^* \boxtimes \mathfrak{V}^*)$, where \mathfrak{V}^* denotes the dual bundle of \mathfrak{V} and $\mathfrak{V}^* \boxtimes \mathfrak{V}^*$ is the outer tensor product bundle (this is the bundle over $M \times M$ having fibre $\mathfrak{V}_p^* \otimes \mathfrak{V}_q^*$ at $(p, q) \in M \times M$, with the obvious projection), so that

$$\begin{aligned} w(Pf \otimes f') &= \int \varphi_{(p,q)}(f(p) \otimes f'(q)) d\mu(p) d\mu(q), \\ w(f \otimes Pf') &= \int \psi_{(p,q)}(f(p) \otimes f'(q)) d\mu(p) d\mu(q) \end{aligned}$$

holds for all $f, f' \in C_0^\infty(\mathfrak{V})$. Here, $d\mu$ denotes the volume measure induced by the metric g . In view of the fact that the projection of $\text{WF}(w)$ to the base manifold yields $\text{sing supp } w$, one can see that, upon defining $w^{(P)}, w_{2(P)} \in (C_0^\infty(\mathfrak{V}) \otimes C_0^\infty(\mathfrak{V}))'$ by

$$\begin{aligned} w^{(P)}(f \otimes f') &:= w(Pf \otimes f'), \\ w_{2(P)}(f \otimes Pf') &:= w(f \otimes Pf'), \quad f, f' \in C_0^\infty(\mathfrak{V}), \end{aligned}$$

w is a bisolution for the wave operator $P \bmod C^\infty$ exactly if

$$\text{WF}(w^{(P)}) = \emptyset \quad \text{and} \quad \text{WF}(w_{2(P)}) = \emptyset.$$

In keeping with that notation, when $w, w' \in (C_0^\infty(\mathfrak{V} \boxtimes \mathfrak{V}))'$ we shall also say that w agrees with $w' \bmod C^\infty$, or

$$w = w' \quad \bmod C^\infty,$$

if $\text{WF}(w - w') = \emptyset$.

Let us now define the set of “null-covectors”

$$\mathcal{N} := \{(q, \xi) \in \text{T}^*M : g^{\sigma\rho}(q)\xi_\sigma\xi_\rho = 0\}. \quad (3.2)$$

Since (M, g) possesses a time orientation, it is useful to introduce the following two disjoint future/past-oriented parts of \mathcal{N} ,

$$\mathcal{N}_\pm := \{(q, \xi) \in \mathcal{N} \mid \pm \xi \triangleright 0\}, \quad (3.3)$$

where $\xi \triangleright 0$ means that the vector $\xi^\mu = g^{\mu\nu}\xi_\nu$ is future-pointing and non-zero.

On the set \mathcal{N} one can introduce an equivalence relation as follows:

Definition 3.1. One defines

$$(q, \xi) \sim (q', \xi')$$

iff there is an affinely parametrized lightlike geodesic γ with $\gamma(t) = q$, $\gamma(t') = q'$ and

$$g^{\sigma\rho}(q)\xi_\rho = \left(\frac{d}{ds}\Big|_{s=t}\gamma(s)\right)^\sigma, \quad g^{\sigma\rho}(q')\xi'_\rho = \left(\frac{d}{ds}\Big|_{s=t'}\gamma(s)\right)^\sigma.$$

That is to say, ξ and ξ' are co-parallel to the lightlike geodesic γ connecting the base points q and q' , and therefore ξ and ξ' are parallel transports of each other along that geodesic.

By $\text{B}(q, \xi) := [(q, \xi)]_\sim$ we will denote the equivalence class associated with $(q, \xi) \in \mathcal{N}$.

With this terminology, we can formulate the propagation of singularities theorem (PST) for wave-operators, which is a consequence of more general results of Dencker [7] together with [12, Lemma 6.5.5]. See also [24] for a more elementary account.

Proposition 3.2. *Let P be a wave operator on $C_0^\infty(\mathfrak{V})$, and suppose that $w \in (C_0^\infty(\mathfrak{V}) \otimes C_0^\infty(\mathfrak{V}))'$ is a bisolution mod C^∞ for P . Then there holds*

$$\text{WF}(w) \subset \mathcal{N} \times \mathcal{N}$$

and

$$(q, \xi; q', \xi') \in \text{WF}(w) \quad \text{with} \quad \xi \neq 0 \quad \text{and} \quad \xi' \neq 0 \quad \Rightarrow \quad \text{B}(q, \xi) \times \text{B}(q', \xi') \subset \text{WF}(w).$$

3.3 Propagators and Cauchy-Problem

As in the previous section, we assume that \mathfrak{V} is a vector bundle with typical fibre \mathbb{C}^r and base manifold M . Again, M comes endowed with a Lorentzian metric g with the property that the spacetime (M, g) is time-orientable and globally hyperbolic. A time-orientation is assumed to have been chosen. Moreover we suppose that there is a fibre wise complex conjugation Γ on \mathfrak{V} , and a wave-operator P operating on $C_0^\infty(\mathfrak{V})$ and commuting with Γ .

An additional structure will be introduced now: We assume that \mathfrak{V} is a hermitean vector bundle. That is, there is a smooth section h in $\mathfrak{V}^* \boxtimes \mathfrak{V}^*$ so that, for each p in M , h_p is a sesquilinear form on \mathfrak{V}_p (this form need not be positive definite). Clearly, h induces an antilinear vector-bundle morphism $\vartheta : \mathfrak{V} \rightarrow \mathfrak{V}^*$ covering the identity via

$$h(v, w) := (\vartheta v)(w), \quad v, w \in \mathfrak{V}_q, \quad q \in M. \quad (3.4)$$

Then one can use h to introduce a non-degenerate sesquilinear form

$$(f, f') := \int_M h(f(q), f'(q)) d\mu(q), \quad f, f' \in C_0^\infty(\mathfrak{V}), \quad (3.5)$$

on $C_0^\infty(\mathfrak{V})$. The volume form $d\mu$ appearing here is that induced by the metric g on M .

We will furthermore make the following assumption:

$$(Pf, f') = (f, Pf'), \quad f, f' \in C_0^\infty(\mathfrak{V}). \quad (3.6)$$

It has been observed in [26, 11] that under the stated assumptions the results of [30] imply the existence of unique advanced and retarded fundamental solutions of P . A similar statement can be deduced from [19, Prop. III.4.1]. We quote this result as part (a) of the subsequent proposition from [26]. Part (b) of this proposition is the statement that the Cauchy-problem for the wave-operator is well-posed. The proof of this statement may either be given by generalizing the classical energy-estimate arguments as given e.g. in [20] for tensor-fields to sections in vector bundles, or by using the arguments in [10] Lemma A.4 to globalize the local version of that statement which is proven e.g. in [19, Prop. III.5.4].

Proposition 3.3. (a) *The wave-operator P possesses unique advanced/retarded fundamental solutions, i.e. there is a unique pair of (continuous) linear maps*

$$E^\pm : C_0^\infty(\mathfrak{Y}) \rightarrow C^\infty(\mathfrak{Y})$$

such that

$$PE^\pm f = E^\pm Pf = f \quad \text{and} \quad \text{supp}(E^\pm f) \subset J^\pm(\text{supp } f), \quad f \in C_0^\infty(\mathfrak{Y}).$$

Moreover, from $\Gamma P = P\Gamma$ it follows that $\Gamma E^\pm = E^\pm \Gamma$, and if P has the hermiticity property (3.6), then it holds that

$$(E^\pm f, f') = (f, E^\mp f'), \quad f, f' \in C_0^\infty(\mathfrak{Y}).$$

(b) *Let Σ be a Cauchy-surface in (M, g) , and let n be the future-pointing unit-normal vector field along Σ . Using Gaussian normal coordinates, n determines by geodesic transport a vector field in a neighbourhood of Σ (the geodesic spray of n) which is also denoted by n . Then, given any pair $f, f' \in C_0^\infty(\mathfrak{Y})$, there is exactly one $\phi \in C^\infty(\mathfrak{Y})$ solving the Cauchy-problem for the wave operator P with data induced by f and f' , i.e. ϕ obeys*

$$(i) \quad P\phi = 0,$$

$$(ii) \quad (\phi - f) \upharpoonright \Sigma = 0, \quad (\nabla_n^{(P)} \phi - f') \upharpoonright \Sigma = 0,$$

where $\nabla^{(P)}$ is the connection induced by P .

3.4 Spin Structures and Spinor Fields

In the present section, we summarize a few basics about manifolds with spin structure and Dirac operators, following Dimock's article [11] to large extent, however generalizing parts therein to spacetime dimensions ≥ 3 while specializing at the same time to Majorana spinors. In this context, we refer the reader to [6].

As before, we suppose that (M, g) is a time-orientable, globally hyperbolic Lorentzian spacetime of dimension m . Additionally, we suppose that (M, g) is "space-orientable", i.e. that each Cauchy-surface is orientable. We suppose that time- and space-orientations have been chosen. Then we define $F(M, g)$ as the bundle of time- and space-oriented g -orthonormal frames on M . That is, $F(M, g)$ consists of m -tuples $(v_0, v_1, \dots, v_{m-1})_{(p)}$ of vectors $v_\mu \in T_p M$, $p \in M$, such that v_0 is timelike and future-pointing, (v_1, \dots, v_{m-1}) is a collection of spacelike vectors having the prescribed spatial orientation, and $g(v_\mu, v_\nu) = \eta_{\mu\nu}$ where $(\eta_{\mu\nu})_{\mu, \nu=0}^{m-1} = \text{diag}(+, -, \dots, -)$ is the m -dimensional Minkowski-metric in a Lorentz frame. The base projection $\pi_F : F(M, g) \rightarrow M$ is given by $(v_0, \dots, v_{m-1})_{(p)} \mapsto p$. $F(M, g)$ has the structure of a principal fibre bundle with structure group $\text{SO}^\uparrow(1, m-1)$, where the arrow signifies that the transformations preserve the time-orientation.

The universal covering group of $\text{SO}^\uparrow(1, m-1)$ is $\text{Spin}^\uparrow(1, m-1)$. Let us denote by

$$\text{Spin}^\uparrow(1, m-1) \ni \boldsymbol{\lambda} \mapsto \Lambda(\boldsymbol{\lambda}) \in \text{SO}^\uparrow(1, m-1)$$

the 2–1 covering projection. Then a *spin structure* for (M, g) is, by definition, a principal fibre bundle $S(M, g)$ with base manifold M ($\pi_S : S(M, g) \rightarrow M$ will denote the base projection) and with structure group $\text{Spin}^\uparrow(1, m-1)$, together with a bundle-homomorphism $\phi : S(M, g) \rightarrow F(M, g)$ preserving the base points, $\pi_F \circ \phi = \pi_S$, and having the property that

$$\phi \circ R_{\boldsymbol{\lambda}}(\mathbf{s}) = R_{\Lambda(\boldsymbol{\lambda})} \circ \phi(\mathbf{s}), \quad \mathbf{s} \in S(M, g).$$

Here, R denotes the right action of the structure group on the corresponding principal fibre bundles. A sufficient criterion for existence of spin-structures is that M is parallelizable; this is for instance the case for all 4-dimensional globally hyperbolic spacetimes.

It is known (cf. [6]) that for the cases $m = 3, 4, 9, 10 \pmod 8$ there are *Majorana algebras* $\mathcal{M}(1, m-1)$, defined as the real-linear subalgebras of $\mathbf{M}(\mathbb{C}^{2^{[m/2]}})$ (the algebra of complex $2^{[m/2]} \times 2^{[m/2]}$ matrices³) which are generated by elements $\{\gamma_\mu : \mu = 0, \dots, m-1\}$ obeying the relations:

$$\begin{aligned} \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu &= 2\eta_{\mu\nu}, \quad \text{and} \\ \gamma_0^* &= \gamma_0, \quad \gamma_k^* = -\gamma_k \quad (k = 1, \dots, m-1), \quad \overline{\gamma_\mu} = -\gamma_\mu \quad (\mu = 0, \dots, m-1), \end{aligned}$$

where $(\cdot)^*$ means taking the hermitean conjugate matrix and $\overline{(\cdot)}$ denotes the matrix with complex conjugate entries. Given a Majorana algebra $\mathcal{M}(1, m-1)$, one can construct a canonical, faithful group endomorphism $\ell : \text{Spin}^\uparrow(1, m-1) \rightarrow \mathcal{M}(1, m-1)$ so that the group multiplication is carried to the matrix product, and with the property that

$$\ell(\boldsymbol{\lambda}) \cdot \gamma_\mu = \Lambda(\boldsymbol{\lambda})_\mu^\nu \gamma_\nu \cdot \ell(\boldsymbol{\lambda}).$$

Therefore ℓ is at the same time a linear representation of $\text{Spin}^\uparrow(1, m-1)$ on $\mathbb{C}^{2^{[m/2]}}$. Thus, given a spin structure and a Majorana algebra, one may form the vector bundle

$$D_\ell M = S(M, g) \times_{\ell} \mathbb{C}^{2^{[m/2]}},$$

the vector bundle associated to $S(M, g)$ and the representation ℓ of its structure group $\text{Spin}^\uparrow(1, m-1)$ on $\mathbb{C}^{2^{[m/2]}}$. It is called the bundle of *Majorana spinors* (corresponding to the Dirac representation ℓ induced by $\mathcal{M}(1, m-1)$). The dual bundle to $D_\ell M$ will be denoted by $D_\ell^* M$.

Remark. It is just a matter of convenience that we restrict our discussion to the case of Majorana spinors. One could work with Dirac spinors as well; then one has to introduce appropriate ‘doublings’ of spinor bundle and Dirac operator. Such an approach has, in the context of quantizing Dirac fields, been favoured elsewhere [11, 21, 27, 29, 35]. By employing somewhat more elaborate notation, one may generalize our results in Chapter 5 to the slightly more general case of Dirac spinors.

3.5 Dirac-operators

The metric-induced connection ∇ on TM naturally gives rise to a connection on the frame bundle $F(M, g)$. Since the Lie-algebras of $\text{Spin}_0^\uparrow(1, m-1)$ and $\text{SO}^\uparrow(1, m-1)$ can be

³ $[m/2]$ denotes the integer part of $m/2$

canonically identified, that connection on $F(M, g)$ induces a connection on $S(M, g)$, from which one obtains a linear connection on $D_\ell M$. We denote the corresponding covariant derivative operator by $\nabla : C^\infty(TM \otimes D_\ell M) \rightarrow C^\infty(D_\ell M)$, $v \otimes f \mapsto \nabla_v f$, without indicating the dependence on the representation ℓ .

Now one can proceed exactly as in [11] and introduce the spinor-tensor γ , the Dirac operator ∇ and the Dirac adjoint u^+ . The spinor-tensor $\gamma \in C_0^\infty(T^*M \otimes D_\ell M \otimes D_\ell^* M)$ is defined by requiring that its components $\gamma_\mu^{a_b}$ with respect to (appropriate) local frames are equal to the matrix elements $(\gamma_\mu)^{a_b}$, and the Dirac operator is introduced by setting in frame components, for a local section $f = f^a e_a \in C_0^\infty(D_\ell M)$,⁴

$$(\nabla f)^a := \eta^{\mu\nu} \gamma_\mu^{a_b} (\nabla_\nu f)^b.$$

The Dirac adjoint $D_\ell M \ni u \mapsto u^+ \in D_\ell^* M$ is a base-point preserving anti-linear bundle morphism defined by setting $(u^+)_a = \overline{u^b} \gamma_{0ab}$ for the dual frame components. This allows to define a hermitean structure h on $D_\ell M$ via

$$h(u, w) := u^+(w), \quad u, w \in D_\ell M,$$

and thus the Dirac adjunction plays the role of ϑ of the last section. One can, moreover, introduce a conjugation Γ on $D_\ell M$ by setting, in any frame, $(\Gamma u)^a = \overline{u^a}$ for the components. Then one finds

$$h(\Gamma u, \Gamma w) = -h(w, u), \quad u, w \in D_\ell M, \quad (3.7)$$

showing that Γ is a skew-conjugation for the hermitean form h and that h is, while non-degenerate, not positive, but rather a conjugate skew-symmetric form (analogous to a symplectic form). As in the last section, h induces a hermitean (now, conjugate skew-symmetric) form on $C_0^\infty(D_\ell M)$ given by

$$(f, f') := \int_M h(f(p), f'(p)) d\mu(p), \quad f, f' \in C_0^\infty(D_\ell M), \quad (3.8)$$

where again $d\mu$ is the volume form induced by the metric g on M , and clearly Γ acts now as skew-conjugation with respect to this hermitean form on $C_0^\infty(D_\ell M)$.

Now if $\mathfrak{m} \geq 0$ is a constant (more generally, it could be a Γ -invariant bundle map of $D_\ell M$ covering the identity), one may introduce a pair of Dirac operators

$$D_\triangleright := \nabla + i\mathfrak{m}, \quad D_\triangleleft := \nabla - i\mathfrak{m}, \quad (3.9)$$

both of which are first-order linear partial differential operators acting on $C^\infty(D_\ell M)$ having the same principal part. Moreover, they have the properties:

$$\Gamma D_\triangleright = -D_\triangleright \Gamma, \quad D_\triangleright D_\triangleleft = D_\triangleleft D_\triangleright, \quad \text{and} \quad (D_\triangleright f, f') = -(f, D_\triangleright f'), \quad f, f' \in C_0^\infty(D_\ell M),$$

and similar relations hold when replacing D_\triangleright by D_\triangleleft . Another property, entailed by the relations (Clifford relations) for the generators of $\mathcal{M}(1, m-1)$, is that $P = D_\triangleright D_\triangleleft$ is a wave operator on $D_\ell M$ which fulfills the hermiticity condition (3.6).

The following proposition is a trivial generalization of similar statements in [11] for the four-dimensional case; we refer to that reference for the proof.

⁴Note that, at this point, the indices μ, ν are frame-indices, while elsewhere they are coordinate-indices

Proposition 3.4. [11] *Let $D_{\triangleright}, D_{\triangleleft}$ be the Dirac operators on $D_\ell M$ defined above. Define*

$$S_{\triangleright}^{\pm} := D_{\triangleright} E^{\pm} \quad \text{and} \quad S_{\triangleright} := S_{\triangleright}^{+} - S_{\triangleright}^{-},$$

where E^{\pm} are the advanced/retarded fundamental solutions of the wave-operator $P = D_{\triangleright} D_{\triangleleft}$. Then it holds that S_{\triangleright}^{\pm} is the unique advanced/retarded fundamental solution of D_{\triangleleft} , i.e. the unique continuous operator from $C_0^{\infty}(\mathfrak{Y})$ to $C^{\infty}(\mathfrak{Y})$ so that

$$D_{\triangleleft} S_{\triangleright}^{\pm} f = S_{\triangleright}^{\pm} D_{\triangleleft} f = f \quad \text{and} \quad \text{supp}(S_{\triangleright}^{\pm} f) \subset J^{\pm}(\text{supp } f), \quad f \in C_0^{\infty}(\mathfrak{Y}).$$

Moreover, it follows that $\Gamma S_{\triangleright}^{\pm} = -S_{\triangleright}^{\pm} \Gamma$, and

$$(S_{\triangleright} f, f') = (f, S_{\triangleright} f') \quad \text{and} \quad (f, S_{\triangleright} f) \geq 0, \quad f, f' \in C_0^{\infty}(\mathfrak{Y}).$$

4 Quantum fields, CAR and CCR

The present section serves to explain what it means that a quantum field satisfies CAR or CCR. First, however, we have to make precise the idea of a vector-valued quantum field on a spacetime (M, g) :

Let \mathfrak{Y} be a vector bundle over the base manifold M , carrying a fibrewise conjugation Γ . A quantum field is then a collection of objects $\{\Phi, \mathcal{D}, \mathcal{H}\}$, where \mathcal{H} is a Hilbert-space, \mathcal{D} is a dense subspace of \mathcal{H} and Φ is an operator valued distribution having domain \mathcal{D} . That is to say, $\Phi(f)$ is for each f in $C_0^{\infty}(\mathfrak{Y})$ a closable operator with domain \mathcal{D} and \mathcal{D} is left invariant under application of $\Phi(f)$. Moreover, for all $\psi, \psi' \in \mathcal{D}$, the map

$$C_0^{\infty}(\mathfrak{Y}) \ni f \mapsto \langle \psi, \Phi(f) \psi' \rangle$$

is in $(C_0^{\infty}(\mathfrak{Y}))'$. We also require that

$$\Phi(\Gamma f) \subset \Phi(f)^* \quad \text{for all } f \in C_0^{\infty}(\mathfrak{Y}),$$

where $\Phi(f)^*$ denotes the adjoint operator of $\Phi(f)$.

Let w be a distribution in $(C_0^{\infty}(\mathfrak{Y} \boxtimes \mathfrak{Y}))'$. One defines the symmetric ($w^{(+)}$) and antisymmetric ($w^{(-)}$) part of w by

$$w^{(\pm)}(f \otimes f') = \frac{1}{2} (w(f \otimes f') \pm w(f' \otimes f))$$

and continuous linear continuation to $C_0^{\infty}(\mathfrak{Y} \boxtimes \mathfrak{Y})$. To say that a quantum field satisfies CAR or CCR amounts to specifying the symmetric or antisymmetric part, respectively of the two-point functions

$$w_2^{(\psi)}(f \otimes f') = \langle \psi, \Phi(f) \Phi(f') \psi \rangle_{\mathcal{H}}, \quad f, f' \in C_0^{\infty}(\mathfrak{Y}),$$

independently of $\psi \in \mathcal{D}$, $\|\psi\| = 1$. (“c-number commutation relations”.) To describe this more concretely, we introduce CAR- and CCR-structures.

CAR case: We assume that there is a complex Hilbert-space $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ carrying a conjugation C , together with a continuous linear map

$$q_{\mathcal{V}} : C_0^{\infty}(\mathfrak{A}) \rightarrow \mathcal{V}$$

having a dense range, such that $C \circ q_{\mathcal{V}} = q_{\mathcal{V}} \circ \Gamma$. Relative to such a *CAR-structure*, we say that the quantum field Φ satisfies the CAR if

$$w_2^{(\psi)(+)}(f \otimes f') = \langle Cq_{\mathcal{V}}(f), q_{\mathcal{V}}(f') \rangle_{\mathcal{V}}, \quad f, f' \in C_0^{\infty}(\mathfrak{A}),$$

holds for all unit vectors $\psi \in \mathcal{D}$.

CCR case: Here we assume that there is a (real-linear) symplectic space $(\mathcal{S}, \sigma(\cdot, \cdot))$ and a real-linear, symplectic map

$$q_{\mathcal{S}} : C_0^{\infty}(\mathfrak{A}^{\circ}) \rightarrow \mathcal{S}.$$

Relative to this *CCR-structure*, we say that the quantum field Φ satisfies the CCR if

$$w_2^{(\psi)(-)}(f \otimes f') = \sigma(q_{\mathcal{S}}(f), q_{\mathcal{S}}(f')), \quad f, f' \in C_0^{\infty}(\mathfrak{A}),$$

holds for all unit vectors $\psi \in \mathcal{D}$.

One can, instead of using quantum fields, alternatively consider states on the Borchers-algebra [2] over the test-section space $C_0^{\infty}(\mathfrak{A})$. Since we have presented this approach in [34], we won't discuss this here. Instead, we very briefly sketch the C^* -algebraic variant of CAR and CCR which shows how quantum fields may be constructed from states on C^* -algebras associated with CAR- or CCR-structures.

We begin with the CAR case. Let a CAR-structure $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}}, C, q_{\mathcal{V}})$ be given. Then one can form the corresponding self-dual CAR-algebra $\mathcal{B}(\mathcal{V}, C)$ [1], which is the C^* -algebra with unit 1 generated by a family of elements $\{B(v) : v \in \mathcal{V}\}$ with the relations

- (i) $v \mapsto B(v)$ is \mathbb{C} -linear,
- (ii) $B(v)^* = B(Cv), \quad v \in \mathcal{V},$
- (iii) $B(v)^*B(w) + B(w)B(v)^* = \langle v, w \rangle_{\mathcal{V}} \cdot 1, \quad v, w \in \mathcal{V}.$

(There is a unique C^* -norm compatible with these relations.) Now let ω be any state, i.e. a positive ($\omega(B^*B) \geq 0$), normalized ($\omega(1) = 1$) linear functional on $\mathcal{B}(\mathcal{V}, C)$. Then let $(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega})$ be the corresponding GNS-representation⁵. It induces a quantum field $\{\Phi, \mathcal{D}, \mathcal{H}\}$ as follows. Take $\mathcal{H} = \mathcal{H}_{\omega}$, and define $\Phi(f)$ by

$$\Phi(f) := \pi_{\omega}(B(q_{\mathcal{V}}(f))), \quad f \in C_0^{\infty}(\mathfrak{A}). \quad (4.1)$$

⁵ We recall here the following fact. Let ω denote a positive, normalized linear functional on a C^* -algebra \mathcal{A} with unit 1. Then there exists a triple $(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega})$, called GNS-representation of ω where \mathcal{H}_{ω} is a complex Hilbert-space, π_{ω} is a $*$ -representation of \mathcal{A} by bounded operators on \mathcal{H}_{ω} , and Ω_{ω} is a unit vector in \mathcal{H} which is cyclic for π_{ω} ($\pi_{\omega}(\mathcal{A})\Omega_{\omega}$ is dense in \mathcal{H}_{ω}), with the property that $\omega(A) = \langle \Omega_{\omega}, \pi_{\omega}(A)\Omega_{\omega} \rangle$ for all $A \in \mathcal{A}$. The triple $(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega})$ is unique up to unitary equivalence.

As domain \mathcal{D} one may take $\mathcal{P}\Omega_\omega$, where \mathcal{P} is the set of all polynomials in the $\Phi(f)$, $f \in C_0^\infty(\mathfrak{Y})$. One could as well take $\mathcal{D} = \mathcal{H}_\omega$ since the $\Phi(f)$ are bounded operators as consequence of the CAR. It is then straightforward to see that this quantum field satisfies the CAR. Note that Φ depends on the chosen state ω , and each state ω induces via (4.1) a quantum field satisfying the CAR.

We can associate with any state ω on $\mathcal{B}(\mathcal{V}, C)$ its two point function ω_2 , defined by

$$\omega_2(f \otimes f') := \langle \Omega_\omega, \Phi(f)\Phi(f')\Omega_\omega \rangle, \quad f, f' \in C_0^\infty(\mathfrak{Y}),$$

with $\Phi(f)$ as in (4.1); then ω_2 is an element of $(C_0^\infty(\mathfrak{Y} \boxtimes \mathfrak{Y}))'$.

Next, we turn to the CCR-case. Let $(\mathcal{S}, \sigma, q_{\mathcal{S}})$ be a CCR-structure. Then let $\mathcal{W}(\mathcal{S}, \sigma)$ be the CCR- or Weyl-algebra associated with the symplectic space (\mathcal{S}, σ) . This is the C^* -algebra with unit 1 generated by a family of elements $\{W(\phi) : \phi \in \mathcal{S}\}$ with relations

- (i) $W(\phi)^* = W(-\phi) = W(\phi)^{-1}$,
- (ii) $W(\phi)W(\psi) = e^{-i/2\sigma(\phi, \psi)}W(\phi + \psi)$.

(Also in this case there is a unique C^* -norm compatible with these relations.) Now let ω be a state on $\mathcal{W}(\mathcal{S}, \sigma)$ with corresponding GNS-representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$. This state is called *regular* if, for each $\phi \in \mathcal{S}$, the unitary group $\mathbb{R} \ni t \mapsto \pi_\omega(W(t\phi))$ is strongly continuous. Consequently, we have for each $\phi \in \mathcal{S}$ a selfadjoint generator $R(\phi)$ so that $\pi_\omega(W(t\phi)) = \exp(itR(\phi))$. However, in order to ensure that there is a dense common invariant domain for all $R(\phi)$, $\phi \in \mathcal{S}$, and moreover, to obtain a quantum field, one needs to impose a stronger regularity condition. We say that ω is *C^∞ -regular* if, for all $N \in \mathbb{N}$, the map

$$\mathbb{R}^N \times C_0^\infty(\mathfrak{Y}^\circ)^N \ni (\vec{t}, \vec{f}) \mapsto \omega(W(q_{\mathcal{S}}(t_1 f_1)) \cdots W(q_{\mathcal{S}}(t_N f_N)))$$

is C^∞ in \vec{t} and if it is, together with all partial \vec{t} -derivatives, continuous in \vec{f} . Note that this requires that $f, f' \mapsto \sigma(q_{\mathcal{S}}(f), q_{\mathcal{S}}(f'))$ is continuous. Given a C^∞ -regular state ω on $\mathcal{W}(\mathcal{S}, \sigma)$, we obtain a quantum field $\{\Phi, \mathcal{D}, \mathcal{H}\}$ from the GNS-representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ via setting $\mathcal{H} = \mathcal{H}_\omega$, $\mathcal{D} = \mathcal{P}\Omega_\omega$ where \mathcal{P} is the set of polynomials in the $R(\phi)$, $\phi \in \mathcal{S}$ and

$$\Phi(f) = R(q_{\mathcal{S}}(f)), \quad f \in C_0^\infty(\mathfrak{Y}^\circ). \quad (4.2)$$

(Then $\Phi(f) = 1/2(\Phi(f + \Gamma f) + i\Phi(i(f - \Gamma f)))$ for all $f \in C_0^\infty(\mathfrak{Y})$.) We remark that, as in the CAR-case, the quantum field depends on the choice of a C^∞ -regular state ω . There exist very many C^∞ -regular states for $\mathcal{W}(\mathcal{S}, \sigma)$ once $f, f' \mapsto \sigma(q_{\mathcal{S}}(f), q_{\mathcal{S}}(f'))$ is continuous, in particular every quasifree state on $\mathcal{W}(\mathcal{S}, \sigma)$ is C^∞ -regular.

Similarly as above, we associate with any C^∞ -regular state ω on $\mathcal{W}(\mathcal{S}, \sigma)$ its two-point function,

$$\omega_2(f \otimes f') := \langle \Omega_\omega, \Phi(f)\Phi(f')\Omega_\omega \rangle, \quad f, f' \in C_0^\infty(\mathfrak{Y}),$$

where $\Phi(f)$ is defined by (4.2). Again ω_2 induces a distribution in $(C_0^\infty(\mathfrak{Y} \boxtimes \mathfrak{Y}))'$.

Finally, we indicate how wave-operators and Dirac operators induce CCR-structures and

CAR-structures, respectively.

Wave operator/CCR case: We assume that \mathfrak{V} is a hermitean vector bundle, with typical fibre \mathbb{C}^r , and base manifold M so that (M, g) is a globally hyperbolic spacetime of dimension $m \geq 3$. Furthermore, we suppose that P is a wave operator acting on the smooth sections in the vector bundle satisfying the hermiticity condition (3.6). Let $E := E^+ - E^-$ where E^\pm are the unique advanced/retarded fundamental solutions of P , cf. Prop. 3.3. Then define

$$\begin{aligned} \mathfrak{S} &:= C_0^\infty(\mathfrak{V}^\circ)/\ker E, \quad q_{\mathfrak{S}}(f) := f + \ker E, \\ \sigma(q_{\mathfrak{S}}(f), q_{\mathfrak{S}}(f')) &:= (f, Ef'), \quad f, f' \in C_0^\infty(\mathfrak{V}^\circ), \end{aligned}$$

where (\cdot, \cdot) is the hermitean form on $C_0^\infty(\mathfrak{V}^\circ)$ introduced in (3.5). We call the thus defined CCR-structure the *CCR structure induced by P* .

Dirac operator/CAR case: Let (M, g) be a globally hyperbolic spacetime of dimension $m = 3, 4, 9, 10 \pmod{8}$ and let $D_\ell M$ be the associated bundle of Majorana spinors. Moreover, let $S_\triangleright = S_\triangleright^+ - S_\triangleright^-$ where S_\triangleright^\pm are the advanced/retarded fundamental solutions of the operator D_\triangleleft , cf. Prop. 3.4. Then define the *CAR structure induced by D_\triangleleft* by setting

$$\begin{aligned} q_{\mathcal{V}} &: C_0^\infty(\mathfrak{V}) \rightarrow C_0^\infty(\mathfrak{V})/\ker S_\triangleright, \quad q_{\mathcal{V}}(f) := f + \ker S_\triangleright, \\ \langle q_{\mathcal{V}}(f), q_{\mathcal{V}}(f') \rangle_{\mathcal{V}} &= (f, S_\triangleright f'), \quad Cq_{\mathcal{V}}(f) := q_{\mathcal{V}}(\Gamma f), \\ \mathcal{V} &:= \text{completion of } C_0^\infty(\mathfrak{V})/\ker S_\triangleright \text{ w.r.t. } \langle \cdot, \cdot \rangle. \end{aligned}$$

Here (\cdot, \cdot) is the hermitean form on $C_0^\infty(\mathfrak{V})$ introduced in (3.8).

Finally, it should be noted that the quantum fields associated with these CAR and CCR structures satisfy the Dirac-equation and the wave-equation, respectively. That is, if $(\mathfrak{S}, \sigma, q_{\mathfrak{S}})$ is the CCR-structure induced by the wave-operator P and the quantum field Φ is defined as in (4.2), then

$$\Phi(Pf) = 0, \quad f \in C_0^\infty(\mathfrak{V}), \quad (4.3)$$

and if $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}}, q_{\mathcal{V}})$ is the CAR structure induced by the Dirac operator D_\triangleleft , and Φ is defined as in (4.1), then

$$\Phi(D_\triangleleft f) = 0, \quad f \in C_0^\infty(\mathfrak{V}). \quad (4.4)$$

5 Hadamard forms, Hadamard states

5.1 Definition of Hadamard forms and Hadamard states

Our next task is to give the definition of Hadamard forms and of Hadamard states. Our definition follows that given by Kay and Wald [25] (for bosonic fields; the formulation for fermionic fields is an adaptation of the approach in [25] together with the notion of Hadamard form for Dirac fields in [31] which in similar form appeared in [27] and [40]). The definition of Hadamard forms in full detail is unfortunately somewhat laborious. We

proceed by first collecting the definitions of various notions entering into the definition of Hadamard forms; however, we relegate the full definition of the coefficient sections determined by the Hadamard recursion relations for the wave operator P (taken from [19]) to the Appendix.

We suppose that we are given a hermitean vector bundle \mathfrak{V} over a globally hyperbolic spacetime manifold (M, g) ($m := \dim M \geq 3$), together with a wave-operator P on $C_0^\infty(\mathfrak{V})$ fulfilling the hermiticity condition (3.6). Γ denotes a fibrewise conjugation on \mathfrak{V} commuting with P .

- (a) *Causally normal related points:* A convex normal neighbourhood in M is an open domain U in M such that for each pair of points $p, q \in U$ there is a unique geodesic segment contained in U which connects p and q . We denote by \mathcal{X} the set of all those $(p, q) \in M \times M$ which are causally related and for which $J^+(p) \cap J^-(q)$ and $J^-(p) \cap J^+(q)$ are contained in a convex normal neighbourhood in M .
- (b) *Causal normal neighbourhoods:* According to [25], an open neighbourhood N of a Cauchy-surface Σ in (M, g) is called a *causal normal neighbourhood* of Σ if Σ is a Cauchy-surface for N and if for each choice of $p, q \in N$ with $p \in J^+(q)$, there exists a convex normal neighbourhood in M in which $J^-(p) \cap J^+(q)$ is contained. It is shown in [25] that each Cauchy-surface possesses causal normal neighbourhoods.
- (c) *Squared geodesic distance, Hadamard coefficient sections:* There is an open neighbourhood \mathcal{U} of \mathcal{X} on which $s(p, q)$, the squared geodesic distance between p and q (signed, such that $s(p, q) > 0$ for p, q spacelike, $s(p, q) \leq 0$ for p, q causally related), is well defined and smooth. Moreover, \mathcal{U} may be chosen so that there are smooth sections $U, V^{(n)}$ and $T^{(n)}$, $n \in \mathbb{N}$, in $C^\infty((\mathfrak{V} \boxtimes \mathfrak{V}^*)_{\mathcal{U}})$ which are uniquely determined by the ‘Hadamard recursion relations’ for the wave operator P , see Appendix A.1 for precise definition (taken from [19]). These are called the *Hadamard coefficient sections* for P . If \mathcal{U} has been chosen in the described way, then we call \mathcal{U} a *regular domain*.
- (d) *N -regularizing functions:* Let N be a causal normal neighbourhood of a Cauchy-surface. A smooth function $\chi : N \times N \rightarrow [0, 1]$ will be called *N -regularizing* if there is a regular domain \mathcal{U} , and an open neighbourhood $\mathcal{U}_* \subset N \times N$ of the set of pairs of causally related points in N with $\overline{\mathcal{U}_*} \subset \mathcal{U}$, such that $\chi \equiv 1$ on \mathcal{U}_* and $\chi \equiv 0$ outside of \mathcal{U} . The sets $\mathcal{U}_*, \mathcal{U}$ are then called the *domain pair* corresponding to χ . It can be shown that N -regularizing functions exist, a proof is given in [32].
- (e) *Time-functions:* A smooth function $t : M \rightarrow \mathbb{R}$ is called a *time-function* if its gradient is a future-directed timelike vector field normalized to 1.

We also have to define some distributions on M . To this end, let N be a causal normal neighbourhood, t a time-function on N , and $\varepsilon > 0$. Moreover, let χ be an N -regularizing function with domain pair $\mathcal{U}_*, \mathcal{U}$. Then we define the smooth function (with $m = \dim M$)

$$\chi G_\varepsilon^{(1)}(x, y) := \beta^{(1)} \chi(x, y) (s(x, y) - i2\varepsilon(t(x) - t(y)) + \varepsilon^2)^{-\frac{m}{2}+1} \quad (5.1)$$

with support in $N \times N$. In case m is even, we also define

$$\chi G_\varepsilon^{(2)}(x, y) := \beta^{(2)} \chi(x, y) \ln (s(x, y) - i2\varepsilon(t(x) - t(y)) + \varepsilon^2) \quad (5.2)$$

where the branch cut of the logarithm is taken along the negative real line. The constants $\beta^{(1)}, \beta^{(2)}$ in the above formulas are given by⁶

$$\beta^{(1)} = \frac{1}{2} \begin{cases} (-1)^{\frac{m+1}{2}} \pi^{\frac{2-m}{2}} \left[\Gamma\left(\frac{4-m}{2}\right) \right]^{-1} & \text{for } m \text{ odd} \\ -\pi^{-\frac{m}{2}} \Gamma\left(\frac{m}{2} - 1\right) & \text{for } m \text{ even} \end{cases}, \quad \beta^{(2)} = (-1)^{\frac{m}{2}} 2^{1-m} \pi^{-\frac{m}{2}} \left[\Gamma\left(\frac{m}{2}\right) \right]^{-1}.$$

We define distributions on $C_0^\infty(M \times M)$ by

$$\chi G^{(2)}(F) = \lim_{\varepsilon \rightarrow 0^+} \iint \chi G_\varepsilon^{(2)}(p, q) F(p, q) d\mu(p) d\mu(q), \quad F \in C_0^\infty(M \times M). \quad (5.3)$$

For an account of some properties of these distributions, see Appendix A.3.

Now we can formulate the notion of Hadamard form:

Definition 5.1. We say that $w \in (C_0^\infty(\mathfrak{V} \boxtimes \mathfrak{V}))'$ is of *Hadamard form on N* for the wave operator P if there are

- an N -regularizing function χ with corresponding domain pair $\mathcal{U}_*, \mathcal{U}$ (implying that the square of the geodesic distance s and the Hadamard coefficient sections $U, T^{(n)}, V^{(n)}, n \in \mathbb{N}$, for P are well-defined and smooth on \mathcal{U}),
- a time-function t ,
- for each $n \in \mathbb{N}$ an $H^{(n)} \in C^n((\mathfrak{V} \boxtimes \mathfrak{V}^*)_{N \times N})$

such that for all $f, f' \in C_0^\infty(\mathfrak{V}_N)$ in case m odd:

$$w(\Gamma f \otimes f') = \chi G^{(1)}((\vartheta f) T^{(n)} f') + \int (\vartheta f)(p) H^{(n)}(p, q) f'(q) d\mu(p) d\mu(q),$$

and for m even:

$$w(\Gamma f \otimes f') = \chi G^{(1)}((\vartheta f) U f') + \chi G^{(2)}((\vartheta f) V^{(n)} f') + \int (\vartheta f)(p) H^{(n)}(p, q) f'(q) d\mu(p) d\mu(q),$$

where we have used abbreviations like

$$((\vartheta f) T^{(n)} f')(p, q) := (\vartheta f)_a(p) T^{(n)a}_b(p, q) f'^b(q)$$

to denote the function in $C_0^\infty(N \times N)$ resulting from contracting $\vartheta f \otimes f'$ pointwise with $T^{(n)}$. Here, ϑ is the antilinear base-point preserving bundle-morphism from \mathfrak{V} onto \mathfrak{V}^* induced by the hermitean form as in (3.4), and we have written ϑ also in places where we should have written ϑ^* in order to simplify notation.

⁶the Γ appearing here denotes the Gamma-function, not the conjugation on the underlying vector bundle introduced before

The notion of Hadamard form seems to depend on the choice of the time-function t and the N -regularizing function χ ; however, that turns out not to be the case. The difference of a distribution w which is of Hadamard form relative to an N -regularizing function χ and a distribution w' of Hadamard form relative to an N -regularizing function χ' is C^∞ because χ and χ' are both equal to 1 in a neighbourhood of the singular support of w and w' . Thus w is Hadamard relative to χ' and w' relative to χ , as well — a different choice of χ may be absorbed into a different choice of the $H^{(n)}$. In Appendix A.3 we will use an argument similar to that in [25] for the case of scalar fields to show that, given a causal normal neighbourhood N of a Cauchy-surface, the definition of Hadamard form is independent of the choice of the time-function t that entered into the definition.

Moreover, a solution w of the wave-equation mod C^∞ which is of Hadamard form has another remarkable property, known as “propagation of the Hadamard form”. We will turn to this in the subsequent section.

We are now ready to present our definition of Hadamard states associated with the wave operator P in the CCR case.

Definition 5.2. Let $(\mathcal{S}, \sigma, q_\sigma)$ be the CCR-structure induced by the wave operator P and ω a C^∞ -regular state on the CCR-algebra $\mathcal{W}(\mathcal{S}, \sigma)$. We say that ω is a *Hadamard state* if there is a causal normal neighbourhood N of a Cauchy-surface in (M, g) so that ω_2 (the two-point function of ω) is of Hadamard form.

The CAR case needs slightly different assumptions. Let (M, g) be a globally hyperbolic spacetime of dimension $m = 3, 4, 9, 10 \pmod{8}$, and let $D_\ell M$ be the corresponding bundle of Majorana spinors, and $D_\triangleright, D_\triangleleft$ as in (3.9).

Definition 5.3. Let $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}}, C, q_{\mathcal{V}})$ be the CAR structure induced by D_\triangleleft , and ω a state on the CAR-algebra $\mathcal{B}(\mathcal{V}, C)$. Then we call ω a *Hadamard state* if there is a causal normal neighbourhood of a Cauchy-surface in (M, g) , and a $w \in C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V})_{N \times N})'$ of Hadamard form on N for the wave operator $P = D_\triangleright D_\triangleleft$ so that

$$\omega_2(f \otimes f') = w(D_\triangleright f \otimes f'), \quad f, f' \in C_0^\infty(\mathfrak{V}_N)$$

holds for the two point function ω_2 of ω .

We note several things in connection with this definition. First, these definitions of Hadamard state seem to depend on the choice of a causal normal neighbourhood N , but the next section will show that this is not the case.

Moreover, for reasons of consistency with the CCR- and CAR-structures one has to check the following necessary conditions.

Lemma 5.4. (a) *In the wave operator/CCR-case: Given a causal normal neighbourhood N , there is a Hadamard form w on N for the wave operator P so that*

$$w^{(-)}(\Gamma f \otimes f') = i(f, E f'), \quad f, f' \in C_0^\infty(\mathfrak{V}_N). \quad (5.4)$$

(b) *In the Dirac operator/CAR-case: Given a causal normal neighbourhood N , there is a Hadamard form w on N for the wave operator $P = D_\triangleright D_\triangleleft$ so that*

$$w^{(+)}(\Gamma D_\triangleright f \otimes f') = -i(f, S_\triangleright f'), \quad f, f' \in C_0^\infty(D_\ell M_N). \quad (5.5)$$

(c) Let N be a causal normal neighbourhood of any Cauchy-surface, and suppose that $w \in (C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V})_{N \times N}))'$ is of Hadamard form for the wave-operator P on N . Then w is a bisolution mod C^∞ for P on N .

Drawing on results of [15, 19], it will be shown in Appendix A.4 that Hadamard forms possess the claimed properties.

However, we should point out that these properties of Hadamard forms, while enough for our purposes later, are really only necessary conditions for the existence of Hadamard states. First, a further condition is imposed by the positivity of a state, $\omega(A^*A) \geq 0$, for all $A \in \mathcal{W}(\mathcal{S}, \sigma)$ or all $A \in \mathcal{B}(\mathcal{V}, C)$. At the level of two-point functions, this implies

$$\omega_2(\Gamma f \otimes f)\omega_2(\Gamma f' \otimes f') \geq |\omega_2(\Gamma f \otimes f')|^2 \quad (5.6)$$

for all test-sections f and f' , both in the CCR and CAR case. Moreover, two-point functions ω_2 are *proper* bisolutions — not just mod C^∞ — for the wave-operator in the CCR case, and for the Dirac operator in the CAR case.

Since one may construct quasifree states on $\mathcal{W}(\mathcal{S}, \sigma)$ or $\mathcal{B}(\mathcal{V}, C)$ from two point functions, the question whether Hadamard states exist is equivalent to the question of whether there are Hadamard forms w which are proper bisolutions of the wave-operator or the Dirac operator satisfying (5.4) or (5.5), respectively, together with the property $w(\Gamma f \otimes f)w(\Gamma f' \otimes f') \geq |w(\Gamma f \otimes f')|^2$ for all test-sections f and f' . That question has been answered in the affirmative for the scalar field case in the work [17]. The argument of [17] rests on a “spacetime deformation argument”, i.e. the property of any globally hyperbolic spacetime to possess a “deformed copy” (\tilde{M}, \tilde{g}) which is again globally hyperbolic and coincides with (M, g) on a causal normal neighbourhood N of any given Cauchy-surface while being ultrastatic in the past of N . On the ultrastatic part of (\tilde{M}, \tilde{g}) , one may construct again the CCR-algebra of the Klein-Gordon field together with a stationary ground state which can be shown to be Hadamard. This state induces via the dynamics of the field (i.e. owing to Prop. 3.3) a state of the Klein-Gordon field on N and thus on (M, g) , and by the propagation of Hadamard form, this state is then found to be Hadamard.

We don’t see any obstruction to generalizing that method to tensor-fields and Dirac fields; however for more general vector fields the problem arises if the vectorbundle \mathfrak{V} on M possesses, in a suitable sense, a “deformed copy” $\tilde{\mathfrak{V}}$ on the deformed copy (\tilde{M}, \tilde{g}) of (M, g) . We won’t consider that problem in our present work.

What seems worth mentioning is that, as we will see in the next section, the positivity condition (5.6) forces the hermitean form h on \mathfrak{V} to be positive, in the wave operator/CCR case.

5.2 Propagation of Hadamard form

In this section we are going to present the propagation of Hadamard form under the dynamics of a wave-operator on sections of a vector bundle. It is a very straightforward generalization of an analogous result for the case of scalar fields, which has been established in a first version by Fulling, Sweeny and Wald [18] and, for the present notion of “global” Hadamard form, by Kay and Wald [25]. The main reason for presenting the propagation of Hadamard form result here is that, in contrast to a claim made in [32]

(regarding the same issue for the scalar field case), it will turn out to be needed in establishing the characterization of Hadamard states in terms of the wavefront sets of their two-point functions. Below, in the proof of Thm. 5.8, we will explain why the argument in [32] (and similarly a related argument in [28]) contains a gap, which will be closed by employing the propagation of Hadamard form.

The assumptions on $P, \mathfrak{V}, \Gamma, (M, g)$ etc. are the same as in Sec. 5.1.

Theorem 5.5. [18, 17, 25] (*Propagation of Hadamard form*)

Let $w \in (C_0^\infty(\mathfrak{V} \boxtimes \mathfrak{V}))'$ be a bisolution mod C^∞ for the wave-operator P . Moreover, assume that there is a Cauchy-surface Σ in (M, g) having a causal normal neighbourhood N so that w is of Hadamard form for the wave-operator on N .

Then, if N' is a causal normal neighbourhood of any other Cauchy-surface Σ' in (M, g) , w is also of Hadamard form for the wave-operator P on N' .

Proof. Since there is essentially no deviation from the proof of this statement given for the scalar case in the works [18, 17, 25], we shall be content with giving only a sketch of the proof.

Let Σ' be another Cauchy-surface with causal normal neighbourhood N' . We assume first that $\Sigma' \subset \text{int } J^+(\Sigma)$. Let $\Sigma'_\# \subset \Sigma'$ have compact closure and set $K' = \text{int } D(\Sigma'_\#) \cap N'$. Then choose any open, relatively compact neighbourhood $\Sigma_\#$, in Σ , of $J^-(\Sigma'_\#) \cap \Sigma$. Denote the set $\text{int } D(\Sigma_\#) \cap N$ by K , and denote the set $\text{int}(J^+(\Sigma) \cap J^-(\Sigma'))$ by $M(\Sigma, \Sigma')$. Then $M(\Sigma, \Sigma')$, endowed with the appropriate restriction of g as spacetime metric, is a globally hyperbolic sub-spacetime of (M, g) . Thus there is a foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ of $M(\Sigma, \Sigma')$ in Cauchy-surfaces. Each Σ_t possesses a causal normal neighbourhood N_t in $M(\Sigma, \Sigma')$, so $\{N_t\}_{t \in \mathbb{R}}$ forms an open covering of $M(\Sigma, \Sigma')$. Now consider two causal normal neighbourhoods \tilde{N} and \tilde{N}' of Σ and Σ' (in M) such that their closures are contained in N and N' , respectively. Then let

$$\mathcal{C} = \text{cl}[(\text{int } D(\Sigma_\#) \cap M(\Sigma, \Sigma')) \setminus \text{cl}(\tilde{N} \cup \tilde{N}')].$$

We write \mathcal{C}° for the open interior of \mathcal{C} ; it is also a globally hyperbolic sub-spacetime. Now \mathcal{C} is a compact subset of $M(\Sigma, \Sigma')$ and hence there is a finite subfamily N_1, \dots, N_k of $\{N_t\}_{t \in \mathbb{R}}$ covering \mathcal{C} . It is not very difficult to see that one may choose such a family with the following properties: (1) $\Sigma_{j+1} \subset \text{int } J^+(\Sigma_j)$ holds for the corresponding Cauchy-surfaces of which the N_j are causal normal neighbourhoods; (2) For all $j = 1, \dots, k-1$ there is some $t(j) \in \mathbb{R}$ with $(N_j \cap N_{j+1}) \cap \mathcal{C}^\circ \supset \Sigma_{t(j)} \cap \mathcal{C}^\circ$; (3) $N_1 \cap \mathcal{C}^\circ$ covers $K \cap \mathcal{C}^\circ$ and $N_k \cap \mathcal{C}^\circ$ covers $K' \cap \mathcal{C}^\circ$ (by enlarging N_1, N_k, \tilde{N} and \tilde{N}' if necessary).

Now by assumption, w is of Hadamard form on N , and thus certainly w is of Hadamard form when restricted to K (that is, w restricted to $C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V}))_{K \times K}$). By construction, $N_1 \cap \mathcal{C}^\circ$ covers the part $K \cap \mathcal{C}^\circ$ of K . On the other hand, $N_1 \cap \mathcal{C}^\circ$ is a globally hyperbolic sub-spacetime of $M(\Sigma, \Sigma')$, so there is a Hadamard form w_1 on $N_1 \cap \mathcal{C}^\circ$. Therefore, on $K \cap \mathcal{C}^\circ$ we have $w = w_1 \text{ mod } C^\infty$. Now $N \setminus \text{cl}(\tilde{N})$ contains a Cauchy-surface $\tilde{\Sigma}$ for $M(\Sigma, \Sigma')$ (owing to the properties of causal normal neighbourhoods). And hence, since w_1 is a Hadamard form on $N_1 \cap \mathcal{C}^\circ$ and thus a bisolution of the wave-operator mod C^∞ , as likewise is w by assumption, this implies that $w = w_1 \text{ mod } C^\infty$ on $\text{int } D(\tilde{\Sigma} \cap \mathcal{C}^\circ)$, as follows by a straightforward generalization of Lemma A.2 in [17]. But this entails that $w = w_1 \text{ mod } C^\infty$ on all of $N_1 \cap \mathcal{C}^\circ$, and thus w is of Hadamard form $N_1 \cap \mathcal{C}^\circ$. From

here onwards, one iterates the just given argument to show inductively that if w is of Hadamard form on $N_j \cap \mathcal{C}^\circ$, then it is also of Hadamard form on $N_{j+1} \cap \mathcal{C}^\circ$, and hence w is of Hadamard form on all $N_j \cap \mathcal{C}^\circ$, $j = 1, \dots, k$; cf. the argument of ‘‘Cauchy-evolution in small steps’’ in [18]. Finally, since $N_k \cap \mathcal{C}^\circ$ covers the part $K' \cap \mathcal{C}^\circ$ of K' , one concludes in a like manner that w is also of Hadamard form on K' . And since the relatively compact set $\Sigma'_\# \subset \Sigma'$ entering in the definition of K' was arbitrary, this shows that w is of Hadamard form on all of N' .

This establishes the statement of the Theorem for the case that $\Sigma' \subset \text{int } J^+(\Sigma)$, but it is obvious that an analogous proof establishes the statement also in the case $\Sigma' \subset \text{int } J^-(\Sigma)$.

Now let Σ' be an arbitrary Cauchy-surface. Then one can choose a Cauchy-surface $\Sigma'' \subset \text{int}(J^-(\Sigma) \cap J^-(\Sigma'))$. One concludes first that w is of Hadamard form on any causal normal neighbourhood N'' of Σ'' , and then that w is of Hadamard form on N' . \square

5.3 Scaling limits

Next we shall determine the short distance scaling limits of Hadamard forms (and thereby, of Hadamard states); this also gives in combination with Prop. 2.8 some first information on their wavefront sets. Some notation needs to be introduced for this purpose.

Let Ω be a convex normal neighbourhood of a point p in M such that \mathfrak{V}_Ω trivializes. Ω can be covered by (inverse) normal coordinates $\xi_{p'}$, centered at p' , for any $p' \in \Omega$. The precise definition of these coordinates is given in Appendix A.2. Fixing $p \in \Omega$, we can now define dilations

$$\delta_\lambda(q) := \xi_p(\lambda \xi_p^{-1}(q)), \quad q \in \Omega, \lambda \in [0, 1]. \quad (5.7)$$

Let $(e_i)_{i=1\dots r}$ be a local frame for \mathfrak{V}_Ω . This frame induces a local bundle morphism D_λ covering δ_λ via

$$D_\lambda(q, v^i e_i(q)) = (\delta_\lambda(q), v^i e_i(\delta_\lambda(q))) \quad (5.8)$$

as well as a bundle morphism

$$R : \mathfrak{V}_\Omega \rightarrow \mathbb{R}^m \times \mathbb{C}^r, \quad (q, v^i(q) e_i(q)) \mapsto (\xi_p^{-1}(q), v^i(q) b_i)$$

where $(b_i)_{i=1\dots r}$ is the standard basis of \mathbb{C}^r . Furthermore, we can express the linear map $\vartheta \circ \Gamma|_p : \mathfrak{V}_p \rightarrow \mathfrak{V}_p^*$ as a matrix with respect to the basis $(e_i|_p)_{i=1,\dots,r}$ of \mathfrak{V}_p and its dual basis in \mathfrak{V}_p^* . This matrix will be denoted by $\Theta = (\Theta_{ab})_{a,b=1}^r$. Then we write

$$((\Theta R^* f) R^* f')(q, q') = \Theta_{ab} f^b(\xi_p(q)) f'^a(\xi_p(q')).$$

Now let $\alpha \in \mathbb{R}$. We define an action of the dilations on test sections by

$$(D_\lambda^{(\alpha)} f)(q) := \lambda^{-\alpha} (D_\lambda^* f)(q), \quad f \in C_0^\infty(\mathfrak{V}_\Omega).$$

We use this action to define scaling limits for distributions as described in Section 2.4.

The following result gives information about the scaling limit of a Hadamard state at $(p, p) \in M \times M$.⁷ For its formulation note that $G_\eta^{(1)}$ stands for the distribution $G^{(1)}$ taken with respect to the flat metric (“ $g = \eta$ ”) on the domain of ξ_p induced by the normal coordinates at p . The proof of this statement will be given in Appendix A.5.

Proposition 5.6. *Let $\alpha_1 = m/2 + 1$, $\alpha_2 = \alpha_1 - 1/2$ and ω be a quasifree Hadamard state fulfilling the CCR or CAR. For the corresponding two-point function ω_2 we have*

$$\begin{aligned} \text{CCR case:} \quad & \lim_{\lambda \rightarrow 0} \omega_2 \left(D_\lambda^{(\alpha_1)} f \otimes D_\lambda^{(\alpha_1)} f' \right) = G_\eta^{(1)} \left((\Theta R^* f) R^* f' \right) =: \omega_2^{(C)}(f \otimes f') \\ \text{CAR case:} \quad & \lim_{\lambda \rightarrow 0} \omega_2 \left(D_{\triangleright} D_\lambda^{(\alpha_2)} f \otimes D_\lambda^{(\alpha_2)} f' \right) = G_\eta^{(1)} \left((\gamma_\mu \partial^\mu \Theta R^* f) R^* f' \right) =: \omega_2^{(A)}(f \otimes f') \end{aligned}$$

The statement says that the scaling limit of the two-point function of a Hadamard state assumes the form of the two-point function for a multicomponent field obeying the massless Klein-Gordon equation (CCR-case) or the massless Dirac equation (CAR) on flat Minkowski space, apart from the appearance of the invertible matrix Θ .

Since we have assumed that ω is a state so that $\omega_2(\Gamma f \otimes f) \geq 0$, Θ cannot be completely arbitrary. In fact, in the CCR-case, $\Theta \cdot (\Gamma \circ \Gamma_0)$ must be a positive definite matrix, and hence the sesquilinear form h must be positive definite, i.e. h is a fibre bundle scalar product. Here, $\Gamma \circ \Gamma_0$ means the matrix obtained in the basis $(e_i|_p)$ from composing the conjugation $\Gamma : \mathfrak{Y}_p \rightarrow \mathfrak{Y}_p$ with the conjugation $\Gamma_0 : v^i e_i|_p \mapsto \bar{v}^i e_i|_p$.

Lemma 5.7. *For the above defined sesquilinear forms $\omega_2^{(C)}$ and $\omega_2^{(A)}$ on $C_0^\infty(\mathfrak{Y}_U)$ there holds*

$$(i) \quad (p, \xi; p, -\xi) \in \text{WF}(\omega_2^{(C)}) \text{ and}$$

$$(ii) \quad (p, \xi; p, -\xi) \in \text{WF}(\omega_2^{(A)})$$

for all $(p, \xi) \in \mathcal{N}_-$.

Proof. The claim (i) is easy to see: Since Θ is an invertible matrix, one can use Lemma 2.6 (for the case of a bundle morphism) to reduce the proof of the statement to the scalar case, where the claimed property is well-known (cf. [33, 32]⁸). To prove (ii), note first that again by Lemma 2.6 it is sufficient to show $(0, v; 0, -v) \in \text{WF}(u_2^{(A)})$ for each past pointing, lightlike v in Minkowski space, where⁹

$$u_2^{(A)}(f \otimes f') = \lim_{\epsilon \rightarrow 0^+} \int \frac{\delta_{ab} \left(\sum_{\mu=0}^{m-1} \gamma_\mu \partial_\mu f \right)^a (y) f'^b(y')}{-(y - y')^2 - 2i\epsilon(y^0 - y'^0) + \epsilon^2} d^m y d^m y', \quad f, f' \in \oplus^r C_0^\infty(\mathbb{R}^m).$$

⁷it is customary to call this also simply the “scaling limit at p ”; this is abuse of language according to the definition of scaling limit in Sec. 2.4: Note that the objects $q \in N$, \mathfrak{X} , D_λ^* of Sec. 2.4 correspond to $(p, p) \in M \times M$, $\mathfrak{Y} \boxtimes \mathfrak{Y}$, $D_\lambda^{(\alpha)} \otimes D_\lambda^{(\alpha)}$ here.

⁸Note however that in these references (p, ξ) is found to lie in \mathcal{N}_+ due to a different sign convention in the definition of a Hadamard form.

⁹we write $y^2 = \eta_{\mu\nu} y^\mu y^\nu$ for the squared Minkowskian distance in coordinates

Assume the wavefront set of $u_2^{(A)}$ were empty at the base-point $(0, 0)$. Then $u_2^{(A)}$ is C^∞ near $(0, 0)$ and we find that

$$\lim_{\lambda \rightarrow 0} \lambda^{-\alpha} u_2^{(A)}(f^{[\lambda]} \otimes f'^{[\lambda]}) = 0, \quad f, f' \in \oplus^r C_0^\infty(\mathbb{R}^m)$$

with $\alpha \leq 2m - 1$ and $f^{[\lambda]}(y) := f(\lambda^{-1}y)$. But $u_2^{(A)}$ is scale invariant, i.e.

$$\lambda^{3-2m} u_2^{(A)}(f^{[\lambda]} \otimes f'^{[\lambda]}) = u_2^{(A)}(f \otimes f')$$

for all $f, f' \in \oplus^r C_0^\infty(\mathbb{R}^m)$, $1 > \lambda > 0$, so that we are forced to conclude that $u_2^{(A)}(f \otimes f') = 0$ for all $f, f' \in \oplus^r C_0^\infty(\mathbb{R}^m)$. This entails

$$\begin{aligned} 0 &= u_2^{(A)}(\gamma_\rho f \otimes f') + u_2^{(A)}(f \otimes \gamma_\rho^T f') \\ &= \lim_{\epsilon \rightarrow 0^+} \int \frac{\delta_{ab} \partial_{y^\rho} f^a(y) f'^b(y')}{-(y - y')^2 - 2i\epsilon(y^0 - y'^0) + \epsilon^2} d^m y d^m y' \end{aligned}$$

for each $\rho = 0, \dots, m - 1$ and all $f, f' \in \oplus^r C_0^\infty(\mathbb{R}^m)$, which implies

$$0 = \lim_{\epsilon \rightarrow 0^+} \int \frac{\delta_{ab} (-\Delta f^a)(y) (-\Delta f'^b)(y')}{-(y - y')^2 - 2i\epsilon(y^0 - y'^0) + \epsilon^2} d^m y d^m y'$$

for all $f, f' \in \oplus^r C_0^\infty(\mathbb{R}^m)$, where Δ is the Euclidean Laplacian in \mathbb{R}^m . But this is clearly a contradiction since $(-\Delta) \otimes (-\Delta)$ is an elliptic differential operator and thus preserves the wavefront set of the distribution

$$f \otimes f' \mapsto \lim_{\epsilon \rightarrow 0^+} \int \frac{\delta_{ab} f^a(y) f'^b(y')}{-(y - y')^2 - 2i\epsilon(y^0 - y'^0) + \epsilon^2} d^m y d^m y'$$

and this is non-empty at coinciding base points as remarked above. Therefore, there are elements $(p, \xi; p, \xi') \in \text{WF}(\omega_2^{(A)})$. However, every such element must be of the form $\xi \in \mathcal{N}_-$, $\xi' = -\xi$ since this is so for $\omega_2^{(C)}$ and $\omega_2^{(A)}$ results from $\omega_2^{(C)}$ by application of a derivative operator. So, there is some $(p, \xi; p, -\xi)$, $\xi \in \mathcal{N}_-$ in $\text{WF}(\omega_2^{(A)})$. Now we use that $u_2^{(A)}$ is invariant under spatial coordinate rotations with respect to $y^0 = 0$ (i.e. rotations in the $y^0 = 0$ -hyperplane) together with Lemma 2.6 to conclude that each $(p, \xi; p, -\xi)$, $\xi \in \mathcal{N}_-$, is contained in $\text{WF}(\omega_2^{(A)})$. \square

5.4 Main Theorem

The following theorem generalizes the results on the equivalence of Hadamard form and microlocal spectrum condition, which have first been given by Radzikowski [32] for the scalar field case, and later by Köhler [27], by Kratzert [29] and by Hollands [21] for the case of Dirac fields, to fields that are sections in vector bundles, and fulfill the CCR or CAR. The arguments used are in part taken from [32] with some adaptations. However, we won't make use of the existence of 'distinguished parametrices' for the wave operator which was established in the scalar field case in [12]. And, as has been mentioned before, there is a gap in the arguments of [32]. A similar gap affects Cor. 1 in [28], and it also

affects the statements in [27, 29, 21] regarding the equivalence of Hadamard form and microlocal spectrum condition since the authors of these works rely on Radzikowski's main argument (as we shall also mostly do). We will explain in Remark (iii) below the statement of the next theorem where this gap occurs, and will repair it in our proof.

We also mention that the approach taken in the references [27, 29, 21] is slightly more general to the extent that, in contrast to our approach, it is not assumed in these references that the Dirac fields are Majorana fields (cf. the Remark at the end of Sec. 3.4). Thus, in these references Hadamard states are not automatically charge-conjugation invariant in the sense that $\omega_2(\Gamma f, \Gamma f') = \omega_2(f', f)$, as is the case here. That situation could be obtained, however, by considering appropriate 'doublings' of the field systems considered in the mentioned references.

The assumptions are the same as in Sec. 5.1.

Theorem 5.8. *Let ω be either:*

- *a C^∞ -regular state on the CCR-algebra $\mathcal{W}(\mathcal{S}, \sigma)$ associated with a CCR-structure induced by a wave operator P .*
- *a state on the CAR-algebra $\mathcal{B}(\mathcal{V}, C)$ associated with a CAR-structure induced by a Dirac operator D_\triangleleft .*

Then it holds that:

- (a) *If there is a causal normal neighbourhood N of a Cauchy-surface in (M, g) so that ω is a Hadamard state on N , then*

$$\text{WF}(\omega_2) = \mathcal{R} \tag{5.9}$$

where

$$\mathcal{R} = \{(q, \xi; q', \xi') \in \mathcal{N}_- \times \mathcal{N}_+ : (q, \xi) \sim (q', -\xi')\}. \tag{5.10}$$

- (b) *Conversely, if (5.9) holds, then ω is a global Hadamard state.*

Remark. (i) As will be obvious from the proof, one also has the following slightly more general statement of part (a): Let w be a bisolution mod C^∞ for the wave-operator, and suppose that w is of Hadamard form on N . Then $\text{WF}(w) = \mathcal{R}$. It is not clear, however, if the converse direction (b) holds for w unless its symmetric or antisymmetric part is suitably fixed (mod C^∞) as is the case for two-point functions of quantum fields fulfilling CAR or CCR.

(ii) It will also be apparent from the proof that (b) holds also under the assumption $\text{WF}(\omega_2) \subset \mathcal{R}$ (and even under the seemingly much weaker assumption $\text{WF}(\omega_2) \subset \mathcal{N}_- \times \mathcal{N}_+$). This proves the claim made in [34] that a two-point function ω_2 of a quantum field fulfilling CAR or CCR and $\text{WF}(\omega_2) \subset \mathcal{R}$ is of Hadamard form and thus $\text{WF}(\omega_2) = \mathcal{R}$.

(iii) The proof of part (a) needs an argument proving that the relation $\text{WF}(\omega_{2N \times N}) \subset \mathcal{R}$ implies $\text{WF}(\omega_2) \subset \mathcal{R}$, where $\omega_{2N \times N}$ denotes the restriction of ω_2 to $C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V})_{N \times N})$. To show this one invokes, as in [32], the propagation of singularities theorem which says

that $(q, \xi; q', \xi') \in \text{WF}(\omega_2)$ implies $B(q, \xi) \times B(q', \xi') \in \text{WF}(\omega_2)$. The argument proving the said implication requires, however, that both bicharacteristics $B(q, \xi)$ and $B(q', \xi')$ really consist of inextendible lightlike geodesics, and this is not the case if either $\xi = 0$ or $\xi' = 0$ since $B(q, 0)$ equals $\{q\}$. Thus, when considering e.g. $(q, \xi; q', 0)$ with q' not in N , then $B(q', 0)$ won't meet N and so one cannot use the propagation of singularities theorem to decide if $(q, \xi; q', 0)$ is in $\text{WF}(\omega_2)$ by knowing that $\text{WF}(\omega_{2N \times N}) \subset \mathcal{R}$. Due to having overlooked this gap, it has been claimed explicitly in [32] that the propagation of Hadamard form result were not needed in order to conclude that $\text{WF}(\omega_2) \subset \mathcal{R}$ once it is known that ω_2 is of Hadamard form on some causal normal neighborhood N of an arbitrary Cauchy surface. As far as we can see, however, the result on the propagation of Hadamard form is needed in order to conclude that pairs $(q, \xi; q', 0)$ or $(q, 0; q', \xi')$ aren't contained in $\text{WF}(\omega_2)$. At least it will prove sufficient to reach at this conclusion.

Proof. (a) Let the element \mathcal{G}_n of $(C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V})_{N \times N}))'$ be defined by

$$\mathcal{G}_n(f \otimes f') = \begin{cases} \chi G^{(1)}((\vartheta f)T^{(n)}f') & \text{for } m \text{ odd} \\ \chi G^{(1)}((\vartheta f)Uf') + \chi G^{(2)}((\vartheta f)V^{(n)}f') & \text{for } m \text{ even,} \end{cases}$$

where χ is an N -regularizing function. Using the arguments of part (i) of the proof of Thm. 5.1 in [32] in combination with Lemma 2.6, it is straightforward to deduce $\text{WF}(\mathcal{G}_n) \subset \mathcal{R} \cap (\text{T}^*N \times \text{T}^*N)$. Thus, if $w \in (C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V})_{N \times N}))'$ denotes a Hadamard form on N , then by the very definition of Hadamard form $w - \mathcal{G}_n$ is given by a C^n -integral kernel, for all $n \in \mathbb{N}$. Therefore one obtains as in the proof of Thm. 5.1 in [32] that $\text{WF}(w) \subset \mathcal{R} \cap (\text{T}^*N \times \text{T}^*N)$.

Denoting by $\omega_{2N \times N}$ the restriction of ω_2 to $C_0^\infty((\mathfrak{V} \boxtimes \mathfrak{V})_{N \times N})$, it follows that

$$\text{WF}(\omega_{2N \times N}) \subset \mathcal{R} \cap (\text{T}^*N \times \text{T}^*N) \quad (5.11)$$

whenever ω_2 is the two-point function of a Hadamard state on N . (For the CCR case this is immediate as in this case $\omega_{2N \times N} = w$ for some Hadamard form w on N . For the CAR case this follows since then $\omega_{2N \times N} = (D_{\triangleright} \otimes 1)w$ for some Hadamard form w on N , and application of differential operators cannot increase the wavefront set.)

For any quasifree state ω fulfilling the CCR or CAR it holds that ω_2 is a bisolution for the wave-operator (it would be sufficient for the subsequent arguments that ω_2 be a bisolution mod C^∞). This means that one can apply the PST, Prop. 3.2, in order to show that (5.11) already implies

$$\text{WF}(\omega_2) \subset \mathcal{R} \quad (5.12)$$

owing to the fact that N is a neighbourhood of a Cauchy-surface Σ : Let $(q, \xi; q', \xi')$ be an element of $\text{WF}(\omega_2)$. Then the first part of Prop. 3.2 shows that ξ, ξ' are both lightlike. As any inextendible lightlike geodesic intersects Σ , we can — provided that *both* ξ and ξ' are non-zero — use the second part of Prop. 3.2 to conclude that $(p, \zeta; p', \zeta') \in \text{WF}(\omega_2)$, where $(p, \zeta; p', \zeta')$ is the (unique) element of $B(q, \xi) \times B(q', \xi')$ with $p, p' \in \Sigma$. But then, because of (5.11), $p = p', \zeta = -\zeta'$ with ζ' future pointing. Thus we conclude that $(q, \xi; q', \xi') \in \mathcal{R}$.

Now we will show that the PST in combination with the propagation of Hadamard form entails that $(q, \xi, q', 0)$ and $(q, 0; q', \xi')$ are absent from $\text{WF}(\omega_2)$. We will give an

indirect proof and thus assume that $\text{WF}(\omega_2)$ contains an element of the form $(q, \xi; q', 0)$. Then there will be a Cauchy-surface Σ' passing through q' ; this Cauchy-surface possesses a causal normal neighbourhood N' . By the propagation of Hadamard form, ω_2 , being a bisolution (mod C^∞ would suffice) for the wave-operator, will be of Hadamard form on N' . Moreover, $\xi \neq 0$, and so there is some point $(p, \zeta) \in \text{B}(q, \xi)$, $\zeta \neq 0$, with $p \in \Sigma'$. Since ω_2 is of Hadamard form on N' , it follows (see above) that $\text{WF}(\omega_2|_{N' \times N'}) \subset \mathcal{R} \cap (\text{T}^*N' \times \text{T}^*N')$, and thus $(p, \zeta; q', 0)$ can only be contained in $\text{WF}(\omega_2)$ if $p = q'$ and $\zeta = 0$. By the PST, this contradicts the assumption that $(q, \xi; q', 0) \in \text{WF}(\omega_2)$.

Thus elements of the form $(q, \xi; q', 0)$ are absent from $\text{WF}(\omega_2)$, and by an analogous argument, also pairs of covectors of the form $(q, 0; q', \xi')$ aren't contained in $\text{WF}(\omega_2)$. Thus we have established the inclusion (5.12).

Now we have to establish the reverse inclusion

$$\text{WF}(\omega_2) \supset \mathcal{R}. \quad (5.13)$$

In order to prove this we use Prop. 5.6 and Lemma 5.7 together with Proposition 2.8, showing that for any Hadamard state ω on the CCR-algebra one has

$$\text{WF}(\omega_2) \supset \{(q, \xi; q, \xi') \in \text{T}_q^*N \times \text{T}_q^*N : \xi \in \mathcal{N}_-, \xi' = -\xi\}.$$

The same result can be derived for any Hadamard state ω on the CAR-algebra.

According to the PST, this implies that

$$\text{WF}(\omega_2) \supset \text{B}(q, \xi) \times \text{B}(q, \xi')$$

for all $\xi, \xi' \in \text{T}_q^*N$ with $\xi \in \mathcal{N}_-, \xi' = -\xi$. Since this holds for all q in the causal normal neighbourhood N of a Cauchy surface, relation (5.13) now follows. Thus we have proved $\text{WF}(\omega_2) = \mathcal{R}$.

(b) Now let ω be a state on the CCR or CAR algebra associated to some (wave or Dirac) operator with the property that (5.9) holds. Let N be a causal normal neighbourhood of any given Cauchy-surface. According to Lemma 5.4, there is, in the CCR-case, a Hadamard form w on N fulfilling (5.4), and in the CAR-case, there is a Hadamard form w on N obeying (5.5). According to part (a) of the proof, it holds in either case,

$$\text{WF}(w) = \mathcal{R} \cap (\text{T}^*N \times \text{T}^*N),$$

so that one obtains

$$\text{WF}(w - \omega_2|_{N \times N}) \subset \mathcal{R} \cap (\text{T}^*N \times \text{T}^*N) \subset \mathcal{N}_- \times \mathcal{N}_+.$$

Now we have in the CCR-case $w^{(-)} - \omega_2^{(-)}|_{N \times N} = 0$. Introducing the flip morphism

$$\iota : M \times M \rightarrow M \times M, \quad (p, q) \mapsto (q, p),$$

and some bundle morphism I covering ι , as well as $u := w - \omega_2|_{N \times N}$, this implies

$$\text{WF}(u) = \text{WF}(u^{(+)}) = \text{WF}(I^*u^{(+)}) = {}^tD\iota\text{WF}(u^{(+)}) = {}^tD\iota\text{WF}(u).$$

But because of the anti-symmetry of the set $\mathcal{N}_- \times \mathcal{N}_+$, its intersection with its image under ${}^tD\iota$, $\mathcal{N}_+ \times \mathcal{N}_-$, is empty, so one finds

$$\text{WF}(w - \omega_{2N \times N}) = \emptyset.$$

The same reasoning applies to the CAR-case, where $w^{(+)} - \omega_{2N \times N}^{(+)} = 0$. Thus ω_2 is shown to be of Hadamard form on N in both cases. This shows that ω is a Hadamard state. \square

A Appendix

A.1 The Hadamard coefficients

In this appendix we give the definition of the Hadamard coefficients for the wave-operator P on the vector-bundle \mathfrak{V} according to Chapter III in [19], adapted to our notation.

Let N be a causal normal neighbourhood of an arbitrary Cauchy-surface, and let χ be an N -regularizing function with support domain $\mathcal{U}_*, \mathcal{U}$. Then for $(x, y) \in \mathcal{U}$, the (signed) square of the geodesic distance between x and y , $s(x, y)$, is well-defined and a smooth function of both arguments. For each $(x, y) \in \mathcal{U}$, denote by $\mathbf{s}_y(x)$ the vector $\text{grad}_x s(x, y)$ in $T_x N$, and define

$$M(x, y) := \frac{1}{2} \square_x s(x, y) - m.$$

Then by Prop. III.1.3 in [19], there is exactly one sequence $\{U_{(k)}\}_{k \in \mathbb{N}_0}$ of sections $U_{(k)} \in C^\infty((\mathfrak{V} \boxtimes \mathfrak{V}^*)_{\mathcal{U}})$ satisfying the differential equations

$$(P \otimes 1)U_{(k-1)}(x, y) + (\nabla_{\mathbf{s}_y(x)}^{(P)} \otimes 1)U_{(k)}(x, y) + (M(x, y) + 2k)U_{(k)}(x, y) \quad (\text{A.1})$$

with the initial conditions

$$U_{(-1)}(x, y) = 0, \quad U_{(0)}^a{}_b(x, x) = \delta^a{}_b,$$

where the latter condition is to be understood with respect to dual frame indices for \mathfrak{V} and \mathfrak{V}^* , respectively, and the differential operators in (A.1) act on the left tensor entry, i.e. with respect to the variable x . (We caution the reader that at this point our notation deviates from that in [19].)

The members of the sequence $\{U_{(k)}\}_{k \in \mathbb{N}_0}$ are called *Hadamard coefficients*.

With this definition, the sections U , $V^{(n)}$ and $T^{(n)}$ in $C^\infty((\mathfrak{V} \boxtimes \mathfrak{V}^*)_{\mathcal{U}})$ appearing in the main text (which are also often referred to as Hadamard coefficients) are given by

$$\begin{aligned} U(x, y) &:= \sum_{k=0}^{(m-4)/2} (4 - m, k)^{-1} U_{(k)}(x, y) s(x, y)^k \\ V^{(n)}(x, y) &:= \left(2, \frac{m}{2} - 1\right) \sum_{k=0}^n \frac{1}{2^k k!} U_{((m-2)/2+k)}(x, y) s(x, y)^k \\ T^{(n)}(x, y) &:= \sum_{k=0}^{n+(m-3)/2} (4 - m, k)^{-1} U_{(k)}(x, y) s(x, y)^k \end{aligned}$$

where the symbol (α, k) is defined by

$$(\alpha, 0) = 1, \quad (\alpha, k) = \alpha(\alpha + 2) \dots (\alpha + 2k - 2).$$

If the wave-operator P in question is symmetric w.r.t. the sesquilinear form (3.5), the corresponding Hadamard coefficients have an additional symmetry property which we shall use below. To state this property, let $\Theta = \vartheta\Gamma$ where ϑ and Γ are the morphisms defined in the main text and define ι to be the flip morphism

$$\begin{aligned} \iota : C^\infty(\mathfrak{Y} \boxtimes \mathfrak{Y}^*) &\rightarrow C^\infty(\mathfrak{Y}^* \boxtimes \mathfrak{Y}), \\ f(p) \otimes \nu(q) &\mapsto \nu(q) \otimes f(p), \quad f \in C^\infty(\mathfrak{Y}), \nu \in C^\infty(\mathfrak{Y}^*). \end{aligned}$$

Note that the map $\Theta : \mathfrak{Y} \rightarrow \mathfrak{Y}^*$ induces a bilinear form $\Theta(v, v') = [\Theta v](v')$ on \mathfrak{Y} , and hence one can introduce its transpose $\Theta^T : \mathfrak{Y} \rightarrow \mathfrak{Y}^*$ by $[\Theta^T v](v') = \Theta(v', v)$. Using frame-indices, we introduce the notation

$$(\Theta^T U_{(k)} \Theta^{-1})_a^b(p, q) = \Theta^T_{ac}(p) U_{(k)}^c{}_d(p, q) (\Theta^{-1})^{db}(q), \quad (\text{A.2})$$

thus defining $\Theta^T U_{(k)} \Theta^{-1} \in C^\infty(\mathfrak{Y}^* \boxtimes \mathfrak{Y})$. With that notation, we have

Lemma A.1. *If P is symmetric, i.e. fulfills (3.6), the section $U_{(k)} - \iota(\Theta^T U_{(k)} \Theta^{-1})$ vanishes faster than any power of $s(p, q)$ on the set $\{(p, q) \in \mathcal{U} \mid s(p, q) = 0\}$, for all $k \in \mathbb{N}$.*

A proof of this fact can be obtained by combining the results of Prop. 4.6 and 4.9 from [19], Chap. III.

A.2 Normal coordinates

We begin with some words on normal coordinates: Let Ω be a convex normal neighbourhood, containing a point p . Then we can cover Ω by normal coordinates $\xi_p : \mathbb{R}^m \rightarrow \Omega$ centered at p as follows: We identify \mathbb{R}^m and $T_p\Omega$, using a basis $v_{(k)}$ of $T_p\Omega$ which fulfills $g_p(v_{(i)}, v_{(j)}) = \eta_{ij}$, by

$$w : \mathbb{R}^m \mapsto T_p\Omega, \quad x = (x_0, \dots, x_{m-1}) \mapsto w(x) = \sum_{i=0}^{m-1} x_i v_{(i)}$$

and let $\xi_p(x) := \exp_p(w(x))$ for all x .

A useful fact about normal coordinates is that

$$-s(p, \xi_p(x)) = \eta(x, x) \quad \text{for } \xi_p(x) \in \Omega. \quad (\text{A.3})$$

Unfortunately, there is no such simple formula for the geodesic distance $s(\xi_p(x), \xi_p(y))$ if both points are different from p . There is, however, a useful approximation to it, which we will have occasion to use below: Let λ be in \mathbb{R} , $\lambda \geq 0$. Then we have

$$-s(\xi_p(\lambda x), \xi_p(\lambda y)) = \lambda^2 \eta(x - y, x - y) + \lambda^4 \phi(x, y), \quad (\text{A.4})$$

where $\phi(x, y)$ is a remainder which is smooth in x, y . For details see [36].

It will be useful to have a symbol for the pullback of functions f in $C_0^\infty(\Omega)$ via normal coordinates. We therefore define

$$\check{f}_p(x) := \begin{cases} f(\xi_p(x)) \det(g_{\xi_p(x)})^{\frac{1}{2}} & \text{for } \xi_p(x) \in \Omega \\ 0 & \text{else} \end{cases}.$$

A.3 Some properties of $\chi G^{(1)}_2$

We can now begin our investigation of the distributions showing up in our definition of a Hadamard form by considering special kinds of distributions on Minkowski space:

Let t be a time function on m dimensional Minkowski space with $t(0) = 0$ and define

$$\tilde{G}^{(1)}_2(f) = \lim_{\varepsilon \rightarrow 0^+} \int f(x) G_\varepsilon^{(1)}(0, x) d^m x, \quad f \in C_0^\infty(\mathbb{R}^m), \quad (\text{A.5})$$

where the functions $G_\varepsilon^{(1)}, G_\varepsilon^{(2)}$ are given by equations (5.1), (5.2), taken in the case of the Minkowski metric (i.e. $-s(x, y) = \eta_{\mu\nu}(x - y)^\mu(x - y)^\nu = \eta(x - y, x - y)$) and with $\chi \equiv 1$. Although the time function t enters the above definition, the distributions $\tilde{G}^{(1)}$ and $\tilde{G}^{(2)}$ do actually not depend on t . More precisely, we have

Lemma A.2.

$$\tilde{G}^{(1)}(f) = \begin{cases} \beta(m+1, m) \int [\theta(-\eta(x, x))\sqrt{-\eta(x, x)} \\ -i \text{sign}(x_0)\theta(\eta(x, x))\sqrt{\eta(x, x)}] \square^{\frac{m-1}{2}} f(x) d^m x & \text{for } m \text{ odd} \\ \beta(m+2, m) \int [\frac{1}{\pi}\eta(x, x) \ln |\eta(x, x)| \\ -i \text{sign}(x_0)\theta(\eta(x, x))\eta(x, x)] \square^{\frac{m}{2}} f(x) d^m x & \text{for } m \text{ even} \end{cases}$$

$$\tilde{G}^{(2)}(f) = \beta(m+2, m) \int \left[\frac{1}{\pi}\eta(x, x) \ln |\eta(x, x)| - i \text{sign}(x_0)\theta(\eta(x, x))\eta(x, x) \right] \times \\ \times (\square + 1 + 2/m) f(x) d^m x \quad (\text{A.6})$$

where $\theta(s) = 1$ for $s \geq 0$ and $\theta(s) = 0$ for $s < 0$, and

$$\beta(\alpha, m) := 2^{1-\alpha} \pi^{(2-m)/2} \left[\Gamma\left(\frac{\alpha-m}{2} + 1\right) \Gamma\left(\frac{\alpha}{2}\right) \right]^{-1}.$$

Proof. As first step, we will prove independence of t by generalizing an argument given in [25] to arbitrary dimensions:

For $\tilde{G}^{(2)}$ note that $G_\varepsilon^{(2)}$ converges for $\varepsilon \rightarrow 0$ to a locally integrable function which does not depend on t anymore. As we can use Lemma B2 of [25] to conclude that we may interchange integration and limit in (A.5), we see that $\tilde{G}^{(2)}$ is indeed a well defined distribution and independent of t .

The limit of $G_\varepsilon^{(1)}$ for $\varepsilon \rightarrow 0$ is not locally integrable, so before we can apply Lemma B2 of [25] to argue as above, we will have to rewrite (A.5), using integration by parts. To this end, introduce coordinates τ, σ, ϑ on \mathbb{R}^m , where

$$\tau(x_0, \dots, x_{m-1}) = x_0, \quad \sigma(x_0, \dots, x_{m-1}) = \sum_{i=1}^{m-1} x_i^2$$

and ϑ stands for some coordinatization of S^{m-2} . In these coordinates

$$\tilde{G}^{(1)}(f) = \lim_{\varepsilon \rightarrow 0^+} \iiint d\tau d\sigma d\vartheta f(\tau, \sigma, \vartheta) \sigma^{\frac{m-3}{2}} (\sigma - \tau^2 - i2\varepsilon t(\tau, \sigma, \vartheta) + \varepsilon^2)^{-\frac{m}{2}+1}$$

In case m is even, we carry out a $m/2 - 1$ -fold partial integration with respect to σ and arrive at

$$\tilde{G}^{(1)}(f) = c \lim_{\varepsilon \rightarrow 0^+} \iiint d\tau d\sigma d\vartheta \ln(\sigma - \tau^2 - i2\varepsilon t + \varepsilon^2) \underbrace{\left(\partial_\sigma \left(\frac{1}{1 - i2\varepsilon \partial_\sigma t} \partial_\sigma \left(\frac{1}{1 - i2\varepsilon \partial_\sigma t} \cdots \sigma^{\frac{m-3}{2}} f \right) \right) \right)}_{\frac{m}{2} - 1 \text{ times}},$$

where c is some constant. Now, the limit of the integrand is locally integrable and turns out to be independent of t , too. Hence Lemma B2 of [25] can be applied to show the desired result.

In case m is odd, we carry out a $(m - 3)/2$ -fold partial integration with respect to σ and arrive at

$$\tilde{G}^{(1)}(f) = c' \lim_{\varepsilon \rightarrow 0^+} \iiint d\tau d\sigma d\vartheta (\sigma - \tau^2 - i2\varepsilon t + \varepsilon^2)^{-1} (\sigma - \tau^2 - i2\varepsilon t + \varepsilon^2)^{\frac{1}{2}} \underbrace{\left(\partial_\sigma \left(\frac{1}{1 - i2\varepsilon \partial_\sigma t} \partial_\sigma \left(\frac{1}{1 - i2\varepsilon \partial_\sigma t} \cdots \sigma^{\frac{m-3}{2}} f \right) \right) \right)}_{\frac{m-3}{2} \text{ times}},$$

where again c' is a suitable constant. Since the limit of the integrand is not locally integrable in τ , we are not ready to apply Lemma B2 of [25] yet. But we have already written the integrand in a suggestive form as to the next partial integration. This turns $(\sigma - \tau^2 - i2\varepsilon t + \varepsilon^2)^{-1}$ into a logarithm and the $(\sigma - \tau^2 - i2\varepsilon t_p + \varepsilon^2)^{1/2}$ to $(\dots)^{-1/2}$, at worst, thus rendering the integrand locally integrable in the limit $\varepsilon \rightarrow 0$. We also get a boundary term at the integration boundary $\sigma = 0$, which is integrable in the limit $\varepsilon \rightarrow 0$ as well. Inspection of these limits shows that they are indeed independent of t , whence $G^{(1)}$ is well defined and independent of t also in this case.

As a consequence of the independence of t , we can write the distributions in the form given in the statement of the Lemma: Using the trivial time-function $t_0(x) := x_0$ and the abbreviation $g_\varepsilon(x) := -\eta(x, x) - i2\varepsilon x_0 + \varepsilon^2$, we get

$$\tilde{G}^{(1)}(f) = \lim_{\varepsilon \rightarrow 0} \begin{cases} \beta(m+1, m) \int \sqrt{g_\varepsilon(x)} \square^{\frac{m-1}{2}} f(x) d^m x & \text{for } m \text{ odd} \\ -\frac{1}{\pi} \beta(m+2, m) \int g_\varepsilon \ln(g_\varepsilon)(x) \square^{\frac{m}{2}} f(x) d^m x & \text{for } m \text{ even} \end{cases}$$

$$\tilde{G}^{(2)}(f) = -\frac{1}{\pi} \beta(m+2, m) \lim_{\varepsilon \rightarrow 0} \int g_\varepsilon \ln(g_\varepsilon)(x) (\square + 1 + 2/m) f(x) d^m x$$

By exchanging the integration and the limit, we finally get the desired result. \square

We also have to introduce the so called *Riesz distributions*, as defined in [19], Chap. II. To this end, let $\alpha > m$ and define the distributions

$$\tilde{R}(\alpha)[f] = -\beta(\alpha, m) \int \text{sign}(x_0) \theta(\eta(x, x)) (\eta(x, x))^{\frac{\alpha-m}{2}} f(x) d^m x, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Note that β is chosen such that $\tilde{R}(\alpha)[\phi] = \tilde{R}(\alpha+2)[\square\phi]$ for all $\alpha > m$. We define the distributions $\tilde{R}(\alpha)$ for all $\alpha \in \mathbb{R}$ by means of this relation.

As before, let Ω denote a causal normal neighbourhood and ξ_p normal coordinates on Ω . We can then define distributions $R^\Omega(\alpha)$ on $C_0^\infty(\Omega \times \Omega)$ by

$$R^\Omega(\alpha)[f \otimes f'] := \int f(p) \tilde{R}(\alpha)[\check{f}'_p] d\mu(p), \quad f, f' \in C_0^\infty(\Omega)$$

and continuous extension to $C_0^\infty(\Omega \times \Omega)$. These so called *Riesz distributions* bear a certain relation to the distributions we are really interested in: Let Ω be a normal neighbourhood, t be a time function on Ω and denote by $G^{(1,\Omega)}$, $G^{(2,\Omega)}$ the distributions on $C_0^\infty(\Omega \times \Omega)$ obtained by setting $\chi \equiv 1$ in the definition of the distributions $\chi G^{(1)}$, $\chi G^{(2)}$, respectively. We caution the reader, that, as a consequence, $G^{(1,\Omega)}$ and $G^{(2,\Omega)}$ are not well defined on $C_0^\infty(M \times M)$ in contrast to $\chi G^{(1)}$, $\chi G^{(2)}$.

Using normal coordinates on Ω (especially their property (A.3)) and interchanging limit and integration, we see that

$$G^{(2,\Omega)}(f \otimes f') = \int f(p) \tilde{G}^{(2)}(\check{f}'_p) dp, \quad f, f' \in C_0^\infty(\Omega).$$

The interchange of limit and integration is valid because the limit $\varepsilon \rightarrow 0$ in $\tilde{G}^{(1/2)}(\check{f}'_p)$ is uniform in p on compact sets. Thus we see that $G^{(1,\Omega)}$, $G^{(2,\Omega)}$ are actually independent of t . Moreover, by comparison with the definition of $R(\alpha)$, using the formulae (A.6), we find that

$$G^{(1,\Omega)(-)} = iR^\Omega(2), \quad \text{and} \quad G^{(2,\Omega)(-)} = iR^\Omega(m). \quad (\text{A.7})$$

A.4 Proof of Lemma 5.4

(i) *Existence of Hadamard forms*

Following Sec. 4.3 in [15], the argument that Hadamard forms for the wave-operator P exist at all on a causal normal neighbourhood N runs as follows:

Assume that $\phi \in C_0^\infty(\mathbb{R})$ has the following properties: $0 \leq \phi(s) \leq 1$, $\phi(s) = 1$ for $|s| \leq 1/2$, and $\phi(s) = 0$ for $|s| \geq 1$. Then one can show (cf. [15, Lemma 4.3.2], [19, Prop. III.2.6.3]) that there exists a strictly increasing and diverging sequence $(\kappa_j)_{j \in \mathbb{N}}$ of natural numbers so that the modified Hadamard coefficient sections $\tilde{V}^{(n)}$ and $\tilde{T}^{(n)}$, which are defined like $V^{(n)}$ and $T^{(n)}$, but with the terms $s(x, y)^k$ replaced by $s(x, y)^k \cdot \phi(\kappa_k s(x, y))$, converge for $n \rightarrow \infty$ uniformly on compact subsets of \mathcal{U} to smooth sections \tilde{V} and \tilde{T} , respectively. Moreover, it holds that for all n ,

$$s(x, y)^{-n}(\tilde{V}(x, y) - V^{(n)}(x, y)) \quad \text{and} \quad s(x, y)^{-n}(\tilde{T}(x, y) - T^{(n)}(x, y))$$

converge to 0 as $s(x, y) \rightarrow 0$. Thus it is not difficult to check that (for m even) $f, f' \mapsto \chi G^{(2)}((\vartheta f)(V^{(n)} - \tilde{V})f')$ is given by a C^m -kernel, and likewise (for m odd) $f, f' \mapsto \chi G^{(1)}((\vartheta f)(T^{(n)} - \tilde{T})f')$ is given by a C^n -kernel. This guarantees the existence of Hadamard forms on N .

(ii) In a next step, one must show that the $H^{(n)} \in C^m((\mathfrak{V} \boxtimes \mathfrak{V}^*)_{N \times N})$ can be chosen such that (5.4), respectively (5.5), are fulfilled. Let us treat the case of relation (5.4) first.

We note that in case $p, q \in N$ lie acausal to each other, there is a neighbourhood \mathcal{V} in $N \times N$ such that E vanishes on \mathcal{V} , and any Hadamard form on \mathcal{V} is C^∞ . One can thus correct the Hadamard form w on \mathcal{V} by a C^∞ integral kernel so as to obtain a new Hadamard form vanishing on \mathcal{V} . Hence, we need only consider the situation in a neighbourhood of causally related points $(p, q) \in N \times N$ (and eventually use a partition of unity argument). Given a pair of causally related points $p, q \in N$, there is, by definition of causal normal neighbourhood, a convex normal neighbourhood Ω containing both p and q .

The importance of the distributions $R^\Omega(\alpha)$ defined above lies in the fact that they show up in the explicit formula for the fundamental solution E^Ω of a wave-operator P on Ω , given in [19]:

$$(f, E^\Omega f') = \begin{cases} R^\Omega(2) [\vartheta(f)T^{(n)}f'] + (f, M^{(n)}f') & \text{for } m \text{ odd} \\ R^\Omega(2) [\vartheta(f)Uf'] + R^\Omega(m)U [\vartheta(f)V^{(n)}f'] + (f, M^{(n)}f') & \text{for } m \text{ even} \end{cases} \quad (\text{A.8})$$

where $U_{(k)}$, $T^{(n)}$, $V^{(n)}$ and U are the sections defined in Appendix A.1 and $M^{(n)} \in C^m((\mathfrak{B} \boxtimes \mathfrak{B}^*)_{\Omega \times \Omega})$ are suitably chosen. We have also used the shorthands

$$\begin{aligned} \vartheta(f)T^{(n)}f'(p, q) &:= \vartheta(f)_a(p)T^a_b(p, q)f'^b(q), \\ (f, M^{(n)}f') &:= \int (\vartheta(f)_a(p)M^a_b(p, q)f'^b(q) d\mu(q)d\mu(p), \quad \text{etc.} \end{aligned}$$

Now let w be a Hadamard form, and let $f, f' \in C_0^\infty(\mathfrak{B}_\Omega)$. Then

$$\begin{aligned} w^{(-)}(\Gamma f \otimes f') = & \\ \frac{1}{2} \begin{cases} G^{(1, \Omega)}(\vartheta(f)\chi T^{(n)}f') + (f, H^{(n)}f') \\ -G^{(1, \Omega)}(\vartheta(\Gamma f')\chi T^{(n)}\Gamma f) - (\Gamma f', H^{(n)}\Gamma f) & \text{for } m \text{ odd} \\ G^{(1, \Omega)}(\vartheta(f)\chi Uf') + G^{(2, \Omega)}(\vartheta(f)\chi V^{(n)}f') + (f, H^{(n)}f') \\ -G^{(1, \Omega)}(\vartheta(\Gamma f')\chi U\Gamma f) - G^{(2, \Omega)}(\vartheta(\Gamma f')\chi V^{(n)}\Gamma f) - (\Gamma f', H^{(n)}\Gamma f) & \text{for } m \text{ even} \end{cases} \end{aligned}$$

Using the fact that Γ is a conjugation in the CCR case, one can compute that (cf. (A.2) for notation)

$$G^{(1, \Omega)}(\vartheta\Gamma(f')\chi T^{(n)}\Gamma(f)) = G^{(1, \Omega)}(\iota(\vartheta(f))\iota(\Theta^T \chi T^{(n)} \Theta^{-1})f')$$

Now we can use the fact that χ is identically 1 on $\{(p, q) \in \Omega \times \Omega \mid s(p, q) = 0\}$ together with Lemma A.1 to the effect that

$$G^{(1, \Omega)}(\vartheta\Gamma(f')\chi T^{(n)}\Gamma(f')) = G^{(1, \Omega)}(\iota(\vartheta(f))\chi T^{(n)}f') \text{ mod } C^\infty.$$

Treating the other terms with a minus sign in a similar way, we arrive at

$$\begin{aligned} w^{(-)}(\Gamma f \otimes f') = & \\ \begin{cases} G^{(1, \Omega)(-)}(\vartheta(f)\chi T^{(n)}f') + (f, H^{(n)}f') \\ - (f, \iota(\Theta^T H^{(n)} \Theta^{-1})f') + (f, K^{(n)}f') & \text{for } m \text{ odd} \\ G^{(1, \Omega)(-)}(\vartheta(f)\chi Uf') + G^{(2, \Omega)(-)}(\vartheta(f)\chi V^{(n)}f') + (f, H^{(n)}f') \\ - (f, \iota(\Theta^T H^{(n)} \Theta^{-1})f') + (f, K^{(n)}f') & \text{for } m \text{ even} \end{cases} \end{aligned}$$

where $K^{(n)} \in C^\infty((\mathfrak{V} \boxtimes \mathfrak{V}^*)_{N \times N})$ reflects the potential asymmetry of $\chi, U_{(k)}$ away from $\{(p, q) \in \Omega \times \Omega \mid s(p, q) = 0\}$. Upon using the identification (A.7) between $R^\Omega(\alpha)$ and $G^{(\dots, \Omega)^{(-)}}$ and comparing with the expression (A.8), one can now see that it is indeed possible to choose the $H^{(n)}$ in such a way that $w^{(-)}(\Gamma f \otimes f') = i(f, E^\Omega f')$. Because of uniqueness of the fundamental solution, we have $E^\Omega = E|_\Omega$, which concludes the proof of the lemma in the CCR case.

For the CAR case, note that Γ acts as a skew-conjugation. Thus, the computation can be carried through as above, and one can choose the sections $H^{(n)}$ of w' such that

$$w'^{(+)}(\Gamma D_{\triangleright} f \otimes f') = i(D_{\triangleright} f, E f'), \quad f, f' \in C_0^\infty(\mathfrak{V}_N),$$

which in turn gives the desired result.

(iii) Finally, we have to show that part (c) of Lemma 5.4 holds. We follow the argument given in the ‘‘Note added in proof’’ in [32]. To this end we note that by part (a), $w^{(-)}$, the antisymmetric part of a Hadamard form w , is always a bisolution mod C^∞ for the wave-operator P . On the other hand, according to Thm. 5.1, part (i), in [32] (cf. the proof of our Thm. 5.8), it holds that $\text{WF}(w) \subset \mathcal{R}$, where the set \mathcal{R} has been defined in (5.9); it is significant that $\mathcal{R} \subset \mathcal{N}_- \times \mathcal{N}_+$. Now since $w^{(-)}$ is a bisolution mod C^∞ for the wave-operator P , it holds that $\text{WF}(w^{(P)^{(-)}}) = \emptyset$, where $w^{(P)}$ has been defined in Sec. 3.2. Thus $\text{WF}(w^{(P)}) = \text{WF}(w^{(P)^{(+)}}) \subset \mathcal{N}_- \times \mathcal{N}_+$. But since $w^{(P)^{(+)'}$ is symmetric, its wavefront set must be invariant under ${}^t D \iota^{-1}$ where $\iota : (p, q) \mapsto (q, p)$ is the ‘flip’ morphism on $N \times N$. That is, one concludes exactly as in part (b) of the proof of Thm. 5.8 that $\text{WF}(w^{(P)^{(+)'})$ must be contained in $(\mathcal{N}_+ \times \mathcal{N}_-) \cap (\mathcal{N}_- \times \mathcal{N}_+) = \emptyset$. Similarly one concludes that $\text{WF}(w_{(P)}^{(+)'})$ is empty, and thus w is a bisolution up to C^∞ for the wave-operator P .

A.5 Scaling limits

In this section, we will prove Prop. 5.6 of the main text, determining the scaling limit of a Hadamard distribution. Let, for the rest of the section, p be some point of M and Ω a convex normal neighbourhood of p , small enough such that \mathfrak{V}_Ω trivializes. Let the morphisms $\delta_\lambda, D_\lambda^{(\alpha)}$ be defined as in Section 5.3. Additionally define the action of dilations on test functions $f \in C_0^\infty(\Omega)$ by $d_\lambda^{(\alpha)} f = \lambda^{-\alpha} f \circ \delta_\lambda^{-1}$. We will also use the shorthand $G_\eta^{(1)}$ for the distribution $G^{(1, \Omega')}$, evaluated on Minkowski space (i.e. $g = \eta, \Omega' = \mathbb{R}^m$).

As the main step of the proof, we will compute the scaling limit of the distributions $G^{(1, \Omega)}, G^{(2, \Omega)}$:

Lemma A.3. *Let $\alpha = m/2 + 1$ and $F \in C^\infty(\Omega \times \Omega)$. Then for all $f, f' \in C_0^\infty(\Omega)$ there holds*

$$\begin{aligned} \lim_{\lambda \rightarrow 0} G^{(1, \Omega)}(F \cdot (d_\lambda^{(\alpha)} f \otimes d_\lambda^{(\alpha)} f')) &= F(p, p) \cdot G_\eta^{(1)}(f \circ \xi_p \otimes f' \circ \xi_p), \\ \lim_{\lambda \rightarrow 0} G^{(2, \Omega)}(F \cdot (d_\lambda^{(\alpha)} f \otimes d_\lambda^{(\alpha)} f')) &= 0. \end{aligned}$$

Proof. Since for each $f \in C_0^\infty(\Omega)$ the support of $d_\lambda^{(\alpha)} f$ is for $\lambda \rightarrow 0$ shrinking to p , it suffices to prove the statement for the case that $F \in C_0^\infty(\Omega \times \Omega)$. We will demonstrate the statement only for simple tensors $F = u \otimes u'$ with $u, u' \in C_0^\infty(\Omega)$ since this results

in slightly simpler notation, but it will be obvious from the argument that general $F \in C_0^\infty(\Omega \times \Omega)$ can be dealt with in exactly the same manner.

We begin by considering $G^{(1,\Omega)}$ in case m is odd: By using the results of App. A.3, we see that

$$G^{(1,\Omega)}(ud_\lambda^{(\alpha)} f \otimes u'd_\lambda^{(\alpha)} f') = c \iint (ud_\lambda^{(\alpha)} f)(q)h(v)\square^{\frac{m-1}{2}}(u'd_\lambda^{(\alpha)} f')_q(v) d^m v d\mu(q)$$

where c is some constant depending on m and

$$h(v) = \theta(-\eta(v, v))\sqrt{-\eta(v, v)} - i \operatorname{sign}(v_0)\theta(\eta(v, v))\sqrt{\eta(v, v)}. \quad (\text{A.9})$$

Now we change the integration variables from $(q, \xi_q^{-1}(q'))$ to $(\xi_p^{-1}(q), \xi_p^{-1}(q'))$, which we denote by (x, y) . We get

$$\begin{aligned} & G^{(1,\Omega)}(ud_\lambda^{(\alpha)} f \otimes u'd_\lambda^{(\alpha)} f') \\ &= c \iint \lambda^{-2\alpha} u(\xi_p(x))f(\xi_p(\frac{x}{\lambda}))h(v(x, y)) \cdot \\ & \quad \cdot \left(\square_y^{\frac{m-1}{2}} [u'(\xi_p(y))f'(\xi_p(\frac{y}{\lambda}))] + \begin{matrix} \text{terms with less than} \\ m-1 \text{ derivatives} \end{matrix} \right) \gamma_{\xi_p(x)}^{\frac{1}{2}} \gamma_{\xi_p(y)}^{\frac{1}{2}} d^m y d^m x \\ &= c \iint \lambda^{m-2} u(\xi_p(\lambda x))f(\xi_p(x))h(v(\lambda x, \lambda y)) \cdot \\ & \quad \cdot \left(\lambda^{1-m} \square_y^{\frac{m-1}{2}} [u'(\xi_p(\lambda y))f'(\xi_p(y))] + O(\lambda^{2-m}) \right) \gamma_{\xi_p(\lambda x)}^{\frac{1}{2}} \gamma_{\xi_p(\lambda y)}^{\frac{1}{2}} d^m y d^m x \end{aligned}$$

where $\gamma_p = |\det(g_p)|$. To compute the limit $\lambda \rightarrow 0$, we have to investigate the behaviour of $\lambda^{-1}h(v(\lambda x, \lambda y))$. We use equation (A.4) and the fact that

$$-v_0(x, y) = x_0 - y_0 + O(s(q, q')) + O(s(p, q')) + O(s(p, q))$$

where we have set $q = \xi_p(x)$, $q' = \xi_p(y)$ (see [36, Chp. II, §9] for details), to conclude

$$\lim_{\lambda \rightarrow 0} \lambda^{-1}h(v(\lambda x, \lambda y)) = h(x - y)$$

with h given by (A.9). Using this, and $\lim_{\lambda \rightarrow 0} \det(g_{\xi_p(\lambda x)}) = \det(g_p) = 1$ we get the desired result.

The case m even is treated exactly the same way, with the exception that there is a term $(x - y)^2 \ln \lambda^2$ in $\lambda^{-1}h(v(\lambda x, \lambda y))$ which seems to blow up for $\lambda \rightarrow 0$. Using partial integration, it is easy to see, though, that the term in $G^{(1,\Omega)}$ resulting from this term in h vanishes for any λ . Therefore, the rest of the argument goes through unchanged.

The argument for $G^{(2,\Omega)}$ runs along the same lines. \square

Proposition 5.6 is now a corollary of the above Lemma: Let U be in $C^\infty(\mathfrak{V} \boxtimes \mathfrak{V}^*)$. In the components of the frame (e_i) used to define D_λ in (5.8), we write

$$[(\vartheta \circ \Gamma D_\lambda^{(\alpha)})UD_\lambda^{(\alpha)} f'](q, q') = \Theta_{ac}(q)(D_\lambda^{(\alpha)} f)^c(q)U^a_b(q, q')(D_\lambda^\alpha f')^b(q')$$

and thus we obtain by the lemma,

$$\lim_{\lambda \rightarrow 0} G^{(1,\Omega)} \left(((\vartheta \circ \Gamma D_\lambda^{(\alpha)} f)UD_\lambda^{(\alpha)} f') \right) = G_\eta^{(1)} \left(((\Theta R^* f)UR^* f') \right),$$

where \mathbf{U} denotes the image of $U|_{(p,p)}$ under $R \otimes R$. The content of Prop. 5.6 concerning the CCR-case is now a consequence of simple properties of the Hadamard coefficients, such as $U_{(0)}(p, p) = 1$. In the CAR case, the appearance of an additional factor λ^{-1} due to the differential operator D_{\triangleright} has to be compensated by a different choice of α , as done in the proposition.

References

- [1] Araki, H., “On quasifree states of CAR and Bogoliubov transformations”, Publ. RIMS **6**, 385 (1970/71)
- [2] Borchers, H.J., “On the structure of the algebra of field operators”, Nuovo Cimento **24**, 214 (1962)
- [3] Bros, J., Buchholz, D., “Towards a relativistic KMS condition”, Nucl. Phys. B **429**, 291 (1994)
- [4] Brunetti, R., Fredenhagen, K., “Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds”, Commun. Math. Phys. **208**, 623 (2000)
- [5] Brunetti, R., Fredenhagen, K., Köhler, M., “The microlocal spectrum condition and Wick polynomials of free fields in curved spacetimes”, Commun. Math. Phys. **180**, 633 (1996)
- [6] Coquereaux, R., “Spinors, reflections and Clifford algebras”, in: *Spinors in physics and geometry*, eds. G. Furlan and A. Trautman, World Scientific, Singapore, 1988
- [7] Dencker, N., “On the propagation of polarization sets for systems of real principal type”, J. Funct. Anal. **46**, 351 (1982)
- [8] Dieudonné, J., *Foundations of Analysis, vol. 3*, Academic Press, New York, 1972
- [9] Dieudonné, J., *Foundations of Analysis, vol. 7*, Academic Press, New York, 1988
- [10] Dimock, J., “Algebras of local observables on a manifold”, Commun. Math. Phys. **77**, 219 (1980)
- [11] Dimock, J., “Dirac quantum fields on a manifold”, Trans. Am. Math. Soc. **269**, 133 (1982)
- [12] Duistermaat, J.J., Hörmander, L., “Fourier integral operators. II”, Acta Mathematica **128**, 183 (1972)
- [13] Fewster, C.J., “A general worldline quantum inequality”, Class. Quantum Grav. **17**, 1897 (2000)
- [14] Flanagan, E.E., Wald, R.M., “Does back reaction enforce the averaged null energy condition in semiclassical gravity?”, Phys. Rev. D **54**, 6233 (1996)
- [15] Friedlander, F.G., *The wave equation on a curved spacetime*, Cambridge University Press, Cambridge, 1975

- [16] Fulling, S.A., *Aspects of quantum field theory in curved spacetime*, Cambridge University Press, Cambridge, 1989
- [17] Fulling, S.A., Narcowich, F.J., Wald, R.M., “Singularity structure of the two-point function in quantum field theory in curved spacetime, II”, *Ann. Phys. (N.Y.)* **136**, 243 (1981)
- [18] Fulling, S.A., Sweeny, M., Wald, R.M., “Singularity structure of the Two-point function in quantum field theory in curved spacetime”, *Commun. Math. Phys.* **63**, 257 (1978)
- [19] Günther, P., *Huygens principle and hyperbolic equations*, Academic Press, Boston, 1988
- [20] Hawking, S.W., Ellis, G.F.R., *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973
- [21] Hollands, S., “The Hadamard condition for Dirac fields and adiabatic states on Robertson-Walker spacetimes”, gr-qc/9906076
- [22] Hörmander, L., *The analysis of linear partial differential operators I*, Springer Verlag, Berlin, 1983
- [23] Hörmander, L., *The analysis of linear partial differential operators III*, Springer Verlag, Berlin, 1986
- [24] Kay, B.S., Radzikowski, M.J., Wald, R.M., “Quantum field theory on spacetimes with a compactly generated Cauchy horizon”, *Commun. Math. Phys.* **183**, 533 (1997)
- [25] Kay, B.S., Wald, R.M., “Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon”, *Phys. Rep.* **207**, 49 (1991)
- [26] Keyl, M., “Quantum field theory and the geometric structure of Kaluza-Klein spacetime”, *Class. Quantum Grav.* **14**, 629 (1997)
- [27] Köhler, M., *The stress energy tensor of a locally supersymmetric quantum field on a curved spacetime*, Dissertation, Hamburg University, 1995. Preprint DESY-95-080, gr-qc/9505014
- [28] Köhler, M., “New examples for Wightman fields on a manifold”, *Class. Quantum Grav.* **12**, 1413 (1995)
- [29] Kratzert, K., “Singularity structure of the two-point function of the free Dirac field on a globally hyperbolic spacetime”, *Annalen Phys.* **9**, 475 (2000)
- [30] Leray, J. *Hyperbolic differential equations*, lecture notes, Institute for Advanced Study, Princeton, N.J., 1953
- [31] Najmi, A.-H., Ottewill, A.C., “Quantum states and the Hadamard form II, Energy minimization for spin 1/2 fields”, *Phys. Rev. D* **12**, 2573 (1984)
- [32] Radzikowski, M.J., “Micro-local approach to the Hadamard condition in quantum field theory in curved spacetime”, *Commun. Math. Phys.* **179**, 529 (1996)
- [33] Reed, M., Simon, B., *Methods of modern mathematical physics II*, Academic Press, San Diego, 1975

- [34] Sahlmann, H., Verch, R., “Passivity and microlocal spectrum condition”, *Commun. Math. Phys.* **214**, 705 (2000)
- [35] Strohmaier, A., “The Reeh-Schlieder property for quantum fields on stationary spacetimes”, *Commun. Math. Phys.* **215**, 105 (2000)
- [36] Synge, J.L., *Relativity: The general theory*, North Holland Publishing Company, Amsterdam 1960
- [37] Taylor, M.E., *Pseudodifferential operators*, Princeton University Press, Princeton, 1981
- [38] Verch, R., “Local definiteness, primarity and quasiequivalence of quasifree Hadamard quantum states in curved spacetime”, *Commun. Math. Phys.* **160**, 507 (1994)
- [39] Verch, R., “Continuity of symplectically adjoint maps and the algebraic structure of Hadamard vacuum representations of quantum fields in curved spacetime”, *Rev. Math. Phys.* **9**, 635 (1997)
- [40] Verch, R., *Scaling analysis and ultraviolet behaviour of quantum field theories in curved spacetime*, Dissertation, Hamburg University, 1996
- [41] Wald, R.M., “The back-reaction effect in particle creation in curved spacetime”, *Commun. Math. Phys.* **54**, 1 (1977)
- [42] Wald, R.M., *General relativity*, University of Chicago Press, Chicago, 1984
- [43] Wald, R.M., *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, Chicago, 1994