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Fundamental strings in Dp-Dq brane systems

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Abstract

We study conformal field theory correlation functions relevant for string diagrams with open strings that stretch between several parallel branes of different dimensions. In the framework of conformal field theory, they involve boundary condition changing twist fields which intertwine between Neumann and Dirichlet conditions. A Knizhnik-Zamolodchikov-like differential equation for correlators of such boundary twist fields and ordinary string vertex operators is derived, and explicit integral formulas for its solutions are provided.

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1. Introduction

D-branes [30] have become the most important ingredient of the new picture of string theory that has emerged in recent years. They have shaped a new understanding of non-perturbative effects in string theory and of low-energy effective theories associated with string theories. In the latter context, systems of many branes are of particular importance, since they provide a natural way to include non-abelian gauge theories into string theory [40]. Systems of several branes of different dimensions, most notably of D1- and D5-branes, play a major role in proposals of how to derive the Bekenstein-Hawking entropy of black holes from string theory [39,27]. More recently, stacks of branes and anti-branes have been reconsidered in connection with a K-theoretic classification of branes [41], based on results concerning tachyon condensation in [37].
Some qualitative features of such systems can be uncovered within a target space approach. But e.g. the process of brane-antibrane annihilation in the last-mentioned application, or the properties of near-extremal black holes involve the analysis of non-BPS states for which the world-sheet approach is better suited, since it does not critically depend on supersymmetry. Computation of CFT correlation functions is indispensable if one wants to deal with problems like Hawking radiation off D1-D5-systems, or for a clean discussion of bound state formation [23].

The world-sheet formulation of string sectors with branes is well known in string theory, although mainly in connection with flat targets. The general setup involves boundary conformal field theory as introduced and developed by Cardy [8,9,10] and first exploited for string theory by Sagnotti [34]. CFT on surfaces with boundaries exhibits a very interesting internal structure and finds interesting applications beyond string theory: For many s-wave dominated scattering processes, the universal behaviour is described by a boundary CFT in two dimensions, irrespective of the dimensionality of the original system. The most famous problem that could be tackled with boundary CFT methods is the Kondo effect in condensed matter physics [1].
In string theory, methods of boundary CFT are not only valuable in the study of situations without the BPS-property, but also to uncover non-classical features like unexpected moduli [37,33] and non-commutative geometry; see e.g. [15,35,36,3,16] and references therein. Moreover, they allow one to analyze D-branes in non-geometric string compactifications such as Gepner models [29,38,32,24,6,13].

In this paper, we ask how to compute CFT correlators describing string amplitudes of arbitrary closed and open string vertex operators in the presence of multiple flat branes in $\mathbb{R}^D$. The open strings involved stretch between two *or more* different branes, which may have *different dimensions*. It appears that no systematic method for the computation of those string diagrams, which contribute to scattering processes in higher orders of the string coupling constant, is available in the literature. See, however, [25] for some sample computations of scattering amplitudes in the presence of a pair of branes.

The world-sheet description requires surfaces with several boundary components. We restrict our attention to diagrams without internal closed string loops so that we can map the world-sheet to the disk or to the upper half-plane, but with different boundary conditions assigned to consecutive intervals on the boundary; see Figure 1. We focus on parallel branes here; thus we can reduce our analysis to a one-dimensional target. Results

for $\mathbb{R}^D$ with $D > 1$ follow by taking tensor products. The boundary state for a p-brane involves $p+1$ Neumann and $D-p-1$ Dirichlet boundary states of single free bosons.
The interesting transitions between boundary conditions in a one-dimensional target are those from Neumann to Dirichlet or vice versa. They are mediated by boundary fields of a special type, namely boundary condition changing *twist fields*.

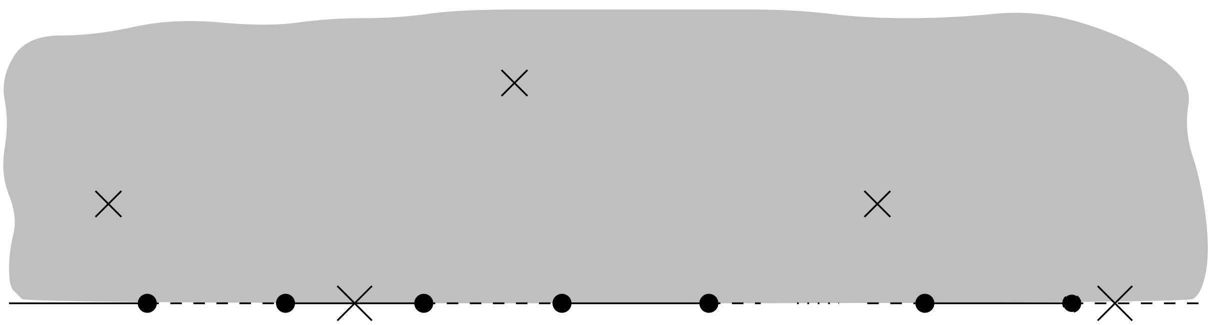

Figure 1: The upper half-plane with a sequence of Neumann (solid intervals) and Dirichlet (dashed intervals) boundary conditions along the real line. The dots between Neumann and Dirichlet intervals mark insertions of boundary condition changing twist fields, while crosses on the boundary or in the interior refer to insertions of ordinary open resp. closed string vertex operators. Such a world-sheet diagram can be understood as Hawking radiation (closed string states) from a system of branes (multiple changes of boundary conditions) with simultaneous inner excitations (open string states).

Conformal boundary conditions which preserve the chiral algebra $\mathcal{W}$ of the theory are parametrized by certain automorphisms Ω of $\mathcal{W}$, together with the amplitudes of one-point functions [32]. If $\mathcal{W}$ is the U(1) current algebra, Ω can act as $\pm\,\mathrm{id}$ on the currents, and the 1-point functions determine the location of a brane. If the boundary condition is constant along the boundary, arbitrary n-point functions can be expressed in terms of the usual conformal blocks of $\mathcal{W}$; see e.g. [8,32,19].
If the gluing conditions described by an automorphism Ω of $\mathcal{W}$ remain constant along the boundary the computation of correlation functions is not, in principle, a difficult problem. Otherwise, the simple Ward identities for the symmetry algebra $\mathcal{W}$ are broken, and one has to find new methods to construct the "twisted chiral blocks" involving a new type of boundary condition changing operators which correspond to twisted rather than ordinary representations of $\mathcal{W}$. It is the aim of this article to develop a convenient formalism for computing such correlation functions in the case of a flat target space.

The plan of this paper is as follows: In the next section, we look at correlation functions which contain just one insertion of a boundary twist field. We shall provide a complete operator construction of the boundary CFT, from which one can derive correlators with an arbitrary number of closed string vertex operators inserted in the bulk. When there are more than two twist fields on the boundary, such techniques are no longer available. Our strategy is then to derive Ward identities for the correlation functions. They will lead us to Knizhnik-Zamolodchikov-like differential equations which describe the effect of moving insertion points for bulk and boundary fields in terms of a flat connection. We explain this idea in Section 3 and exploit it in the fourth section to give explicit integral formulas for the correlators. While some of the technical steps in setting up the Knizhnik-Zamolodchikov equations are rather involved, parts of our final results can be related to electrostatics. Section 5 comments on possible generalizations and applications.

2. Operator formalism for a single twist field insertion

As a simple example, we consider open strings propagating freely in the target $\mathbb{R}$, with Dirichlet boundary conditions imposed at one end of the string and Neumann boundary conditions at the other. We thus have to deal with a free bosonic field $X(t,\sigma)$ defined for space variables $\sigma \in [0,\pi]$ and subject to

$$\partial_t X(t,0) = 0 \qquad , \qquad \partial_\sigma X(t,\pi) = 0$$

for all $t \in \mathbb{R}$. Mapping the strip to the upper half-plane $\mathbb{H}$ by $z = \exp(t + i\sigma)$, $X(z,\bar{z})$ satisfies Dirichlet boundary conditions for $z \in \mathbb{R}_{>0}$ and Neumann boundary conditions for $z \in \mathbb{R}_{<0}$:

$$X(z,\bar{z}) \;=\; \mathrm{x}_0 \;\text{ for }\; z = \bar{z} > 0 \quad , \quad (\partial - \bar{\partial})\, X(z,\bar{z}) \;=\; 0 \;\text{ for }\; z = \bar{z} < 0 \ . \tag{1}$$

Our task is to compute correlation functions in this bosonic theory which involves two insertions of twist fields on the boundary (see [12] for an early treatment of that problem). Conformal symmetry allows us to place these boundary condition changing operators at $x_1 = 0$ and $x_2 = \infty$.

We propose to construct operators $X(z,\bar{z})$ satisfying (1), as well as open and closed string vertex operators, and then to derive differential equations on the correlation functions from the algebraic properties of these operators and from the symmetries of the theory. First, we have to determine the space our fields are to act on.

2.1 The spectrum of boundary twist fields. The space in question is spanned by excited states of open strings stretching between a Neumann and a Dirichlet boundary condition – hence the name "boundary condition changing operators" for the boundary fields uniquely associated to these states. There is a relatively simple technique to determine the spectrum of boundary condition changing operators that intertwine between two constant conformal boundary conditions $B_1 = (\Omega_1, \alpha_1)$ and $B_2 = (\Omega_2, \alpha_2)$ in some boundary CFT. The state space of this boundary theory is denoted by $\mathcal{H}_{12}$. Because of the state-field correspondence, the spectrum of boundary condition changing operators is described through the partition function of the boundary theory,

$$Z_{12}(q) \;=\; \mathrm{Tr}_{\mathcal{H}_{12}}\, q^{H^{(H)}} \qquad \text{where} \qquad H^{(H)} \;=\; L_0 - \frac{c}{24} \ .$$

By an interchange of space and time coordinates ("world-sheet duality"), the open string 1-loop diagram underlying Z_{12} may be viewed as a closed string tree diagram, i.e.

$$Z_{12}(q) \;=\; \langle B_1|\, \tilde{q}^{\frac{1}{2}H^{(P)}}\, |B_2\rangle \quad \text{where} \quad H^{(P)} \;=\; L_0^{(P)} + \overline{L}_0{}^{(P)} - \frac{c}{12}\ ,$$

where $\tilde{q} = \exp(-2\pi i/\tau)$ is related to the variable $q = \exp(2\pi i\tau)$ as usual. The closed strings propagate between the *boundary states* $|B_i\rangle = |\alpha_i\rangle_{\Omega_i}$ associated with the boundary conditions B_i. They allow to transfer boundary conditions from the upper half-plane (where the Hamiltonian is $H^{(H)}$) into a CFT on the full plane (with Hamiltonian $H^{(P)}$), see [26,9].

The boundary states implementing Dirichlet and Neumann conditions along the whole boundary are of course well known, see e.g. [7]. Let $a_n^{(P)}$, $\overline{a}_n^{(P)}$ be two commuting sets of oscillator modes (in the plane CFT) with standard commutation relations. The ground states $|k\rangle$ of their Fock spaces are labeled by the momentum $k \in \mathbb{R}$. Neumann and Dirichlet boundary states are given by

$$\begin{aligned} |\,N\,\rangle &= \frac{1}{\sqrt{2}}\ \exp\{\,-\sum_{n\geq 1}\frac{1}{n}\, a_{-n}^{(P)}\,\overline{a}_{-n}^{(P)}\,\}\ |0\rangle\ , \\ |\,D(\mathrm{x}_0)\,\rangle &= \int dk\ e^{ik\mathrm{x}_0}\ \exp\{\sum_{n\geq 1}\frac{1}{n}\, a_{-n}^{(P)}\,\overline{a}_{-n}^{(P)}\,\}\ |k\rangle\ , \end{aligned} \tag{2}$$

where $\mathrm{x}_0 \in \mathbb{R}$, as in (1), denotes the location of the "D-brane", i.e. $\hat{x}\,|D(\mathrm{x}_0)\rangle = \mathrm{x}_0\,|D(\mathrm{x}_0)\rangle$ for the center of mass coordinate $\hat{x}$.
If we build the boundary state $|p\rangle$ for a p-brane in $\mathbb{R}^D$ as a tensor product of $|N\rangle$ ($p+1$ times) and $|D(\mathrm{x}_0^i)\rangle$ ($i = p+2,\dots,D$) from eq. (2), the partition function $Z_{pp}(q)$ counts boundary fields that do not change the boundary condition. They describe excitations of open strings attached to the p-brane. These open string vertices have the form $\varepsilon_{\mu_1\dots\mu_n}\partial X^{\mu_1}\cdots\partial X^{\mu_n}e^{ikX}$ with certain Lorentz tensors $\varepsilon_{\mu_1\dots\mu_n}$ and with momentum k parallel to the Neumann directions. The case $n=1$ (where the polarization ε_μ is transversal) contains the massless modes: gauge fields living on the brane world-volume.

The partition function of the theory with Neumann boundary conditions on one side and Dirichlet on the other follows from the boundary states as explained above,

$$\begin{aligned} Z_{ND}(q) &= \ \mathrm{Tr}_{\mathcal{H}_{ND}}\ q^{H_{ND}} \ = \langle\, N\,|\, \tilde{q}^{L_0-\frac{c}{24}}\,|\, D(\mathrm{x}_0)\,\rangle \\ &= \ \frac{1}{\sqrt{2}}\langle 0|\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}\,a_n\,\overline{a}_n}\ \tilde{q}^{L_0-\frac{c}{24}}\ e^{\sum_{m=1}^{\infty}\frac{1}{m}\,a_{-m}\,\overline{a}_{-m}}\ |0\rangle\ . \end{aligned} \tag{3}$$

Orthonormality of the Fock ground states implies that only the contribution from the vacuum sector survives in the second line. In particular, Z_{ND} is independent of the parameter x_0. Computation of the vacuum expectation value above is straightforward. The result can be written as

$$Z_{ND}(q) = q^{\frac{1}{48}}\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})^{-1} = \frac{1}{\eta(q)}\ \sum_{n=1}^{\infty} q^{\frac{1}{4}(n-\frac{1}{2})^2}\ . \tag{4}$$

Our main conclusion concerns the conformal weights of the boundary fields that can induce a transition between Dirichlet and Neumann type boundary conditions. The lowest weight that appears is $h = \frac{1}{16}$. Above this value, the spectrum of conformal weights has half-integer spacings. The boundary condition changing operator with conformal weight $h = \frac{1}{16}$ corresponding to the lowest-energy state $|\sigma\rangle$ in the whole sector $\mathcal{H}_{ND}$ will be called $\sigma(x)$. We will also refer to $\sigma(x)$ as a "twist field" since the sum of irreducible Virasoro characters in (4) can alternatively be regarded as the character of a twisted U(1) representation. The absence of a vacuum state and the half-integer energy grading are symptoms for the fact that the jump from Neumann to Dirichlet destroys the simple U(1) Ward identities that are present in a boundary CFT with constant Neumann or Dirichlet condition all along the boundary. See [21] for general results about twist fields and partition functions in boundary conformal field theory.
It will be our main concern in the following to find "substitutes" for the broken Ward identities, namely twisted Knizhnik-Zamolodchikov equations.

2.2 Construction of the basic fields. In order to construct a field $X(z,\bar{z})$ obeying the boundary conditions (1) on $\mathcal{H} = \mathcal{H}_{ND}$, we introduce a set of oscillator modes a_r labelled by half-integers $r \in \mathbb{Z} + \frac{1}{2}$. (We drop the superscript $^{(H)}$ for operators of the upper half-plane theory.) They are supposed to obey the relations

$$[\, a_r \, , \, a_s \,] \;=\; r\, \delta_{r,-s} \quad , \quad a_r^* \;=\; a_{-r} \quad .$$

Creation operators $a_r,\ r < 0$, generate the Fock space $\mathcal{H}$ out of the ground state $|\sigma\rangle$, which is annihilated by modes a_r with index $r > 0$. All the fields we shall consider act on this state space $\mathcal{H}$. It is simple to verify that the decomposition $X(z,\bar{z}) = X(z) - \overline{X}(\bar{z})$ of the bosonic field yields the desired properties if

$$X(z) \;=\; \mathrm{x}_0 + i \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{a_r}{r}\, z^{-r} \quad , \quad \overline{X}(\bar{z}) \;=\; i \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{a_r}{r}\, \bar{z}^{-r} \quad .$$

To make the square root well defined, we have to introduce a branch cut in the plane, which extends from $x = 0$ to $-\infty$. Once the bosonic field is known, we obtain chiral currents as

$$J(z) \;:=\; i\,\partial X(z,\bar{z}) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} a_r\, z^{-r-1} \quad , \quad \overline{J}(\bar{z}) \;:=\; i\,\bar{\partial} X(z,\bar{z}) = \;-\sum_{r\in\mathbb{Z}+\frac{1}{2}} a_r\, \bar{z}^{-r-1} \quad .$$

Finally, the components $T(z)$ and $\overline{T}(\bar{z})$ of the stress energy tensor are given by

$$T(z) \;=\; \lim_{w\to z} \frac{1}{2} \left(J(w)J(z) - \frac{1}{(w-z)^2} \right) \quad ,$$

and likewise for $\overline{T}(\bar{z})$. Since T and $\overline{T}$ are quadratic in J and $\overline{J}$, they satisfy the usual boundary condition $T(z) = \overline{T}(\bar{z})$ all along the real line $\mathrm{Im}z = 0$. By the usual arguments [8] this implies that the modes

$$L_n \;:=\; \int_{C_+} \frac{dz}{2\pi i}\, z^{n+1}\, T(z) + \int_{C_-} \frac{d\bar{z}}{2\pi i}\, \bar{z}^{n+1}\, \overline{T}(\bar{z})$$

obey commutation relations of the Virasoro algebra with central charge $c = 1$. Here $C_+ - C_-$ is a closed oriented contour surrounding the origin, with C_+ contained in the upper half-plane and C_- contained in the lower half-plane. The commutation relation between L_n and a_r is easily checked to be of the form

$$[\, L_n \,,\, a_r \,] \;=\; -r\; a_{n+r} \quad .$$

It is convenient to introduce two generating fields T and J by the formal sums

$$\mathsf{T}(w) \;=\; \sum_{n\in\mathbb{Z}} L_n\; w^{-n-2} \quad , \quad \mathsf{J}(w) \;=\; \sum_{r\in\mathbb{Z}+\frac{1}{2}} a_r\; w^{-r-1} \quad .$$

One may think of T as being defined on the entire complex plane with $\mathsf{T}(w) = T(w)$ in the upper half-plane, $\mathrm{Im}w > 0$, and $\mathsf{T}(w) = \overline{T}(w)$, for all w with $\mathrm{Im}w < 0$. The generating field J naturally lives on the two-fold branched cover of the complex plane defined by $\omega^2 = w$. By introducing the branch cut from $x = 0$ to $-\infty$ we have specified a coordinate patch on this surface with the local coordinate being denoted by w. In this chart, $\mathsf{J}(w) = J(w)$ for $\mathrm{Im}w > 0$ and $\mathsf{J}(w) = -\overline{J}(w)$ for $\mathrm{Im}w < 0$.

The commutation relations for the modes L_n, a_r with the bosonic field $X(z,\bar z)$ can be expressed in terms of T and J as follows

$$[\, \mathsf{T}(w) \,,\, X(z,\bar z) \,] = \partial X(z,\bar z)\; \delta(z-w) + \bar\partial X(z,\bar z)\; \delta(\bar z - w) \quad ,$$

$$[\, \mathsf{J}(w) \,,\, X(z,\bar z) \,] = i\,\delta(z-w) \;+\; i\,\delta(\bar z - w) \quad ,$$

where

$$\delta(z-w) \;:=\; \frac{1}{z}\sum_{n\in\mathbb{Z}} \Big(\frac{z}{w}\Big)^{n} \;=\; \frac{1}{z}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big(\frac{z}{w}\Big)^{r} \quad .$$

We state two simple consequences of these formulas that shall be important below. We split J and T into two parts $\mathsf{J}(w) = \mathsf{J}_>(w) + \mathsf{J}_<(w)$ and $\mathsf{T}(w) = \mathsf{T}_>(w) + \mathsf{T}_<(w)$ such that

$$\mathsf{J}_>(w) \;:=\; \sum_{r\geq 1/2} a_r\; w^{-r-1} \quad , \quad \mathsf{T}_>(w) \;:=\; \sum_{n\geq -1} L_n\; w^{-n-2} \quad .$$

In the next subsection, we will use the commutation relations between the singular parts $\mathsf{T}_>, \mathsf{J}_>$ of the generating fields T, J and the bosonic field $X(z,\bar z)$:

$$[\, \mathsf{J}_>(w) \,,\, X(z) \,] = -\Big(\frac{z}{w}\Big)^{1/2} \frac{i}{w-z} \;, \qquad [\, \mathsf{T}_>(w) \,,\, X(z) \,] = \frac{1}{w-z}\; \partial_z X(z) \;. \tag{5}$$

Moreover, we will need the following lemma, a proof of which is given in Appendix A.

Lemma 1: *One may rewrite the generating field $\mathsf{T}(w)$ in terms of the objects $\mathsf{J}_>(w)$ and $\mathsf{J}_<(w)$, namely*

$$\mathsf{T}(w) \;=\; \frac{1}{2}\big(\, \mathsf{J}_<(w)\mathsf{J}(w) + \mathsf{J}(w)\mathsf{J}_>(w) \,\big) + \frac{1}{16}\frac{1}{w^2} \quad . \tag{6}$$

2.3 Bulk and boundary primary fields. Our next aim is to construct primary bulk and boundary fields. Here, the latter term refers to open string vertex operators which can be inserted on $\mathbb{R}_{<0}$ or $\mathbb{R}_{>0}$ without changing the boundary condition. They are in one-to-one correspondence to states in $\mathcal{H}_{DD}$ or $\mathcal{H}_{NN}$, *not* to states in $\mathcal{H}_{ND}$. We will see that bulk fields can be regarded as products of such boundary fields $\Psi_{\mathrm{g}}(z)$. Therefore we discuss these "chiral fields" first – admitting arbitrary complex insertion points, not just $z \in \partial\mathbb{H}$. The fields $\Psi_{\mathrm{g}}(z)$ are labeled by a real parameter g and enjoy the properties

$$\begin{aligned} [\,\mathsf{T}(w)\,,\,\Psi_{\mathrm{g}}(z)\,] \;&=\; \partial_z\Psi_{\mathrm{g}}(z)\;\delta(x-w) + h\,\Psi_{\mathrm{g}}(z)\;\partial_z\delta(z-w) \quad , \\ [\,\mathsf{J}(w)\,,\,\Psi_{\mathrm{g}}(z)\,] \;&=\; \mathrm{g}\,\Psi_{\mathrm{g}}(z)\;\delta(z-w) \quad . \end{aligned}$$

We have used the definition of the formal δ-function specified above and $h = \frac{1}{2}\mathrm{g}^2$. For the commutators of the $\mathsf{T}_>(w), \mathsf{J}_>(w)$ with the fields $\Psi_{\mathrm{g}}(x)$, this implies

$$[\,\mathsf{T}_>(w)\,,\,\Psi_{\mathrm{g}}(z)\,] \;=\; \frac{1}{w-z}\;\partial_z\Psi_{\mathrm{g}}(z) + \frac{h}{(w-z)^2}\;\Psi_{\mathrm{g}}(z) \quad , \tag{7}$$

$$[\,\mathsf{J}_>(w)\,,\,\Psi_{\mathrm{g}}(z)\,] \;=\; \left(\frac{z}{w}\right)^{1/2}\frac{\mathrm{g}}{w-z}\;\Psi_{\mathrm{g}}(z) \quad . \tag{8}$$

Lemma 2: *The unique solution (up to normalization) $\Psi_{\mathrm{g}}(z)$ to the requirements (7,8) is given by*

$$\Psi_{\mathrm{g}}(z) \;=\; \left(\frac{i}{2}\right)^h z^{-h}\, e^{i\mathrm{g}X_<(z)}\; e^{i\mathrm{g}X_>(z)} \qquad \text{where} \qquad X_>(z) \;=\; i\sum_{r>0}\frac{a_r}{r}z^{-r}$$

and $X_<(z) = X(z) - X_>(z)$. Note that $\Psi_{\mathrm{g}}(z)$ is normal-ordered, i.e., the annihilators $a_r, r>0$, appear to the right of the creation operators.

A proof can be found in Appendix A.
Our next aim is to describe the U(1)-primary bulk fields $\phi_{\mathrm{g}}(z,\bar{z})$. By definition, they obey the following commutation relations with respect to $\mathsf{J}_>$ and $\mathsf{T}_>$,

$$\begin{aligned} [\,\mathsf{T}_>(w)\,,\,\phi_{\mathrm{g}}(z,\bar{z})\,] = &\frac{1}{w-z}\;\partial\phi_{\mathrm{g}}(z,\bar{z}) + \frac{h}{(w-z)^2}\;\phi_{\mathrm{g}}(z,\bar{z}) \\ &+\frac{1}{w-\bar{z}}\;\bar{\partial}\phi_{\mathrm{g}}(z,\bar{z}) + \frac{h}{(w-\bar{z})^2}\;\phi_{\mathrm{g}}(z,\bar{z})\;, \end{aligned} \tag{9}$$

$$[\,\mathsf{J}_>(w)\,,\,\phi_{\mathrm{g}}(z,\bar{z})\,] = \left(\frac{z}{w}\right)^{1/2}\frac{\mathrm{g}}{w-z}\;\phi_{\mathrm{g}}(z,\bar{z}) - \left(\frac{\bar{z}}{w}\right)^{1/2}\frac{\mathrm{g}}{w-\bar{z}}\;\phi_{\mathrm{g}}(z,\bar{z})\;. \tag{10}$$

Note that each term from eqs. (7,8) appears a second time with z being replaced by the variable $\bar{z}$. One can easily work out commutation relations between the full generating elements $\mathsf{T}(w), \mathsf{J}(w)$ and the bulk primary fields $\phi_{\mathrm{g}}(z,\bar{z})$. It is obvious from our discussion of boundary fields that bulk fields $\phi_{\mathrm{g}}(z,\bar{z})$ can be written as products of chiral vertex operators,

$$\phi_{\mathrm{g}}(z,\bar{z}) \;=\; \Psi_{\mathrm{g}}(z)\,\Psi_{-\mathrm{g}}(\bar{z}) \quad .$$

The formulas we have reviewed here would enable us to perform a direct computation of arbitrary correlations functions $G(\vec{z})$ for bulk-fields $\phi_{\rm g}(z,\bar{z})$ with two twist fields σ inserted at $x=0$ and $x=\infty$,

$$G(\vec{z}) \ := \ \langle\sigma|\, \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)|\sigma\rangle \quad \text{where} \quad \phi_\nu(z_\nu,\bar{z}_\nu) \ = \ \phi_{{\rm g}_\nu}(z_\nu,\bar{z}_\nu) \ .$$

The calculations would proceed by moving all annihilation operators to the right until they act on the ground state $|\sigma\rangle$. The same techniques would apply if there are extra boundary fields $\Psi_{\rm g}(x)$ inserted in addition to the bulk fields $\phi_{\rm g}(z,\bar{z})$. Since we will develop another, more generally applicable approach to the computation of correlation functions below, we do not enter details here.

Before we conclude this subsection, we would like to derive *bulk-boundary operator product expansions* which allow to expand our bulk fields $\phi_{\rm g}(z,\bar{z})$ in terms of boundary operators [10]. The essential idea is that, in the presence of a boundary, bulk fields split into products of chiral vertex operators inserted at points which are obtained from each other by reflection at the real axis and with opposite charges ("method of image charges"). In order to obtain concrete formulas, we first rewrite the bulk fields $\phi_{\rm g}(z,\bar{z}) = \Psi_{\rm g}(z)\Psi_{-{\rm g}}(\bar{z})$ in terms of $X_>(z,\bar{z}) = X_>(z) - \overline{X}_>(\bar{z})$ and $X_<(z,\bar{z}) = X_<(z) - \overline{X}_<(\bar{z})$,

$$\phi_{\rm g}(z,\bar{z}) = \ \left(\frac{i}{2}\right)^{2h} (z\bar{z})^{-h} \left(\frac{\sqrt{z}+\sqrt{\bar{z}}}{\sqrt{z}-\sqrt{\bar{z}}}\right)^{2h} \ e^{i{\rm g}X_<(z,\bar{z})} \ e^{i{\rm g}X_>(z,\bar{z})} \quad . \tag{11}$$

We have used the expression in Lemma 2 and then normal-ordered the right hand side with the help of the BCH formula, which leads to the additional $\sqrt{z}$- and$\sqrt{\bar{z}}$-dependent factor.

Lemma 3 (bulk-boundary OPE): *For arguments $z = x+iy$ close to the boundary, i.e., $y>0$ small, the operators $\phi_{\rm g}(z,\bar{z})$ can be expanded in a series involving boundary primary fields, with leading asymptotics*

$$\phi_{\rm g}(z,\bar{z}) \sim \frac{e^{i{\rm g}{\rm x}_0}}{y^{2h}} \ \ \mathbf{1} \qquad\qquad \text{for} \quad x>0 \ \ ,$$

$$\phi_{\rm g}(z,\bar{z}) \sim y^{2h} \ \Psi_{2{\rm g}}(x) \qquad \text{for} \quad x<0 \ \ ,$$

where $\mathbf{1}$ *is the identity field.*

Proof: Let us begin with the case $x>0$ in which $\sqrt{z}-\sqrt{\bar{z}} \to 0$ as y becomes very small.

$$\begin{aligned} \phi_{\rm g}(z,\bar{z}) &= \left(\frac{i}{2}\right)^{2h} (z\bar{z})^{-h} \left(\frac{z+\bar{z}+2\sqrt{z\bar{z}}}{z-\bar{z}}\right)^{2h} \ e^{i{\rm g}X_<(z,\bar{z})} e^{i{\rm g}X_>(z,\bar{z})} \\ &\sim \left(\frac{i}{2}\right)^{2h} x^{-2h} \ \left(\frac{2x}{iy}\right)^{2h} \ \mathbf{1} \ = \ \frac{1}{y^{2h}} \, \mathbf{1} \ \ . \end{aligned}$$

We have also used that $X_<(x,x) = \mathrm{x}_0$ and $X_>(x,x) = 0$ for $x > 0$, which is a direct consequence of the Dirichlet boundary condition.

If $x < 0$ and y tends to zero, the sum $\sqrt{z}+\sqrt{\bar z}$ vanishes and we can estimate the behaviour of $\phi_{\mathrm{g}}(z,\bar z)$ according to

$$\begin{aligned}\phi_{\mathrm{g}}(z,\bar z) &= \left(\frac{i}{2}\right)^{2h} (z\bar z)^{-h} \left(\frac{z-\bar z}{z+\bar z-2\sqrt{z\bar z}}\right)^{2h} e^{i\mathrm{g}X_<(z,\bar z)} e^{i\mathrm{g}X_>(z,\bar z)} \\ &\sim \left(\frac{i}{2}\right)^{2h} x^{-2h} \left(\frac{iy}{2x}\right)^{2h} e^{2i\mathrm{g}X_<(x)} e^{2i\mathrm{g}X_>(x)} \;=\; y^{2h}\;\Psi_{2\mathrm{g}}(x) \quad . \end{aligned}$$ ∎

Observe that boundary condition changing operators themselves do not arise from the bulk-boundary OPE of bulk fields. Let us finally note that operator product expansions for bulk fields or for boundary fields can be worked out with the same techniques. Those for bulk fields $\phi_{\mathrm{g}}(z,\bar z)$ of course agree with the usual OPE of primary fields in the bulk.

2.4 Correlation functions and the Knizhnik-Zamolodchikov equation. This subsection contains the main result of this section, namely a derivation of the Knizhnik-Zamolodchikov equation for correlation functions of bulk and boundary primaries in the presence of a transition from Dirichlet to Neumann boundary conditions at the origin. Let us look more closely at correlation functions containing n chiral fields $\Psi_{\mathrm{g}}(z)$,

$$F(\vec z) \;:=\; \langle\sigma|\,\Psi_1(z_1)\dots\Psi_n(z_n)|\sigma\rangle \quad \text{where} \quad \Psi_\nu(z_\nu) \;:=\; \Psi_{\mathrm{g}_\nu}(z_\nu) \quad .$$

Before we start, let us state two elementary formulas for the action of $\mathsf{J}_>(w)$, $\mathsf{T}_>(w)$ on the ground state $|\sigma\rangle$:

$$\mathsf{T}_>(w)\;|\sigma\rangle \;=\; \left(\frac{1}{w^2}\,h_\sigma + \frac{1}{w}\,L_{-1}\right)\;|\sigma\rangle \quad , \tag{12}$$

$$\mathsf{J}_>(w)\;|\sigma\rangle \;=\; \sum_{r\geq 1/2} a_r\,|\sigma\rangle\,w^{-r-1} \;=\; 0 \quad , \tag{13}$$

where h_σ is the conformal weight of the state $|\sigma\rangle$. We will recover the equation $h_\sigma = \frac{1}{16}$ in a moment. The object $\langle\sigma|$ dual to $|\sigma\rangle$ obeys the relations $\langle\sigma|\,\mathsf{J}_<(w) = 0 = \langle\sigma|\,\mathsf{T}_<(w)$.

A first differential equation is obtained by inserting the generating field $\mathsf{T}(w)$ into the correlation function:

$$\langle\sigma|\,\mathsf{T}(w)\Psi_1(z_1)\dots\Psi_n(z_n)\,|\sigma\rangle \;=\; \langle\sigma|\,\mathsf{T}_>(w)\Psi_1(z_1)\dots\Psi_n(z_n)\,|\sigma\rangle$$

$$= \left[\,\sum_{\nu=1}^{n}\left(\frac{1}{w-z_\nu}\,\partial_\nu \;+\; \frac{h_\nu}{(w-z_\nu)^2}\right) + \frac{h_\sigma}{w^2}\,\right]\;F(\vec z) + \frac{1}{w}\;\langle\sigma|\,\Psi_1(z_1)\dots\Psi_n(z_n)\,L_{-1}\,|\sigma\rangle \quad .$$

Here, we have commuted $\mathsf{T}_>(w)$ through the fields $\Psi_\nu(z_\nu)$ until it acts on the ground state $|\sigma\rangle$ so that we can use formula (12).

Now we want to compute the same correlation function with the help of the affine Sugawara construction, i.e. by exploiting eq. (6):

$$
\begin{aligned}
&\langle\sigma|\,\mathsf{T}(w)\Psi_1(z_1)\dots\Psi_n(z_n)\,|\sigma\rangle \\
&= \frac{1}{2}\,\langle\sigma|\,\mathsf{J}_>(w)\mathsf{J}_>(w)\Psi_1(z_1)\dots\Psi_n(z_n)\,|\sigma\rangle + \frac{1}{16w^2}\,\langle\sigma|\,\Psi_1(z_1)\dots\Psi_n(z_n)\,|\sigma\rangle \\
&= \frac{1}{2}\sum_{\nu=1}^{n}\Big(\frac{z_\nu}{w}\Big)^{1/2}\frac{\mathrm{g}_\nu}{w-z_\nu}\;\langle\sigma|\,\mathsf{J}_>(w)\Psi_1(z_1)\dots\Psi_n(z_n)\,|\sigma\rangle + \frac{1}{16w^2}\,F(\vec{z}) \\
&= \left[\frac{1}{2}\sum_{\nu,\mu}\frac{\sqrt{z_\nu z_\mu}}{w}\frac{\mathrm{g}_\nu\mathrm{g}_\mu}{(w-z_\nu)(w-z_\mu)} + \frac{1}{16w^2}\right]\;F(\vec{z}) \quad .
\end{aligned}
$$

Comparison with our first formula for the insertion of $\mathsf{T}(w)$ yields $h_\sigma = \frac{1}{16}$. From the residue at $w = 0$ we get

$$
\langle\sigma|\,\Psi_1(z_1)\dots\Psi_n(z_n)\,L_{-1}\,|\sigma\rangle \;=\; \frac{1}{2}\sum_{\nu,\mu}\frac{\mathrm{g}_\nu\mathrm{g}_\mu}{\sqrt{z_\nu z_\mu}}\;F(\vec{z}) \quad .
$$

Finally, from the residue at $w = z_\nu$ we obtain

$$
\partial_{z_\nu}\,F(\vec{z}) \;=\; \left[-\frac{h_\nu}{z_\nu} \;+\; \sum_{\mu\neq\nu}\sqrt{\frac{z_\mu}{z_\nu}}\;\frac{\mathrm{g}_\nu\mathrm{g}_\mu}{z_\nu - z_\mu}\right]\;F(\vec{z}) \;. \tag{14}
$$

This is the *Knizhnik-Zamolodchikov equation* we were after. Note that the terms in square brackets determine a flat connection, as in the ordinary Knizhnik-Zamolodchikov equation.

We can solve (14) by a simple coordinate transformation. In fact, if we introduce coordinates $u_\nu = \sqrt{z}_\nu$ and the function $F_u(u_1,\dots,u_n) = \prod_\nu u_\nu^{h_\nu} F(u_1^2,\dots,u_n^2)$, then the system (14) of first order differential equations becomes

$$
\partial_{u_\nu}\,F_u(u_1,\dots,u_n) = \bigg(-\frac{\mathrm{g}_\nu^2}{2u_\nu} \;+\; \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{n}\Big[\frac{\mathrm{g}_\nu\mathrm{g}_\mu}{u_\nu - u_\mu} - \frac{\mathrm{g}_\nu\mathrm{g}_\mu}{u_\nu + u_\mu}\Big]\bigg)\,F_u(u_1,\dots,u_n)\;. \tag{15}
$$

This equation is formally identical to the usual Knizhnik-Zamolodchikov equation with $2n$ fields of charges $\pm\mathrm{g}_\nu$ inserted at the points $\pm u_\nu$. The solution to (15) is given by

$$
F_u(u_1,\dots,u_n) = \kappa\cdot\prod_{\nu=1}^{n} u_i^{-\frac{\mathrm{g}_\nu^2}{2}} \prod_{1\leq\nu<\mu\leq n}\Big(\frac{u_\nu - u_\mu}{u_\nu + u_\mu}\Big)^{\mathrm{g}_\nu\mathrm{g}_\mu} \quad . \tag{16}
$$

The free parameter κ can be determined from the boundary condition of the bosonic field X on the positive real line, i.e. $\kappa = \kappa(\mathrm{x}_0)$ depends on the position x_0 of the D-brane.

3. Twisted Knizhnik-Zamolodchikov equation for multiple transitions

In the following, we study n-point functions of a free bosonic field theory on the half-plane with several insertions of twist operators placed along the boundary. In the corresponding string diagrams, open strings stretch between three or more branes of various dimensions. As long as only one DN-jump occurs, a simple Hilbert space formulation of the boundary CFT is available, and we can solve, in principle, for correlation functions by purely algebraic techniques, as indicated in the last section. In the presence of many boundary condition changing twist fields, it may be simpler to resort to OPE methods and to the theory of complex functions on higher genus Riemann surfaces, and this is the approach we pursue in the present section. We begin with a very brief review of relevant input from the theory of hyperelliptic surfaces. We then discuss Ward identities in the second subsection. The latter allow us to derive a system of linear first order differential equations for the correlation functions similar to the Knizhnik-Zamolodchikov equations. Note that free bosons on higher genus surfaces without boundaries (i.e. higher loop diagrams of closed strings propagating in a flat target) have been studied in great detail in [4,31].

3.1 Hyperelliptic surfaces. Our aim is to investigate a scenario in which a bosonic field $X(z,\bar{z})$ is defined on the upper half-plane with boundary conditions switching between Dirichlet and Neumann at $2g+2$ points $x_i, i = 1,\ldots,2g+2$, on the boundary. Without loss of generality, we shall assume that $x_{2g+2} = \infty =: x_0$. To be more precise, we impose Dirichlet boundary conditions in the intervals $]x_i, x_{i+1}[$ for i odd and Neumann boundary conditions along the rest of the boundary, i.e.,

$$X(z,\bar{z}) \;=\; \mathrm{x}_0^k \quad \text{for} \quad z \;=\; \bar{z} \;\in\; D_k \;:=\;]x_{2k-1}, x_{2k}[$$

and

$$\partial_y X(z,\bar{z}) \;=\; 0 \quad \text{for} \quad z \;=\; \bar{z} \;\in\; N_k \;:=\;]x_{2k-2}, x_{2k-1}[\quad .$$

The variable y is defined through $z = x + iy$, and $k = 1,\ldots,g+1$. In terms of the chiral currents $J(z) = i\,\partial X(z,\bar{z})$ and $\overline{J}(\bar{z}) = i\,\bar{\partial} X(z,\bar{z})$, these conditions become

$$J(x) \;=\; -\overline{J}(x) \quad \text{for} \quad x \;\in\; D_k \;, \quad \text{and} \quad J(x) \;=\; \overline{J}(x) \quad \text{for} \quad x \;\in\; N_k \quad .$$

As in the previous section, it is convenient to work with a single field J that contains all information about the two chiral currents J and $\overline{J}$. Such a field necessarily lives on a two-fold branched cover of the complex w-plane, namely on the *hyperelliptic surface*, M, of genus g which is described by the equation

$$\omega^2 \;=\; P(w) \;:=\; \prod_{i=1}^{2g+1} (w - x_i) \quad .$$

Introducing branch cuts along the intervals $N_k = [x_{2k-2}, x_{2k-1}]$, we obtain a particular coordinate patch of this surface with local coordinate w. In this chart, $\mathsf{J}(w)$ satisfies $\mathsf{J}(w) = J(w)$ for $\mathrm{Im}w > 0$ and $\mathsf{J}(w) = -\overline{J}(w)$ for $\mathrm{Im}w < 0$. The Virasoro field T obeys the gluing condition $T(x) = \overline{T}(x)$, all along the boundary, since it is quadratic in the currents. Consequently, the generating field $\mathsf{T}(w)$ is defined on the complex w-plane and coincides with T (resp. $\overline{T}$) on the upper (resp. lower) half-plane.

The coordinate w on the complex plane lifts to a meromorphic function of degree 2, also denoted by w, on the hyperelliptic surface M. This function defines a two-fold covering of the sphere branched over $2g+2$ points $Q_1, \ldots, Q_{2g+2}$, where $w(Q_i) = x_i$. A basis for the space of holomorphic 1-forms on M is then given by

$$\omega_k \ := \ \frac{w^{k-1} dw}{\sqrt{P(w)}} \quad \text{for} \quad k = 1, \ldots, g \ .$$

It will be convenient to work with a canonical homology basis $\{\gamma_k,\ \tilde{\gamma}_k\}$ on M chosen as in Figure 2. We denote by Ω_{kl} the period of the 1-form ω_l along the cycle γ_k, i.e.,

$$\Omega_{kl} := \oint_{\gamma_k} \omega_l = \oint_{\gamma_k} \frac{w^{l-1} dw}{\sqrt{P(w)}} \ .$$

The basis of holomorphic 1-forms, $\{\zeta_k\}$, dual to the canonical homology basis $\{\gamma_k,\ \tilde{\gamma}_k\}$ is defined by the equation

$$\oint_{\gamma_k} \zeta_l = \delta_{kl} \ .$$

In particular, we have the relation

$$\omega_k = \sum_{l=1}^{g} \ \Omega_{lk} \ \zeta_l \ ,$$

and the matrix Ω is invertible. The *period matrix* τ is given by

$$\tau_{kl} := \oint_{\tilde{\gamma}_k} \zeta_l \ ,$$

and it is known to be symmetric and to have positive definite imaginary part. The surface M has an anti-holomorphic involution induced by complex conjugation on the complex plane. In terms of the functions w and $\omega = \sqrt{P(w)}$, it can be written as

$$(w, \omega) \to (\bar{w}, \bar{\omega}) \ .$$

This involution will be used to extend the theory from the upper half-plane to the lower half-plane while taking care of the boundary conditions on the real axis. There is a second holomorphic involution that interchanges the two sheets of M and that can be written as

$$(w, \omega) \to (w, -\omega) \ .$$

This involution is used when passing from the sphere with cuts to its cover M.

3.2 The Ward identities. To begin with, we introduce the correlations that we plan to investigate below. Besides the primary bulk fields $\phi_{\mathrm{g}}(z,\bar{z}) := \exp(i\mathrm{g}X(z,\bar{z}))$, they involve $2g+2$ boundary twist fields inserted at the points x_i, which induce changes between Dirichlet and Neumann boundary conditions. From our discussion in the previous section we know that there is an infinite number of boundary condition changing operators that we could insert. But it is sufficient to study the fields $\sigma(x)$ of conformal weight $h = \frac{1}{16}$, since all others can be obtained out of $\sigma(x)$ by OPE with chiral fields. Thus, our discussion deals with correlators of the form

$$G(\vec{z},\vec{x}) \;=\; \langle \phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\;\sigma(x_1)\ldots\sigma(x_{2g+1})\rangle \tag{17}$$

where we use the notation $\phi_\nu = \phi_{\mathrm{g}_\nu}$, and where the boundary field $\sigma(x_{2g+2}) = \sigma(\infty)$ is absorbed in the notation $\langle\ldots\rangle = \langle\sigma|\ldots|0\rangle$, with $|0\rangle$ denoting the vacuum state. We could insert further boundary fields $\Psi_{\mathrm{g}}(z)$. Their interpretation depends on whether they are inserted in an interval with Dirichlet or Neumann boundary conditions. In the former case, they could induce jumps in the Dirichlet parameters x_0^k if they are associated with open strings stretching between branes at different positions. A boundary operator $\Psi_{\mathrm{g}}(z)$ inserted in one of the Neumann intervals, on the other hand, creates an open string which has both ends on the same brane (a Euclidean 1-brane, in our case) and moves with some definite momentum along its world-volume.

As far as Ward identities are concerned, correlation functions of such boundary fields are actually more fundamental, since one may split the bulk fields $\phi_{\mathrm{g}}(z,\bar{z})$ into a product of $\Psi_{\mathrm{g}}(z)$ and $\Psi_{-\mathrm{g}}(\bar{z})$. For this reason, most of our investigations below involve the correlation functions

$$F(\vec{z},\vec{x}) \;=\; \langle \Psi_1(z_1)\ldots\Psi_n(z_n)\;\sigma(x_1)\ldots\sigma(x_{2g+1})\rangle \quad , \tag{18}$$

from which the correlators $G(\vec{z},\vec{x})$ can be reconstructed.

Our analysis will make essential use of the Mittag-Leffler theorem, and hence it is based on the study of singularities in correlation functions. The latter are encoded in the operator product expansions between chiral fields and the bulk and boundary fields appearing in (17,18). For the Virasoro field T one has the standard expansions:

$$\mathsf{T}(w)\,\Psi_{\mathrm{g}}(z) \sim \left[\frac{h_{\mathrm{g}}}{(w-z)^2} + \frac{1}{w-z}\,\partial_z\right]\;\Psi_{\mathrm{g}}(z) \quad , \tag{19}$$

$$\mathsf{T}(w)\;\sigma(x) \sim \left[\frac{h_\sigma}{(w-x)^2} + \frac{1}{w-x}\,\partial_x\right]\;\sigma(x) \quad . \tag{20}$$

Here and in the following, the symbol $\sim$ means "equal up to terms which are regular as $w \to z$". According to the rule $\phi_{\mathrm{g}}(z,\bar{z}) \approx \Psi_{\mathrm{g}}(z)\Psi_{-\mathrm{g}}(\bar{z})$, the operator product expansions of T with ϕ_{g} contain further terms in which z is replaced by $\bar{z}$ (note that $h_{\mathrm{g}} = h_{-\mathrm{g}}$). These formulas may be compared with equations (7) and (12) in Section 2. For our correlation functions, eqs. (19,20) imply

$$\langle \mathsf{T}(w)\,\Psi_1(z_1)\ldots\Psi_n(z_n)\;\sigma(x_1)\ldots\sigma(x_{2g+1})\rangle$$

$$= \left[\;\sum_{\nu=1}^{n}\left(\frac{h_\nu}{(w-z_\nu)^2} + \frac{1}{w-z_\nu}\,\frac{\partial}{\partial z_\nu}\right) + \sum_{i=1}^{2g+1}\left(\frac{h_\sigma}{(w-x_i)^2} + \frac{1}{w-x_i}\,\frac{\partial}{\partial x_i}\right)\right]\;F(\vec{z},\vec{x})\;. \tag{21}$$

The situation is more subtle for the current $\mathsf{J}(w)$, which we recall is only well-defined on a surface of genus g. More precisely, $\mathsf{J}(w)dw$ is a meromorphic 1-form on a hyperelliptic surface. For the operator product expansion of $\mathsf{J}(w)$ with the field $\Psi_{\rm g}(z)$, we shall use

$$\mathsf{J}(w)\,\Psi_{\rm g}(z)\,dw \;\sim\; \frac{\rm g}{\sqrt{P(w)}}\frac{\sqrt{P(z)}}{w-z}\;\Psi_{\rm g}(z)\,dw \quad . \tag{22}$$

Indeed, the right hand side has a first order pole at $w = z$ with residue g and is regular otherwise. Equation (22) generalizes formula (8) in Section 2.2. To determine the operator product expansion between J and the twist field $\sigma(x)$, we observe that the leading contribution

$$\mathsf{J}(w)\,\sigma(x) \;\sim\; (w-x)^{h_\tau - 1 - h_\sigma}\;\tau(x) + \dots$$

must involve a field τ of conformal weight $h_\tau = h_\sigma + 1/2 + \mathbb{Z}$. Otherwise, the expansion would not be consistent with the periodicity properties of J close to the branch point at $w = x$. Among the boundary condition changing operators, there is one field with conformal weight $h_\tau = \frac{1}{2} + \frac{1}{16}$ which gives rise to the most singular contribution diverging with $(w-x)^{-1/2}$, cf. the spectrum (4) computed above.
After multiplication of the previous equation with dw, we are supposed to study the right hand side in the local coordinate $\xi = \sqrt{w-x}$. The outcome is rather simple: The form $(w-x)^{-1/2}dw = 2d\xi$ on the right hand side is regular at $\xi = 0$ so that we conclude

$$\mathsf{J}(w)\,\sigma(x)\,dw \;\sim\; 0 \quad , \tag{23}$$

i.e., the singular part of the operator product expansion between $\mathsf{J}(w)dw$ and the twist field $\sigma(x)$ vanishes.

As we will see, the following important formula is a consequence of these operator product expansions:

$$\langle \mathsf{J}(w)\,\Psi_1(z_1)\dots\Psi_n(z_n)\;\sigma(x_1)\dots\sigma(x_{2g+1})\rangle$$

$$= \left[\;\sum_{\nu=1}^{n}\;\frac{{\rm g}_\nu}{\sqrt{P(w)}}\frac{\sqrt{P(z_\nu)}}{w-z_\nu} + \sum_{k=1}^{g}\frac{\alpha_k(\vec z,\vec x)\,w^{k-1}}{\sqrt{P(w)}}\;\right]\;F(\vec z,\vec x) \quad , \tag{24}$$

where

$$\alpha_k(\vec z,\vec x) \;=\; \sum_{l=1}^{g}\Omega^{-1}_{kl}\left(i\,\Delta^k{\rm x}_0 - \sum_{\nu=1}^{n}{\rm g}_\nu B^k(z_\nu)\right) \quad ,$$

and the parameters $\Delta^k{\rm x}_0 := {\rm x}_0^k - {\rm x}_0^{k+1}$ are obtained from the values of the bosonic field at the boundary.

To derive the formula (24) we exploit the fact that a meromorphic 1-form on a compact Riemann surface is determined by its principal part up to some holomorphic 1-from. Operator product expansions, on the other hand, contain all information about the principal part. Hence, from eqs. (22,23) we conclude that

$$\langle \mathsf{J}(w)\, \Psi_1(z_1) \dots \Psi_n(z_n)\; \sigma(x_1) \dots \sigma(x_{2g+1}) \rangle$$

$$= \sum_{\nu=1}^{n} \frac{\mathrm{g}_\nu}{\sqrt{P(w)}} \frac{\sqrt{P(z_\nu)}}{w - z_\nu}\, F(\vec{z}, \vec{x}) \;+\; \sum_{k=1}^{g} \frac{\beta_k(\vec{z}, \vec{x})\, w^{k-1}}{\sqrt{P(w)}} \quad . \tag{25}$$

Note that the insertion points z_ν and x_i parametrize a whole family of meromorphic 1-forms, and the coefficients β_k may depend on them. Actually, we can determine this dependence completely. To this end we integrate the above equation along a loop γ_k which surrounds the interval $[x_{2k}, x_{2k+1}]$, as shown in Figure 2. On the right hand side of our equation, this integral may be expressed in terms of the matrix Ω and

$$B^k(z) \;:=\; \oint_{\gamma_k} \frac{1}{\sqrt{P(w)}} \frac{\sqrt{P(z)}}{w - z}\; dw \quad . \tag{26}$$

Note that the matrix elements Ω_{kl} and the functions $B^k(z)$ depend on the insertion points x_i. With these conventions we find

$$\oint_{\gamma_k} dw \;(\text{r.h.s. of (25)}) \;=\; \sum_{\nu=1}^{n} \mathrm{g}_\nu\, B^k(z_\nu)\, F(\vec{z}, \vec{x}) + \sum_{l=1}^{g} \Omega_{kl}\, \beta_l \quad .$$

Next, let us analyze the integral over the left hand side of equation (25). By a deformation, we can make the integration contour γ_k symmetric under a reflection $\gamma \to \bar{\gamma}$ along the real line. Now, we may split γ_k into two parts $\gamma_k^>, \gamma_k^<$ with the property $\mathrm{Im}\gamma_k^> \geq 0$ and $\mathrm{Im}\gamma_k^< \leq 0$, so that each piece lies entirely in one of the half-planes. With these conventions, our contour can be written as a composition $\gamma_k = \gamma_k^> \circ \gamma_k^<$ which obeys $\bar{\gamma}_k^< = -\gamma_k^>$. If we recall, in addition, that the field J coincides with J on the upper and with $-\overline{J}$ on the lower half-plane we deduce

$$\oint_{\gamma_k} \mathsf{J}(w)\; dw \;=\; \int_{\gamma_k^>} J(w)\, dw + \int_{\gamma_k^>} \overline{J}(\bar{w})\, d\bar{w} \;=\; i \int_{\gamma_k^>} dX(w, \bar{w}) \;=\; i\, (\mathrm{x}_0^k - \mathrm{x}_0^{k+1}) \quad .$$

In the penultimate step, we have expressed the currents through the bosonic field X by $J(w) = i\, \partial X(w, \bar{w})$ and $\overline{J}(\bar{w}) = i\, \bar{\partial} X(w, \bar{w})$. The contour integral over the differential dX is finally determined by the values of X at the boundary. For the integration over the left hand side of (25), this result implies that

$$\oint_{\gamma_k} dw \;(\text{l.h.s. of (25)}) \;=\; i\, (\mathrm{x}_0^k - \mathrm{x}_0^{k+1})\, F(\vec{z}, \vec{x}) \;=:\; i\, \Delta^k \mathrm{x}_0\, F(\vec{z}, \vec{x}) \quad .$$

Putting all this together, we arrive at the following formula for the function $\beta_k(\vec{z}, \vec{x})$:

$$\beta_k(\vec{z}, \vec{x}) \;=\; \sum_{l=1}^{g} \Omega_{kl}^{-1} \left(i\, \Delta^l \mathrm{x}_0 - \sum_{\nu=1}^{n} \mathrm{g}_\nu B^l(z_\nu) \right)\; F(\vec{z}, \vec{x}) \;=:\; \alpha_k(\vec{z}, \vec{x})\; F(\vec{z}, \vec{x}) \quad .$$

The functions α_k introduced here depend on the insertion points z_ν, x_i, the charges g_ν and the values x_0^k of the bosonic field at the boundary. Additional information, e.g., on the unknown function $F(\vec{z}, \vec{x})$, is not needed. This concludes our derivation of eq. (24).

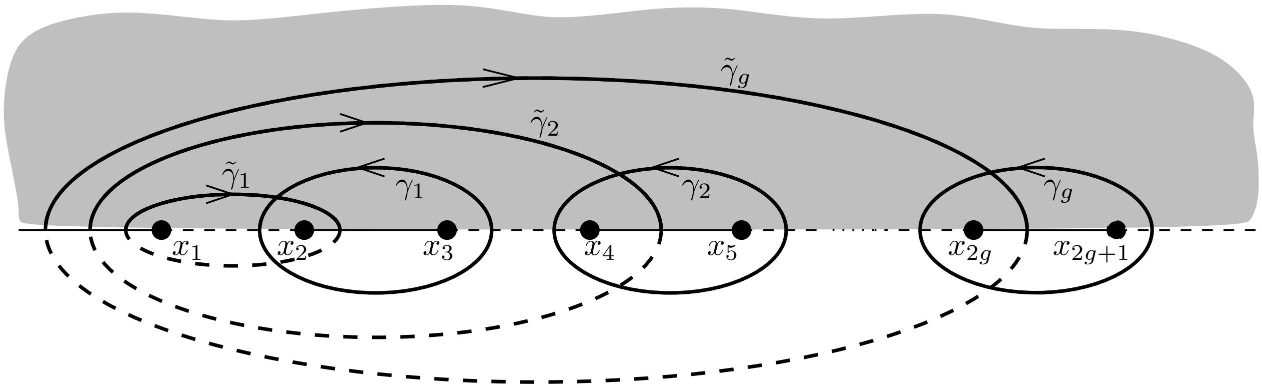

Figure 2: The curves γ_k run counterclockwise around the Neumann cuts on the real line. The curves $\widetilde{\gamma}_k$ run clockwise and close on the second sheet through Neumann intervals.

3.3 The twisted Knizhnik-Zamolodchikov equations. We now start to exploit formula (24) together with the Sugawara construction of the energy-momentum tensor T to compute the effect of inserting the field T into our correlation functions. If we include one extra field $\Psi_{\mathrm{g}}(u)$ into eq. (24), differentiate with respect to u and set $\mathrm{g} = 0$, we obtain as a consequence of $\mathsf{J}(u) = \frac{1}{\mathrm{g}} \partial_u \psi_{\mathrm{g}}(u)|_{\mathrm{g}=0}$

$$\langle \mathsf{J}(w)\, \mathsf{J}(u)\, \Psi_1(z_1) \dots \Psi_n(z_n)\ \sigma(x_1) \dots \sigma(x_{2g+1}) \rangle$$

$$= \Bigg[\sum_{\nu,\mu=1}^{n} \frac{\mathrm{g}_\nu \mathrm{g}_\mu}{\sqrt{P(u)}\sqrt{P(w)}} \frac{\sqrt{P(z_\nu)}}{w - z_\nu} \frac{\sqrt{P(z_\mu)}}{u - z_\mu} + \sum_{k=1}^{g} \sum_{\nu=1}^{n} \frac{\alpha_k(\vec{z}, \vec{x})\, w^{k-1}}{\sqrt{P(u)}\sqrt{P(w)}} \frac{\mathrm{g}_\nu \sqrt{P(z_\nu)}}{w - z_\nu}$$

$$+ \sum_{k,l=1}^{g} \frac{\alpha_k \alpha_l w^{k-1} u^{l-1}}{\sqrt{P(w)}\sqrt{P(u)}} + \frac{1}{\sqrt{P(w)}} \frac{d}{du} \left(\frac{\sqrt{P(u)}}{w - u} - \Omega^{-1}_{kl} B^l(u) w^{k-1} \right) \Bigg]\ F(\vec{z}, \vec{x}) \quad .$$

At this stage we can subtract the term $1/(w-u)^2$, multiply by a factor 1/2 and perform the limit $u \to w$. A short and elementary computation gives

$$\langle \mathsf{T}(w)\, \Psi_1(z_1) \dots \Psi_n(z_n)\ \sigma(x_1) \dots \sigma(x_{2g+1}) \rangle$$

$$= \Bigg[\frac{1}{2} \sum_{\nu,\mu=1}^{n} \frac{\mathrm{g}_\nu \mathrm{g}_\mu}{P(w)} \frac{\sqrt{P(z_\nu)}}{w-z_\nu} \frac{\sqrt{P(z_\mu)}}{w-z_\mu} + \sum_{k=1}^{g} \sum_{\nu=1}^{n} \frac{\alpha_k(\vec{z},\vec{x})\, w^{k-1}}{P(w)} \frac{\mathrm{g}_\nu \sqrt{P(z_\nu)}}{w-z_\nu}$$

$$+ \frac{1}{2} \sum_{k,l=1}^{g} \frac{\alpha_k \alpha_l w^{k+l-2}}{P(w)} - \frac{\sqrt{P(w)}''}{4\sqrt{P(w)}} - \frac{1}{2} \sum_{k,l=1}^{g} \frac{\Omega^{-1}_{kl} B^l(w)' w^{k-1}}{\sqrt{P(w)}} \Bigg] \; F(\vec{z},\vec{x}) \quad .$$

The same correlator has been computed in eq. (21) directly with the help of operator product expansions between the Virasoro field T and Ψ_{g}, σ from (19,20). Comparison of the residues at $w = z_\nu$ in the two different expressions gives the z-components of the Knizhnik-Zamolodchikov equations:

$$\partial_{z_\nu} F(\vec{z},\vec{x}) \; = \; \left[\sum_{\mu\neq\nu}^{n} \frac{\sqrt{P(z_\mu)}}{\sqrt{P(z_\nu)}} \frac{\mathrm{g}_\nu \mathrm{g}_\mu}{z_\nu - z_\mu} - \sum_{i} \frac{h_\nu}{z_\nu - x_i} + \sum_{k=1}^{g} \frac{\mathrm{g}_\nu \alpha_k \, z_\nu^{k-1}}{\sqrt{P(z_\nu)}} \right] \; F(\vec{z},\vec{x}) \; .$$

If we restrict to the case of two twist field insertions, i.e. $g = 0$, the last term vanishes. Putting $x_1 = 0$ and hence $P(w) = w$, formula (14) is recovered.

Computation of the residues at $w = x_i$ yields a formula for the derivative of F with respect to the position x_i of twist fields. Using that

$$\mathrm{Res}_{x_i} \left(\frac{\sqrt{P(w)}''}{4\sqrt{P(w)}} \right) \; = \; \frac{1}{8} \sum_{j\neq i} \frac{1}{x_i - x_j}$$

and

$$\mathrm{Res}_{x_i} \left(\sum_{k,l=1}^{g} \frac{\Omega^{-1}_{kl} B^l(w)' w^{k-1}}{\sqrt{P(w)}} \right) \; = \; \sum_{k,l=1}^{g} \Omega^{-1}_{kl} \int_{\gamma_l} \frac{1}{2} \frac{x_i^{k-1}}{\sqrt{P(\xi)}} \frac{1}{\xi - x_i} \; d\xi$$

we obtain

$$\partial_{x_i} F(\vec{z},\vec{x}) \; = \; \Bigg[\frac{1}{2\prod_{j\neq i}(x_i - x_j)} \left(\sum_{\nu=1}^{n} \frac{\mathrm{g}_\nu \sqrt{P(z_\nu)}}{x_i - z_\nu} + \sum_{k=1}^{g} \alpha_k(\vec{z},\vec{x}) \; x_i^{k-1} \right)^2$$

$$- \frac{1}{8} \sum_{j\neq i} \frac{1}{x_i - x_j} - \frac{1}{4} \sum_{k,l=1}^{g} \Omega^{-1}_{kl} \oint_{\gamma_l} \frac{x_i^{k-1}}{\sqrt{P(\xi)}(\xi - x_i)} \; d\xi \; \Bigg] \; F(\vec{z},\vec{x}) \quad . \qquad (27)$$

To conclude this section, we come back to the original correlators $G(\vec{z},\vec{x})$ of bulk-fields $\phi_{\mathrm{g}}(z,\bar{z})$ and boundary twist fields $\sigma(x_i)$. The differential equations they obey will be formulated with the help of the following functions:

$$\omega_0(z,w) \; = \; \frac{1}{\sqrt{P(z)}} \left[\frac{\sqrt{P(w)}}{z-w} - \frac{\sqrt{P(\bar{w})}}{z-\bar{w}} \right] - \sum_{k,l=1}^{g} \frac{z^{k-1}}{\sqrt{P(z)}} \; \Omega^{-1}_{kl} \left(B^l(w) - B^l(\bar{w}) \right) \; ,$$

$$\lambda_0(z) \; = \; -\frac{1}{2} \sum_{i=1}^{2g+1} \frac{1}{z - x_i} - \frac{1}{\sqrt{P(z)}} \frac{\sqrt{P(\bar{z})}}{z - \bar{z}} - \sum_{k,l=1}^{g} \frac{z^{k-1}}{\sqrt{P(z)}} \; \Omega^{-1}_{kl} \left(B^l(z) - B^l(\bar{z}) \right) \; ,$$

$$\sigma_\Delta(z) \; = \; \sum_{k,l=1}^{g} \frac{z^{k-1}}{\sqrt{P(z)}} \; \Omega^{-1}_{kl} \, \Delta^l \mathrm{x}_0 \quad .$$

Taking into account the equation $\phi_{\rm g}(z,\bar z) = \Psi_{\rm g}(z)\Psi_{-\rm g}(\bar z)$, we conclude that the correlation functions $G(\vec z,\vec x)$ must satisfy the following set of first order linear differential equations:

$$\partial_\xi G(\vec z,\vec x) \;=\; A_\xi \; G(\vec z,\vec x) \quad \text{for all} \quad \xi = z_\nu, \bar z_\nu, x_i \tag{28}$$

with the connection matrices A_{z_ν}, $A_{\bar z_\nu}$ and A_{x_i} being defined by

$$A_{z_\nu} \;=\; \sum_{\mu\neq\nu} {\rm g}_\nu {\rm g}_\mu \omega_0(z_\nu, z_\mu) + {\rm g}_\nu^2 \lambda_0(z_\nu) + i g_\nu \sigma_\Delta(z_\nu) \quad , \tag{29}$$

$$A_{\bar z_\nu} \;=\; \sum_{\mu\neq\nu} {\rm g}_\nu {\rm g}_\mu \overline{\omega_0(z_\nu, z_\mu)} + {\rm g}_\nu^2 \overline{\lambda_0(z_\nu)} + i g_\nu \overline{\sigma_\Delta(z_\nu)} \quad , \tag{30}$$

$$A_{x_i} \;=\; \frac{1}{2}\lim_{x\to x_i} (x-x_i)\left(\sum_{\nu=1}^{n} {\rm g}_\nu \omega_0(x, z_\nu) + i\sigma_\Delta(x)\right)^2$$

$$- \frac{\partial_i H_i(x_i)}{8H_i(x_i)} - \frac{1}{4}\sum_{k,l=1}^{g} \Omega^{-1}_{kl} \oint_{\gamma_l} \frac{x_i^{k-1}}{\sqrt{P(\xi)}(\xi - x_i)} \; d\xi \quad . \tag{31}$$

We have introduced the function $H_i(x_i) := \prod_{j\neq i}(x_i - x_j)$. Note that the term in brackets has a simple pole at $x = x_i$ which we cancel by the extra factor $x - x_i$ before performing the limit. Equations (28) through (31) constitute the main result of this section.

4. Construction of correlation functions

It remains to reconstruct the correlation functions $G(\vec z,\vec x)$ from the system of linear first order differential equations that we obtained in the previous section. Integration of the equations is, in principle, straightforward, but it leaves one constant factor undetermined. The latter is found explicitly in terms of the boundary conditions. Moreover, we shall manage to express the correlators $G(\vec z,\vec x)$ in terms of rather elementary building blocks.

4.1 Integration of the z-connection. To begin with, we simplify our problem by fixing the insertion points of the boundary twist fields $\sigma(x_i)$ and considering only the dependence of $G_{\vec x}(\vec z) = G(\vec z,\vec x)$ on the positions $(z_\nu, \bar z_\nu)$ of bulk fields. This means that we have to integrate the z-connection $A_{z_\nu} dz_\nu + A_{\bar z_\nu} d\bar z_\nu$ defined in eqs. (29,30) . The result will be written with the help of two functions $G_0(z,w)$ and $S_0(w)$ which are given by

$$G_0(z,w) \;:=\; 2\ {\rm Re} \int_x^z \omega_0(\xi, w)\, d\xi \quad , \tag{32}$$

$$S_0(z) \;:=\; \lim_{v\to x}\left[\; 2\ {\rm Re}\int_v^z \lambda_0(\xi)\, d\xi - \log(v-\bar v) - {\rm Re}\log P(v)\;\right] \quad . \tag{33}$$

Here, the point x is chosen to lie in the Dirichlet-interval $]x_1, x_2[$. The integrand in eq. (32) is regular on the real axis and hence the integral is well defined. One can easily see that its value neither depends on the starting point $x \in]x_1, x_2[$ nor on the choice of the curve γ from x to z. In contrast, the integrand $\lambda_0(\xi)$ in our definition of $S_0(z)$ has a simple pole on the real axis. This is the reason why we subtract the divergent term $\log(v-\bar{v})$ before taking the limit $v \to x$. The contribution $-\mathrm{Re}\log P(v)$ is added to render the whole function independent of the integration contour and, in particular, of the starting point x. More details on the definition of $S_0(z)$ and a discussion of its properties can be found in Appendix B.

The two functions we have just defined possess a number of abstract properties that characterize them uniquely. First of all, it is not difficult to see that $G_0(z,w)$ is simply a Green's function on the upper half-plane, i.e., it obeys

$$\Delta_z G_0(z,w) \;=\; 4\pi\delta(z-w) \qquad \text{for} \quad \mathrm{Im} z > 0$$

and the boundary conditions

$$G_0(z,w) \;=\; 0 \quad \text{for} \quad z \;\in\; D_i \quad , \qquad \frac{\partial}{\partial \mathrm{Im} z}\, G_0(z,w) \;=\; 0 \quad \text{for} \quad z \;\in\; N_i \quad .$$

A standard computation shows that $G_0(z,w)$ is symmetric in its arguments, $G_0(z,w) = G_0(w,z)$.

The function $S_0(z)$, on the other hand, is harmonic throughout the whole upper half-plane, i.e. $\Delta_z S_0(z) = 0$. It diverges at the boundary with a leading singularity of the form

$$S_0(z) \;\sim\; \mp \log|z-\bar{z}| \;+\; \ldots \quad \text{for} \quad \mathrm{Re}\, z \;\in\; \begin{cases} D_i \\ N_i \end{cases} \quad . \tag{34}$$

We are now in a position to integrate the differential equations (29,30) for the correlators of bulk fields $\phi_{\mathrm{g}}(z,\bar{z})$. The result is given by

$$\begin{aligned}\log G_{\vec{x}}(\vec{z}) \;&=\; \int_{\vec{w}}^{\vec{z}} (A_{z_\nu} d\xi_\nu + A_{\bar{z}_\nu} d\bar{\xi}_\nu) \quad + \quad \Lambda(\vec{w}) \\ &=\; \frac{1}{2} \sum_{\substack{\nu,\mu=1 \\ \nu\neq\mu}}^{n} \mathrm{g}_\nu \mathrm{g}_\mu\, G_0(z_\nu, z_\mu) + \sum_{\nu=1}^{n} \mathrm{g}_\nu^2\, S_0(z_\nu) + \sum_{\nu=1}^{n} \frac{i\mathrm{g}_\nu}{4\pi} \sum_{i=1}^{g+1} \int_{D_i} \mathrm{x}_0^i\, \frac{\partial}{\partial \mathrm{Im}\xi}\, G_0(\xi, z)\; d\xi \;.\end{aligned} \tag{35}$$

In the first line we have chosen some arbitrary curve in the configuration space of n particles in the upper half-plane starting at points w_ν with $\mathrm{Im} w_\nu > 0$. The integration "constant" $\Lambda(\vec{w})$ depends on the choice of the starting point and has to be fixed such that the resulting function $\log G_{\vec{x}}(\vec{z})$ satisfies the desired boundary conditions. In passing to the second line, we have extended the integration to $w_\nu = x$ on the boundary and inserted the definitions (32,33) of the functions G_0 and S_0. Then we use the auxiliary formula

$$\sum_{i=1}^{g+1} \frac{1}{4\pi} \int_{D_i} \mathrm{x}_0^i \frac{\partial}{\partial \mathrm{Im}\xi}\, G_0(z,\xi)\, d\xi \;=\; 2\mathrm{Re} \sum_{k=1}^{g} \int_x^z \frac{\xi^{k-1}}{\sqrt{P(\xi)}}\, \Omega_{kl}^{-1} \Delta^l \mathrm{x}_0\; d\xi \;+\; \mathrm{x}_0^1 \quad .$$

To prove this formula one should notice that the function on the l.h.s. of the equation is harmonic in the upper half-plane, that it satisfies Neumann boundary conditions along the intervals N_k and that it approaches the constant values x_0^k for $z \in D_k$. By explicit computation, one can establish the same behaviour for the function on the r.h.s. Since these properties are sufficient to determine the functions uniquely, the desired equation follows. We employ it to bring the third term of eq. (35) into a form that allows the most explicit control of the boundary behaviour and hence is quite appropriate for fixing the remaining $\Lambda(x)$.

4.2 Integration of the x-connection. Before we address the integration of the full connection (29,30,31) below, we investigate another simplified situation in which there are no bulk fields present. Consequently, the charges g_ν in the equation (27) can be set to zero, and we are confronted with the problem of solving the following equation for $Z_\Delta(\vec{x}) := G(\vec{x})$:

$$2H_i\,\partial_i \log Z_\Delta(\vec{x}) \;=\; \left(\sum_{k=1}^{g} \Omega^{-1}_{kl}\, i\Delta^l \mathrm{x}_0\; x_i^{k-1}\right)^2 - \frac{1}{4}\,\partial_i H_i - \frac{1}{2}\sum_{k,l=1}^{g} \Omega^{-1}_{kl} \oint_{\gamma_l} \frac{x_i^{k-1} H_i}{\sqrt{P(\xi)}(\xi - x_i)}\, d\xi\;,$$

where H_i denotes the function $H_i = \prod_{j\neq i}(x_i - x_j)$ as before, and $\Delta = \{\Delta^l \mathrm{x}_0\}$. These differential equations were solved by Zamolodchikov in [42]. Here we simply quote the final result:

$$Z_\Delta(\vec{x}) \;=\; \prod_{i>j}^{2g+1} (x_i - x_j)^{-\frac{1}{8}}\; \det{}^{-\frac{1}{2}}(\Omega)\; e^{\frac{i}{8\pi}\sum_{k,l=1}^{g} \Delta^k \mathrm{x}_0 \Delta^l \mathrm{x}_0\, \tau_{kl}}\;.$$

4.3 The correlation functions $G(\vec{z}, \vec{x})$. The results of the previous two subsections can be combined into explicit expressions for the correlators $G(\vec{z}, \vec{x})$ of bulk fields $\phi_{\mathrm{g}}(z, \bar{z})$ and boundary twist fields $\sigma(x)$. To see this we note that solutions of the differential equations (28) can be found by integrating the connection 1-form $A_{w_\nu} dw_\nu + A_{\bar{w}_\nu} d\bar{w}_\nu + A_{\xi_i} d\xi_i$ along an arbitrary curve $\gamma(t) = (\gamma_{\vec{z}}(t), \gamma_{\vec{x}}(t)),\ t \in [0,1]$, that ends at the point $(\vec{z}, \vec{x})$ in the $(2n + 2g + 1)$-dimensional real configuration space.

With some care (see the first subsection) we can start the integration with all insertion points being on the real axis. Furthermore, we may choose γ such that all twist-fields are moved to their final position at $\vec{x}$ *before* we begin moving the bulk fields into their desired locations.

In more mathematical terms this means that γ consists of two parts $\gamma = \gamma^{(2)} \circ \gamma^{(1)}$ with $\partial_t \gamma^{(1)}_{\vec{z}}(t \in [0, \frac{1}{2}]) = 0$ and $\partial_t \gamma^{(2)}_{\vec{x}}(t \in [\frac{1}{2}, 1]) = 0$. As long as $t \leq \frac{1}{2}$, the bulk fields are located at points $w_\nu = \gamma_\nu(t)$ belonging to the first Dirichlet interval $D_1 =]x_1, x_2[$. This implies $\omega_0(x, \gamma_\nu(t)) = 0$ for $t \in [0, \frac{1}{2}]$, so that one term in our expression (31) for A_{x_i} drops out. Hence, we are precisely in the situation considered in the previous subsection, and the integration of our connection 1-form over γ in the interval $t \in [0, \frac{1}{2}]$ gives $Z_\Delta(\vec{x})$. When we continue the integration to $t = 1$, we add the expression for $\log G_{\vec{x}}(\vec{z})$ computed in eq. (35). After some rewriting, the following result for the correlations function $G(\vec{z},\vec{x})$ is obtained:

$$G(\vec{z},\vec{x}) \;=\; Z_\Delta(\vec{x})\;\exp\left(\sum_{\nu=1}^{n} \mathrm{g}_\nu^2 S_0(z_\nu)\right)\;\exp\left(\sum_{\nu=1}^{n} i\mathrm{g}_\nu \Phi_{\mathrm{x}_0}^{(n-1)}(z_\nu)\right) \quad . \tag{36}$$

Here, $\Phi_{\mathrm{x}_0}^{(n-1)}(z_\nu)$ denotes the potential that is created by $n-1$ charges g_μ at points $z_\mu \neq z_\nu$ in the presence of the boundary with mixed boundary conditions,

$$\Phi_{\mathrm{x}_0}^{(n-1)}(z_\nu) \;=\; \frac{1}{2}\int_{\mathrm{Im}w>0} d^2w\; G_0(z_\nu,w)\sum_{\mu\neq\nu}\mathrm{g}_\mu\,\delta(w-z_\mu)+\frac{1}{4\pi}\sum_{i=1}^{g+1}\int_{x_{2i-1}}^{x_{2i}} \mathrm{x}_0^i\,\frac{\partial}{\partial \mathrm{Im}\xi}\,G_0(\xi,z_\nu)\,d\xi \;\;.$$

It is quite instructive to interpret each of the three factors in our final expression for the correlation function $G(\vec{z},\vec{x})$ directly within conformal field theory. For the moment, let us specify the number of bulk fields in G by some extra superscript, i.e. we shall write $G(\vec{z},\vec{x}) = G^{(n)}(\vec{z},\vec{x})$. Now consider the object

$$\Phi(z_\nu) \;:=\; \frac{1}{i\mathrm{g}_\nu}\log\left(\frac{G^{(n)}(\vec{z},\vec{x})}{\exp(\mathrm{g}^2 S_0(z_\nu))\,G^{(n-1)}(\vec{z}',\vec{x})}\right)$$

where $\vec{z}'$ denotes the set of $n-1$ bulk coordinates $z_\mu,\ \mu\neq\nu$. It is easy to determine the behaviour of $\Phi(z_\nu)$ as a function of the bulk coordinate z_ν from the bulk and bulk-boundary operator product expansions of the fields $\phi_{\mathrm{g}}(z_\nu,\bar{z}_\nu)$, cf. Subsect. 2.3:

$$\Delta_{z_\nu}\,\Phi(z_\nu) \;=\; 2\pi\sum_{\mu\neq\nu}\,\mathrm{g}_\mu\;\delta(z_\nu - z_\mu)$$

$$\Phi(z_\nu) \;=\; \mathrm{x}_0^i \quad\text{for}\quad z_\nu\;\in\;D_i\;\;,\qquad \frac{\partial}{\partial\mathrm{Im}z_\nu}\,\Phi(z_\nu)\;=\;0\quad\text{for}\quad z\;\in\;N_i\;\;.$$

These properties characterize the function $\Phi(z_\nu)$ uniquely, and by standard formulas from electrostatics we obtain that $\Phi(z_\nu) = \Phi_{\mathrm{x}_0}^{(n-1)}(z_\nu)$. An iteration of this construction along with $G^{(0)}(\vec{x}) = Z_\Delta(\vec{x})$ leads to our product formula (36) for the correlation function $G(\vec{z},\vec{x})$.
The three factors can be interpreted as follows: $Z_\Delta(\vec{x})$ is a "partition function" corresponding to some line charge distribution provided by the twist fields alone; the term

$$\sum_{\nu=1}^{n} g_\nu\;\Phi_{\mathrm{x}_0}^{(n-1)}(z_\nu)$$

is the electrostatic potential corresponding to a configuration of n point particles with charges $g_1,\ldots,g_n$ located at the points $z_1,\ldots,z_n$ and line charges distributions along the Dirichlet intervals. The term

$$\sum_{\nu=1}^{n} g_\nu^2\;S_0(z_\nu)$$

can be interpreted as a renormalized electrostatic self-energy of the point particles located at $z_1, \ldots, z_n$.

4.4 Path integral approach and extensions. Before we conclude, let us briefly sketch how the theory can be formulated in the path integral approach. This is important for the following two reasons: First, as long as one is only interested in free bosons, the path integral approach is a powerful alternative to our analysis above involving Knizhnik-Zamolodchikov connections. The path integral formulation allows for a more direct computation of the correlation functions (but it does not easily extend to non-abelian group targets.) Secondly, using path integrals we will be able to describe rather easily possible extensions of our analysis to compact targets and to D-branes with B-fields.

To begin with, we consider once more the familiar situation of a non-compact 1-dimensional target and Dirichlet parameters x_0^i. We denote by G the Green's function of the Laplacian on the upper half-plane,

$$\Delta_z G(z,w) \;=\; \delta(z-w) \;,$$

subject to the boundary conditions

$$\begin{aligned} G(x,w) =&\; 0 \quad \text{ for } x \in D \;, \\ \partial_y \, G(x,w) =&\; 0 \quad \text{ for } x \in N \;. \end{aligned}$$

The Gaussian measure with covariance G and mean 0 is denoted by μ_G, and we use χ for the corresponding random variable. With the help of the Dirichlet parameters x_0^i, we define a real-valued function ξ on the upper half-plane by

$$\begin{aligned} \Delta\xi \;&=\; 0 \;, \\ \xi|_{D_i} \;&=\; \mathrm{x}_0^i \;, \quad \partial_y\xi|_{N_i} \;=\; 0 \;, \quad i \;=\; 1, \ldots, g+1 \;. \end{aligned}$$

The effect of the Dirichlet parameters is incorporated through a shift of the random variable χ by ξ which gives us the bosonic field $X = \chi + \xi$. It appears when we construct the basic fields of the theory, namely the vertex operators

$$\varphi_g(z,\bar z) \;=\; : e^{igX(z,\bar z)} : \;.$$

In this framework we could recover the correlation functions above by integrating products of fields $\varphi_g(z,\bar z)$ using the Gaussian measure.

When the free boson X takes values in a circle S^1, the boundary conditions depend both on Dirichlet parameters x_0^i and on Neumann parameters denoted by y_0^i. The Neumann parameters determine the Dirichlet parameters of the T-dual theory and can be thought of as the strength of constant Wilson lines turned on along a Neumann direction. In Sect. 3, we have worked in a local chart with the Neumann intervals cut out. Information on Neumann parameters is, therefore, lost unless we take a second chart into account which has cuts along the Dirichlet intervals. As above, we can derive Knizhnik-Zamolodchikov equations for each chart; the full correlation functions of the compactified theory are to be built up from the respective solutions in such a way that the boundary conditions are met.

The path integral computation of correlators is rather easy to adjust to the compactified situation: if we set Neumann parameters to zero and restrict attention to the fields $\varphi_g(z,\bar z)$, we can use precisely the same formulas as above.

In order to incorporate non-vanishing Neumann parameters y_0^i and to compute more general correlators for fields $\varphi^{(c)}_{g,\bar g}(z,\bar z)$ with $(g,\bar g)$ taken from an even, self-dual Lorentzian lattice, we introduce a real-valued function η on the upper half-plane defined by

$$\begin{aligned} \Delta\eta &= 0\ , \\ \eta|_{N_i} &= \mathrm{y}_0^i\ , \quad \partial_y\,\eta|_{D_i} = 0\ , \quad i = 1,\dots,g+1\ . \end{aligned}$$

Then, we set

$$\varpi_{g,z}(w) := gG'(w,z) + \eta(w)\ ,$$

where G' denotes the Green's function of the Laplacian on the upper half-plane with interchanged Dirichlet and Neumann boundary conditions, i.e.,

$$\begin{aligned} \Delta G'(w,z) &= \delta(w-z)\ , \\ \partial_y\, G'(x,z) &= 0 \quad \text{for } x\in D\ , \\ G'(x,z) &= 0 \quad \text{for } x\in N\ . \end{aligned}$$

We now introduce the *disorder operators* $D_g(z,\bar z)$ satisfying

$$D_g(z,\bar z)F[dX] = F[dX + *d\varpi_{g,z}]\ ,$$

for any functional F. The left- resp. right-moving chiral vertex operators can then be written as

$$\psi_g(z) = \varphi_{\frac{g}{2}}(z,\bar z)D_{\frac{g}{2}}(z,\bar z)\ , \qquad \bar\psi_{\bar g}(\bar z) = \varphi_{\frac{\bar g}{2}}(z,\bar z)D_{-\frac{\bar g}{2}}(z,\bar z)\ ;$$

see also [17] for more details and for an application to soliton quantization in 2-dimensional theories. The basic fields of the compactified theory are products

$$\varphi^{(c)}_{g,\bar g}(z,\bar z) = \psi_g(z)\bar\psi_{\bar g}(\bar z)$$

where $(g,\bar g)$ lies in some even, self-dual Lorentzian lattice. For another approach to the rational compactified boson, the reader is referred to [22].

Another extension would involve the appearance of B-fields on our D-branes. This has attracted some interest recently, because of its relation with non-commutative geometry, see e.g. [15,35,36, 3,16] and references therein. Non-trivial B-fields can only exist if one of our branes is at least 2-dimensional. For simplicity, we shall focus on a pair of a Dp-and a

D0-brane. The field strength on the Dp-brane will be denoted by B. In terms of boundary conditions for a multi-component free bosonic field, the situation is described as follows

$$\partial_t X^a(t,0) \;=\; 0 \quad \text{and} \quad \partial_\sigma X^a(t,\pi) \;=\; B^a_b\, \partial_t X^b(t,\pi) \quad \text{for} \quad a,b = 1,\ldots,p \;\;.$$

The spectrum of the associated boundary condition changing operators and the Green's functions in the presence of two twist fields have been discussed at various places (see e.g. [36,11]). Our techniques from Sections 3 and 4 allow to extend such investigations to the case of multiple twist insertions. Instead of giving the details here, we simply state how one has to adjust the path integral computation to the new scenario. This is rather easy: All it requires is to replace the function G above by some matrix valued Green's function $G_B = (G_B^{ab})$. The latter is a Green's function for the Laplacian $\mathbf{1}_p\,\Delta$ on the upper half-plane ($\mathbf{1}_p$ denotes the p-dimensional identity matrix), subject to the boundary conditions

$$G_B^{ab}(x,w) = \; 0 \qquad\qquad \text{for } x \in D \;,$$

$$\partial_y\, G_B^{ab}(x,w) = \; iB^a_c\; \partial_x G_B^{cb}(x,w) \quad \text{for } x \in N \;.$$

With the help of this function, the calculation of correlators proceeds as before.

5. Outlook

We have succeeded in decomposing the complete bulk and boundary correlators in the presence of DN-transitions into functions with rather natural interpretations – both from the point of view of electrostatics and from the CFT perspective. This is useful for carrying out the remaining step in the computation of string amplitudes, namely the integration over insertion points of fields on the world-sheet. The calculation of such string amplitudes gives effective actions involving a hyper-multiplet χ which comes with the twist fields. To leading order, the bosonic part of these actions can be found in [14, 27,25]. Multiple twist insertions allow to compute higher order corrections.

When we turn on a B-field, the string amplitudes may be described through field theories on some non-commutative space. It was suggested in [36] that these theories are related to some model on an ordinary commutative space through a complicated non-linear transformation. This statement can be checked order by order in the effective description. After the appropriate (but straightforward) extension to non-vanishing B-fields, the considerations presented above may be used to perform a similar analysis for theories which contain a hyper-multiplet χ.

Keeping the bulk insertions fixed, the sequence of correlators with arbitrarily many twist field insertions can be viewed as building blocks of the perturbation series of a relevant perturbation by the twist field. This "tachyon condensation" is responsible, e.g., for the formation of D0-D2 bound states, as discussed in [23]. Upon integrating over twist field insertion points in the one-point functions $Z_\Delta(\vec{x})\,\exp\{g^2 S_0(z)\}$, one would arrive at one-point functions which characterize the boundary theory after tachyon condensation. Sen's approach [37] and the results of [33] allow one to circumvent the relevant boundary flow and to replace it by a combination of marginal bulk and boundary deformations. However, some questions as to the equivalence of both procedures remain open, and it might be useful to have an independent check of these methods. The correlation functions constructed here provide a starting point.

For applications to superstring theory, it is mandatory to extend our analysis to free fermions. This does not pose serious problems, since systems of an even number of fermions can be bosonized.

Problems of the type of our free boson problem are encountered in general boundary CFT as soon as the "parent" CFT on the plane admits different boundary conditions. For some general results on the rational case, see [20,21,22]. The spectrum of boundary condition changing operators can be derived as in Sect. 2, once boundary states for the "constant" boundary conditions are known. Again, the computation of correlators becomes non-trivial if boundary conditions with different gluing automorphisms are combined. In non-abelian WZW models, which constitute and important generalization of the free boson case, the Sugawara construction can be exploited in a similar fashion as for the free boson and leads to twisted, non-abelian Knizhnik-Zamolodchikov equations. The partition functions counting BCCOs in non-abelian boundary WZW models are linear combinations of the twining characters investigated in [18] (see also [5] and references therein). Apart from the models with affine Lie algebra symmetry, there is the rather large class of so-called "quasi-rational CFTs" [28] on the plane for which generalizations of Knizhnik-Zamolodchikov equations exist even without a Sugawara form for the energy-momentum tensor [2]. It might be interesting to see how such structures extend to boundary CFT.

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Appendix

A: Proofs of Lemma 1 and Lemma 2.

Proof of Lemma 1: We start from the usual expression for T in terms of J and rewrite it until we can perform the limit $w_1 \to w_2$.

$$\mathsf{J}(w_1)\,\mathsf{J}(w_2) - \frac{1}{(w_1 - w_2)^2}$$

$$= \mathsf{J}_<(w_1)\mathsf{J}(w_2) + \mathsf{J}(w_2)\mathsf{J}_>(w_1) + [\,\mathsf{J}_>(w_1)\,,\,\mathsf{J}(w_2)\,] - \frac{1}{(w_1-w_2)^2}$$

$$= \mathsf{J}_<(w_1)\mathsf{J}(w_2) + \mathsf{J}(w_2)\mathsf{J}_>(w_1) + \frac{\frac{1}{2}\left(\frac{w_1}{w_2}\right)^{1/2} + \frac{1}{2}\left(\frac{w_2}{w_1}\right)^{1/2} - 1}{(w_2-w_1)^2}\ .$$

We can now perform the limit $w_1 \to w_2 =: w$ to recover the generating field $\mathsf{T}(w)$ from the last formula

$$\mathsf{T}(w) = \frac{1}{2}\big(\,\mathsf{J}_<(w)\mathsf{J}(w) + \mathsf{J}(w)\mathsf{J}_>(w)\,\big) + \frac{1}{16}\frac{1}{w^2}$$

where we used $\displaystyle \lim_{u\to 1} \frac{u^{1/2}+u^{-1/2}-2}{2(1-u)^2} \ = \ \frac{1}{8}\ .$ ∎

Proof of Lemma 2: The derivation of the commutation relation with $\mathsf{J}_>(w)$ is straightforward. So, let us turn directly to the calculation of the commutator with $\mathsf{T}_>(w)$. Recall that $h = g^2/2$. Then,

$$\begin{aligned}
&[\,\mathsf{T}_>(w)\,,\,\Psi_{\rm g}(z)\,] \\
&= \left(\frac{i}{2z}\right)^h \Big([\mathsf{T}_>(w), e^{i{\rm g}X_<(z)}]\, e^{i{\rm g}X_>(z)} + e^{i{\rm g}X_<(z)}\,[\mathsf{T}_>(w), e^{i{\rm g}X_>(z)}]\Big) \\
&= \left(\frac{i}{2z}\right)^h e^{i{\rm g}X_<(z)} \Big(i{\rm g}\,[\mathsf{T}_>(w), X_<(z)] - h\,[X_<(x), [X_<(x), \mathsf{T}_>(w)]\,] \\
&\qquad + \quad i{\rm g}\,[\mathsf{T}_>(w), X_>(z)] + h\,[X_>(z), [X_>(z), \mathsf{T}_>(w)]\,]\Big)\, e^{i{\rm g}X_>(z)} \\
&= \left(\frac{i}{2z}\right)^h e^{i{\rm g}X_<(z)} \Bigg(i{\rm g}\,[\mathsf{T}_>(w), X(z)] + h\,\Big(\sum_{r,s<0,n\geq -1} w^{-n-2}z^n\,\delta_{r+s,-n} \\
&\qquad - \sum_{r,s>0,n\geq -1} w^{-n-2}z^n\,\delta_{r+s,-n}\Big)\Bigg)\, e^{i{\rm g}X_>(z)} \\
&= \left(\frac{i}{2z}\right)^h e^{i{\rm g}X_<(z)} \left(\frac{i{\rm g}}{w-z}\ \partial_z X(z) + h\,\Big(\sum_{n\geq 1}\ w^{-n-2}z^n\,n - \frac{1}{wz}\Big)\right) e^{i{\rm g}X_>(z)} \\
&= \frac{1}{w-z}\ \left(\frac{i}{2z}\right)^h \partial_z\left(e^{i{\rm g}X_<(z)}e^{i{\rm g}X_>(z)}\right) + h\,\left(\frac{z}{w}\frac{1}{(w-z)^2} - \frac{1}{wz}\right)\Psi_{\rm g}(z) \\
&= \frac{1}{w-z}\ \partial_z\Psi_{\rm g}(z) + \frac{1}{w-z}\frac{h}{z}\Psi_{\rm g}(z) + h\,\left(\frac{1}{(w-z)^2} - \frac{1}{z(w-z)}\right)\Psi_{\rm g}(z) \\
&= \frac{1}{w-z}\ \partial_z\Psi_{\rm g}(z) + \frac{h}{(w-z)^2}\ \Psi_{\rm g}(z)\quad .
\end{aligned}$$

In the process of this computation we have inserted the commutation relation between L_n, a_r and eq. (5). The rest involves only standard algebraic manipulations. ∎

B: The function $S_o(z)$.

In this appendix we want to explain a number of properties of the function $S_0(z)$ that is introduced in Section 4.1. To show that the limit $\lim_{v\to x}$ exists, we insert the definition of $\lambda_0(\xi)$ into eq. (33) . After splitting off all non-singular terms in λ_0 we obtain:

$$
\begin{aligned}
S_0(z) &= \lim_{v\to x}\left[\ 2\ \mathrm{Re}\int_v^z \lambda_0(\xi)\,d\xi - \log(v-\bar{v}) - \mathrm{Re}\log P(v)\ \right] \\
&= \lim_{v\to x}\left[\ -2\ \mathrm{Re}\int_v^z \frac{1}{\xi-\bar{\xi}}\,d\xi - \log(v-\bar{v})\ + \mathrm{reg}_{v\to x}\ \right] \\
&= \lim_{v\to x}\left[\ \log(v-\bar{v}) - \log(z-\bar{z}) - \log(v-\bar{v}) + \mathrm{reg}'_{v\to x}\ \right] \quad .
\end{aligned}
$$

Since the singularity from the integral cancels against the term $\log(v-\bar{v})$, the limit can be taken.

Our second aim is to understand that $S_0(z) \equiv S_0^x(z)$ does not depend on the choice of x. Let us displace x by some small amount $a \in \mathbb{R}$ such that $x+a$ is still in the Dirichlet interval D_1. Comparison of $S_0^x(z)$ and $S_0^{x+a}(z)$ gives

$$
\begin{aligned}
&S_0^x(z) - S_0^{x+a}(z) \\
&= \lim_{v\to x}\left[\ 2\ \mathrm{Re}\int_v^{v+a} \lambda_0(\xi)\,d\xi - \mathrm{Re}\log P(v)\ + \mathrm{Re}\log P(v+a)\ \right] \\
&= \lim_{v\to x}\left[-2\mathrm{Re}\int_v^{v+a}\sum_{i=1}^{2g-1}\frac{\frac{1}{2}}{\xi-x_i}\,d\xi\ +\ \mathrm{Re}\log\frac{P(v+a)}{P(v)}\ \right] \\
&= \lim_{v\to x}\left[-\mathrm{Re}\sum_{i=1}^{2g+1}\left(\log(v+a-x_i)-\log(v-x_i)\right)\ +\ \mathrm{Re}\log\frac{P(v+a)}{P(v)}\ \right]\ =\ 0 \quad .
\end{aligned}
$$

In passing to the second line we omitted all terms in the integrand which vanish when ξ comes close to the real axis.

Finally, we investigate the behaviour of $S_0(z)$ at the boundary. Basically, one repeats the analysis we have sketched above in our discussion of $\lim_{v\to x}$. If the end-point z of our integration approaches one of the Dirichlet intervals, this leads to the singularity $\sim -\log|z-\bar{z}|$. In the argument one needs that the quotient $\sqrt{P(z)/P(\bar{z})}$ in front of the singular term $1/(z-\bar{z})$ satisfies $\lim_{z\to x}\sqrt{P(z)/P(\bar{z})} = 1$ for $x \in D_k$. This is no longer true when z is sent to the real axis in one of the Neumann intervals N_k. In fact, upon moving x from a Neumann into a Dirichlet interval, the polynomial $P(x)$ changes sign, causing the quotient $P(z)/P(\bar{z})$ to surround the origin of the complex plane once. After taking the square root we conclude that $\lim_{z\to x}\sqrt{P(z)/P(\bar{z})} = -1$ for $x \in N_k$ and hence $S_0(x) \sim \log|z-\bar{z}|$ near the Neumann intervals.

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