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On the Stability of the Kerr Metric

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Abstract: The reduced (in the angular coordinate φ) wave equation and Klein–Gordon equation are considered on a Kerr background and in the framework of C^0-semigroup theory. Each equation is shown to have a well-posed initial value problem, i.e., to have a unique solution depending continuously on the data. Further, it is shown that the spectrum of the semigroup's generator coincides with the spectrum of an operator polynomial whose coefficients can be read off from the equation. In this way the problem of deciding stability is reduced to a spectral problem and a mathematical basis is provided for mode considerations. For the wave equation it is shown that the resolvent of the semigroup's generator and the corresponding Green's functions can be computed using spheroidal functions. It is to be expected that, analogous to the case of a Schwarzschild background, the quasinormal frequencies of the Kerr black hole appear as *resonances*, i.e., poles of the analytic continuation of this resolvent. Finally, stability of the solutions of the reduced Klein–Gordon equation is proven for large enough masses.

1. Introduction

Linear stability of the Schwarzschild black hole was demonstrated by Kay and Wald [14] who showed the boundedness of all solutions of the wave equation corresponding to C^∞ data of compact support. Their proof rests on the positivity of the conserved energy.

The problem is more subtle for Kerr space time. A conserved energy exists, but the energy density is negative inside the ergosphere. Hence the total energy could be finite while the field still might grow exponentially in parts of the spacetime. Papers by Press and Teukolsky [22], Hartle and Wilkins [8], and Stewart [28] make the absence of exponentially growing normal modes very plausible. Whiting [31] has proven that there are no such modes, and in his proof he showed that normal modes grow at most linearly in time. Recent numerical evolution calculations [16, 17] for slowly and fast rotating Kerr black holes show no sign of exponential growth. In the case of massive scalar perturbations of Kerr results of Damour, Deruelle and Ruffini [3], Zouros and

Eardley [34], and Detweiler [3] point to the existence of unstable modes. These modes are very slowly growing with growth times similar to the age of the universe. This fact complicates the numerical detection of such modes.

Here we consider the reduced (in the angular coordinate φ) wave equation and Klein–Gordon equation on a Kerr background and in the framework of C^0-semigroup theory. For this the mathematical framework from [2] is used. For each equation it is shown that the initial value problem is well-posed, i.e., has a unique solution which depends continuously on the data. Further, it is shown that the spectrum of the semigroup's generator coincides with the spectrum of an operator polynomial whose coefficients can be read off from the equation. In this way the problem of deciding stability is reduced to a spectral problem. For the wave equation it is shown that the resolvent of the semigroup's generator and the corresponding Green's functions can be computed using spheroidal functions. It is to be expected that, analogous to the case of a Schwarzschild background, the quasinormal frequencies of the Kerr black hole appear as poles of the analytic continuation of this resolvent. Finally, the stability of the background with respect to reduced massive perturbations is proven for large enough masses. This is done by applying an abstract stability criterium from [2].

The Kerr metric in Boyer-Lindquist coordinates t, r, θ, φ is given by

$$g = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar\sin^2\theta}{\Sigma} dt d\varphi - \frac{\Sigma}{\triangle} dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2}{\Sigma}\right) \sin\theta d\varphi^2, \tag{1}$$

where M is the mass, $a \in [0, M]$ is the rotational parameter,

$$\triangle := r^2 - 2Mr + a^2, \quad \Sigma := r^2 + a^2\cos^2\theta. \tag{2}$$

The coordinates are constrained by $-\infty < t < +\infty$, $r_+ < r < +\infty$, $-\pi < \varphi < \pi$ and $0 < \theta < \pi$, where

$$r_+ := M + \sqrt{M^2 - a^2}. \tag{3}$$

As a little reminder on the Kerr geometry we give the following basic facts relevant for the discussion of the wave equation. The coordinate vector field $\partial/\partial r$ becomes singular at $r = r_+$. This value of the radial coordinate marks the event horizon for Kerr spacetime. The coordinate vectorfield $\partial/\partial t$ is *null* on the ergosphere

$$r = M + \sqrt{M^2 - a^2\cos^2\theta}, \tag{4}$$

is *spacelike* inside and *timelike* outside. So *t is not a time coordinate inside the ergosphere* and therefore one might think that Boyer-Lindquist coordinates are unsuitable for a stability discussion. It turns out that this is not the case for the methods (from semigroup theory) of this paper. Finally, the Kerr metric is globally hyperbolic outside the horizon and hence the Cauchy problem for the scalar wave equation is well posed for data on any Cauchy surface. This result is not used in this paper. Existence, uniqueness and continuous dependence of the solutions on the initial data is proved here, too.

The reduced wave equation governing solutions of the form $\psi(t, r, \theta, \phi) = \exp(im\varphi)$ $u(t, r, \theta)$, where m runs through all integers, is given by

$$\frac{\partial^2 u}{\partial t^2} + \left[\left(\frac{(r^2+a^2)^2}{\triangle} - a^2 \sin^2\theta \right)^{-1} \cdot \left(i\, \frac{4mMar}{\triangle} \frac{\partial u}{\partial t} - \frac{\partial}{\partial r} \triangle \frac{\partial}{\partial r} - \frac{m^2 a^2}{\triangle} - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta} \right) \right] u = 0. \quad (5)$$

A first inspection shows that

$$0 < \frac{4mMar}{\triangle} \left(\frac{(r^2+a^2)^2}{\triangle} - a^2 \sin^2\theta \right)^{-1} \leqslant \frac{ma}{Mr_+} \quad (6)$$

on $\Omega := (r_+, \infty) \times (0, \pi)$. Hence the coefficient multiplying $i\,\partial u/\partial t$ is real-valued, strictly positive and bounded. Moreover the coeffcients multiplying the derivatives in r and θ are real-valued and vanish at the horizon. As a consequence (5) *is singular at all points on the boundary of* Ω.

The structure of this paper is as follows. Section 2 contains the used conventions. Section 3 gives an initial value formulation for (5). There the equation is interpreted as an abstract equation

$$\begin{pmatrix} u \\ v \end{pmatrix}' = -G \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} -v \\ (A+C)u + iBv \end{pmatrix} \quad (7)$$

for a differentiable function ${}^t(u, v)$ assuming values in an appropriate Hilbert space Y and in particular in the domain of a linear operator G. Here $'$ denotes a Hilbert space derivative. The linear operators $A+C$ and B will be read off from (5). B is the maximal multiplication operator in X given by the function multiplying $i\,\partial u/\partial t$. The auxiliary operator C is a suitable negative multiple of the identity operator on X. The definition of A is more involved. A preliminary form $A_0 + C$ of $A + C$ is given by the differential operator enclosed in square brackets. It has as a domain all complex-valued functions on Ω which are twice continuously differentiable and have a compact support in Ω. Moreover X is chosen such that $A_0 + C$ is symmetric. It will be obvious that this operator is in addition semibounded (from below). In the next step $A + C$ will be defined as the *Friedrichs extension* of $A_0 + C$. Using this existence, uniqueness, and continuous dependence of the solutions on the initial data (7) follow from abstract theorems derived in [2]. In Sect. 4 the domain of A will be investigated further. This is done for two reasons. Firstly, to make sure that it contains functions having a reasonable behaviour, both, on the axis of symmetry and on the horizon. Secondly, such information is needed as a basis for Sect. 5. There the resolvent of G is constructed using spheroidal functions. Section 6 discusses the reduced Klein–Gordon equation. The well-posedness of the initial value problem is shown. Further, the stability of the solutions is shown for large enough masses. Section 7 contains the discussion. The Appendix gives auxiliary theorems used in the computation of the resolvent.

2. Conventions

The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the natural numbers (including zero), all real numbers and all complex numbers, respectively.

To ease understanding we follow common abuse of notation and don't differentiate between coordinate maps and coordinates. For instance, interchangeably r will denote some real number greater than r_+ or the coordinate projection onto the open interval (r_+, ∞). The definition used will be clear from the context. In addition we assume composition of maps (which includes addition, multiplication, etc.) always to be maximally defined. So for instance the addition of two maps (if at all possible) is defined on the intersection of the corresponding domains.

For each $k \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0\}$ and each non-trivial open subset M of $\mathbb{R}^n$ the symbol $C^k(M, \mathbb{C})$ denotes the linear space of k-times continuously differentiable complex-valued functions on M. Further $C_0^k(M, \mathbb{C})$ denotes the subspace of $C^k(M, \mathbb{C})$ consisting of those functions which in addition have compact support in Ω.

Throughout the paper Lebesgue integration theory is used in the formulation of [24]. Compare also Chapter III in [11] and Appendix A in [30]. To improve readability we follow common usage and don't differentiate between an almost everywhere (with respect to the chosen measure) defined function f and the associated equivalence class (consisting of all almost everywhere defined functions which differ from f only on a set of measure zero). In this sense $L^2_C(M, \rho)$, where ρ is some strictly positive real-valued continuous function on M, denotes the Hilbert space of complex-valued, square integrable (with respect to the measure $\rho\, d^n x$) functions on M. The scalar product $\langle | \rangle$ on $L^2_C(M, \rho)$ is defined by

$$\langle f | g \rangle := \int_M f^* g \, \rho \, d^n x, \tag{8}$$

for all $f, g \in L^2_C(M, \rho)$, where * denotes complex conjugation on $\mathbb{C}$. It is a standard result of functional analysis that $C_0^k(M, \mathbb{C})$ is dense in $L^2_C(M, \rho)$.

Finally, throughout the paper standard results and nomenclature of operator theory is used. For this compare standard textbooks on Functional analysis, e.g., [23, Vol. I], [24,32].

3. Basic Choices and First Consequences

As the basic Hilbert space X for (7) we chose

$$X := L^2_C\left(\Omega, g^{00}\sqrt{-|g|}\right), \tag{9}$$

where $|g|$ denotes the determinant of the matrix g_{ab}. Note that

$$g^{00}\sqrt{-|g|} = \left(\frac{(r^2 + a^2)^2}{\triangle} - a^2 \sin^2\theta\right) \sin\theta \tag{10}$$

is singular at the horizon. Hence the elements of X vanish there in the mean. In the limit case $a = 0$ this measure reduces to the standard one often used for the stability discussion of the Schwarzschild metric [29,13].

The operator B is chosen as the maximal multiplication operator in X by the function multiplying $i\, \partial u/\partial t$ in (5). Since that function is bounded and positive real-valued, B is a bounded linear and positive self-adjoint operator on X given by

$$Bf = \frac{4mMar}{\triangle}\left(\frac{(r^2 + a^2)^2}{\triangle} - a^2 \sin^2\theta\right)^{-1} f \tag{11}$$

for every $f \in X$. The operator $A_0 + C$ is defined by

$$(A_0 + C)f := D^2_{r\theta} f \tag{12}$$

for all $f \in C^2_0(\Omega, \mathbb{C})$, where we set for every $f \in C^2(\Omega, \mathbb{C})$,

$$D^2_{r\theta} f := \left(\frac{(r^2+a^2)^2}{\triangle} - a^2 \sin^2\theta \right)^{-1} \cdot \\ \left(-\frac{\partial}{\partial r} \triangle \frac{\partial}{\partial r} - \frac{m^2 a^2}{\triangle} - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta} \right) f. \tag{13}$$

Then $A_0 + C$ is in particular linear and (using partial integration) symmetric. Further again by partial integration it is easy to see that $A_0 + C$ is semibounded with the lower bound $-\alpha$, where $\alpha := m^2 a^2/(4M^2 r_+^2)$. Note that this bound approaches $-\infty$ for $|m| \to \infty$, which would suggest that the unreduced wave equation on Kerr background would have no stable initial value problem. Also note that it approaches 0 for $a \to 0$ which is the optimal bound for Schwarzschild.

In the next step we define $C := -(\alpha + \varepsilon)$, where $\varepsilon > 0$ is assumed to have the dimension l^{-2}. The exact value of ε does not influence the results in any essential way. Finally, we define A as the Friedrichs extension of A_0. As a consequence A is a densely-defined, linear, selfadjoint and semibounded operator having the same lower bound as A_0, i.e., ε.

The objects X, A, B and C are easily seen to satisfy Assumptions 1 and 4 of [2]. Applying the results of that paper gives

Theorem 1. (i) *By*

$$Y := D(A^{1/2}) \times X \tag{14}$$

and

$$(\xi|\eta) := \langle A^{1/2}\xi_1 | A^{1/2}\eta_1 \rangle + \langle \xi_2 | \eta_2 \rangle \tag{15}$$

for all $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in Y$ *there is defined a complex Hilbert space* $(Y, (\,|\,))$.

(ii) *The operators* G *and* $-G$ *defined by*

$$G(\xi, \eta) := (-\eta, (A + C)\xi + iB\eta) \tag{16}$$

for all $\xi \in D(A)$ *and* $\eta \in D(A^{1/2})$ *are infinitesimal generators of strongly continuous semigroups* $T_+ : [0, \infty) \to L(Y, Y)$ *and* $T_- : [0, \infty) \to L(Y, Y)$*, respectively.*

(iii) *For all* $t \in [0, \infty)$*:*

$$|T_\pm(t)| \leqslant \exp(\|C\| \, \|A^{-1/2}\| t)\,, \tag{17}$$

where $|\,|$, $\|\,\|$ *denote the operator norm for* $L(Y, Y)$ *and* $L(X, X)$*, respectively.*

(iv) *For every* $t_0 \in \mathbb{R}$ *and every* $\xi \in D(A) \times D(A^{1/2})$ *there is a uniquely determined differentiable map* $u : \mathbb{R} \to Y$ *such that*

$$u(t_0) = \xi \tag{18}$$

and

$$u'(t) = -Gu(t) \tag{19}$$

for all $t \in \mathbb{R}$. *Here* $'$ *denotes differentiation of functions assuming values in* Y. *Moreover this* u *is given by*

$$u(t) := \begin{cases} T_+(t)\xi & \text{for } t \geqslant 0 \\ T_-(-t)\xi & \text{for } t < 0 \end{cases} \tag{20}$$

for all $t \in \mathbb{R}$.

(v) $\lambda \in \mathbb{C}$ *is a spectral value, eigenvalue of* iG *if and only if*

$$A + C - \lambda B - \lambda^2 \tag{21}$$

is not bijective and not injective, respectively.

(vi) *For any* λ *from the resolvent set of* iG *and any* $\eta = (\eta_1, \eta_2) \in Y$ *one has:*

$$(iG - \lambda)^{-1}\eta = (\xi, i(\lambda\xi + \eta_1))\,, \tag{22}$$

where

$$\xi := (A + C - \lambda B - \lambda^2)^{-1}\,[(B + \lambda)\eta_1 - i\eta_2]\,. \tag{23}$$

Equation (19) is the interpretation of (5) used in this paper. In this sense (iv) shows the well-posedness of the initial value problem for (5), i.e., the existence and uniqueness of the solution and its continuous dependence on the initial data. Moreover (20) gives a representation of the solution and (iii) gives a rough bound for its growth in time. In general, this bound is not strong enough to imply stability of the solutions to (5). Part (v) reduces the determination of the generator's spectrum to the determination of the spectrum of the operator polynomial $A + C - \lambda B - \lambda^2$, $\lambda \in \mathbb{C}$ [19,26]. Moreover (vi) does the same for the resolvents. Further, [2] gives the following stability criteria:

Theorem 2. (i) *If*

$$\langle \xi | (A + C)\xi \rangle + \frac{1}{4}\langle \xi | B\xi \rangle^2 \;\geqslant 0$$

for all $\xi \in D(A)$ *with* $\|\xi\| = 1$, *then the spectrum of* iG *is real.*

(ii) *If* $A + C - (b/2)B - (b^2/4)$ *is positive for some* $b \in \mathbb{R}$, *then the spectrum of* iG *is real and there are* $K \geqslant 0$ *and* $t_0 \geqslant 0$ *such that*

$$|u(t)| \leqslant Kt$$

for all $t \geqslant t_0$.

Here $\| \, \|$, $| \, |$ denote the induced norm on $(X, \langle | \rangle)$ and $(Y, (\, | \,))$, respectively. Note that the reality of the generator's spectrum would exclude the existence of exponentially growing mode solutions of (5). It seems that these criteria are *not* strong enough to prove stability of the solutions of (5).[1] But later on (ii) will be used to conclude stability for the corresponding Klein–Gordon equation for cases where the mass exceeds some given bound depending on m. Note that the positivity of $A_0 + C$ would imply stability via (ii). On first sight positivity of $A_0 + C$ seems unlikely because of the negative potential term $-m^2 a^2/\triangle$ in (13). On the other hand it is well-known that the occurrence of such a term can be due to the chosen representation space for $A_0 + C$. In addition the domain of this operator is very much restricted by the condition that its elements have compact support in Ω. Since Ω is open it follows that the support of such a function has a strictly positive distance from the boundary. In the theory of Schrödinger operators it is well-known from so-called "Poincare-Sobolev inequalities" that the kinetic energy associated with such a state can exceed a negative potential energy. See, e.g., [33] or for a simple example [23] Vol. II, Example 1 in Chapter X.3. Indeed such inequalities were found, but only ones leading to a positive potential term with asymptotic behaviour $\sim \triangle^{-\beta}$ for $r \to r_+$, where $0 \leqslant \beta < 1$. So none of them was found to be strong enough to show positivity of $A + C$. Indeed the apparent absence of better estimates lead to the impression that $A + C$ is indeed negative. If this is really true it should be easy to prove using the results on the domain of $A + C$ from the next section. This point has not been investigated further, because the negativity alone would not give any further information on the stability of the solutions of (5).

4. Investigation of the Domain of $A + C$

In this section the domain of $A + C$ will be further investigated. This is done for two reasons. Firstly, to make sure that it contains functions having a reasonable behaviour, both, on the axis of symmetry and on the horizon. It turns out that this is indeed the case. In particular, as it should be the case, functions of the form $f(r) P^l_m(\cos\theta)$, where $f \in C^2_0(I_r, \mathbb{C})$ and P^l_m, $l = |m|, |m| + 1, \ldots$, are the usual generalized Legendre polynomials are found to be in the domain of $A + C$. Secondly such information is needed as a basis for the construction of the resolvent of G in the next section.

We do not give a full characterization of $D(A + C)$ here. Instead more modestly sufficient conditions are given on functions $f(r)$ and $g(\theta)$ which guarantee that the product $f(r)g(\theta)$ is in $D(A + C)$. These conditions will turn out to be sufficient as a basis for the next section. They are as follows:

Theorem 3. *For this denote $I_r := (r_+, \infty)$ and $I_\theta := (0, \pi)$ and define*

$$X_r := L^2_{\mathbb{C}}(I_r, r^4/\triangle), \quad X_\theta := L^2_{\mathbb{C}}(I_\theta, \sin\theta), \tag{24}$$

and for every $f \in C^2(I_r, \mathbb{C})$ and $g \in C^2(I_\theta, \mathbb{C})$,

$$D^2_r f := \frac{\triangle}{r^4}\left[-\left(\triangle f'\right)' - \frac{m^2 a^2}{\triangle} f\right], \quad D^2_\theta \, g := -\frac{1}{\sin\theta}\left(\sin\theta \, g'\right)' + \frac{m^2}{\sin^2\theta}\, g. \tag{25}$$

[1] In the following discussion the trivial cases $a = 0$, i.e., the case of a Schwarschild background, and $m = 0$ corresponding to purely axial perturbations, are excluded. Of course, for these stability of the solutions can be concluded from Theorem 2(ii).

Let be $f \in C^2(I_r, \mathbb{C}) \cap X_r$ *and* $g \in C^2(I_\theta, \mathbb{C}) \cap X_\theta$ *such that*

$$D_r^2 f \in X_r \quad \textit{and} \quad D_\theta^2 g \in X_\theta \tag{26}$$

and for $m = 0$ *in addition such that*

$$\lim_{\theta \to 0} \sin\theta\, g'(\theta) = \lim_{\theta \to \pi} \sin\theta\, g'(\theta) = 0. \tag{27}$$

Then $f(r)g(\theta) \in D(A + C)$ *and*

$$(A + C) f(r)g(\theta) = D_{r\theta}^2 f(r)g(\theta). \tag{28}$$

Proof. First it follows from the obvious inequalities

$$\frac{r^4}{\triangle} \leqslant \frac{(r^2 + a^2)^2}{\triangle} - a^2 \sin^2\theta \leqslant \frac{4M^2}{r_+^2}\, \frac{r^4}{\triangle}, \tag{29}$$

that $L^2_{\mathbb{C}}(\Omega, r^4 \sin\theta/\triangle)$ and X are identical as sets and that the associated norms on that set are equivalent. A further consequence of (29) along with partial integration is the fact that $f(r)g(\theta)$ is in the domain $D((A_0 + C)^*)$ of the adjoint $(A_0 + C)^*$ of $A_0 + C$ and in particular that

$$(A_0 + C)^* f(r)g(\theta) = D_{r\theta}^2 f(r)g(\theta). \tag{30}$$

Hence $f(r)g(\theta) \in D((A_0 + C)^*)$ if and only if there is a sequence $h_0, h_1, \ldots$ of elements of $C_0^2(\Omega, \mathbb{C})$ converging to $f(r)g(\theta)$ and such that for every given $\varepsilon > 0$ there is $\nu_0 \in \mathbb{N}$ such that for all $\mu, \nu \in \mathbb{N}$ satisfying $\mu \geqslant \nu_0$ and $\nu \geqslant \nu_0$:

$$|\langle h_\mu - h_\nu | (A_0 + C + \alpha)(h_\mu - h_\nu)\rangle| < \varepsilon. \tag{31}$$

In the following the existence of such a sequence will be shown. Basic for this is the following inequality valid for all $u \in C_0^2(I_r, \mathbb{C})$ and $v \in C_0^2(I_\theta, \mathbb{C})$:

$$\begin{aligned} \langle u(r)v(\theta) | (A_0 + C + \alpha) u(r)v(\theta)\rangle &\leqslant \left(\int_{r_+}^{\infty} |u|^2 dr\right)\left(\int_0^{\pi} \sin\theta\, v^* D_\theta^2 v\, d\theta\right) \\ &+ \left(\int_{r_+}^{\infty} r^4 \triangle^{-1} u^* (D_r^2 + m^2a^2/r_+^4) u\, dr\right)\left(\int_0^{\pi} \sin\theta\, |v(\theta)|^2\, d\theta\right) \\ &\leqslant \left(\int_{r_+}^{\infty} r^4 \triangle^{-1} u^* (D_r^2 + m^2a^2/r_+^4 + r_+^{-2}) u\, dr\right)\left(\int_0^{\pi} \sin\theta\, v^* (D_\theta^2 + 1) v\, d\theta\right). \end{aligned} \tag{32}$$

Here some elementary estimates have been used along with the positivity of $D_r^2 + m^2a^2/r_+^4$ on $C_0^2(I_r, \mathbb{C}) \subset X_r$. Since $A + C + \alpha$ is in particular positive also the following inequality is valid for all $u_1, u_2 \in C_0^2(I_r, \mathbb{C})$ and $v_1, v_2 \in C_0^2(I_\theta, \mathbb{C})$:

$$\begin{aligned} &|\langle u_1(r)v_1(\theta) - u_2(r)v_2(\theta) | (A_0 + C + \alpha)[u_1(r)v_1(\theta) - u_2(r)v_2(\theta)]\rangle| \\ &= \|(A + C + \alpha)^{1/2}[u_1(r) - u_2(r)]v_1(\theta) + (A + C + \alpha)^{1/2} u_2(r)[v_1(\theta) - v_2(\theta)]\|^2 \\ &\leqslant 2\langle [u_1(r) - u_2(r)]v_1(\theta) | (A_0 + C + \alpha)[u_1(r) - u_2(r)]v_1(\theta)\rangle \\ &\quad + 2\langle u_2(r)[v_1(\theta) - v_2(\theta)] | (A_0 + C + \alpha) u_2(r)[v_1(\theta) - v_2(\theta)]\rangle, \end{aligned} \tag{33}$$

where $\alpha' := m^2a^2/r_+^4 + r_+^{-2}$. Since f is in the domain of the Friedrichs extension of D_r^2 on $C_0^2(I_r, \mathbb{C}) \subset X_r$ there is a sequence $f_0, f_1, \dots$ of elements of $C_0^2(I_r, \mathbb{C})$ converging to $f(r)$ and such that for every given $\varepsilon > 0$ there is $\nu_0 \in \mathbb{N}$ such that for all $\mu, \nu \in \mathbb{N}$ satisfying $\mu \geqslant \nu_0$ and $\nu \geqslant \nu_0$:

$$\int_{r_+}^{\infty} r^4 \triangle^{-1} \left(f_\mu - f_\nu\right)^* (D_r^2 + \alpha')(f_\mu - f_\nu)\, dr < \varepsilon. \tag{34}$$

Obviously, by an argument analogous to (33) this implies that the sequence

$$\int_{r_+}^{\infty} r^4 \triangle^{-1} f_\nu^* (D_r^2 + \alpha') f_\nu \, dr, \quad \nu \in \mathbb{N} \tag{35}$$

is bounded. Moreover since g is in the domain of the Friedrichs extension of D_θ^2 on $C_0^2(I_\theta, \mathbb{C}) \subset X_\theta$ there is a sequence $g_0, g_1, \cdots$ of elements of $C_0^2(I_\theta, \mathbb{C})$ converging to $g(\theta)$ and such that for every given $\varepsilon > 0$ there is $\nu_0 \in \mathbb{N}$ such that for all $\mu, \nu \in \mathbb{N}$ satisfying $\mu \geqslant \nu_0$ and $\nu \geqslant \nu_0$:

$$\int_0^{\pi} \sin\theta \, \left(g_\mu - g_\nu\right)^* (D_\theta^2 + 1)\, (g_\mu - g_\nu)\, d\theta < \varepsilon. \tag{36}$$

Here too, by an argument analogous to (33) this implies that the sequence

$$\int_0^{\pi} \sin\theta \, g_\nu^* \, (D_\theta^2 + 1)\, g_\nu \, d\theta, \quad \nu \in \mathbb{N} \tag{37}$$

is bounded. Finally, because of

$$\begin{aligned} &|\langle f_\mu(r) g_\mu(\theta) - f_\nu(r) g_\nu(\theta) | (A_0 + C + \alpha)[f_\mu(r) g_\mu(\theta) - f_\nu(r) g_\nu(\theta)]\rangle| \\ &\leqslant 2 \left(\int_{r_+}^{\infty} r^4 \triangle^{-1} \left(f_\mu - f_\nu\right)^* (D_r^2 + \alpha')(f_\mu - f_\nu)\, dr \right) \\ &\quad \cdot \left(\int_0^{\pi} \sin\theta \, g_\mu^* (D_\theta^2 + 1) g_\mu \, d\theta \right) \\ &\quad + 2 \left(\int_{r_+}^{\infty} r^4 \triangle^{-1} f_\nu^* (D_r^2 + \alpha') f_\nu \, dr \right) \\ &\quad \cdot \left(\int_0^{\pi} \sin\theta \, \left(g_\mu - g_\nu\right)^* (D_\theta^2 + 1) \left(g_\mu - g_\nu\right) \, d\theta \right), \end{aligned} \tag{38}$$

the sequence $h_0, h_1, \dots$ defined by

$$h_\nu := f_\nu(r) g_\nu(\theta), \quad \nu \in \mathbb{N} \tag{39}$$

has the required properties.

In the proof we have used facts on the Sturm-Liouville operators D_r^2 and D_θ^2. Now, for the reader's convenience these will be given. For this define the (obviously) linear, symmetric and semibounded operators A_{r0}, $A_{\theta 0}$ in X_r and X_θ, respectively, by

$$A_{r0} f := D_r^2 f, \quad A_{\theta 0} g := D_\theta^2 g, \tag{40}$$

for every $f \in C_0^2(I_r, \mathbb{C})$ and every $g \in C_0^2(I_\theta, \mathbb{C})$. Then one has the following

Lemma 4. (i) *A_{r0} is essentially self-adjoint.*

(ii) *$A_{\theta 0}$ is essentially self-adjoint for $m > 0$. For $m = 0$, the Friedrichs extension of $A_{\theta 0}$ is given by the closure of the operator $A_{\theta F}$ defined by $A_{\theta F} f := D_\theta^2 g$ for every $g \in C^2(I_\theta, \mathbb{C}) \cap X_\theta$ satisfying (27) together with the condition that $D_\theta^2 g \in X_\theta$. For all m the spectrum of the Friedrichs extension of $A_{\theta 0}$ is given by $\{|m|(|m| + 1), (|m| + 1)(|m| + 2), \dots\}$.*

Proof. (i) For this define the auxiliary Sturm–Liouville operator $\hat{A}_{r\,0}$ in X_r by

$$\hat{A}_{r\,0} f := -\frac{\triangle}{r^4} \left(\triangle f'\right)' \tag{41}$$

for every $f \in C_0^2(I_r, \mathbb{C})$. Obviously, $\hat{A}_{r\,0}$ is densely-defined, linear, symmetric and positive. Moreover since $-m^2 a^2/r^4$ is bounded on I_r, it follows by the Rellich–Kato theorem (see, e.g, Theorem X.12 in Volume II of [23]) that A_{r0} is essentially self-adjoint if and only if $\hat{A}_{r0}$ is essentially self-adjoint. Now, the equation $\left(\triangle f'\right)' = 0$ has nonvanishing constants as solutions. Since these are not in X_r at both ends of I_r, it follows that $\hat{A}_{r0}$ is in the limit point case, both, at r_+ and at $+\infty$. Hence $\hat{A}_{r0}$ is essentially self-adjoint (see, e.g., [30]). Finally, from this follows (i). (ii) This statement is, of course, well-known. □

5. Computation of the Generator's Resolvent

In the following the resolvent of G will be determined for spectral parameters λ which are non-real and at the same time such that $ia\lambda$ is not an exceptional value.[2] Note that because of Theorem 1 (vi), the resolvent of G can be derived from the inverses of the operator polynomial $A + C - \lambda B - \lambda^2$ which are given in (ii) of the following theorem on a dense subset of X.

Theorem 5. *Let λ be a non-real element of the resolvent set of iG which moreover is such that $ia\lambda$ is not an exceptional value. For each $m \in \mathbb{Z}$ let*

$$ps_l^m(\cos\theta, -a^2\lambda^2), \quad l = |m|, |m| + 1, |m| + 2, \cdots \tag{42}$$

be the basis[3] of X_θ consisting of spheroidal eigenfunctions of $D_\theta^2 + \lambda^2 a^2 \sin^2\theta$ corresponding to the eigenvalues

$$\lambda_{|m|}^m(-a^2\lambda^2), \lambda_{|m|+1}^m(-a^2\lambda^2), \dots, \tag{43}$$

respectively.[4] Finally, let $g \in C_0(I_r, \mathbb{C})$, $m \in \mathbb{Z}$ and $l \in \{|m|, |m| + 1, \dots\}$. Then

(i) *the subset of X consisting of all finite linear combinations of elements of the form*

$$h(r,\theta) g(r) p_l^m(\cos\theta, -a^2\lambda^2), \tag{44}$$

[2] For the definition of these values see [20].

[3] In the sense that the span of these functions is dense in X_θ. Note that these functions are *not* orthogonal in general. Instead this sequence and the sequence consisting of its complex conjugates form a *biorthogonal* Basis of X_θ. See Theorem 4 in Sect. 3.23 of [20].

[4] For the definition of the functions ps_l^m see [20].

where

$$h := \frac{r^4}{\triangle}\left(\frac{(r^2+a^2)^2}{\triangle} - a^2\sin^2\theta\right)^{-1}, \tag{45}$$

and g, l run through the elments of $C_0(I_r,\mathbb{C})$ and $\{|m|, |m|+1, |m|+2, \dots\}$, respectively, is dense in X;

(ii)

$$\begin{aligned}(A + C - \lambda B - \lambda^2)^{-1} h(r,\theta) g(r) p_l^m(\cos\theta, -a^2\lambda^2) \\ = f_r(r) p_l^m(\cos\theta, -a^2\lambda^2),\end{aligned} \tag{46}$$

where $f_r \in C^2(I_r,\mathbb{C}) \cap X_r$ is such that $D_r^2 f_r \in X_r$ and moreover satisfies

$$D_{r\lambda}^2 f_r + \lambda_l^m(-a^2\lambda^2)(\triangle/r^4) f_r = g. \tag{47}$$

Here for every $\phi \in C^2(I_r,\mathbb{C})$,

$$D_{r\lambda}^2\phi := -\frac{\triangle}{r^4}\left(\triangle\phi'\right)' - \frac{1}{r^4}\left[(ma + 2\lambda Mr)^2 + \lambda^2\triangle(\triangle + 4Mr)\right]\phi. \tag{48}$$

Proof. First we notice that h is C^∞ on Ω and satisfies as a consequence of (29,)

$$r_+^2/(4M^2) \leqslant h \leqslant 1. \tag{49}$$

Hence the maximal multiplication operator T_h by h in X is defined on the whole of X, is bijective and its inverse is given by the maximal multiplication operator $T_{1/h}$ which is defined on the whole of X, too, by the function $1/h$ in X. Obviously, the subset of X consisting of all finite linear combinations of elements of the form

$$g(r) p_l^m(\cos\theta, -a^2\lambda^2), \tag{50}$$

where $g \in C_0(I_r,\mathbb{C})$ and $l = |m|, |m|+1, |m|+2, \dots$ is dense in X. Hence this is true for the subset of X consisting of all finite linear combinations of elements of the form

$$h(r,\theta) g(r) p_l^m(\cos\theta, -a^2\lambda^2), \tag{51}$$

where $g \in C_0(I_r,\mathbb{C})$ and $l = |m|, |m|+1, |m|+2, \dots$, too. In the following let g be some element of $C_0(I_r,\mathbb{C})$ and l be some element of $\{|m|, |m|+1, |m|+2, \dots\}$. We will compute the element $f \in X$ satisfying

$$\left(A + C - \lambda B - \lambda^2\right) f(r,\theta) = h(r,\theta) g(r) p_l^m(\cos\theta, -a^2\lambda^2). \tag{52}$$

We note that by Theorem 2

$$\begin{aligned}&\left(A + C - \lambda B - \lambda^2\right) f_r(r) p_l^m(\cos\theta, -a^2\lambda^2) \\ &\quad = h(r,\theta)\left[\left(D_{r\lambda}^2 f_r + \lambda_l^m(-a^2\lambda^2)(\triangle/r^4) f_r\right)(r)\right] p_l^m(\cos\theta, -a^2\lambda^2)\end{aligned} \tag{53}$$

for every $f_r \in C^2(I_r,\mathbb{C}) \cap X_r$ such that

$$D_r^2 f_r \in X_r. \tag{54}$$

In the following we construct such a f_r which satisfies in particular (47). Then by the bijectivity of $A + C - \lambda B - \lambda^2$ we conclude that

$$f(r,\theta) = f_r(r)p_l^m(\cos\theta, -a^2\lambda^2). \tag{55}$$

For this construction we need some auxiliary solutions f_1, f_2, f_3 and f_4 of the homogeneous equation associated with (47), i.e.,

$$f_r'' + \frac{2(r-M)}{\triangle}f_r' + \left[\frac{(ma+2\lambda Mr)^2}{\triangle^2} + \lambda^2\left(1+\frac{4Mr}{\triangle}\right) + \frac{s}{\triangle}\right]f_r = 0, \tag{56}$$

where $s := \lambda_l^m(-a^2\lambda^2)$, having special asymptotic behaviour at the singular point $r = r_+$ and at $+\infty$. First, by defining $\bar{f}_r := \triangle^{1/2} f_r$ and by introducing the new independent variable r_*,

$$r_* := \sqrt{r(r+4M)} + 2M\ln\left((\sqrt{r+4M}+\sqrt{r})^2/(4M)\right), \tag{57}$$

one gets a homogeneous first order system for $\bar{f}_r$ and $d\bar{f}_r/dr_*$ which is equivalent to (56) and which satisfies the assumptions of Theorem 4 in the Appendix. From this theorem follows the existence of linear independent continuously differentiable solutions $(\bar{f}_{r1}, d\bar{f}_{r1}/dr_*)$ and $(\bar{f}_{r2}, d\bar{f}_{r2}/dr_*)$ of the system along with continuously differentiable functions R_1 and R_2 such that

$$\begin{aligned}
&\bar{f}_{r1}(r_*) = e^{i\lambda r_*}\left(1+R_{11}(r_*)\right), \quad \frac{d\bar{f}_{r1}}{dr_*}(r_*) = e^{i\lambda r_*}\left(i\lambda + R_{12}(r_*)\right),\\
&\bar{f}_{r2}(r_*) = e^{-i\lambda r_*}\left(1+R_{21}(r_*)\right), \quad \frac{d\bar{f}_{r2}}{dr_*}(r_*) = e^{-i\lambda r_*}\left(-i\lambda + R_{22}(r_*)\right),\\
&\lim_{r_*\to\infty}|R_1(r_*)| = \lim_{r_*\to\infty}|R_2(r_*)| = 0.
\end{aligned} \tag{58}$$

In the following denote by f_{r1}, f_{r2} the solutions of (56) corresponding to $(\bar{f}_{r1}, d\bar{f}_{r1}/dr_*)$ and $(\bar{f}_{r2}, d\bar{f}_{r2}/dr_*)$, respectively. Morover define

$$f_{rR} := \begin{cases} f_{r1} \text{ for } \mathrm{Im}(\lambda) > 0 \\ f_{r2} \text{ for } \mathrm{Im}(\lambda) < 0\,. \end{cases} \tag{59}$$

Then it follows by (58) that $\phi f_{rR} \in C^2(I_r, \mathbb{C}) \cap X_r$ and $D_r^2(\phi f_{rR}) \in X_r$ for every $\phi \in C^2(I_r, \mathbb{R})$ which is identically 0 for $r < r_0$ and identically 1 for $r > r_1$, where $r_0, r_1 \in I_r$ are such that $r_0 < r_1$, but otherwise arbitrary. For the second step by defining $g_1 := f_r/\triangle$ and $g_2 := f'$ one gets a homogeneous first order system for (g_1, g_2) which is equivalent to (56) and which satisfies the assumptions of Corollary 5 in the Appendix. From this corollary follows the existence of linear independent continuously differentiable solutions (g_{11}, g_{12}) and (g_{21}, g_{22}) of the system along with continuously differentiable functions R_3 and R_4 such that

$$\begin{aligned}
&g_{11}(r) = (r-r_+)^{-\sigma_1}[1+R_{31}(r)],\\
&g_{12}(r) = (r-r_+)^{-\sigma_1}[-i(ma+2\lambda Mr_+)+R_{32}(r)],\\
&g_{21}(r) = (r-r_+)^{-\sigma_2}[1+R_{41}(r)],\\
&g_{22}(r) = (r-r_+)^{-\sigma_2}[i(ma+2\lambda Mr_+)+R_{42}(r)],\\
&\lim_{r_*\to r_+}|R_3(r_*)| = \lim_{r_*\to r_+}|R_3(r_*)| = 0,
\end{aligned} \tag{60}$$

where

$$\sigma_1 := \left[\sqrt{M^2 - a^2} + i\,((ma/2) + \lambda M r_+)\right] / \sqrt{M^2 - a^2},$$
$$\sigma_2 := \left[\sqrt{M^2 - a^2} - i\,((ma/2) + \lambda M r_+)\right] / \sqrt{M^2 - a^2}. \tag{61}$$

In the following denote by f_{r3}, f_{r4} the solutions of (56) corresponding to (g_{11}, g_{12}) and (g_{21}, g_{22}), respectively. Moreover define

$$f_{rL} := \begin{cases} f_{r3} \text{ for } \mathrm{Im}(\lambda) > 0 \\ f_{r4} \text{ for } \mathrm{Im}(\lambda) < 0\,. \end{cases} \tag{62}$$

Then it follows by (60) that $\phi f_{rL} \in C^2(I_r, \mathbb{C}) \cap X_r$ and $D_r^2(\phi f_{rL}) \in X_r$ for every $\phi \in C^2(I_r, \mathbb{R})$ which is identically 1 for $r < r_0$ and identically 0 for $r > r_1$, where $r_0, r_1 \in I_r$ are such that $r_0 < r_1$, but otherwise arbitrary.

In the next step we notice that f_{rR} and f_{rL} are linear independent, because otherwise we would get a contradiction to the assumed bijectivity of $A + C - \lambda B - \lambda^2$. Hence the Wronski determinant W of f_{rR} and f_{rL},

$$W := \triangle \left(f_{rL}\, f'_{rR} - f'_{rL}\, f_{rR} \right), \tag{63}$$

is constant and different from 0. Therefore we can define

$$f_r(r) := -\frac{f_{rR}(r)}{W} \int_{r_+}^{r} f_{rL}(r')g(r')dr' - \frac{f_{rL}(r)}{W} \int_{r}^{\infty} f_{rR}(r')g(r')dr' \tag{64}$$

for all $r \in I_r$.

It follows from the foregoing results on f_{rL} and f_{rR} and from a simple computation that $f_r \in C^2(I_r, \mathbb{C}) \cap X_r$, $D_r^2 f_r \in X_r$ and that f_r satisfies (47). Finally, from the bijectivity of $A + C - \lambda B - \lambda^2$ we conclude (46). □

6. The Case of the Klein–Gordon Equation

Compared to the wave equation considered in the previous sections, the only change in this case is that the operator C has to be substituted by $C' := C + (m_0^2/g^{00})$, where m_0 denotes the mass of the field and m_0^2/g^{00} is the maximal multiplication operator in X, which is defined on the whole of X as well as bounded, since this function is easily seen to be bounded on Ω. The other objects X, A and B stay the same. Again it is easy to verify that X, A, B and C' satisfy Assumptions 1 and 4 of [2]. As a consequence one has theorems analogous to Theorem 1 and Theorem 2. They imply the well-posedness of the initial value problem, i.e., the existence and uniqueness of the solution and its continuous dependence on the initial data. Further, via the analogue of Theorem 2 (ii), Theorem 7 below implies for masses satisfying (69), that the spectrum of the corresponding generator is real and that the norm of the solutions grow at most linearly in time. In particular there are no exponentially growing modes in these cases.

Lemma 6. *Let B' be a bounded linear and self-adjoint operator on X. Then $A + B'$ is identical to the Friedrichs extension of $A_0 + B'$.*

Proof. First, since B' is bounded linear and self-adjoint on X, it follows that

$$(A_0 + B')^* = A_0^* + B'. \tag{65}$$

Hence the domain of the Friedrichs extension $(A_0 + B')_F$ of $A_0 + B'$ is given by those elements f from $D(A_0^*)$ for which there is a sequence $f_0, f_1, \dots$ in $D(A_0)$ converging to f and such that for every $\delta > 0$ there is a corresponding $\nu_0 \in \mathbb{N}$ such that for all $\mu, \nu \in \mathbb{N}$,

$$|\langle f_\mu - f_\nu | (A_0 + B' + \|B'\|)(f_\mu - f_\nu)\rangle| < \delta \tag{66}$$

if, both, $\mu > \mu_0$ and $\nu > \nu_0$. Since (66) implies

$$|\langle f_\mu - f_\nu | A_0(f_\mu - f_\nu)\rangle| < \delta, \tag{67}$$

it follows that f is an element of $D(A)$, too. Further, (65) implies

$$(A_0 + B')_F f = (A_0 + B')^* f = Af + B'f. \tag{68}$$

Hence $A + B'$ is a linear self-adjoint (by the Rellich-Kato theorem, see, e.g, Theorem X.12 in Volume II of [23]) extension of $(A_0 + B')_F$ and, finally, since $(A_0 + B')_F$ is self-adjoint, $(A_0 + B')_F = A + B'$. □

Theorem 7. *Define* $b := ma/(Mr_+)$ *and let be*

$$m_0 \geqslant \frac{|m|a}{2Mr_+}\sqrt{1 + \frac{2M}{r_+} + \frac{a^2}{r_+^2}}. \tag{69}$$

Then

$$A + C + m_0^2/g^{00} + (b/2)B - b^2/4 \tag{70}$$

is positive.

Proof. Because of the preceding lemma it is enough to prove that

$$\langle f | (A_0 + C + m_0^2/g^{00} + (b/2)B - b^2/4)f\rangle \geqslant 0 \tag{71}$$

for all $f \in C_0^2(\Omega, \mathbb{C})$. Now let f be such an element. Since its support supp(f) is a compact subset of Ω there are $r_0 > r_+$ and $r_1 > r_0$ such that supp$(f) \subset J \times (0, \pi) \subset \Omega$, where $J := (r_0, r_1)$. In a first step one gets by partial integration, Fubini's theorem and Lemma 4 (ii),

$$\begin{aligned}
&\langle f | (A_0 + C) f\rangle \\
&\quad = \int_\Omega \sin\theta\, f^* \left(-\frac{\partial}{\partial r}\triangle\frac{\partial}{\partial r} - \frac{m^2 a^2}{\triangle} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta}\right) f\, dr\, d\theta \\
&\quad = \int_0^\pi \left[\int_{r_0}^{r_1} f_\theta^* \left(-\frac{d}{dr}\triangle\frac{d}{dr} - \frac{m^2a^2}{\triangle}\right) f_\theta\, dr\right] \sin\theta\, d\theta \\
&\qquad + \int_{r_0}^{r_1} \left[\int_0^\pi \sin\theta\, f_r^* \left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \frac{m^2}{\sin^2\theta}\right) f_r\, d\theta\right] dr \\
&\quad \geqslant \int_0^\pi \left[\int_{r_0}^{r_1} f_\theta^* \left(|m|(|m|+1) - \frac{m^2a^2}{\triangle}\right) f_\theta\, dr\right] \sin\theta\, d\theta.
\end{aligned}$$

Further using

$$\langle f|f/g^{00}\rangle \geqslant \int_0^\pi \left[\int_{r_0}^{r_1} r^2\,|f_\theta|^2\,dr\right]\sin\theta d\theta, \tag{72}$$

$$\langle f|Bf\rangle = 4mMa\int_0^\pi \left[\int_{r_0}^{r_1} \frac{r}{\triangle}\,|f_\theta|^2\,dr\right]\sin\theta d\theta, \tag{73}$$

$$\langle f|f\rangle \leqslant \int_\Omega \frac{(r^2+a^2)^2}{\triangle}\,\sin\theta\,|f|^2\,dr\,d\theta, \tag{74}$$

we get

$$\begin{aligned}
&\langle f|\left(A_0 + C + m_0^2/g^{00} + (b/2)B - b^2/4\right) f\rangle\\
&\geqslant \int_0^\pi \left[\int_{r_0}^{r_1} f_\theta^* \left(|m|(|m|+1) - \frac{m^2a^2}{r_+^2}\frac{r-r_+}{r-r_-} + m_0^2 r^2\right.\right.\\
&\qquad\left.\left. -\frac{m^2a^2}{4M^2r_+^2}(r^2+2Mr+a^2)\right) f_\theta\,dr\right]\sin\theta\,d\theta\\
&\geqslant \int_0^\pi \left[\int_{r_0}^{r_1} f_\theta^* \left(|m| + m^2\left(1-\frac{a^2}{r_+^2}\right)\right.\right.\\
&\qquad\left.\left. +\frac{m^2a^2}{4M^2r_+}\cdot\left[(r_+^2+2Mr_+ + a^2)r^2 - r_+^2(r^2+2Mr+a^2)\right]\right) f_\theta\,dr\right]\sin\theta\,d\theta \geqslant 0.
\end{aligned} \tag{75}$$

Hence the positivity of $A_0 + C + m_0^2/g^{00} + (b/2)B - b^2/4$ follows. □

7. Discussion

The reduced (in the angular coordinate φ) wave equation and Klein–Gordon equation were considered on a Kerr background and in the framework of C^0-semigroup theory. Each equation was shown to have a well-posed initial value problem, i.e., to have a unique solution depending continuously on the data. Further, it was proved that the spectrum of the semigroup's generator coincides with the spectrum of an operator polynomial whose coefficients can be read off from the equation. In this way the problem of deciding stability is reduced to a spectral problem. In addition a mathematical basis is provided for mode considerations.[5] For the wave equation it was shown that the resolvent of the semigroup's generator and the corresponding Green's functions can be computed using spheroidal functions. It is to be expected that, analogous to the case of a Schwarzschild background, the quasinormal frequencies of the Kerr black hole appear as *resonances*, i.e., poles of the analytic continuation of this resolvent. Finally, stability of the background with respect to reduced massive perturbations was proved for masses exceeding a given bound (see (69)).

[5] To give an example for this claim, say, we would be able to show that the unstable spectrum of G consists of discrete eigenvalues and that the corresponding eigenstates seperate in the way assumed by Whiting. Then, via the results of this paper, Whiting's result [31] on the absence of exponentially growing modes would imply the stability of the solutions of the wave equation.

It is interesting to compare the last result to earlier results of Detweiler in [4], Damour, Deruelle, Ruffini in [3] and Zouros, Eardley in [34]. These make the existence of exponentially growing modes for the massive Klein–Gordon equation very plausible. They found approximate unstable modes in the superradient regime, i.e., with frequencies ω satisfying $Re(\omega) < ma/(2Mr_+)$. These modes become stable when this condition is violated. The approximations made in these papers lead to further restrictions. It turns out that the assumption of, both, these restrictions and the bound (69) derived here is *incompatible* with the assumption of superradience. Hence the stability result here does not contradict the results in these papers, but is complementary instead.[6] Moreover it suggests that the negation of an inequality of the form of (69) (or some equivalent form) *is* the superradient condition. For this it should be noted that with some effort and along the lines of this paper it may be possible to improve (69), i.e., to decrease the bound. For this the Poincare–Sobolev inequalities mentioned at the end of Section 3 should be helpful.

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8. Appendix

The following theorem used in Sect. 5 was first proved by Dunkel in [5] (compare also [18,1,9]).

Theorem 8. *Let $n \in \mathbb{N} \setminus \{0\}$, $a \in \mathbb{R}$, $I := [a, \infty)$ and $I_0 := (a, \infty)$. In addition let A_0 be a diagonalizable complex $n \times n$ matrix and $e'_1, \dots, e'_n$ be a basis of $\mathbb{C}^n$ consisting of eigenvectors of A_0. Further, for each $j \in \{1, \dots, n\}$ let λ_j be the eigenvalue corresponding to e'_j and P_j be the matrix representing the projection of $\mathbb{C}^n$ onto $\mathbb{C}.e'_j$ with respect to the canonical basis of $\mathbb{C}^n$. Finally, let A_1 be a continuous map from I into the complex $n \times n$ matrices $M(n \times n, \mathbb{C})$ such that A_{1jk} is Lebesgue integrable for each $j, k \in 1, \dots, n$.*

Then there is a C^1 map $R : I_0 \to M(n \times n, \mathbb{C})$ with $\lim_{t\to\infty} R_{jk}(t) = 0$ for each $j, k \in 1, \dots, n$ and such that $u : I_0 \to M(n \times n, \mathbb{C})$ defined by

$$u(t) := \sum_{j=1}^{n} \exp(\lambda_j t) \cdot (E + R(t)) \cdot P_j \tag{76}$$

for all $t \in I_0$ (where E is the $n \times n$ unit matrix), maps into the invertible $n \times n$ matrices and satisfies

$$u'(t) = (A_0 + A_1(t)) \cdot u(t) \tag{77}$$

for all $t \in I_0$.

This theorem has the following

Corollary 9. *Let $n \in \mathbb{N} \setminus \{0\}$; $a, t_0 \in \mathbb{R}$ with $a < t_0$; $\mu \in \mathbb{N}$; $\alpha_\mu := 1$ for $\mu = 0$ and $\alpha_\mu := \mu$ for $\mu \neq 0$. In addition let A_0 be a diagonalizable complex $n \times n$ matrix and $e'_1, \dots, e'_n$ be a basis of $\mathbb{C}^n$ consisting of eigenvectors of A_0. Further, for each $j \in$*

[6] The author is very grateful to J. L. Friedman for directing his attention to this fact.

$\{1, \ldots, n\}$ *let* λ_j *be the eigenvalue corresponding to* e'_j *and* P_j *be the matrix representing the projection of* $\mathbb{C}^n$ *onto* $\mathbb{C}.e'_j$ *with respect to the canonical basis of* $\mathbb{C}^n$*. Finally, let* A_1 *be a continuous map from* (a, t_0) *into the complex* $n \times n$ *matrices* $M(n \times n, \mathbb{C})$ *for which there is a number* $c \in (a, t_0)$ *such that the restriction of* A_{1jk} *to* $[c, t_0)$ *is Lebesgue integrable for each* $j, k \in 1, \ldots, n$*.*

Then there is a C^1 *map* $R : (a, t_0) \to M(n \times n, \mathbb{C})$ *with* $lim_{t \to 0} R_{jk}(t) = 0$ *for each* $j, k \in 1, \ldots, n$ *and such that* $u : (a, t_0) \to M(n \times n, \mathbb{C})$ *defined by*

$$u(t) := \begin{cases} \sum_{j=1}^{n} (t_0 - t)^{-\lambda_j} \cdot (E + R(t)) \cdot P_j & \text{for } \mu = 0 \\ \sum_{j=1}^{n} \exp(\lambda_j (t_0 - t)^{-\mu}) \cdot (E + R(t)) \cdot P_j & \text{for } \mu \neq 0 \end{cases} \tag{78}$$

for all $t \in (a, t_0)$ *(where* E *is the* $n \times n$ *unit matrix), maps into the invertible* $n \times n$ *matrices and satisfies*

$$u'(t) = \left(\frac{\alpha_\mu}{(t_0 - t)^{\mu+1}} A_0 + A_1(t) \right) \cdot u(t) \tag{79}$$

for each $t \in (a, t_0)$*.*

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