Bogomol'nyi limit for magnetic vortices in a rotating superconductor

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This work is the sequel to a previous investigation of stationary and cylindrically symmetric vortex configurations for simple models representing an incompressible nonrelativistic superconductor in a rigidly rotating background. In the present paper, we carry out our analysis with a generalized Ginzburg-Landau description of the superconductor, which provides a prescription for the radial profile of the normal density within the vortex. Within this framework, it is shown that the Bogomol'nyi limit condition marking the boundary between type I and type II behavior is unaffected by the rotation of the background.

I. INTRODUCTION

The present work is the sequel to a previous investigation[1] on the energy of vortices in a rotating superconductor, where it was shown in particular that the magnetic and kinetic contributions to the energy density that are proportional to the background angular velocity remarkably cancel. The motivation which brought us to study vortices in rotating superconductors is the study of the interior of neutron stars, more specifically their inner core which is believed to contain a proton superconductor. Although we worked out the macroscopic description of an array of magnetic vortices and superfluid vortices in a general relativistic framework[2] necessary for refined analysis of neutron stars, thus generalizing the earlier work of Mendell and Lindblom[3] in a Newtonian approach, we have here restricted our analysis to a Newtonian framework for simplicity.

In our previous work,[1] we left undetermined the explicit structure of the vortex core. One of the purposes of the present work is to provide a specification for the profile of the condensate particle density, based on a Ginzburg-Landau-type approach. Our treatment will, however, not be restricted just to the widely used standard Ginzburg-Landau description, but will also be valid for a generalized version (more readily justifiable by heuristic considerations[4,5]), leaving arbitrary the coupling constant g_c that enters the gradient energy density.

The second purpose of this work is to reexamine the question of the Bogomol'nyi limit[6] in the context of a superconductor in a rotating background. The conclusion will be that the usual boundary between type I and type II superconductors remains unaffected by the rotation of the background.

Before entering into the details of this work, let us recall the essential features of the model and define the relevant quantities. The superconducting matter will consist of a charged superfluid component, represented by a locally variable number density n_s of bosonic particles characterized by an effective mass m, a charge q, and an ordinary component of opposite charge which locally compensates the charge of the first component. The essential property that distinguishes the superfluid constituent from ordinary matter is that its momentum is directly related to the phase variable φ (a scalar with period 2π) of the boson condensate, according to the expression

$$m\vec{v}+q\vec{A}=\hbar\vec{\nabla}\varphi. \tag{1}$$

In this formula, $\hbar$ is the Dirac-Planck constant, $\vec{A}$ is the magnetic vector potential, and $\vec{v}$ the velocity of the Bose condensate. The whole system will be described, in addition to the relation (1), by the Maxwell equation

$$\vec{\nabla}\times\vec{B}=4\pi\vec{j}, \tag{2}$$

where $\vec{B}$ is the magnetic field, related to the magnetic potential vector by the usual relation $\vec{B}=\vec{\nabla}\times\vec{A}$, and $\vec{j}$ is the electric current, which consists of the sum of the currents due to the condensate component and to the ordinary component. The ordinary component current will be supposed to be that of a rigidly rotating fluid. To be able to solve the coupled system of equations, one needs a prescription concerning the spatial evolution of the condensate particle number density. It will be given by an energy minimization principle, using the energy functional corresponding to a generalized Ginzburg-Landau approach.

The plan of the paper will be the following. In Sec. II, we shall use the cylindrical symmetry and introduce new variables to simplify the system of coupled equations. Section III will be devoted to the energy minimization principle, which will give a prescription for the determination of n_s. And finally, Sec. IV will deal with the Bogomol'nyi limit condition.

II. SYSTEM OF COUPLED EQUATIONS

The scenarios we shall consider will be of the usual kind, in which each individual vortex is treated as a stationary, cylindrically symmetric configuration consisting of a rigidly rotating background medium with uniform angular velocity Ω, say, together with a charged superfluid constituent in a state of differential rotation with a velocity v, which tends at large distance towards the rigid rotation value given by Ωr, where r is the cylindrical radial distance from the axis. It will be supposed that the superfluid particle number density n_s is a monotonically increasing function of the cylindrical radius variable r, tending asymptotically to a constant value n_∞, say, at large distances from the axis. It will be supposed that the local charge density is canceled by the background so that there is no electric field, but that there is a magnetic induction field with magnitude B and direction parallel to the axis, whose source is the axially oriented electromagnetic current whose magnitude j will be given by

$$j = qn_s(v - \Omega r). \tag{3}$$

The relevant Maxwellian source equation for the magnetic field (2) will have the familiar form

$$\frac{dB}{dr} = -4\pi j. \tag{4}$$

The other relevant Maxwellian equation is the one governing the axial component A (which in an appropriate gauge will be the only component) of the electromagnetic potential covector, which will be related to the magnetic induction by

$$\frac{d(rA)}{dr} = rB. \tag{5}$$

The essential property distinguishing the superconducting case from its ''normal'' analog is the London flux quantization condition (1), which in the present context (where all physically relevant quantities depend only on the cylindrical radius r) will be expressible in the well-known form[7]

$$mv + qA = \frac{N\hbar}{r}, \tag{6}$$

where N is the phase winding number, which must be an integer.

Before proceeding, it will be useful to take advantage of the possibility of transforming the preceding system of equations to a form that is not just linear but also homogeneous, by replacing the variables v, B, A by corresponding variables $\mathcal{V}$, $\mathcal{B}$, $\mathcal{A}$, which are defined by

$$\mathcal{V} = v - \Omega r, \tag{7}$$

$$\mathcal{B} = B - B_L, \tag{8}$$

$$\mathcal{A} = A - \frac{1}{2} r B_L, \tag{9}$$

where B_∞ is the uniform background magnetic field value that would be generated by a rigidly rotating superconductor, which is given by the London formula

$$B_L = -\frac{2m}{q}\Omega, \tag{10}$$

obtained by combining Eqs. (6) and (5) in the special case of rigid corotation, i.e., with $v = \Omega r$.

In terms of these new variables Eq. (5) will be transformed into the form

$$\frac{d(r\mathcal{A})}{dr} = r\mathcal{B}, \tag{11}$$

while the other differential Eq. (4) will be transformed into the form

$$\frac{d\mathcal{B}}{dr} = -4\pi j, \tag{12}$$

in which, rewriting Eq. (3), we shall have

$$j = qn_s\mathcal{V}. \tag{13}$$

Finally, the flux quantization condition (6) will be converted into the form

$$m\mathcal{V} + q\mathcal{A} = \frac{N\hbar}{r}, \tag{14}$$

which can be used to transform Eq. (11) into

$$\frac{m}{qr}\frac{d(\mathcal{V}r)}{dr} = -\mathcal{B}. \tag{15}$$

The advantage of this reformulation is that unlike v, B, and A, the new variables $\mathcal{V}$, $\mathcal{B}$, and $\mathcal{A}$ are subject just to homogeneous boundary conditions, which are simply that they all tend to zero as $r \to \infty$.

III. ENERGY MINIMIZATION PRINCIPLE

The equations of the previous section are not sufficient by themselves to fully determine the system. In order to specify the radial distribution of the condensate particle number density n_s we will use an energy minimization principle based on a model in which the condensate energy density is postulated to be given as the sum of a gradient contribution and a potential energy contribution by an expression of the form

$$\mathcal{E}_{\rm con} = \mathcal{E}_{\rm grad} + V, \tag{16}$$

where the contribution $\mathcal{E}_{\rm grad}$ is proportional to the square of the gradient of n_s with a coefficient that, like the potential energy contribution V, is given as an algebraic function of n_s by some appropriate ansatz. The use of such a model as a fairly plausible approximation is justifiable by heuristic considerations[4] that motivate the use of an ansatz of what we shall refer to as the Ginzburg type, according to which the energy contribution is postulated to have the form

$$\mathcal{E}_{\rm grad} = \frac{g_c^2\hbar^2}{8mn_s}\left(\frac{dn_s}{dr}\right)^2, \tag{17}$$

where g_c is a dimensionless coupling constant, while the potential energy density V is given in terms of some constant proportionality factor $\mathcal{E}_c$, say, by the formula

$$V = \mathcal{E}_c\left(1 - \frac{n_s}{n_\infty}\right)^2, \tag{18}$$

which provides a particularly convenient ansatz for interpolation in the theoretically intractable intermediate region between the comparatively well understood end points of the allowed range $0 \leqslant n_s \leqslant n_\infty$. The constant $\mathcal{E}_c$ is interpretable as representing the maximum condensation energy density. Its value is commonly expressed in terms of the corresponding critical value H_c, say, representing the strength of the maximum magnetic field that can be expelled from the superconductor by the Meissner effect, to which it is evidently related by the formula

$$\mathcal{E}_c=\frac{H_c^2}{8\pi}. \tag{19}$$

The total energy density associated with a vortex will be of the general form

$$\mathcal{E}=\mathcal{E}_{\rm mag}+\mathcal{E}_{\rm kin}+\mathcal{E}_{\rm con}, \tag{20}$$

where $\mathcal{E}_{\rm mag}$, $\mathcal{E}_{\rm kin}$, and $\mathcal{E}_{\rm con}$ are, respectively, the magnetic, kinetic, and condensate energy contributions. More precisely, $\mathcal{E}_{\rm mag}$ is the extra magnetic energy density arising from a nonzero value of the phase winding number N, i.e., the local deviation from the magnetic energy density due just to the uniform field B_L (associated with the state of rigid corotation characterized by the given background angular velocity Ω), namely,

$$\mathcal{E}_{\rm mag}=\frac{B^2}{8\pi}-\frac{B_L^2}{8\pi}, \tag{21}$$

while $\mathcal{E}_{\rm kin}$ is the corresponding deviation of the kinetic energy from that state of rigid corotation, namely,

$$\mathcal{E}_{\rm kin}=\frac{m}{2}n_s(v^2-\Omega^2r^2). \tag{22}$$

It is convenient for many purposes to express such a model in terms of a dimensionless amplitude ψ that varies in the range $0\leqslant\psi\leqslant 1$ according to the conventional specification

$$n_s=\psi^2 n_\infty. \tag{23}$$

Within the general category of Ginzburg-type models as thus described, the special case of the standard kind of Ginzburg-Landau model is characterized more specifically by the postulate that the gradient coupling constant g_c should be exactly equal to unity. This ansatz has the attractive feature of allowing the theory to be neatly reformulated in terms of a complex variable $\Psi\equiv\psi e^{i\varphi}$ where φ is the phase that appears in Eq. (1) in a manner that is evocative of the Schrödinger model for a single particle. Indeed, it is easy to verify that in the case of $g_c=1$, the gradient term (17) and the kinetic term in Eq. (22) can be rewritten, using Eqs. (1) and (23), to give the usual Ginzburg-Landau-type gradient term, i.e.,

$$\frac{\hbar^2}{8mn_s}(\vec{\nabla}n_s)^2+\frac{m}{2}n_s\vec{v}^2=\frac{\hbar^2 n_\infty}{2m}|\vec{\mathcal{D}}\Psi|^2, \tag{24}$$

where the covariant derivative is defined as

$$\vec{\mathcal{D}}\equiv\vec{\nabla}-\frac{iq}{\hbar}\vec{A}. \tag{25}$$

However, although there are physical reasons[4] for expecting that g_c should be comparable with unity, the seductive supposition that it should exactly satisfy the Landau condition $g_c=1$ is more dubious.[5] This more specialized ansatz will not be needed for the work that follows, which applies to the generalized Ginzburg category with no restriction on the parameter g_c.

Using the expression

$$8\pi(\mathcal{E}_{\rm mag}+\mathcal{E}_{\rm kin})=\mathcal{B}^2+4\pi\frac{m}{q}j\mathcal{V}+\frac{B_L}{r}\frac{d}{dr}(r^2\mathcal{B}) \tag{26}$$

for the first two terms in the combination (20), it can be seen that the total energy density arising from the presence of the vortex will be given by

$$\mathcal{E}=\frac{\mathcal{B}^2}{8\pi}+\frac{H_c^2}{8\pi}(1-\psi^2)^2+\frac{n_\infty}{2m}\left[\left(g_c\hbar\frac{d\psi}{dr}\right)^2+(m\mathcal{V}\psi)^2\right]+\frac{B_L}{8\pi r}\frac{d(r^2\mathcal{B})}{dr}. \tag{27}$$

The equation governing the distribution of the condensate particle number density n_s is obtained by requiring that the integral of the energy density (27) be stationary with respect to variation of n_s or, equivalently, of ψ, which gives the field equation for the latter in the form

$$\frac{g_c^2\hbar^2}{mr}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=m\mathcal{V}^2\psi-\frac{4}{n_\infty}\mathcal{E}_c\psi(1-\psi^2). \tag{28}$$

When this equation is satisfied, it can be seen that the energy density (27) will reduce to a value given by

$$\mathcal{E}=\frac{\mathcal{B}^2}{8\pi}+\frac{H_c^2}{8\pi}(1-\psi^4)+\frac{1}{r}\frac{d}{dr}\left(B_L\frac{r^2\mathcal{B}}{8\pi}+\frac{g_c^2\hbar^2 n_\infty r}{4m}\frac{d\psi^2}{dr}\right), \tag{29}$$

in which the last term is a divergence that goes out when integrated, so that for the total energy per unit length one is left simply with

$$U=\frac{1}{8\pi}\int[\mathcal{B}^2+H_c^2(1-\psi^4)]dS. \tag{30}$$

IV. BOGOMOL'NYI INEQUALITY

We shall now try to rewrite the energy density associated with the vortex in a different form. Let us begin by writing the relation

$$\left(g_c\hbar\frac{d\psi}{dr}\right)^2+(m\mathcal{V}\psi)^2=\left(g_c\hbar\frac{d\psi}{dr}\mp m\mathcal{V}\psi\right)^2\pm g_c\hbar q\psi^2\mathcal{B}\pm\frac{g_c\hbar m}{qn_\infty}\frac{d(rj)}{rdr}, \tag{31}$$

which can be obtained by rewriting $\psi(d\psi/dr)$ in terms of dj/dr and $d\mathcal{V}/dr$, the latter term being transformed by use of the Maxwell equation (15). In the first term on the right hand side of Eq. (31), one takes the minus sign if $\mathcal{V}$ is positive, the plus sign otherwise. We are thus able to obtain a Bogomol'nyi type reformulation of Eq. (27), which is given by

$$\mathcal{E}=\frac{g_c\hbar q n_\infty}{2m}|\mathcal{B}|+\frac{1}{2\pi}\left(\frac{|\mathcal{B}|}{2}+\frac{\pi g_c\hbar q n_\infty}{m}(\psi^2-1)\right)^2+\frac{n_\infty}{2m}\left(g_c\hbar\frac{d\psi}{dr}-m|\mathcal{V}|\psi\right)^2+\left(\frac{H_c^2}{8\pi}-\frac{\pi g_c^2\hbar^2q^2n_\infty^2}{2m^2}\right)\times(1-\psi^2)^2+\frac{1}{r}\frac{d}{dr}\left(\frac{B_L r^2\mathcal{B}}{8\pi}+\frac{g_c\hbar r|j|}{2q}\right), \tag{32}$$

which generalizes the original version[6] by inclusion of the term with the coefficient B_L, which allows for the effect of the background rotation velocity Ω. Since this extra term is just the divergence of a quantity that vanishes both on the axis and in the large distance limit, it gives no contribution to the corresponding integral expression, which therefore has the same form as the usual Bogomol'nyi relation for the non rotating case.[8] More generally, performing the Bogomol'nyi trick (31) just for a fraction f of the combined kinetic and gradient contribution, one can see that for a vortex with (relative) magnetic flux

$$\Phi=\int \mathcal{B}dS, \tag{33}$$

the total vortex energy per unit length and per unit of flux will be expressible in the form

$$U=\frac{fH_c}{4\pi\kappa}|\Phi|+\frac{H_c^2}{8\pi}\left(1-\frac{f^2}{\kappa^2}\right)\int(1-\psi^2)^2dS+\frac{H_c^2}{8\pi}\int\left(\frac{|\mathcal{B}|}{H_c}-\frac{f(1-\psi^2)}{\kappa}\right)^2dS+\frac{fn_\infty}{2m}\int\left(g_c\hbar\frac{d\psi}{dr}-m|\mathcal{V}|\psi\right)^2dS+\frac{(1-f)n_\infty}{2m}\int\left[\left(g_c\hbar\frac{d\psi}{dr}\right)^2+(m\mathcal{V}\psi)^2\right]dS, \tag{34}$$

in which κ is a dimensionless constant given by the definition

$$\kappa=\frac{mH_c}{2g_c\pi\hbar q n_\infty}. \tag{35}$$

It will be convenient to introduce the so-called London penetration length λ, defined by the expression

$$\lambda^2=\frac{m}{4\pi q^2 n_\infty}, \tag{36}$$

and the usual flux quantum

$$\phi=\frac{2\pi\hbar}{q}\quad\text{so that }\ \Phi=N\phi, \tag{37}$$

where the integer N is the so-called winding number of the vortex. This allows us to express the critical field H_c as

$$H_c=g_c\kappa\frac{\phi}{4\pi\lambda^2}. \tag{38}$$

It is to be observed that all the terms in expression (34) will be non-negative provided the quantity f is chosen not only so as to lie in the range $0\leqslant f\leqslant 1$ but also so as to satisfy $f\leqslant\kappa$, a requirement that will be more restrictive if $\kappa\leqslant 1$. In the latter case we can maximize the first term on the right of Eq. (34) by choosing $f=\kappa$, thereby incidentally eliminating the second term, so that we obtain the lower limit

$$\kappa\leqslant 1\Rightarrow\frac{U}{|\Phi|}\geqslant\frac{H_c}{4\pi}. \tag{39}$$

If $\kappa\geqslant 1$, we shall be able to choose $f=1$, thereby eliminating the final term in Eq. (34), so that we obtain the inequality

$$\kappa\geqslant 1\Rightarrow\frac{U}{|\Phi|}\geqslant\frac{H_c}{4\pi\kappa}. \tag{40}$$

More particularly it can be seen that choosing $f=1$ will eliminate both the second term and the last term on the right of Eq. (34) in the special Bogomol'nyi limit case characterized the condition

$$\kappa=1. \tag{41}$$

[Readers should be warned that much of the relevant literature[9,8] follows a rather awkward tradition in which the symbol κ is used for what in the present notation scheme would be denoted by $\sqrt{2}\kappa$, which instead of Eq. (41) makes the critical condition come out to be $\kappa=1/\sqrt{2}$.] In this Bogomol'nyi limit, the energy per unit flux per unit length, Eq. (34), will be minimized by imposing the conditions

$$g_c\hbar\frac{d\psi}{dr}=m|\mathcal{V}|\psi \tag{42}$$

(with the sign adjusted so as to make the right hand side positive), and

$$|\mathcal{B}|=\frac{g_c\phi}{4\pi\lambda^2}(1-\psi^2). \tag{43}$$

which (as when the background is nonrotating[10]) will automatically guarantee the solution of the field equations in this case, annihilating the last two terms in Eq. (34) so that one is left simply with

$$\frac{U}{|\Phi|}=\frac{H_c}{4\pi}. \tag{44}$$

The qualitative distinction between what are known[11] as Pippard-type or type I superconductors on the one hand and as London-type or type II superconductors on the other hand is based on the criterion of whether or not, for a given total flux, the energy will be minimized by gathering the flux in a small number of vortices, each with large winding number N, or by separating the flux in a large number of vortices, each just with unit winding number. Within the framework of our analysis, a model may be characterized as type I if the derivative with respect to $|\Phi|$ of $U/|\Phi|$ is always negative, i.e., if $dU/d|\Phi|<U/|\Phi|$, in which case the vortices will effectively be mutually attractive, and as being of type II if the derivative with respect to $|\Phi|$ of $U/|\Phi|$ is always positive, i.e., if $dU/d|\Phi|>U/|\Phi|$, in which case the vortices will effectively be mutually repulsive. One can of course envisage the possibility of models that are intermediate in the sense of having a sign for the derivative of $U/|\Phi|$ that depends on N, so that the minimum is obtained for some large but finite value of the winding number.

What can be seen directly from Eq. (44) is that within the category of models characterized by the Ginzburg-Landau ansatz, the special Bogomol'nyi limit case lies precisely on the boundary between type I and type II, since it evidently satisfies the exact equality

$$4\pi\lambda^2 H_c = g_c\phi \Rightarrow \frac{dU}{d|\Phi|} = \frac{U}{|\Phi|}, \tag{45}$$

for all values of $|\Phi|$. The implication of our work is that the well-known conclusion[9,8] that there is neither repulsion nor attraction between Ginzburg model vortices in the Bogomol'nyi limit case will remain valid even in the presence of a rotating background. (It has also been shown to be generalizable to cases where self-gravitation is allowed for in a general relativistic framework.[12,13])

Since the change of variables performed in Sec. II has transformed the system to a representation that is formally identical to that of the case with a nonrotating background, it can also be concluded that the conclusions of the pioneering analysis of Kramer[9] will remain valid, i.e., that in the sense of the preceding paragraph the system will be of the Pippard kind (type I), if $\kappa < 1$, i.e.,

$$4\pi\lambda^2 H_c < g_c\phi \Rightarrow \frac{dU}{d|\Phi|} < \frac{U}{|\Phi|}, \tag{46}$$

and that it will be of the London kind (type II), if $\kappa > 1$, i.e., if

$$4\pi\lambda^2 H_c > g_c\phi \Rightarrow \frac{dU}{d|\Phi|} > \frac{U}{|\Phi|}, \tag{47}$$

at least as long as the winding number N and the parameters κ and g_c are not too far from the neighborhood of unity, in which the numerical investigations have been carried out. It does not seem that the possibility of an intermediate scenario, with $U/|\Phi|$ minimized by a winding number N that is finite but larger that 1, can occur within the framework of Ginzburg-type models except perhaps for parameter values that are too extreme to be of likely physical relevance.

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