



## The Principal $SO(1, 2)$ Subalgebra of a Hyperbolic Kac–Moody Algebra<sup>★</sup>

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**Abstract.** The analog of the principal  $SO(3)$  subalgebra of a finite-dimensional simple Lie algebra can be defined for any hyperbolic Kac–Moody algebra  $\mathfrak{g}(A)$  associated with a symmetrizable Cartan matrix  $A$ , and coincides with the non-compact algebra  $SO(1, 2)$ . We exhibit the decomposition of  $\mathfrak{g}(A)$  into representations of  $SO(1, 2)$ . With the exception of the adjoint  $SO(1, 2)$  algebra itself, all of these representations are unitary. We compute the Casimir eigenvalues; the associated ‘exponents’ are complex and noninteger.

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### 1. Introduction

Over the past years it has become clear that infinite-dimensional Lie algebras [1,2] play an increasingly important role in modern theoretical physics, and string theory in particular. The close links between string vertex operators and Kac–Moody algebras are well known [3–5]. In particular, there has been mounting evidence that Kac–Moody algebras of indefinite type and generalized Kac–Moody (Borcherds) algebras might appear in the guise of duality symmetries in string and M theory.

Within the context of gravity and supergravity, indications of a more concrete realization of indefinite, and especially hyperbolic, Kac–Moody algebras have emerged very recently from results concerning the generic behavior of solutions of Einstein’s field equations near a spacelike cosmological singularity (see [6,7] and references therein). More specifically, the diagonal gravitational metric degrees of freedom can be identified with the Cartan subalgebra of some underlying indefinite Kac–Moody algebra, such that the resulting dynamics is elegantly described as a massless, relativistic, perfectly elastic billiard moving linearly within the

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associated fundamental Weyl chamber and bouncing off its walls. The presence of chaotic oscillations in this motion then becomes correlated to the hyperbolicity of this algebra.

Possibly not unrelated is the fact that Kac–Moody algebras have also appeared in Toda field theories (see [8,9] and references therein). Hyperbolic Toda theories were considered in [10]. Like the Toda theories based on finite-dimensional Lie algebras, and in contradistinction to the affine Toda theories, they are conformally invariant due to the existence of a Weyl vector for the finite and hyperbolic cases. As also shown in [10], a special feature of the strictly hyperbolic Toda theories is the absence of higher-order (spin 2 and above) local conserved charges, indicating the nonintegrability of these models.

As is well known, [1,2] every Kac–Moody algebra is defined in terms of a Cartan matrix  $A_{ij}$  ( $1 \leq i, j \leq r$ ) satisfying the properties listed on page 1 of [1], and a set of generating elements  $\{e_i, f_i, h_i\}$  subject to a set of generating relations (Chevalley–Serre presentation). The algebra itself can always be represented in the form

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad (1)$$

where  $\mathfrak{h}$  is the Cartan subalgebra (whose dimension equals the rank  $r$ ), and where  $\mathfrak{n}_-$  and  $\mathfrak{n}_+$  are the triangular subalgebras consisting of the independent multiple commutators of the  $e_i$ 's and  $f_i$ 's, respectively. The most interesting (and least known) algebras are the ones associated with indefinite and nondegenerate Cartan matrices. Among them, *hyperbolic* Kac–Moody algebras are distinguished by the extra requirement that the deletion of any node from the Dynkin diagram leaves a subalgebra which is either affine or finite. Unfortunately, beyond the usual ‘Dynkinology’, they remain shrouded in mystery. Root multiplicities are known in closed form only for levels  $|\ell| \leq 2$  [11,12], and in implicit form also for  $\ell = \pm 3$  [13]. While the Lie algebra elements at levels 0 and  $\pm 1$  are completely under control, explicit representations of the root space elements beyond low levels have been worked out only for a few examples and exhibit an exceedingly complicated structure [14,15]. Especially in view of their conjectured applications, understanding the structure of hyperbolic Kac–Moody algebras remains a major challenge.

In this Letter we take a step in this direction by generalizing a tool that has proved to be of great use in the study of finite-dimensional simple Lie groups, namely the concept of the principal  $\text{SO}(3)$  subgroup [17] (its Lie algebra is distinguished amongst all those  $\text{SO}(3)$  subalgebras in the complete classification of [18] by being maximal<sup>\*</sup>). The generalization of this concept to hyperbolic Kac–Moody algebras rests on the fact that, among the set of all Cartan matrices, the finite and hyperbolic ones are singled out by the property that the entries of the associated inverse Cartan

<sup>\*</sup>For an early application of this classification in physics, see [19]; embeddings of the special superalgebra  $\text{OSp}(1|2)$  into larger superalgebras were studied in [20].

matrix have a definite sign, i.e. for all  $i, j \in \{1, \dots, r\}$

$$\begin{aligned} A_{ij}^{-1} &\geq 0 && \text{for finite Cartan matrices,} \\ A_{ij}^{-1} &\leq 0 && \text{for hyperbolic Cartan matrices.} \end{aligned} \tag{2}$$

For definiteness and simplicity of notation, only symmetric Cartan matrices,  $A$ , are considered, and the real roots are assumed to have length  $\sqrt{2}$ ; however, the extension of our results to the nonsimply laced algebras based on symmetrizable Cartan matrices is straightforward. The second inequality in (2) above follows from  $A_{ij}^{-1} = \Lambda_i \cdot \Lambda_j$ , and the fact that the fundamental weights  $\Lambda_i$  all lie in the forward lightcone for hyperbolic Kac–Moody algebras.

The above properties have no analogue for affine Cartan matrices (which by definition are not invertible), and in general will also fail for nondegenerate indefinite Cartan matrices, where the entries of  $A^{-1}$  can assume both signs. We believe that we have thus found another indication of the privileged status enjoyed by the hyperbolic algebras among the indefinite Kac–Moody algebras.

The main new insight of the present work is that, due to the relative switch in sign between finite and hyperbolic Cartan matrices  $A$  in (2), the compact  $SO(3)$  associated with a finite-dimensional Lie group is replaced by the noncompact group  $SO(1, 2)$  for hyperbolic Kac–Moody algebras. With the exception of the adjoint representation, the finite-dimensional representations of  $SO(3)$  are accordingly replaced by infinite-dimensional ones. We will exhibit the basic features arising when the algebra  $\mathfrak{g}(A)$  is decomposed into representations of  $SO(1, 2)$ . The fact that the ‘exponents’ of  $\mathfrak{g}(A)$  now come out to be complex and irrational is presumably related to the fact that hyperbolic Kac–Moody algebras do not admit any polynomial Casimir invariants other than the quadratic Casimir–Kac operator [16].

## 2. The Principal $SO(3)$ Subalgebra of a Finite Kac–Moody Algebra

This algebra exists for every Kac–Moody algebra defined by a positive definite Cartan matrix  $A_{ij}$  (it is a standard result that the positive definiteness of  $A$  implies that  $\mathfrak{g}(A)$  is finite-dimensional [1,2]). It is constructed by means of the Weyl vector  $\rho$ , which is defined to obey  $\rho \cdot \alpha_i = 1$  for all simple roots  $\alpha_i$ . An explicit formula is  $\rho = \sum_j \Lambda_j$  where  $\Lambda_j$  are the fundamental weights satisfying  $\alpha_i \cdot \Lambda_j = \delta_{ij}$ . The diagonal generator of the principal  $SO(3)$  is defined by\*

$$J_3 := \rho \cdot H \implies [J_3, E_\alpha] = \text{ht}(\alpha) E_\alpha, \tag{3}$$

where  $\text{ht}(\alpha)$  denotes the height of the root  $\alpha$  and  $E_\alpha$  the generator(s) associated with the root  $\alpha$ . Then, since the number of simple roots equals the dimension of the Cartan subalgebra,  $r$ , there always exist linear combinations of the step operators for the

\*See [5] for our notations and conventions.

simple roots  $E_{\alpha_i}$  and  $E_{-\alpha_i}$

$$J^+ = \sum n_i E_{\alpha_i}, \quad J^- = \sum n_i E_{-\alpha_i} \quad (4)$$

such that

$$[J_3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = J_3. \quad (5)$$

With respect to the principal SO(3) algebra, the Lie algebra  $\mathfrak{g}(A)$  decomposes into  $r$  irreducible representations of spin  $s_j$

$$\mathfrak{g}(A) = \bigoplus_{j=1}^r \mathfrak{g}^{(s_j)}, \quad (6)$$

where  $\mathfrak{g}^{(s_j)}$  carries the  $(2s_j + 1)$ -dimensional irreducible representation of SO(3), and the  $r$  ‘spins’  $s_j$  are known as the exponents of  $\mathfrak{g}(A)$ . In particular,  $\mathfrak{g}^{(0)}$  is empty, while the adjoint representation  $\mathfrak{g}^{(1)}$  is just the principal SO(3) subalgebra itself. Thus, the smallest exponent is always  $s_1 = 1$ . The importance of the principal SO(3) is due to the fact that the exponents  $s_j$  contain essential information about the Lie group. For instance, the orders of the invariant Casimir operators are given by the numbers  $s_j + 1$ ; thus, the representation  $s_1 = 1$  is always associated with the quadratic Casimir invariant. Furthermore, the group (co)homology is specified by the Poincaré polynomial  $\prod_j (1 - x^{2s_j+1})$  [17].

It is straightforward to re-express the SO(3) generators directly in terms of the Chevalley basis  $e_i \equiv E_{\alpha_i}, f_i \equiv E_{-\alpha_i}$  and  $h_i \equiv \alpha_i \cdot H$ . Using  $\Lambda_i = \sum_j A_{ij}^{-1} \alpha_j$  in (3) we readily obtain

$$J_3 = \sum_j p_j h_j, \quad J^+ = \sum_j n_j e_j, \quad J^- = \sum_j n_j f_j, \quad (7)$$

where

$$p_i = \sum_j A_{ij}^{-1} > 0, \quad n_i = \sqrt{p_i}. \quad (8)$$

The strict positivity of  $p_i$  for all  $i$  follows from (2) and the nondegeneracy of  $A_{ij}$ . The SO(3) algebra can now be directly verified from the standard Chevalley–Serre presentation.

### 3. The Principal SO(1,2) Subalgebra of a Hyperbolic Kac–Moody Algebra

Because the Weyl vector exists also for certain infinite-dimensional Kac–Moody algebras, it is natural to extend the above considerations to Kac–Moody algebras whose Cartan matrices  $A$  are no longer positive definite. However, the mere existence of a Weyl vector by itself is not sufficient; rather, it is the fact that the entries of the inverse Cartan matrix are of the same sign which ensures that the construction

can be carried through. For the hyperbolic case the expression for  $J_3$  still takes the same form  $\sum_{i,j} A_{ij}^{-1} h_j$  as before. But, taking account of the relative sign switch in (2) and insisting that  $p_j$  still denotes a positive quantity, we now have, instead of (7) and (8),

$$J_3 = - \sum_j p_j h_j, \quad (9)$$

with

$$p_i := - \sum_j A_{ij}^{-1} > 0, \quad n_i := \sqrt{p_i}. \quad (10)$$

As a consequence of the extra minus signs in these definitions, (5) is replaced by

$$[J_3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = -J_3. \quad (11)$$

Because the Hermiticity properties of the Chevalley generators are the same as before, we see that the compact  $SO(3)$  has been replaced by a noncompact  $SO(1, 2)$ . Evidently, an analogous definition cannot work for affine Cartan matrices, whose inverse does not exist; likewise, it fails for Kac–Moody algebras where the signs of the  $p_i$ 's alternate. The consistency of the above definition may be traced back in part to the fact that the Weyl vector is timelike ( $\rho^2 < 0$ ) and an element of the forward lightcone in root space for hyperbolic Cartan matrices (this property actually holds for all indefinite algebras of rank  $\leq 25$ , provided they are obtained by the procedure of overextension [5]).

As before it is possible to decompose the algebra  $\mathfrak{g}(A)$  into irreducible representations of the principal subalgebra. However, in accordance with the noncompactness of  $SO(1, 2)$ , all the irreducible representations occurring will now be infinite-dimensional and unitary, with the exception of the adjoint representation consisting of the subalgebra  $SO(1, 2)$  itself, which is neither. This will be explained below.

In this context, according to standard definitions, [1], unitary means that the representation space possesses a Hermitian scalar product, denoted  $(x, y)$  with the properties that (i) the actions of  $e_i$  and  $f_i$  are mutually adjoint, while that of  $h_i$  is selfadjoint, i.e. for all  $x, y \in \mathfrak{g}(A)$

$$([e_i, x], y) = (x, [f_i, y]) \quad \text{and} \quad ([h_i, x], y) = (x, [h_i, y]) \quad (12)$$

and (ii) the scalar product is positive definite.

Here the representation space is the vector space of the algebra  $\mathfrak{g}(A)$ , with  $\mathfrak{g}(A)$  acting on itself by adjoint action. Because the Cartan matrix  $A$  is assumed symmetric the algebra possesses a standard invariant bilinear form  $\langle \cdot, \cdot \rangle$ , generalizing the Cartan–Killing form. A natural candidate for the Hermitian scalar product

(extending that familiar in angular momentum theory) is given by ([1], chapter 2),

$$(x, y) := -\langle x | \theta(y) \rangle, \quad (13)$$

where  $\theta$  is the Cartan involution

$$\theta(e_i) = -f_i, \quad \theta(f_i) = -e_i, \quad \theta(h_i) = -h_i. \quad (14)$$

It is easy to check that with this definition  $e_i$  and  $f_i$  are indeed mutually adjoint while  $h_i$  is selfadjoint but the question of positive definiteness is more subtle. There is a rather general theorem, [22], certainly applicable to hyperbolic algebras with symmetric Cartan matrix, that states that  $\mathfrak{g}(A)$  as a vector space decomposes into orthogonal subspaces consisting of the Cartan subalgebra and subspaces associated with each root. All subspaces are positive definite with respect to (13) except for the Cartan subalgebra for which the scalar product reduces to the indefinite one already met in talking of scalar products between roots and weights.

Having verified the desired adjointness properties of the Chevalley generators, it follows immediately from (7) that  $J_3$  is selfadjoint while  $J^+$  and  $J^-$  are mutually adjoint. Furthermore, the norms of these elements are easily calculated to be

$$(J_3, J_3) = \rho^2 = -\sum_j p_j < 0 \quad \text{and} \quad (J^-, J^-) = (J^+, J^+) = \sum_j p_j > 0.$$

Thus the adjoint representation of  $\text{SO}(1, 2)$  is indeed not unitary. The reason is that the Weyl vector  $\rho$  is inside the forward light cone. Since all vectors orthogonal to it are space-like the Hermitian scalar product restricted to this subspace within the Cartan subalgebra orthogonal to the Weyl vector is positive definite. Because the decomposition of  $\mathfrak{g}(A)$  into irreducible representations is into orthogonal subspaces this is the reason that all the components except the three dimensional one are unitary.

#### 4. Irreducible Representations of $\text{SO}(1, 2)$

Next we examine which sorts of unitary representations of  $\text{SO}(1, 2)$  occur.

Because of the adjoint action the spectrum of  $J_3$  is integral, that is  $\exp(2\pi i J_3)$  equals unity, the representations arising must be what is sometimes called single valued, as well as unitary. As usual, the irreducible representations of  $\text{SO}(1, 2)$  are labeled (in part) by the eigenvalues of the  $\text{SO}(1, 2)$  Casimir operator

$$\begin{aligned} \mathcal{Q} &= J_3 J_3 - J^+ J^- - J^- J^+ \\ &= J_3(J_3 - 1) - 2J^+ J^- = J_3(J_3 + 1) - 2J^- J^+. \end{aligned} \quad (15)$$

When evaluating this Casimir on a given element  $x \in \mathfrak{g}(A)$  we will always understand the adjoint action

$$\text{ad}_{\mathcal{Q}}(x) := [J_3, [J_3, x]] - [J^+, [J^-, x]] - [J^-, [J^+, x]] \quad (16)$$

Besides the non-unitary finite-dimensional representations such as the three-

dimensional one,  $SO(1, 2)$  possesses two different kinds of unitary infinite-dimensional representations [21]. The so-called discrete series representations with Casimir eigenvalue  $\mathcal{Q} = s(s-1) > 0$  are characterized by the existence of a lowest (highest) weight state obeying  $J^-|s, s\rangle = 0$  (or  $J^+|-s, -s\rangle = 0$ ); the states of the representation are then given by  $|s, m\rangle$  (or  $|-s, -m\rangle$ ) for  $m = s, s+1, \dots$ . Because we are interested only in the adjoint action (16) we encounter only single-valued representations, namely ones obeying  $\exp(2\pi i J_3) = 1$ . Hence for the discrete series occurring in the decomposition of  $\mathfrak{g}(A)$ ,  $s$  is always a (positive) integer — in fact, we will see below that actually  $s \geq 2$ . The continuous representations split into principal and supplementary series representations, with the respective Casimir eigenvalues obeying

$$\begin{aligned} \text{principal series: } & -\infty < \mathcal{Q} < -\frac{1}{4}, \\ \text{supplementary series: } & -\frac{1}{4} < \mathcal{Q} < 0. \end{aligned} \tag{17}$$

These inequalities are easily deduced from (15) and the positivity requirements  $(x, [J^\pm, [J^\mp, x]]) > 0$  for  $x \neq 0$ , implying  $\mathcal{Q} < m(m \pm 1)$  for all integer-spaced  $m \in E_0 + \mathbf{Z}$ . For the supplementary series  $m$  is never an integer, and therefore the latter representations will not occur in our analysis since  $\exp(2\pi i J_3)$  cannot equal unity for them.

To analyze the representation content of  $\mathfrak{g}(A)$ , let us first consider those on spaces intersecting the Cartan subalgebra  $\mathfrak{h}$ . Apart from  $J_3$ , there are  $(r-1)$  linearly independent combinations belonging to principal series representations. For any linear combination  $\sum_j c_j h_j$  we have

$$\text{ad}_{\mathcal{Q}} \left( \sum_j c_j h_j \right) = -2 \left[ J^-, \left[ J^+, \sum_j c_j h_j \right] \right] = -2 \sum_{i,j} c_i A_{ij} p_j h_j. \tag{18}$$

Setting  $c_j = p_j$  and using  $\sum_j A_{ij} p_j = -1$  for all  $i$ , we obtain

$$\text{ad}_{\mathcal{Q}}(J_3) = -2 \sum_{i,j} p_i A_{ij} p_j h_j = +2J_3 \tag{19}$$

as expected for the adjoint representation of  $SO(1, 2)$ . We have already mentioned that the latter is the only finite-dimensional representation arising. The coefficients of the  $(r-1)$  orthogonal linear combinations satisfy

$$\left( \sum_i c'_i h_i, \sum_j p_j h_j \right) = \sum_{i,j} c'_i A_{ij} p_j = 0 \implies \sum_j c'_j = 0. \tag{20}$$

It is not difficult to see that these orthogonal combinations are of positive norm because the Weyl vector is timelike, and therefore any vector  $\sum_j c'_j \alpha_j$  orthogonal to it must be spacelike. We can now generate the full representations by multiply commuting  $\sum_j c'_j h_j$  with  $J^+$  and  $J^-$ , where the  $(r-1)$  mutually orthogonal linear combinations  $\sum_j c'_j h_j$  are determined by diagonalizing the  $SO(1, 2)$  Casimir operator

(18). Since none of these commutators vanishes, these representations extend simultaneously into  $\mathfrak{n}_-$  and  $\mathfrak{n}_+$ . For instance, commuting once with  $J^+$  we obtain

$$x: \left[ J^+, \sum_j c'_j h_j \right] = - \sum_{i,j} c'_i A_{ij} n_j e_j \tag{21}$$

and one easily checks that  $(x, x) > 0$ . The positivity of the remaining states in the representation then follows from the theorem mentioned above which is indeed proven by induction on the height of the roots [22]. Because the (integer) eigenvalues of  $J_3$  are bounded neither from below nor from above, we conclude that the orthogonal complement of  $J_3$  in  $\mathfrak{h}$  must belong to  $(r - 1)$  principal series representations.

By contrast, the discrete series representations are entirely contained in the triangular subalgebras  $\mathfrak{n}_+$  or  $\mathfrak{n}_-$ . The lowest weight representations are built on states of the form

$$v^{(s)} = \sum_{j_1 \dots j_s} c_{j_1 \dots j_s} [e_{j_1}, \dots, [e_{j_{s-1}}, e_{j_s}] \dots], \quad [J^-, v^{(s)}] = 0, \tag{22}$$

by repeated application of  $J^+$ . In an analogous fashion, the highest-weight states are obtained by acting on the states

$$v^{(-s)} = \sum_{j_1 \dots j_s} c_{j_1 \dots j_s} [f_{j_1}, \dots, [f_{j_{s-1}}, f_{j_s}] \dots], \quad [J^+, v^{(-s)}] = 0, \tag{23}$$

with  $J^-$ . From (3) it is immediately obvious that the lowest weight states indeed have spin  $s^*$ . Because the space spanned by the generators  $e_i$  is of dimension  $r$ , there are no new representations at that level (corresponding to spin  $s = 1$ ). Likewise, for  $s = 2$ , the number of independent Lie algebra elements of type  $[e_i, e_j]$ , that is corresponding to roots of height two, equals the number of links in the Dynkin diagram, which is at most  $r$  for hyperbolic diagrams. Hence, at most one new representation starts with  $s = 2$ , and that only if the diagram has a loop rather than a tree structure. Thus only the spins  $s = 2, 3, 4, \dots$  with corresponding Casimir eigenvalues  $\mathcal{Q} = s(s - 1)$  occur in the discrete series representations, whose unitarity follows again by the general theorem. We emphasize that the discrete series representations have no analog in the finite-dimensional case, where all representations appearing in the decomposition of the Lie algebra intersect the Cartan subalgebra nontrivially.

### 5. Casimir Eigenvalues

We now wish to calculate the Casimir eigenvalues of the principal series representations occurring in the decomposition of  $\mathfrak{g}(A)$  for some concrete examples.

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\*Alternatively, we may use the formula

$$[h_i, [e_{j_1}, \dots, [e_{j_{s-1}}, e_{j_s}] \dots]] = \left( \sum_{k=1}^s A_{ijk} \right) [e_{j_1}, \dots, [e_{j_{s-1}}, e_{j_s}] \dots]$$

which is easily proved by induction.



These representations are the analogues of the  $r$  representations occurring in the decomposition of a finite-dimensional Lie algebra. Of these, the finite-dimensional adjoint representation with  $Q = +2$  is present in both the finite and the infinite-dimensional case, and is unitary for  $SO(3)$ , and nonunitary for  $SO(1, 2)$ . The remaining  $(r - 1)$  representations belonging to the principal series must satisfy the bound  $Q < -\frac{1}{4}$ . Setting

$$Q = s_j(s_j - 1) \quad \text{for } j = 2, \dots, r, \tag{24}$$

we have

$$s_j = \frac{1}{2} + i\lambda_j. \tag{25}$$

The resulting  $(r - 1)$  values of  $s_j$  can be viewed as the analogs of the exponents in the finite-dimensional case, but they are now complex and noninteger.

From (18), we infer that the Casimir eigenvalues of the adjoint and principal series representations are identical with the eigenvalues of the nonsymmetric real matrix  $-2A_{ij}p_j$  (no summation on  $j$ ). This matrix was actually introduced already in [10], albeit for the (slightly different) purpose of determining the location of ‘resonances’ in the associated strictly hyperbolic Toda models. Besides the infinitely many strictly hyperbolic Kac–Moody algebras of rank two\* there are altogether eleven such algebras of rank three and four, see e.g. [23]. The relevant eigenvalues are listed in Table 3 of [10], and with the benefit of hindsight are now easily recognized to be just the  $SO(1, 2)$  Casimir eigenvalues for these algebras.

The simplest hyperbolic algebra obtained by over-extension (hence containing an affine subalgebra) is  $AE_3$ , which was first studied in [11]. In this case,  $\{p_j\} = (\frac{9}{2} | 5 | 2)$ , and therefore

$$A_{ij}p_j = \begin{pmatrix} 9 & -10 & 0 \\ -9 & 10 & -2 \\ 0 & -5 & 4 \end{pmatrix}.$$

The eigenvalues of  $-2A_{ij}p_j$  are given by

$$Q = 2, -2(12 \pm \sqrt{54}). \tag{26}$$

In a similar manner one determines the Casimir eigenvalues of the hyperbolic algebras  $AE_n$  for  $n > 3$ .

For the maximally extended hyperbolic algebra  $E_{10}$ , we have

$$\{p_j\} = (30 | 61 | 93 | 126 | 160 | 195 | 231 | 153 | 76 | 115), \tag{27}$$

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\*With Cartan matrices (for  $mn > 4$ )

$$A_{ij} \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$$

and the matrix  $A_{ij}p_j$  is

$$\begin{pmatrix} 60 & -61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & 122 & -93 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -61 & 186 & -126 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -93 & 252 & -160 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -126 & 320 & -195 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -160 & 390 & -231 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -195 & 462 & -153 & 0 & -115 \\ 0 & 0 & 0 & 0 & 0 & 0 & -231 & 306 & -76 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -153 & 152 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -231 & 0 & 0 & 230 \end{pmatrix}$$

The eigenvalues of  $-2A_{ij}p_j$  can be determined numerically. Besides the expected eigenvalue  $Q = 2$ , we find the nine values

$$\begin{aligned} Q &= -45.86857088 \\ &-221.4130766 \\ &-290.5176114 \\ &-438.1539904 \\ &-594.5608986 \\ &-714.1355888 \\ &-1025.0975582 \\ &-1507.5072788 \end{aligned} \tag{28}$$

The determination of the ‘exponents’ (25) is now an elementary exercise. The complexity of these numbers is related to the nonexistence of polynomial invariants in the enveloping algebra of  $\mathfrak{g}(A)$  other than the quadratic Casimir–Kac element [1,2]. However, as shown in [16], there do exist transcendental invariant functions on the Cartan subalgebra. The precise link between them and the exponents exhibited above remains to be elucidated, however.

## 6. Outlook

Whereas the principal series representations are uniquely determined by diagonalizing the  $SO(1, 2)$  Casimir operator, it is less easy to describe the spectrum of discrete series representations. Certainly the number of highest (or lowest) weight states will increase exponentially with the height and (negative) length of the roots, leaving an equally growing arbitrariness in the number of ways they can be combined into linearly independent and mutually orthogonal elements.

On the other hand, we expect the states belonging to the principal series representations to play a distinguished role, and to provide a new way of ‘probing’ hyperbolic Kac–Moody algebras. Usually, the hyperbolic algebras which arise as over-extensions of affine algebras, are decomposed w.r.t. to the level [11,12] (defined

as the eigenvalue of the central charge operator of the underlying affine algebra), viz.

$$\mathfrak{g}(A) = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}^{[\ell]}(A). \quad (29)$$

Because the generators  $J^\pm$  always have a contribution from the over-extended root, we see that their action does not preserve the level. For this reason, in any of the  $(r - 1)$  principal series representations, there will be states mixing an arbitrary (but given) number of levels. Therefore the decompositions w.r.t. to level and w.r.t. to  $SO(1, 2)$  are extremely oblique relative to one another.

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