HOW MASSLESS ARE MASSLESS FIELDS IN AdS_d

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Abstract

Massless fields of generic Young symmetry type in AdS_d space are analyzed. It is demonstrated that in contrast to massless fields in Minkowski space whose physical degrees of freedom transform in irreps of $o(d-2)$ algebra, AdS massless mixed symmetry fields reduce to a number of irreps of $o(d-2)$ algebra. From the field theory perspective this means that not every massless field in flat space admits a deformation to AdS_d with the same number of degrees of freedom, because it is impossible to keep all of the flat space gauge symmetries unbroken in the AdS space. An equivalent statement is that, generic irreducible AdS massless fields reduce to certain reducible sets of massless fields in the flat limit. A conjecture on the general pattern of the flat space limit of a general AdS_d massless field is made. The example of the three-cell "hook" Young diagram is discussed in detail. In particular, it is shown that only a combination of the three-cell flat-space field with a graviton-like field admits a smooth deformation to AdS_d.

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1 Introduction

The aim of this paper is to demonstrate some peculiarities of generic free massless fields in anti-de Sitter (AdS) space of an arbitrary space-time dimension d. The main conclusion will be that, in general, an irreducible AdS_d massless field does not classify according to irreducible representations of the flat space massless little algebra $o(d-2)$, but reduces to a certain set of irreducible flat space massless fields. The pattern of necessary flat-space massless fields will be given. Another (rather unexpected) manifestation of this fact is that not every massless field in flat space admits a deformation to AdS_d with the same number of degrees of freedom, since it is impossible to keep all the flat space gauge symmetries unbroken in the AdS_d space. This phenomenon does not take place, though, for all those types of massless fields that appear in the usual low-energy massless sectors of the superstring models and supergravities, because it holds only for the representations of the space-time symmetries described by non-rectangular Young diagrams. For the same reason it cannot be observed in AdS_4 higher spin gauge theories [1, 2] (all massless fields in AdS_4 are described by one-row Young diagrams). The effect discussed in this paper takes place for $d \geq 6$.

In superstring theory all types of representations appear at the higher massive levels. The study of higher spin gauge theory has two main motivations (see e.g. [2, 3]): Firstly to overcome the well-known barrier of $N \leq 8$ in $d = 4$ supergravity models and, secondly, to investigate if there is a most symmetric phase of superstring theory that leads to the usual string theory as a result of a certain spontaneous breakdown of higher spin gauge symmetries. These two motivations lead in fact into the same direction because, as shown for the $d = 4$ case [4, 5] higher spin gauge theories require infinite collections of higher spin gauge fields with infinitely increasing spins. Another important feature discovered in [6] is that gauge invariant higher spin interactions require the cosmological constant λ^2 of the background AdS space to be non-zero to compensate the extra length dimensions carried by the higher derivative interactions required by the higher spin gauge symmetries. (In this perspective λ plays the rôle analogous to α' in superstring theory). The fact that higher spin theories require an AdS background was regarded as rather surprising until it was realized that it plays a distinguished rôle in the superstring theory as well [7].

To investigate a possible relationship between the superstring theory and higher spin theories one has to build the higher spin gauge theory in higher dimensions, $d > 4$ (e.g. $d = 10, 11, ...$). A conjecture on the possible form of the higher spin symmetries and equations of motion for higher spin spin gauge fields was made in [8, 9] as a certain generalization of the $d = 4$ results [10, 11] which were proved to describe interactions of all $d = 4$ massless fields.

A generalization like this to higher dimensions is not straightforward because of the use of certain auxiliary twistor type variables. As a starting point, it is therefore important to analyze more carefully the notion of a general massless field in AdS_d. This is the main goal of this paper.

Another motivation comes from the flat space analysis of certain massless (nonsupersymmetric) triplets in $d = 11$ in [12],[13], the dimension of M-theory, where it was shown that there exists an infinite collection of triplets of higher spin fields having equal numbers of bosonic and fermionic degrees of freedom. These triplets show some remarkable

properties. For the first four Dynkin indices the bosonic and the fermionic numbers match up. This phenomenon is rather special for 11 dimensions and follows from the fact that the little group $SO(9)$ is an equal-rank subgroup of F_4, bringing in exceptional groups into the picture. If these triplets have anything to do with higher spin gauge field theories and/or M-theory it is an interesting question whether it is possible to extend the analysis of [13] to the AdS case. That question in fact triggered this investigation.

2 Massless Unitary Representations in AdS_d

AdS_d is a d-dimensional space-time with signature $(d-1,1)$ and the group of motions $SO(d-1,2)$. It is most useful to identify AdS_d with the universal covering space of the appropriate hyperboloid embedded into Minkowski space-time with signature $(d-1,2)$. Physically meaningful relativistic fields in AdS_d are classified according to the lowest weight unitary representations of $o(d-1,2)$. Unitarity implies compatibility with quantum mechanics, while lowest weight of a unitary representation guarantees that the energy is bounded from below.

2.1 General Facts

The commutation relations of $o(d-1,2)$ are

$$[M_{\hat{m}\hat{n}}, M_{\hat{k}\hat{l}}] = i(\eta_{\hat{n}\hat{k}}M_{\hat{m}\hat{l}} - \eta_{\hat{m}\hat{k}}M_{\hat{n}\hat{l}} - \eta_{\hat{n}\hat{l}}M_{\hat{m}\hat{k}} + \eta_{\hat{m}\hat{l}}M_{\hat{n}\hat{k}}), \tag{1}$$

where $\eta_{\hat{n}\hat{m}} = (-,+,\ldots+,-)$ is the flat metric in the $(d-1,2)$ space, $\hat{m},\hat{n},\hat{k},\hat{l} = 0 \div d$. The generators $M_{\hat{m}\hat{n}}$ are Hermitian. Let us choose the following basis in the algebra:

$$t_{\pm}^a = \frac{1}{2}(M_0{}^a \pm iM_d{}^a), \tag{2}$$

$$E = M_{0d}, \qquad L^{ab} = -iM^{ab}, \tag{3}$$

where $a,b = 1 \div d-1$. The commutation relations (1) take the form

$$[E, t_{\pm}^a] = \pm t_{\pm}^a, \tag{4}$$

$$[t_-^a, t_+^b] = \frac{1}{2}(E\delta^{ab} - L^{ab}), \tag{5}$$

$$[L_{ab}, t_{\pm c}] = \delta_{bc}t_{\pm a} - \delta_{ac}t_{\pm b}, \tag{6}$$

$$[L_{ab}, L_{ce}] = \delta_{bc}L_{ae} - \delta_{ac}L_{be} - \delta_{be}L_{ac} + \delta_{ae}L_{bc} \tag{7}$$

with all other commutators vanishing. The hermiticity conditions are

$$E^{\dagger} = E, \qquad (t_{\pm}^a)^{\dagger} = t_{\mp}^a, \qquad (L^{ab})^{\dagger} = -L^{ab}. \tag{8}$$

The generators E and L^{ab} can be identified with the energy and angular momenta, respectively, and span the Lie algebra $o(2) \oplus o(d-1)$ of the maximal compact subgroup of the AdS group $SO(2, d-1)$. The non-compact generators $t_{\pm}^a$ are combinations of AdS translations and Lorentz boosts. The commutation relations are explicitly Z-graded with

$t_\pm^a$ having grade ± 1. E is the grading operator. The lowest weight unitary representations are now constructed in the standard fashion (for a review of the AdS_4 case see e.g. [14] and [15]) starting with the vacuum space $|E_0, \mathbf{s}\rangle$ that is annihilated by t_-^a

$$t_-^a |E_0, \mathbf{s}\rangle = 0 \tag{9}$$

and forms a unitary representation of the compact subalgebra $o(2) \oplus o(d-1)$ that means in particular that

$$E|E_0, s\rangle = E_0 |E_0, \mathbf{s}\rangle . \tag{10}$$

Here $\mathbf{s}$ denotes the type of representation of $o(d-1)$ carried by the vacuum space: $\mathbf{s} = (s_1, \ldots s_\nu)$ with $\nu = [\frac{d-1}{2}]$. $\mathbf{s}$ is a generalized spin characterizing the representation. In terms of Young tableaux s_i is the number of cells in the i-th row of the Young tableaux. Since we are talking about representations of orthogonal algebras, the corresponding tensors are traceless. We will here not discuss the self-dual representations that can be singled out with the aid of the Levi-Civita symbol[1]. Note that $\mathbf{s}$ describes a finite-dimensional representation of $o(d-1)$. Field-theoretically this corresponds to a finite-component field carrying a finite spin.

The full representation of the Lie algebra $o(d-1,2)$, denoted in [16] $D(E_0, \mathbf{s})$, is spanned by the vectors of the form

$$t_+^{a_1} \ldots t_+^{a_k} |E_0, \mathbf{s}\rangle \tag{11}$$

for all k. The states with fixed k are called level-k states. Note that states with pairwise different k are orthogonal as a consequence of (4). As a result, the analysis of unitarity can be reformulated as the check of the positivity of the norms of the finite-dimensional subspaces at every level. For sufficiently large generic E_0 it is intuitively clear that the representation $D(E_0, \mathbf{s})$ is irreducible and unitary, i.e. all norms are strictly positive. Such representations are identified with massive representations of AdS_d.

Let us emphasize that the elements of the module (11) can be identified with the modes of a one-particle state in the corresponding free quantum field theory. The elements of the AdS_d algebra $o(d-1,2)$ are then realized as bilinears in the quantum fields. Note that at the level of equations of motion the light-cone field-theoretical realization of generic AdS_d massive representations for arbitrary E_0 and $\mathbf{s}$ has been developed recently in [17].

We see that massive states are classified by the parameter E_0 (which is the analog of mass) and a representation of $o(d-1)$. This picture is in agreement with the standard description of massive relativistic fields in flat space-time in terms of the Wigner little group $SO(d-1)$.

If E_0 gets sufficiently small, the norm cannot stay positive as is most obvious from the following consequence of (5)

$$[t_{-a}, t_+^a] = \frac{d-1}{2} E , \tag{12}$$

which implies that for negative E_0 some level 1 states cannot have positive norm for a positive-definite vacuum subspace. There is therefore a boundary of the unitarity region

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[1]The self-dual and antiself-dual representations which appear for odd d are usually distinguished by a sign of the $s_{\frac{d-1}{2}}$.

$E_0 = E_0(\mathbf{s}) > 0$ such that some states acquire negative norm for $E_0 < E_0(\mathbf{s})$. Obviously, these states should have zero norm for $E_0 = E_0(\mathbf{s})$.

Starting from the inside of the unitarity region decreasing E_0 for some fixed $\mathbf{s}$ one approaches the boundary of the unitarity region, $E_0 = E_0(\mathbf{s})$. Some zero-norm vectors then appear for $E_0 = E_0(\mathbf{s})$. These necessarily should have vanishing scalar product to any other state. (Otherwise, one can build a negative norm state that is in contradiction with the assumption that we are at the boundary of the unitarity region). Therefore, the zero-norm states form an invariant subspace called a singular submodule. By factoring out this subspace one is left with a unitary representation which is "shorter" than the generic massive representation.

The resulting "shortened" unitary representations correspond either to massless fields [18] or to singletons [19] and doubletons [20],[21] identified with the conformal fields at the boundary of the AdS space [19]. The fact that a singular submodule can be factored out admits an interpretation as some sort of a gauge symmetry (true gauge symmetry for the case of massless fields, or independence of bulk degrees of freedom for singletons and doubletons). For this reason we choose this definition of masslessness for all fields in AdS_d except for the scalar and spinor massless matter fields which are not associated with any gauge symmetry principle and singletons[2].

The analysis of positive definiteness of the scalar product of level-1 states was done in [22] for an arbitrary even d and an arbitrary type of representation of $\mathbf{s}$ carried by the vacuum state. The final result is

$$E_0 \geq E_0(\mathbf{s}) \,, \tag{13}$$

$$E_0(\mathbf{s}) = s_1 + d - t_1 - 2 \,, \tag{14}$$

where t_1 is the number of rows of the maximal length s_1, i.e.

$$s_1 = \ldots = s_{t_1-1} = s_{t_1} > s_{t_1+1} \geq s_{t_1+2} \geq \ldots \geq s_\nu \,.$$

It can be shown that the same result is true for odd d (provided that s_ν is replaced by $|s_\nu|$). Note that this bound for $d = 4$ was originally found in [18]. For the case $d = 5$ see [23] and references therein.

Massless representations are "shorter" than massive ones classified according to the parameter E_0 (equivalent of mass) and a representation of $o(d-1)$. In flat space, massless fields are classified according to the representation of the massless little group $SO(d-2)$. The question we address here is whether or not the shortening in AdS_d can be interpreted in terms of irreducible representations of $SO(d-2)$. We will show that the answer is no for a generic representation. This will be demonstrated both at the algebraic level using the language of singular vectors and at the field-theoretical level focusing on the simplest nontrivial massless field with $\mathbf{s} = (2, 1, 0, \ldots, 0)$. The most important field-theoretical conclusion is that a generic irreducible massless field in AdS_d decomposes into a collection of massless fields in the flat limit. In that sense, a massless field in the AdS_d is generically "less massless" than elementary massless field in flat space. An

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[2]Singleton-type fields live at the boundary of AdS_d and cannot be interpreted as bulk massless fields. Presumably, all singletons except for scalar and spinor correspond to maximally antisymmetrized representations of the AdS_d algebra equivalent to their duals. In particular, this is true for the second rank antisymmetric tensor representation in the AdS_5 case that can be identified with the field strength of the Yang-Mills fields of the $N = 4$ SYM at the boundary of AdS_5.

important consequence of this fact is that not every massless field in flat space can be deformed into AdS geometry.

2.2   Singular Vectors

It is useful to reformulate the problem in terms of singular vectors. Since the energy E is bounded from below for the whole representation, the singular submodule spanned by zero-norm states is itself a lowest weight representation. Therefore it contains at least one nontrivial subspace $|E_0', \mathbf{s}'\rangle$ that has the properties analogous to (9) and (10),

$$t_-^a |E_0', \mathbf{s}'\rangle = 0 \tag{15}$$

and

$$E|E_0', \mathbf{s}'\rangle = E_0'|E_0', \mathbf{s}'\rangle, \tag{16}$$

i.e. it forms some irreducible representation $\mathbf{s}'$ of $o(d-1)$. Obviously,

$$E_0' = E_0 + k \tag{17}$$

if $|E_0', \mathbf{s}'\rangle$ belongs to the level-k subspace.

Such spaces $|E_0', \mathbf{s}'\rangle$ we will call singular vacuum spaces while any of their elements will be called a singular vector[3]. Clearly, singular vacuum spaces form representations of the algebra $o(2) \oplus o(d-1)$ and therefore decompose into a direct sum of irreducible representations of $o(d-1)$ on different levels. The standard situation is with a single irreducible singular vacuum space. The singular module as a whole then has the structure

$$t_+^{a_1} \dots t_+^{a_k} |E_0', \mathbf{s}'\rangle. \tag{18}$$

The factorization to a unitary irreducible representation is equivalent to identifying all these vectors to zero.

Let us now consider the example of a vacuum state $|E_0, \mathbf{s}\rangle$ with $\mathbf{s} = (s_1, s_2, 0, \dots, 0)$ corresponding to a two-row Young diagram,

$$\tag{19}$$

with $0 \le s_2 \le s_1$.

This means that $|E_0, \mathbf{s}\rangle$ can be realized as a tensor $v_{a_1 \dots a_{s_2}, b_1 \dots b_{s_1}}(E_0)$ which is symmetric both in the indices a and in the indices b, satisfying the antisymmetry property

$$v_{a_2 \dots a_{s_2} \{b_{s_1}+1, b_1 \dots b_{s_1}\}}(E_0) = 0, \tag{20}$$

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[3]This terminology is very closely although not exactly coinciding with that used for the Verma module construction when irreducible vacuum subspaces are one-dimensional because the grade zero subalgebra, namely the Cartan subalgebra, is Abelian.

which implies that symmetrization over any $s_1 + 1$ indices a and/or b gives zero. The tensor is traceless, which means that contraction of any two indices with the $o(d-1)$ invariant flat metric η_{ab} gives zero. Taking (20) into account it is enough to require

$$\eta^{b_1 b_2} v_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1}}(E_0) = 0 \, . \tag{21}$$

The level one states are

$$t_+^c v_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1}}(E_0) \, . \tag{22}$$

These states form a reducible representation of $o(d-1)$ (the tensor product of the vector representation with the representation (19)). For generic d, s_2 and s_1 it contains five irreducible components: two Young diagrams with one cell less (index c is contracted to either one of the indices a or one of the indices b) and three diagrams with one cell more: adding one cell to the first, second or an (additional) third row. Our problem therefore is to check whether any of these irreducible representations can be a singular vacuum space, i.e.

$$t_-^e \Pi_\alpha (t_+^c v_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1}}(E_0)) \equiv 0 \tag{23}$$

for some E_0 (Π_α is a projector to one or another irreducible component in (22)). Let us consider the two representations with cells cut. The appropriate projections are given by the following formulae describing irreducible $o(d-1)$ tensors

$$v^1_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1-1}} = t_+^c \{ v_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1-1}c}(E_0) + \frac{s_2}{s_1 - s_2 + 1} v_{ca_1 \ldots a_{s_2-1}, b_1 \ldots b_{s_1-1}a_{s_2}}(E_0) \} \tag{24}$$

(symmetrizations within each of the groups of indices a and b are assumed) and

$$v^2_{a_1 \ldots a_{s_2-1}, b_1 \ldots b_{s_1}} = t_+^c v_{ca_1 \ldots a_{s_2-1}, b_1 \ldots b_{s_1}}(E_0) \, . \tag{25}$$

Elementary computations give that

$$
\begin{aligned}
t_{-c} v^1_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1-1}} &= \frac{1}{2}(E_0 - (d + s_1 - 3)) \Big(v_{a_1 \ldots a_{s_2}, b_1 \ldots b_{s_1-1}c}(E_0) \\
&+ \frac{s_2}{s_1 - s_2 + 1} v_{a_1 \ldots a_{s_2-1}c, b_1 \ldots b_{s_1-1}a_{s_2}}(E_0) \Big)
\end{aligned}
\tag{26}
$$

and

$$t_{-c} v^2_{a_1 \ldots a_{s_2-1}, b_1 \ldots b_{s_1}} = \frac{1}{2}(E_0 - (d + s_2 - 4)) v_{a_1 \ldots a_{s_2-1}c, b_1 \ldots b_{s_1}}(E_0) \, . \tag{27}$$

As a result, singular vectors appear at

$$E_0^1 = d + s_1 - 3 \tag{28}$$

and

$$E_0^2 = d + s_2 - 4 \, . \tag{29}$$

A few comments are now in order.

The cases $s_1 = s_2$ and $s_1 > s_2$ are different because $v^1 \equiv 0$ for $s_1 = s_2$ as a consequence of the antisymmetry property (20). For $s_1 > s_2$ both v^1 and v^2 are nontrivial.

As expected, the values of the "singular" energies (28) and (29) are in agreement with the general analysis of [22], where it was also shown that only the representation resulting

from the factorization of the singular submodule with the highest energy of a singular vector is unitary (this fact is natural from the singular vector description: unitarity can be preserved only when the boundary of the unitarity region is approached; this implies the highest E_0). We therefore conclude that unitary massless particles appear for

$$E_0 = d + s_1 - 3 \qquad for \quad s_1 > s_2 \tag{30}$$

and

$$E_0 = d + s_1 - 4 \qquad for \quad s_1 = s_2 \,. \tag{31}$$

The analysis in terms of singular vectors is simple enough but can be simplified further with the aid of the technique proposed in [24] with the tensors corresponding to various Young diagrams realized as certain subspaces of an appropriate Fock space. This technique is explained in section 4.1 since we will use it in the field-theoretical part of the paper. We here use the tensor language to make most clear the interpretation in terms of the representations of the massless little algebra $o(d-2)$. One can analogously investigate singular spaces with the boxes added to make sure that they have negative energies and therefore do not play a rôle in our analysis.

We expect that the analysis of higher levels does not affect our conclusions. One reason is that the appearance of singular vectors at higher levels within the unitarity region would imply existence of higher spin gauge fields with gauge transformations having more than one derivative acting on a gauge parameter. The general analysis of massless fields in flat space-time of an arbitrary dimension [25] shows that this does not take place.

3 Flat Space Pattern of AdS Massless fields

Let $A_{a_1...a_{s_1},b_1...b_{s_2},...}$ be an irreducible tensor of $o(d-1)$ $(a, b... = 1 \div d-1)$ of a specific symmetry type. Let n^a be a nonzero vector of $o(d-1)$. $o(d-2)$ can then be identified with the stability subalgebra of $o(d-1)$ that leaves n^a invariant. If the tensor $A_{a_1...a_{s_1},b_1...b_{s_2},...}$ is orthogonal to n^a with respect to all possible contractions of indices

$$n^{a_1} A_{a_1...a_{s_1},b_1...b_{s_2},...} = 0 \,, \qquad n^{b_1} A_{a_1...a_{s_1},b_1...b_{s_2},...} = 0 \ldots \tag{32}$$

then it describes a representation of $o(d-2)$ of the same symmetry pattern. If some contractions with n^a are nonzero, one can decompose the $o(d-1)$ tensor $A_{a_1...a_{s_1},b_1...b_{s_2},...}$ into irreducible representations of $o(d-2)$ with the aid of the projection operators constructed from n^a or, in other words, performing dimensional reduction. For example, for a vector,

$$A^a = A^{\|a} + A^{\perp a} \,, \qquad A^{\perp a} = A^a - \frac{n^a n_b}{n_c n^c} A^b \,, \qquad A^{\|a} = \frac{n^a n_b}{n_c n^c} A^b \,. \tag{33}$$

The analysis of singular vectors in section 2.2 admits a similar interpretation. Indeed, let us interpret t_+^a as a vector n^a analogous to the momentum operator in flat-space field theory (note that the operators t_+^a commute with themselves). The fact that a singular vector appears means that some contractions of t_+^a with the vacuum vector decouple from the spectrum and therefore are equivalent to zero. If all possible contractions of t_+^a

would decouple this would mean that the $o(d-1)$ vacuum representation would reduce to the $o(d-2)$ of the same symmetry. Since the energies (28) and (29) are different this cannot be true simultaneously. Therefore, when a singular vector is present, the "reduced representation" is effectively smaller than an irreducible representation of $o(d-1)$ but may be larger than the corresponding irreducible representation of $o(d-2)$, containing a number of irreducible representations of $o(d-2)$.

Let us note that the energy in AdS_d is measured in units of the inverse AdS radius λ that was set equal to unity in our analysis. Reintroducing λ and taking the flat limit $\lambda \to 0$, all energies of singular vectors tend to zero. This means that in the flat limit different singular vectors may decouple simultaneously and therefore a natural possibility consists of the flat space reduction to one (totally t_+^a orthogonal) representation of $o(d-2)$, in agreement with the standard analysis [26] of massless representations of the Poincare' algebra. However such massless representations of the Poincare' algebra may not admit a deformation to a representation of the AdS algebra with $\lambda \neq 0$. At the field-theoretical level this means that it will not be possible to preserve all necessary gauge symmetries for $\lambda \neq 0$. This phenomenon is demonstrated in the field theoretical example in section 4.2. The deformation will be possible however, if one starts with an appropriate collection of massless fields in flat space dictated by the "incomplete" dimensional reduction via decoupling of singular vectors. The main aim of this section is to formulate a conjecture on the pattern of massless fields in flat space compatible with the deformation to AdS_d.

Let us consider an arbitrary Young diagram with row lengths $s_1 \geq s_2 \geq s_3 \dots$. For our analysis it is more convenient to build Young diagrams not from rows as elementary entities but from rectangular blocks of an arbitrary height t and length s
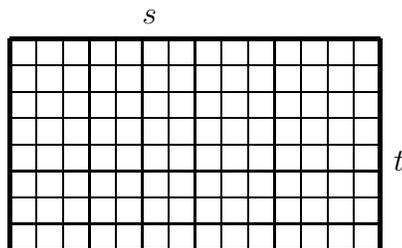
$$(34)$$

In other words, a block is a Young diagram composed of t rows of equal length s (equivalently, from s columns of equal height t). A general Young diagram is a combination of blocks with decreasing lengths (equivalently, heights)

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$$\tag{35}$$
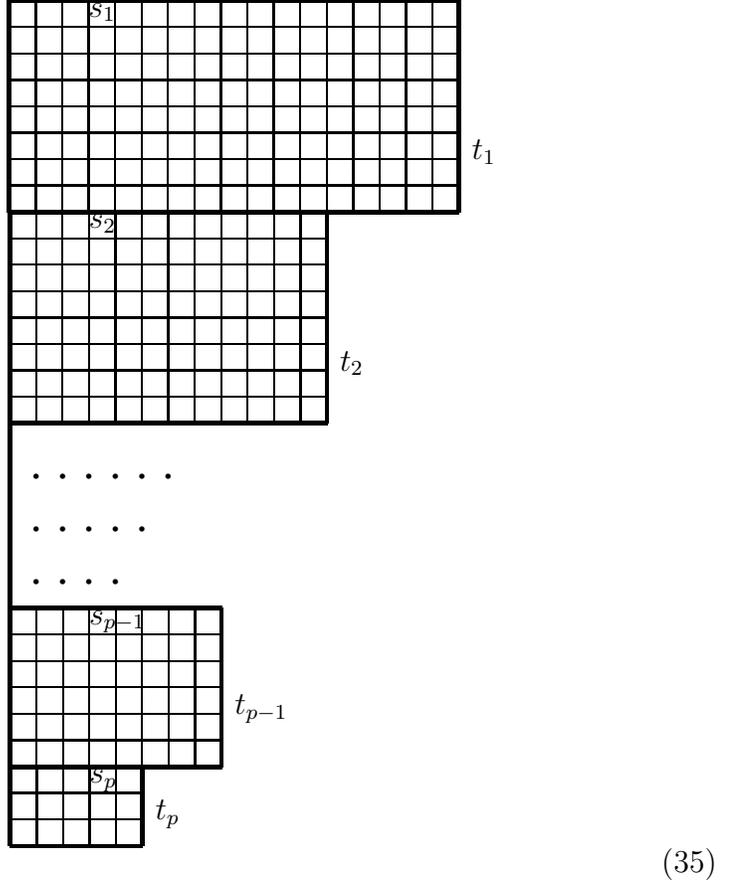
In these terms, a Young diagram $Y[(s_i, t_i)]$ is described by a set of pairs of positive integers (s_i, t_i) with $s_1 > s_2 > s_3 > \ldots s_p > 0$ and arbitrary t_i such that

$$\sum_{i=1}^{p} t_i \leq \frac{1}{2}(d-1). \tag{36}$$

In other words, s_1 is the maximal row length in the Young diagram, while t_1 is the number of rows of the length s_1. s_2 is the maximal row length of the remaining rows and t_2 is the number of rows of length s_2. Note that an elementary block $Y[(s, t)]$ is described in these terms by a single pair of integers (s, t).

Let us now address the question what is the result of a dimensional reduction to one dimension less of a general diagram $Y[(s_i, t_i)]$. Every cell can be identified with some vector index. It can either be aligned along n^a or along the $d - 2$ perpendicular directions. In the first case we cancel a cell, while in the second case we keep it. There cannot be more than s_1 indices along n^a because symmetrization with respect to more than s indices gives identically zero by the definition of a Young diagram. But any number of indices from 0 to s_1 can be chosen to take the extra value $d - 1$. Therefore any number of cells from 0 to s_1 can be canceled. Of course only such cancelings are allowed that result in a Young diagram. (If not, any resulting tensor is identically zero.)

Let us consider some examples.

First consider a representation (e.g. tensor T) described by the Young diagram $Y[(s, t)]$ which is itself an elementary block. Since components of tensors along n_a
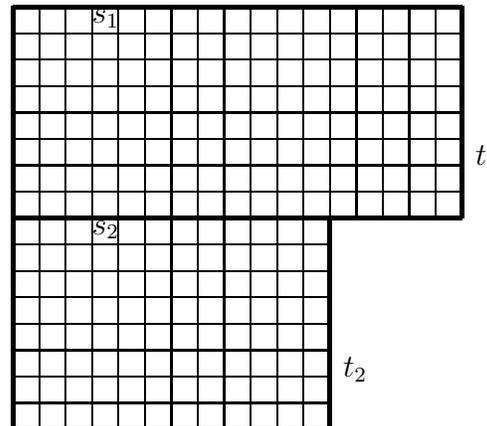
are automatically symmetrized because the tensor $N^{a_1 a_2 \ldots a_k} = n^{a_1} n^{a_2} \ldots n^{a_k}$ is totally symmetric, we can use the symmetry properties of the Young diagrams to reduce any contraction with $N^{a_1 a_2 \ldots a_k}$ of the original tensor to a contraction of $N^{a_1 a_2 \ldots a_k}$ with the bottom row of the block. As a result, dimensional reduction will lead to a number of tensors resulting from cutting an arbitrary number of cells in the bottom row of the block; every tensor appears once (note that the condition that the tensor is traceless does not affect this analysis since the reduced $o(d-2)$ tensors are also assumed to be traceless).

$$\begin{array}{c}
\boxed{\begin{array}{c} s_1 \\ \quad \\ \quad \\ \quad \\ \quad \end{array}} \; t_1 - 1 \\
n_1
\end{array}$$
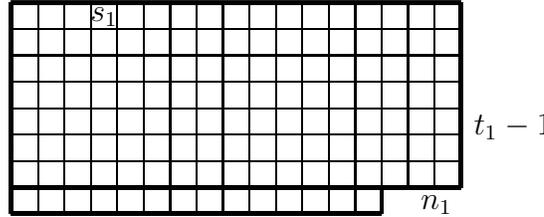
(37)

In other words, the dimensional reduction of the block $Y[(s,t)]$ gives rise to the following representations of $o(d-2)$: $Y[(s,t)]$ (all indices are orthogonal to n^a), $Y[(s,t-1)]$ (a maximal possible number s of indices is contracted) and all diagrams $Y[(s,t-1);(s_1,1)]$ which consist of two blocks with the bottom block having an arbitrary length $0 < s_1 < s$ and height 1.

Now, consider a representation T described by a Young diagram $Y[(s_1,t_1);(s_2,t_2)]$ composed from two blocks.

$$\begin{array}{c}
\boxed{\begin{array}{c} s_1 \\ \quad \\ \quad \end{array}} \; t_1 \\
\boxed{\begin{array}{c} s_2 \\ \quad \\ \quad \end{array}} \; t_2
\end{array}$$

(38)

Again, dimensional reduction means that one can cut some cells from the bottom rows of the upper and lower blocks (all cuts inside a block are equivalent by the properties of the Young diagrams to cutting its bottom line). But now, one cannot cut an arbitrary number of boxes in the top block because the cut line cannot be shorter than the length of the second block s_2. (Such tensors vanish identically.) One can, however, take away an arbitrary number of cells from the bottom line of the second box. The rule therefore is: take away an arbitrary number n_1 such that $s_1 - s_2 \geq n_1 \geq 0$ from the bottom line

of the top block and take away an arbitrary number n_2 such that $s_2 \geq n_2 \geq 0$ of cells from the bottom line of the bottom block


$$(39)$$
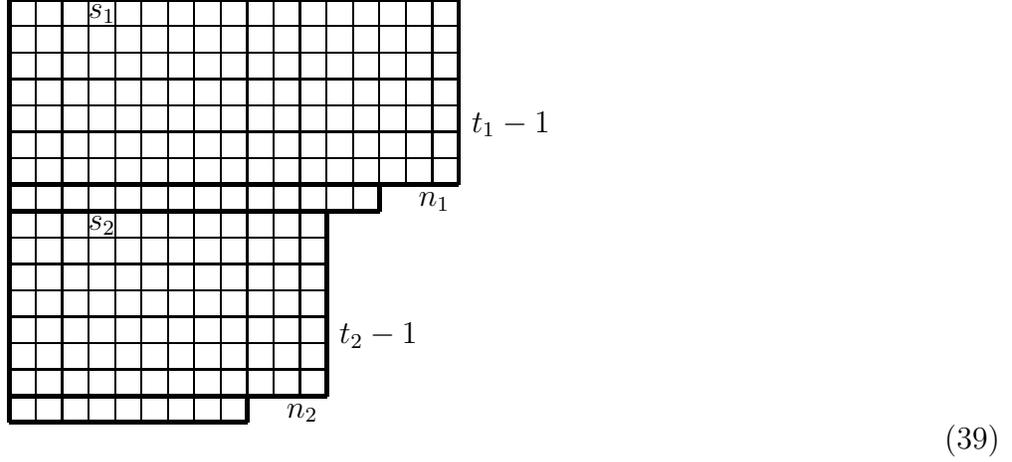
Note that with this prescription we have $n_1 + n_2 \leq s_1$ in accordance with the general argument that one cannot cut a number of cells exceeding the maximal row length in the Young diagram. The pattern of the dimensionally reduced representation therefore consists of four-block diagrams $Y[(s_1, t_1 - 1); (s'_1, 1); (s_2, t_2 - 1); (s'_2, 1)]$ with arbitrary integers s'_1 and s'_2 such that

$$s_1 > s'_1 > s_2 > s'_2 > 0 \tag{40}$$

and their degenerate versions described by the three-block diagrams
$Y[(s_1, t_1 - 1); (s_2, t_2); (s'_2, 1)]$, $Y[(s_1, t_1); (s_2, t_2 - 1); (s'_2, 1)]$, $Y[(s_1, t_1 - 1); (s'_1, 1); (s_2, t_2)]$,
$Y[(s_1, t_1 - 1); (s'_1, 1); (s_2, t_2 - 1)]$ and two-block diagrams $Y[(s_1, t_1); (s_2, t_2)]$,
$Y[(s_1, t_1 - 1); (s_2, t_2)]$, $Y[(s_1, t_1); (s_2, t_2 - 1)]$, $Y[(s_1, t_1 - 1); (s_2, t_2 + 1)]$.

Analogously one proceeds for Young diagrams built from a larger number of blocks. The final result is that the dimensional reduction of a general Young diagram to one dimension less consists of the Young diagrams of the form (every diagram appears once; see e.g. [26] and references therein):

$$\tag{41}$$
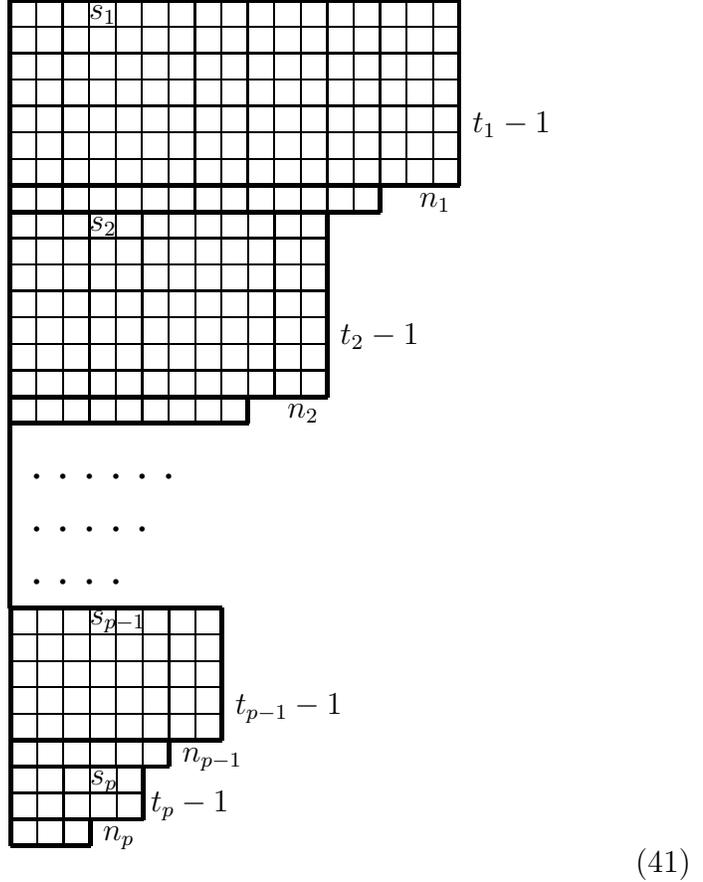
One is allowed to take away any numbers n_i of cells from the ith block provided that

$$0 \leq n_i \leq s_i - s_{i+1} \tag{42}$$

(with the convention that s_j corresponding to the "next to last" block equals zero).

Let us now formulate the final result concerning a pattern of flat space massless fields that admit a unitary deformation to AdS_d.

Conjecture. *Consider a AdS_d massless field characterized by a diagram (35). Take away any number of cells n_i satisfying the conditions (42) of the bottom lines of all blocks except for the top one, i.e. require $n_1 = 0$. Any diagram that appears as a result describes some irreducible representation of the massless little algebra $o(d-2)$ corresponding to some flat space massless field that should be present in the full set compatible with the AdS_d geometry and unitarity.*

Note that massless fields corresponding to arbitrary irreducible representations of flat space massless little algebra $o(d-2)$ were considered in [25].

A few comments are now in order.

The role of the upper block is singled out by the unitarity condition: only singular vectors corresponding to contractions of t_+^a to the upper block have maximal energies and describe unitary representations [22]. Therefore canceling out a box from the upper block corresponds to pure gauge (i.e. singular vector) components that decouple. Non-unitary (i.e. ghost containing) sets of fields can be obtained by a similar procedure with one of the lower blocks remaining untouched instead of the top one as in the unitary case.

13

For Young diagrams being themselves elementary blocks (i.e. $t_i = 0$ for $i > 0$) only one $o(d-2)$ representation appears, described by the same block. This means that elementary block massless fields in AdS_d classify according to irreducible representations of $o(d-2)$ as in the flat case (i.e. no additional massless fields should be added to deform to AdS_d). In fact all examples of massless fields that appear in supergravity and low energy string theory are described by elementary blocks (specifically, either by single rows, or by single columns). That is why the phenomenon discussed in this paper was not observed before. Note also that for the well studied case of lower dimensions $d \leq 4$, only block-type massless representations are nontrivial (propagating) and therefore this phenomenon does not occur either.

The spectrum of flat-space massless fields to which an elementary AdS massless field decomposes is non degenerate, i.e. all the representations of $o(d-2)$ are pairwise different.

Some standard massless (gauge) fields may be needed as ingredients of AdS massless fields with nontrivial diagrams. For example, for the representation $Y[(2,1),(1,1)]$ a graviton-type flat space massless field $Y[(2,1)]$ will be present (see example in section 4.2). It is tempting to speculate that this may correspond to a nontrivial deformation of gravity to the AdS geometry in the presence of other fields. Analogously one can find a totally symmetric spin $s_1 \geq 2$ field corresponding to the diagram $Y[(s_1,1)]$ among the fields resulting from the decomposition of the AdS_d field $Y[(s_1,1);(s_2,1)]$, $0 \leq s_2 \leq s_1$. This is to say that AdS_d massless field corresponding to $Y[(s_1,1);(s_2,1)]$ decomposes into the following irreps of $o(d-2)$ algebra

$$Y[(s_1,1);(s_2,1)]_{AdS} \rightarrow Y[(s_1,1)] \oplus \sum_{s=1}^{s_2} \oplus Y[(s_1,1);(s,1)], \tag{43}$$

where each term under summation appears just once.

An important consequence of the analysis of this section is that totally antisymmetric gauge tensors (i.e. differential forms) corresponding to the diagrams Y[(1,t)] can never appear as a result of a decomposition of a certain irreducible (unitary) AdS_d massless field in the flat limit. The space of differential forms (including the spin one gauge fields) therefore is closed with respect to the deformation to AdS_d.

4 Field Theoretical Example

Now let us explain what happens from the field-theoretic perspective. In fact, it has been observed already in [22] at the level of equations of motion in the Lorentz gauge that some of the redundant gauge symmetries expected in the flat-space description are absent in the AdS case. Here we analyze the problem at the Lagrangian level focusing on the explicit comparison with the flat-space limit.

4.1 Flat Space

Let us consider the simplest nontrivial example of a massless field having the symmetry properties of a non-block diagram with three cells $Y[(2,1);(1,1)]$

$$(44)$$

In flat space the massless field of this symmetry type was described in [27]. To begin with, let us reformulate the results of these authors in a somewhat different, although equivalent way. We take the representation with the field $\Phi_{m_1 m_2, n}$ being a symmetric tensor in m_1 and m_2, satisfying the condition that full symmetrization with respect to all three indices gives zero

$$\Phi_{\{m_1 m_2, m_3\}} = 0 \, . \tag{45}$$

(The authors of [27] used an equivalent representation with explicit antisymmetry in two indices). The Lagrangian can be chosen to be of the form

$$
\begin{aligned}
L \ = \ & \frac{1}{2} \Phi_{m_1 m_2, n} \Box \Phi^{m_1 m_2, n} - \Phi_{m m_1, n} \partial^{m_1} \partial_{m_2} \Phi^{m m_2, n} - \frac{1}{2} \Phi_{m_1 m_2, n_1} \partial^{n_1} \partial_{n_2} \Phi^{m_1 m_2, n_2} \\
& - \frac{3}{4} \Phi^{m_1}{}_{m_1, n} \Box \Phi_{m_2}{}^{m_2, n} + \frac{3}{4} \Phi^{m}{}_{m, n} \partial_{m_1} \partial_{m_2} \Phi^{m_1 m_2, n} + \frac{3}{4} \Phi^{m_1 m_2, n} \partial_{m_1} \partial_{m_2} \Phi^{m}{}_{m, n} \\
& + \frac{3}{4} \Phi^{m_1}{}_{m_1, n_1} \partial^{n_1} \partial_{n_2} \Phi_{m_2}{}^{m_2, n_2} \, .
\end{aligned}
\tag{46}
$$

The corresponding action is invariant under the gauge transformations

$$\delta_{as} \Phi^{m_1 m_2, n} = \frac{1}{2} (\partial^{m_1} \Lambda_{as}^{m_2 n} + \partial^{m_2} \Lambda_{as}^{m_1 n}) \, , \tag{47}$$

$$\delta_{sym} \Phi^{m_1 m_2, n} = \frac{1}{2} (\partial^{m_1} \Lambda_{sym}^{m_2 n} + \partial^{m_2} \Lambda_{sym}^{m_1 n}) - \partial^n \Lambda_{sym}^{m_1 m_2} \tag{48}$$

with antisymmetric gauge parameter $\Lambda_{as}^{mn}(x)$ and symmetric gauge parameter $\Lambda_{sym}^{mn}(x)$,

$$\Lambda_{as}^{mn}(x) = -\Lambda_{as}^{nm}(x) \, , \qquad \Lambda_{sym}^{mn}(x) = \Lambda_{sym}^{nm}(x) \, . \tag{49}$$

In [27] it was proved that the Lagrangian (46) describes a physical massless field in flat space corresponding to the irreducible representation $Y[(2, 1); (1, 1)]$ of the massless little algebra $o(d-2)$. Because this Lagrangian is fixed by the gauge transformations one can say that it is the gauge invariance with respect to the both gauge transformations (47) and (48) that ensures irreducibility of the massless field.

Let us now introduce a notation that simplifies computation. In curved space-time it is convenient to use fiberwise fields

$$\Phi^{m_1 m_2 m_3} \equiv e_{\underline{m}_1}{}^{m_1} e_{\underline{m}_2}{}^{m_2} e_{\underline{m}_3}{}^{m_3} \Phi^{\underline{m}_1 \underline{m}_2 \underline{m}_3} \, , \tag{50}$$

where $e_{\underline{m}}{}^m$ is the vielbein of an appropriate space-time (e.g., Minkowski or AdS) [4].

It is most convenient to formulate the action in terms of the following Fock-type generating function [24]

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \Phi_{m_1 m_2, n} \alpha_1^{m_1} \alpha_1^{m_2} \alpha_2^n |0\rangle \, , \tag{51}$$

---

[4]Tangent space (fiber) indices m, n and target space (base) indices $\underline{m}, \underline{n}$ take the values $0, 1, \dots d-1$.

where α_A^m and $\bar{\alpha}_B^n$ are auxiliary creation and annihilation operators

$$[\bar{\alpha}_A^m, \alpha_B^n] = \eta^{mn}\delta_{AB}, \qquad [\alpha_A^m, \alpha_B^n] = 0, \qquad [\bar{\alpha}_A^m, \bar{\alpha}_B^n] = 0, \tag{52}$$

and $|0\rangle$ is a Fock vacuum

$$\bar{\alpha}_A^m|0\rangle = 0. \tag{53}$$

The indices $A, B, C, E = 1, 2$ label two sets of oscillators.

The fact that we deal with the Young diagram (44) is equivalent to imposing the following constraints on the generating function $|\Phi\rangle$

$$N_{11}|\Phi\rangle = 2|\Phi\rangle, \tag{54}$$

$$N_{22}|\Phi\rangle = |\Phi\rangle, \tag{55}$$

$$N_{12}|\Phi\rangle = 0, \tag{56}$$

where we use the notation

$$N_{AB} \equiv \alpha_A^m \bar{\alpha}_{Bm}, \qquad P_{AB} \equiv \alpha_A^m \alpha_{Bm}, \qquad \bar{P}_{AB} \equiv \bar{\alpha}_A^m \bar{\alpha}_{Bm}. \tag{57}$$

The constraints (54) and (55) tell us that the oscillators α_1^m and α_2^m occur twice and once, respectively, on the right hand side of eq.(51). The constraint (56) is equivalent to the condition (45).

The Lorentz covariant derivative for the representation $|\Phi\rangle$ takes the form

$$D_{\underline{m}} \equiv \partial_{\underline{m}} + \frac{1}{2}\omega_{\underline{m}mn}M^{mn}, \qquad M^{mn} = \sum_{A=1,2}(\alpha_A^m\bar{\alpha}_A^n - \alpha_A^n\bar{\alpha}_A^m), \tag{58}$$

where $\omega_{\underline{m}}{}^{mn}$ is the Lorentz connection of space, while M^{mn} forms a representation of the Lorentz algebra $so(d-1,1)$. In the sequel we will often use the notation

$$D_A \equiv \alpha_A{}^m D_m, \qquad \bar{D}_A \equiv \bar{\alpha}_A{}^m D_m, \qquad D_m \equiv e_m{}^{\underline{m}}D_{\underline{m}}. \tag{59}$$

The flat-space Lagrangian (46) now takes the form

$$L = \frac{1}{2}\langle\Phi|\Box - D_1\bar{D}_1 - D_2\bar{D}_2 - \frac{3}{4}P_{11}\Box\bar{P}_{11} + \frac{3}{4}(P_{11}\bar{D}_1^2 + D_1^2\bar{P}_{11} + P_{11}D_2\bar{D}_2\bar{P}_{11})|\Phi\rangle, \tag{60}$$

where D_A, $\bar{D}_B$ and $\Box = D^m D_m$ are defined via (58) and (59) with the flat space vielbein and Lorentz connection

$$e_{\underline{m}}{}^n = \delta_{\underline{m}}^n, \qquad \omega_{\underline{m}}{}^{kn} = 0.$$

The Lagrangian (60) is invariant under two gauge symmetries generated by the antisymmetric gauge parameter Λ_{as}^{mn} and the symmetric gauge parameter Λ_{sym}^{mn} which can be conveniently described as Fock vectors

$$|\Lambda_{as}\rangle = \frac{1}{\sqrt{2}}\Lambda_{as\,mn}\alpha_1^m\alpha_2^n|0\rangle, \qquad |\Lambda_{sym}\rangle = \frac{1}{\sqrt{2}}\Lambda_{sym\,mn}\alpha_1^m\alpha_1^n|0\rangle.$$

Now the gauge transformations (47), (48) take the form

$$\delta_{as}|\Phi\rangle = D_1|\Lambda_{as}\rangle, \tag{61}$$

$$\delta_{sym}|\Phi\rangle = (\frac{1}{2}D_1N_{21} - D_2)|\Lambda_{sym}\rangle. \tag{62}$$

4.2 Gauge Symmetries in AdS_d

Let us now analyze the situation in the AdS case. The Lorentz covariant derivatives (58) are no longer commuting but satisfy the commutation relationships

$$[D_{\underline{m}}, D_{\underline{n}}] = -\lambda^2 M_{\underline{mn}}, \tag{63}$$

$$D_{\underline{m}}\alpha_{\underline{n}A} - D_{\underline{n}}\alpha_{\underline{m}A} = D_{\underline{m}}\bar{\alpha}_{\underline{n}A} - D_{\underline{n}}\bar{\alpha}_{\underline{m}A} = 0 \tag{64}$$

with the convention

$$\alpha_{\underline{n}A} = e_{\underline{n}}{}^{n}\alpha_{nA}, \qquad \bar{\alpha}_{\underline{n}A} = e_{\underline{n}}{}^{n}\bar{\alpha}_{nA}. \tag{65}$$

The condition (64) is just the standard zero torsion condition

$$D_{\underline{m}}e_{\underline{n}}{}^{a} - D_{\underline{n}}e_{\underline{m}}{}^{a} = 0 \tag{66}$$

while (63) is the equation of the AdS space. The covariant D'Alembertian is

$$\mathcal{D}^2 \equiv D_m^2 + \omega_m{}^{mn}D_n, \tag{67}$$

where the second term accounts for D_m being rotated as a tangent vector. With these conventions the covariant derivatives D_A and $\bar{D}_B$ satisfy a number of useful relationships summarized in the Appendix.

As in the flat case we will analyze gauge symmetries with totally symmetric and totally antisymmetric gauge parameters Λ_{sym}^{mn} and Λ_{as}^{mn}. It is sometimes convenient to combine them into a gauge parameter $\Lambda_{m,n}$ having no definite symmetry properties

$$|\Lambda\rangle = \Lambda_{m,n}\alpha_1^m\alpha_2^n|0\rangle.$$

The symmetric and antisymmetric parts can be singled out as

$$|\Lambda_{sym}\rangle \equiv |S\rangle = N_{12}|\Lambda\rangle, \tag{68}$$

$$|\Lambda_{as}\rangle \equiv (1 - \frac{1}{2}N_{12}N_{21})|\Lambda\rangle. \tag{69}$$

The gauge transformation $\delta|\Phi\rangle$ which respects the constraints (54)-(56) is

$$\delta_\Lambda|\Phi\rangle = D_1|\Lambda\rangle - D_2|\Lambda_{sym}\rangle. \tag{70}$$

It can equivalently be written as a combination of the gauge transformations with symmetric and antisymmetric gauge parameters

$$\delta_\Lambda|\Phi\rangle = \delta_{as}|\Phi\rangle + \delta_{sym}|\Phi\rangle \tag{71}$$

with the gauge transformations of the form (61) and (62) but now with the derivatives D_1 and D_2 as in AdS_d.

Next we analyze whether there exists a Lagrangian that generalizes (46) (equivalently, (60)) to AdS_d. The most general deformation of (60) to the AdS case without higher derivatives is of the form

$$L^{\Phi\Phi} = \frac{1}{2}\langle\Phi|\mathcal{D}^2 - f\lambda^2 - D_1\bar{D}_1 - D_2\bar{D}_2 + \frac{3}{4}P_{11}(-\mathcal{D}^2 + g\lambda^2)\bar{P}_{11}$$

$$+ \frac{3}{4}(P_{11}\bar{D}_1^2 + D_1^2\bar{P}_{11} + P_{11}D_2\bar{D}_2\bar{P}_{11})|\Phi\rangle, \tag{72}$$

where f and g are arbitrary parameters. A straightforward but rather tedious computation with the use of the identities collected in the Appendix leads to the following result

$$
\delta S^{\Phi\Phi} = -\lambda^2 \int_{AdS_d} e\Big[\langle\Phi|(f+3)D_1 + \frac{3}{2}(d-5-g)P_{11}\bar{D}_1|\Lambda\rangle
$$

$$
+ \langle\Phi|(6-3d-f)D_2 + 3(d-3)P_{21}\bar{D}_1 + \frac{3}{2}(1-d-g)P_{21}D_1\bar{P}_{11})|S\rangle\Big]. \quad (73)
$$

From this expression it is clear that the freedom in the parameters f and g is not enough to warrant an action invariant under both types of symmetries. The best one can do is to find a Lagrangian invariant either with respect to the gauge symmetry with the parameter $|\Lambda_{as}\rangle$ or the one with $|\Lambda_{sym}\rangle$. Note that at the level of equations of motion in the Lorentz gauge an analogous phenomenon was observed for redundant gauge symmetries in [22]. Since, according to the general analysis of unitary representations of AdS_d in [22], the case with gauge invariance with respect to $|\Lambda_{sym}\rangle$ does not lead to unitary dynamics, we focus on the Lagrangian possessing the $|\Lambda_{as}\rangle$ invariance. From (73) it is obvious that this is achieved by setting

$$
f = -3, \qquad g = d - 5, \quad (74)
$$

since the antisymmetric part of the gauge parameter enters only via the first term. Thus, we set

$$
L^{\Phi\Phi} = \frac{1}{2}\langle\Phi|\mathcal{D}^2 + 3\lambda^2 - D_1\bar{D}_1 - D_2\bar{D}_2 + \frac{3}{4}P_{11}(-\mathcal{D}^2 + \lambda^2(d-5))\bar{P}_{11}
$$

$$
+ \frac{3}{4}(P_{11}\bar{D}_1^2 + D_1^2\bar{P}_{11} + P_{11}D_2\bar{D}_2\bar{P}_{11})|\Phi\rangle. \quad (75)
$$

Because one of the gauge symmetries is lost, the Lagrangian (75) describes more degrees of freedom than the original flat-space Lagrangian we started with. This is in agreement with the general conclusion of Sect.3 that physical d.o.f. of massless AdS fields may not be described by an irreducible representation of $o(d-2)$. The conjecture of Sect.3 suggests that the flat space and the AdS dynamics can match only once one starts with specific (reducible) collections of fields in flat space. From the general analysis of Sect.3 it follows that, in order to make the AdS deformation consistent for the case under consideration, one has to add a massless spin two field analogous to a graviton field.

Let us therefore introduce the field χ^{mn} symmetric in indices m, n, described by the Fock vector

$$
|\chi\rangle \equiv \chi_{mn}\alpha_1^m\alpha_1^n|0\rangle.
$$

Since this field should describe a massless spin 2 field in the flat limit, it has its own gauge symmetry with the gauge parameter

$$
|\xi\rangle \equiv \xi_m\alpha_1^m|0\rangle. \quad (76)
$$

The idea is that starting from the sum of the free Lagrangians $L^{\Phi\Phi} + L^{\chi\chi}$ one should add cross terms $L^{\Phi\chi}$ which (i) reestablish all (appropriately deformed by λ-dependent

terms) gauge symmetries with the parameters $|\Lambda\rangle$ and $|\xi\rangle$ and (ii) tend to zero in the flat limit. It turns out that this is indeed possible. The final result is that the action

$$S = \int_{AdS_d} e[L^{\Phi\Phi} + L^{\Phi\chi} + L^{\chi\chi}], \tag{77}$$

where

$$L^{\Phi\chi} = \frac{3}{2}(d-3)\lambda\langle\Phi| - 2D_2 + 2P_{12}\bar{D}_1 + P_{11}D_2\bar{P}_{11}|\chi\rangle, \tag{78}$$

$$L^{\chi\chi} = \frac{3}{2}(d-3)\langle\chi|\mathcal{D}^2 + d\lambda^2 - D_1\bar{D}_1 - \frac{1}{2}P_{11}(\mathcal{D}^2 + \lambda^2)\bar{P}_{11}$$

$$+ \frac{1}{2}(P_{11}\bar{D}_1^2 + D_1^2\bar{P}_{11})|\chi\rangle \tag{79}$$

is invariant under the gauge transformations of the form

$$\delta|\Phi\rangle = D_1|\Lambda\rangle - D_2|S\rangle + \lambda(P_{12} - P_{11}N_{21})|\xi\rangle, \tag{80}$$

$$\delta|\chi\rangle = D_1|\xi\rangle + \lambda|S\rangle. \tag{81}$$

Note that the Lagrangian $L^{\Phi\chi}$ is proportional to λ and tends to zero in the flat limit. Therefore, as expected, the action reduces in the flat limit to the sum of two actions for the irreducible fields. In the AdS case, however, the cross term $L^{\Phi\chi}$ becomes nontrivial so that the system does not decompose into a sum of elementary subsystems. Another comment is that according to (81) the field $|\chi\rangle$ becomes a Stueckelberg field that can be gauged away for $\lambda \neq 0$. The resulting gauge fixed action is nothing but the action (75) invariant under the gauge symmetry with antisymmetric gauge parameter. Therefore it describes properly the irreducible AdS representation. The gauge fixing $|\chi\rangle = 0$ is impossible however for $\lambda = 0$. This is why the naive flat limit of the action (75) describes not two fields but only one in agreement with [27]. This phenomenon can be interpreted as some sort of nonanalyticity of the flat limit exhibited already at the free field level.

We expect that one can analogously find a deformation of the flat space Lagrangian (60) to AdS_d by adding an antisymmetric second rank gauge tensor. This would correspond to keeping the symmetry with symmetric parameters. However, because the corresponding representation of the AdS algebra is not unitary, the resulting gauge invariant Lagrangian is expected to have a wrong relative sign of the kinetic terms of the elementary flat-space Lagrangian (i.e., incompatible with unitarity). A similar phenomenon is expected to be true for more complicated Young diagrams: only the sets of fields predicted in Sect.3 will have all signs of kinetic terms of the flat-space Lagrangians correct.

The equations of motion for the fields $|\Phi\rangle$ and $|\chi\rangle$ that follow from the Lagrangian (77) can be reduced to the form

$$\left(\mathcal{D}^2 - D_1\bar{D}_1 - D_2\bar{D}_2 + \frac{1}{2}D_1^2\bar{P}_{11} + D_2D_1\bar{P}_{12} - \lambda^2 P_{11}\bar{P}_{11} - 2\lambda^2 P_{12}\bar{P}_{12} + 3\lambda^2\right)|\Phi\rangle$$

$$+\lambda\Big((d-3)(D_1N_{21} - 2D_2) - P_{12}\bar{D}_1 + P_{11}N_{21}\bar{D}_1 + (P_{12}D_1 - P_{11}D_2)\bar{P}_{11}\Big)|\chi\rangle = 0, \tag{82}$$

$$(\mathcal{D}^2 - D_1\bar{D}_1 + \frac{1}{2}D_1^2\bar{P}_{11} - \frac{1}{2}P_{11}\bar{P}_{11} + d\lambda^2)|\chi\rangle + \lambda(\bar{D}_2 - D_1\bar{P}_{12})|\Phi\rangle = 0. \qquad (83)$$

By imposing the Lorentz gauge $\bar{D}_i|\Phi\rangle = 0$, the tracelessness condition $\bar{P}_{ij}|\Phi\rangle = 0$ and the condition $|\chi\rangle = 0$, we are left with

$$(\mathcal{D}^2 + 3\lambda^2)|\Phi\rangle = 0. \qquad (84)$$

There is a leftover symmetry with the parameter $|S\rangle$ satisfying certain differential conditions. Taking into account that one can identify the Stueckelberg field $|\chi\rangle$ with $|S\rangle$ one can derive these conditions from the equations of motion for $|\chi\rangle$ in the gauge $\bar{D}_i|\chi\rangle = 0$, $\bar{P}_{ij}|\chi\rangle = 0$

$$(\mathcal{D}^2 + d\lambda^2)|S\rangle = 0. \qquad (85)$$

Let us compare these results with the equations for the gauge field corresponding to the AdS_d massless representation $D(E_0, \mathbf{s})$

$$(\mathcal{D}^2 - E_0(E_0 + \lambda - d\lambda) + \lambda^2 \sum_{A=1,2} s_A)|\Phi\rangle = 0, \qquad (86)$$

and the conditions on the leftover gauge parameter $|S\rangle$

$$\left(\mathcal{D}^2 - \lambda^2(s_2 - 2)(s_2 - 3 + d) + \lambda^2 \sum_{A=1,2} s_A - \lambda^2\right)|S\rangle = 0 \qquad (87)$$

found in [22]. For the case under consideration the E_0 and $\mathbf{s}$ are

$$E_0 = \lambda(d-1), \qquad \mathbf{s} = (2, 1, 0, \ldots, 0).$$

Plugging these values into (86) (87) we indeed arrive at the equations (84) and (85).

5 Conclusions

We have shown that generic irreducible massless (gauge) fields in AdS_d in the flat limit decompose into nontrivial sets of irreducible flat space massless fields. These sets are, however, smaller than the result of a dimensional reduction to one less dimension of a corresponding massive field. In that sense AdS_d massless fields are "less massless" than flat space massless fields. We made a conjecture on the pattern of the flat-space reduction of a generic AdS_d massless field. From this conjecture it follows that there is a unique nontrivial situation when a flat space spin two massless field appears as a result of a nontrivial reduction of AdS_d massless field with mixed symmetry properties. This example has been considered in detail. It is tempting to speculate that there may exist some new version of gravity associated with this type of field.

On the other hand we have argued that totally antisymmetric tensors can never result from the flat limit decomposition of other types of AdS_d unitary representations. In other words, the space of differential forms is closed with respect to the flat space limit decomposition.

An interesting problem for the future is to generalize these results to the supersymmetric cases to analyze generic AdS supermultiplets in higher dimensions and, in particular, in AdS_{11}. Another problem is to consider the multiplets occurring in [12] to see how they group themselves in the case of AdS.

Acknowledgments

Appendix. Algebra of commutators

In this appendix we collect some formulas that are used in the computations of the section 4.2.

$$[\bar{D}_A, D_B] = \delta_{AB}(\mathcal{D}^2 + \lambda^2 \sum_C N_{CC}) + \lambda^2(1-d)N_{BA} + \lambda^2 \sum_C (P_{BC}\bar{P}_{AC} - N_{CA}N_{BC})\,, \quad (88)$$

$$[D_A, D_B] = \lambda^2 \sum_C (P_{BC}N_{AC} - P_{AC}N_{BC})\,,$$

$$[\bar{D}_A, \bar{D}_B] = \lambda^2 \sum_C (N_{CB}\bar{P}_{AC} - N_{CA}\bar{P}_{BC})\,,$$

$$[\mathcal{D}^2, D_A] = \lambda^2(1-d)D_A + 2\lambda^2 \sum_C (P_{AC}\bar{D}_C - D_C N_{AC})\,,$$

$$[\mathcal{D}^2, \bar{D}_A] = \lambda^2(d-1)\bar{D}_A + 2\lambda^2 \sum_C (N_{CA}\bar{D}_C - D_C\bar{P}_{AC})\,,$$

where $\mathcal{D}^2$ in (88) is a covariant D'Alembertian operator (67). These formulas can be derived by straightforward but sometimes lengthy calculation. The derivation of the following relationships:

$$[N_{AB}, N_{CE}] = \delta_{BC}N_{AE} - \delta_{AE}N_{CB}\,,$$

$$[\bar{D}_A, N_{BC}] = \delta_{AB}\bar{D}_C\,, \qquad [N_{AB}, D_C] = \delta_{BC}D_A\,,$$

$$[\bar{D}_A, P_{BC}] = \delta_{AB}D_C + \delta_{AC}D_B\,,$$
$$[\bar{P}_{AB}, D_C] = \delta_{BC}\bar{D}_A + \delta_{AC}\bar{D}_B\,,$$

$$[\bar{P}_{AB}, P_{CE}] = \delta_{BC}N_{EA} + \delta_{BE}N_{CA} + \delta_{AC}N_{EB} + \delta_{AE}N_{CB} + d(\delta_{BC}\delta_{AE} + \delta_{BE}\delta_{AC})\,.$$

is elementary.

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