Dimensionally Reduced SYM_4 as Solvable Matrix Quantum Mechanics

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We study the quantum mechanical model obtained as a dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills theory to a periodic light cone "time". After mapping the theory to a cohomological field theory, the partition function (with periodic boundary conditions) regularized by a massive term appears to be equal to the partition function of the twisted matrix oscillator. We show that this partition function perturbed by the operator of the holonomy around the time circle is a tau function of Toda hierarchy. We solve the model in the large N limit and study the universal properties of the solution in the scaling limit of vanishing perturbation. We find in this limit a phase transition of Gross-Witten type.

SPhT/t99/072
LPTENS-99/25

July, 1999

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1. Introduction

The supersymmetric Yang-Mills (SYM) theories have rich physical content and their quantitative analysis is in general as difficult as in the usual, nonsupersymmetric, gauge theories. However they often contain, unlike the purely bosonic YM theories, specific sectors, which can be analysed exactly and where the supersymmetry leads to a nilpotent (topological) symmetry [1]. The dimensionally reduced versions of the SYM theory even allow various massive deformations conserving this symmetry.

In refs. [2] and [3] this symmetry (in the zero dimensional reductions of SYM) was applied for the calculation of the (bulk part of the) Witten index for ensembles of N 0-branes in 4, 6 and 10 dimensions, justifying the conjectures related to the existence of bound states of zero-branes [4]. In [5] the method was applied to study certain correlators of BPS states or, in other words, of perturbations of the original reduced SYM theories, which preserve part of the supersymmetry. In the case of the zero dimensional reduction of $\mathcal{N} = 1$ SYM theory, the large N limit was studied exactly using the method of [6] or by the corresponding integrability properties allowing to write an explicit (KP) differential equation for the partition function. One of the unexpected results was that, in the large N limit, the physical quantities exhibit an essential singularity at $\lambda = 0$, where λ is the coupling of the massive perturbation. The large N limit of the dimensionally reduced SYM theories is also interesting because it may reveal part of the structure of the nonreduced theories, due to the Eguchi-Kawai mechanism.

In this paper we study the one dimensional reduction of the $\mathcal{N} = 1$ $SU(N)$ pure Yang-Mills theory. Unlike the well known and widely used 1d reduction to the usual physical time [6], [7], or [8], we will retain the "time" along the light cone direction compactified on a circle of radius β. Only this reduction allows the direct use of Witten's localization principle [1][1]. In order to get rid of the zero modes of the bosonic fields we will deform the theory by a massive perturbation corresponding to the $O(2)$ twisting of the boundary conditions on the time circle with respect to a subgroup of the euclidean symmetry $O(4)$. The SYM theory reduced in this way appears to be identical to the compactified hermitian matrix oscillator with $SU(N)$-twisted boundary conditions. The twisting angles are related to a global mode of the (time-like) gauge field. This model can be further reduced to that of a unitary (already time-independent) twist matrix. We find that the model is integrable in the sense that its partition function is a tau-function of Toda hierarchy, i.e. it obeys a chain of nonlinear Toda equations.

The model can be solved exactly and rather explicitly in the large N limit. The solution of the corresponding saddle point equation and its physical consequences in the limit of vanishing perturbation represent the main result of this paper. The solution is parametrized in terms of elliptic functions. The analysis of the solution as a function of the two parameters $\beta\epsilon$ and λ, where β is the compactification length, ϵ is the strength of the massive perturbation, and λ is the twist coupling reveals the following phenomena:
1. In the double limit $\epsilon \to 0$, $\lambda \to 0$ the free energy is universal (under certain deformations) function of the ratio ϵ/λ.
2. In the limit of vanishing massive perturbation ($\epsilon \to 0$), the observables exhibit an essential singularity $\sim \exp\left(-\frac{\text{const}}{\epsilon}\right)$.

[1] Our argument follows essentially the construction proposed in [9,10], which allows to lift by one the dimension of the spacetime without loosing the supersymmetry. Technically speaking, our procedure of dimensional reduction replaces the Euclidean spacetime by a point and at the same time introduces the "time" dimension τ. The latter might be interpreted as a lightlike dimension of the original spacetime, but we do not know to what extent this interpretation is justified.

3. The analytical continuation of the model at the point $\epsilon\beta = i\pi$ (inverted oscillator) shows the scaling of the $c = 1$ compactified noncritical string theory. This value of the compactification length $\epsilon\beta$ corresponds to the Kosterlitz-Thouless critical point (see appendix C).

The paper is organized as follows. In section 2 we describe the reduction of the partition function of the one-dimensionally reduced $\mathcal{N} = 1$ SYM_4 theory to that of the reduced twisted matrix oscillator, by the use of the supersymmetry and the localization theorem. We then reduce the configuration space of the model to the set of the eigenvalues of the unitary twist matrix. In section 3 we find that the partition function of our model is a τ function of the Toda integrable hierarchy and write the differential equations satisfied by the partition grand canonical function. In section 4 we give an exact solution of the saddle point equation for the large N limit of the model in terms of elliptic parametrization; the calculations are presented in Appendix A. In section 5 we study the limit of small massive perturbation. We find an universal expression for the free energy in presence of a source for the Wilson loops, in the scaling limit $\epsilon \to 0$ and $\lambda \to 0$. We analyse the properties of the solution, especially in the small compactification radius limit and near the curve of the Gross-Witten type transitions. Section 6 is devoted to conclusions. In Appendix C we give the solution of the analytic continuation of our model to imaginary time $\epsilon\beta = i\pi$, which is the Kosterlitz-Thouless point for the corresponding $c = 1$ noncritical string.

2. Definition of the model and its reduction to one-dimensional matrix quantum mechanics

In this section we will show that the dimensionally reduced $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $SU(N)$ can be mapped to one-dimensional matrix quantum mechanics. The dimensional reduction consists in replacing the 4-dimensional (Euclidean) space-time by a single lightlike "time".

Let us first give the generalization of the argument of [5] to the case of one dimension. We start with the the $\mathcal{N} = 1$ $SU(N)$ SYM_4 containing 4 bosonic matrix fields A_μ $(\mu = 0, 1, 2, 3)$, and 4 real fermionic fields Ψ_α $(\alpha = 1, \ldots, 4)$. After performing a Wick rotation $x_0 = -ix_4$, the action of the Euclidean theory can be written as

$$\mathcal{S} = \int d^4x \mathrm{Tr} \left(-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}\ \Psi^T (\nabla_4 + \vec{\gamma} \cdot \vec{\nabla}) \Psi \right), \tag{2.1}$$

where $\nabla_\mu = i\partial_\mu + A_\mu$ is the covariant derivative and the gamma-matrices are represented as direct products of Pauli matrices: $\gamma_i = \sigma_i \times \sigma_i$ $(i = 1, 2, 3)$. The gauge group $SU(N)$ acts to all fields in the adjoint representation. let us assume that all fields depend only on the time-like coordinate

$$\tau = x_3 - ix_4, \tag{2.2}$$

which parametrizes a circle with radius β. The resulting model is a matrix quantum mechanics containing four bosonic and four fermionic matrix variables.

We will evaluate the functional integral for this one-dimensional matrix model by mapping it to a cohomological field theory, which will allow to apply Witten's localization argument [11]. Let us redefine the fields as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 + iA_4 \\ A_3 - iA_4 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ \phi \\ \bar{\phi} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \eta \\ \chi \end{pmatrix} \tag{2.3}$$

Then the action (2.1) can be written as a BRST exact form. The BRST transformation Q acts on the complex of fields $\Phi = \{X_a, \phi, \bar{\phi}, H; \psi_a, \eta, \chi\}$ (where $H = i[X_1, X_2]$ is considered as an auxilliary field) as

$$\begin{aligned} QX_a = \psi_a, \qquad & Q\psi_a = [i\partial_\tau + \phi, X_\alpha] \qquad (a = 1, 2) \\ Q\bar{\phi} = \eta, \qquad & Q\eta = [i\partial_\tau + \phi, \bar{\phi}] \\ Q\chi = H, \qquad & QH = [i\partial_\tau + \phi, \chi] \\ & Q\phi = 0. \end{aligned} \tag{2.4}$$

Namely, the action

$$\begin{aligned} \mathcal{S} = \int_0^\beta d\tau \mathrm{Tr}\Big(& iH[X_1, X_2] + \tfrac{1}{2}H^2 + [X_a, i\partial_\tau + \phi][X_a, \bar{\phi}] + \tfrac{1}{2}[i\partial_\tau + \phi, \bar{\phi}]^2 \\ & + \frac{1}{2}\chi\epsilon^{ab}[X_a\psi_b] + [X_a, \eta]\psi_a - [\bar{\phi}, X_a][i\partial_\tau + \phi, X_a] + \frac{1}{2}\chi[i\partial_\tau + \phi, \chi] + [\psi_a, \bar{\phi}]\psi_a \Big) \end{aligned}$$

can be written as

$$\begin{aligned} S \quad &= Q\int_0^\beta d\tau \mathrm{Tr}\mathcal{V}(\Phi) \\ \mathcal{V}(\Phi) &= \tfrac{1}{2}\eta[i\partial_\tau + \phi, \bar{\phi}] + \chi(H - i[X_1, X_2]) + \sum_{a=1}^{2}\psi_a[X_a, \bar{\phi}]. \end{aligned} \tag{2.5}$$

The square Q^2 of this transformation represents the "time" gauge transformation generated by ϕ, $\nabla_\tau = [i\partial_\tau + \phi, \ ]$. Hence Q is nilpotent on the gauge-invariant quantities. The ghost number of the fields is -2 for $\bar{\phi}$, -1 for η and χ, 0 for X_a and H, $+1$ for ψ_a, and $+2$ for ϕ.

The functional integral with respect to the BRST complex of fields Φ

$$\mathcal{Z}_N(\beta, g) = \int \frac{\mathcal{D}\Phi}{\mathrm{Vol}(\mathcal{G})} e^{-\frac{1}{g}\mathcal{S}[\Phi]} \tag{2.6}$$

(where the integration measure is normalized by the volume of the gauge group $\mathcal{G}$) can be therefore evaluated using the Witten's localization argument [11]. Namely, the integral is saturated by the BRST critical points $Q\Phi = 0$. More strictly, we have to integrate over a continuous critical manifold, because of the zero modes of $\phi, \chi, \bar{\phi}$. The zero modes are elliminated by adding, following [11], a Q-exact term to the action (2.5) by changing the action to $\mathcal{S} + \delta\mathcal{S}$, with

$$\delta\mathcal{S} = tQ\int_0^\beta d\tau \mathrm{Tr}\chi\bar{\phi}. \tag{2.7}$$

We can also discard from the very beginning, the term $\frac{1}{2}\eta[i\partial_\tau + \phi, \bar{\phi}]$. As before, H can be integrated out by setting $H = t\bar{\phi} + [X_1, X_2]$ in (2.5). The advantage of introducing the perturbation (2.7) is that for $t \neq 0$, the fields $\bar{\phi}, \chi$, and η can be integrated out. However, the perturbed integral does not coinside in general with the original one because of the new fixed points "flowing in from the infinity " when one perturbes to $t \neq 0$

[11]. The correct statement is that there exists a class of BRST-invariant operators whose (nonnormalized) expectation values coincide in the original and the perturbed theory. The first such operator is

$$\omega = \int_0^\beta d\tau \mathrm{Tr}(-[i\partial_\tau + \phi, X_1]X_2 + \psi_1\psi_2) = Q\int_0^\beta d\tau \epsilon^{ab}\mathrm{Tr}(\psi_a X_b). \tag{2.8}$$

The second is any $SU(N)$-invariant function $f(\Omega)$ of the holonomy factor around the circle

$$\Omega = \hat{T}\exp\left(i\int_0^\beta d\tau\phi(\tau)\right). \tag{2.9}$$

The functional integral for the expectation value

$$\langle e^\omega f(\Omega)\rangle = \int \frac{\mathcal{D}\Phi}{\mathrm{Vol}(\mathcal{G})} e^{-\frac{1}{g}(\mathcal{S}+\delta\mathcal{S})+\omega} f(\Omega) \tag{2.10}$$

does not depend on the coupling t. Indeed, taking the derivative in t and integrating by parts, we find zero, since the integrand vanishes at infinity due to the factor e^ω. Therefore we can take the limit $t\to\infty$, after which the integral (2.10) gets localized near the zeros of $H, \bar{\phi}, \chi, \eta$. (In particular, the partition function does not depend on the gauge coupling g.) Now we can calculate the partition function (2.6) understood as the average of the identity operator. Since ω has ghost number +2, due to the ghost number conservation

$$\mathcal{Z}_N(\beta, g) \equiv \langle 1\rangle = \langle e^\omega\rangle. \tag{2.11}$$

The above argument has been applied recently by F. Sugino [12] in order to calculate the partition function of the four-dimensional $\mathcal{N}=1$ SYM reduced to a two-dimensional torus, with periodic boundary conditions for all fields. Our case is slightly more subtle, because of the zero modes of the fields X_a. These zero modes will be elliminated, as in [3] and later in [5], namely by deforming the BRST operator in the definition of the action (2.5). Let us first notice that after the redefinition of the fields the theory is still invariant under the $O(2)$ rotations in the directions orthogonal to the light cone:

$$X_1 + iX_2 \to e^{i\epsilon}(X_1 + iX_2), \quad \psi_1 + i\psi_2 \to e^{i\epsilon}(\psi_1 + i\psi_2).$$

This allows to construct another BRST operator, which squares to a linear combination of a gauge transformation and an $O(2)$ rotation. The twisted BRST charge Q_ϵ acts as

$$\begin{aligned} Q_\epsilon X_\alpha &= \psi_\alpha, & Q_\epsilon\psi_\alpha &= [i\partial_\tau + \phi, X_\alpha] + i\epsilon\varepsilon^{\alpha\beta}X_\beta, \\ Q_\epsilon\bar{\phi} &= \eta, & Q_\epsilon\eta &= [i\partial_\tau + \phi, \bar{\phi}] \\ Q_\epsilon\chi &= H, & Q_\epsilon H &= [i\partial_\tau + \phi, \chi] \\ & & Q_\epsilon\phi &= 0. \end{aligned} \tag{2.12}$$

The modification of the supercharge is equivalent to changing the action (2.5) and the operator (2.8) as

$$\begin{aligned} \mathcal{S} &\to \mathcal{S} + 2i\epsilon\int d\tau\mathrm{Tr}(\bar{\phi}[X_1, X_2]) \\ \omega &\to \omega - \frac{\epsilon}{2}\int_0^\beta d\tau\mathrm{Tr}(X_1^2 + X_2^2). \end{aligned} \tag{2.13}$$

In the limit $t \to \infty$ the integral gets localized near the zeros of $H, \bar{\phi}, \chi, \eta$, leaving the place to the action

$$S = \int d\tau \mathrm{Tr}\left(-i[i\partial_\tau + \phi, X_1]X_2 - \tfrac{1}{2}\epsilon(X_1^2 + X_2^2) + \psi_1\psi_2\right), \tag{2.14}$$

and the ψ's can then be integrated out. Finally, the integration over X_2 gives the partition function of the matrix oscillator (with the coordinate $X_1 \equiv X$) in presence of the one-dimensional gauge field $\phi(\tau)$

$$\mathcal{Z}_N(\beta, g, \epsilon) = \int \frac{\mathcal{D}\phi(\tau)\mathcal{D}X(\tau)}{\mathrm{Vol}\mathcal{G}} \exp\left(-\frac{1}{2}\mathrm{Tr}\int_0^\beta d\tau \left(\frac{1}{\epsilon}[i\partial_\tau + \phi, X]^2 + \epsilon X^2\right)\right), \tag{2.15}$$

with periodic boundary conditions $X(\beta) = X(0), \quad \phi(\beta) = \phi(0)$. It is clear that the integral depends on β and ϵ only through the product $\epsilon\beta$. We will absorb ϵ in β,

$$\epsilon\beta \to \beta, \tag{2.16}$$

remembering that the perturbation is lifted in the limit $\beta \to 0$.

The functional integral over the field ϕ can be written, after fixing a gauge $\partial_\tau\phi = 0$, as an integral over the unitary matrix[2] representing holonomy factor defined by (2.9), namely $\Omega = e^{i\beta\phi}$ normalized by the volume of $U(N)$. The holonomy factor enters the functional integral over X as the twisted boundary condition:

$$X(\beta) = \Omega^+ X(0)\Omega. \tag{2.17}$$

The integral over X can be performed exactly, and the integral over the unitary matrix Ω reduces to an integral over its eigenvalues $e^{i\theta_1}, ..., e^{i\theta_N}$ (which are defined up to a permutation, hence a combinatorial factor $1/N!$). The partition function is therefore given by the N-fold integral[3]

$$\mathcal{Z}_N(\beta) = \frac{1}{N!}\oint \prod_{k=1}^N \frac{d\theta_k}{2\pi} \frac{\prod_{i\neq j}\sin[\frac{1}{2}(\theta_i - \theta_j)]}{\prod_{i,j}\sin[\frac{1}{2}(\theta_i - \theta_j + i\beta)]} \tag{2.18}$$

where $\theta_i = \beta\phi_i$.

The expectation value we are calculating is a deformation of the Witten index

$$\mathcal{Z}_N(\beta) = \mathrm{Tr}(-)^F e^{-\beta H} e^{i\beta\epsilon J}, \tag{2.19}$$

[2] It is assumed that the integration contour for the eigenvalues of ϕ is chosen along the real axis. In this case $\bar{\phi}$ should be taken anti-hermitian, see the the discussion in [11].

[3] This happens to be exactly the partition function of the one-dimensional gas studied by Michel Gaudin in 1966 [13]; it was extensively used in [14] to study the compactified 1+1 dimensional string theory via matrix quantum mechanics; in relation to the actual SYM theory this formula was communicated to us by N. Nekrasov.

where the $(-)^F$-factor is included in order to impose the periodic boundary conditions on the fermionic fields, and the trace is twisted by an $O(2)$ rotation $e^{i\beta\epsilon J}$ in the (12) plane. The twisting of the BRST charge $Q \to Q_\epsilon$ does not change locally the functional integral, but it does change the boundary conditions for the fields.

Now we are at the most subtle point of the reduction procedure, which deserves to be discussed in more detail. Considering the fields ϕ and $\bar{\phi}$ as two independent fields imply that the integration over them is understood as contour integration. The twisting separates the poles and zeroes of the integrand and allows to evaluate the integral by the residue theorem, in complete similarity with the calculation of [3] for the zero dimensional model. Since the integrand does not depend on the variable

$$\bar{\theta} = \frac{\theta_1 + ... + \theta_N}{N},$$

the contour integral with respect to this variable would give zero. In fact, the integration with respect to this variable should be excluded because this is one of the normalizable zero modes of the original fields and the measure $\mathcal{D}\Phi$ should contain a product of delta functions of the bosonic and fermionic zero modes. In particular, the normalized zero mode of ϕ is

$$\phi^{(0)} = \frac{1}{\sqrt{N\beta}} \int_0^\beta d\tau \mathrm{Tr}\phi(\tau) = \sqrt{N\beta}\ \bar{\theta}.$$

For a more detailed discussion see [15]. Therefore the measure in (2.18) contains a delta function $\delta(\phi^{(0)}) \sim \delta(\theta_1 + ... + \theta_N)$, which suppresses the contour integration with respect to $\bar{\theta}$. The integral over θ's is normalized by the volume of the residual global gauge group. The introduction of the delta function should respect this normalization. Thus we have to insert

$$2\pi\delta(\bar{\theta}) = 2\pi N\delta(\theta_1 + ... + \theta_N).$$

Now we can integrate, after representing the integrand as a determinant using the Cauchy identity, by using the residues theorem. The integral is equal to the sum of the identical contributions of the $(N-1)!$ cyclic permutations in the expansion of the determinant

$$\begin{aligned}
\mathcal{Z}_N(\beta) &= (-)^{N-1}\frac{(N-1)!}{N!}\oint \frac{d\theta_1}{2\pi}...\frac{d\theta_N}{2\pi} 2\pi N\delta(\theta_1 + \ldots \theta_N) \prod_{k=1}^{N} \frac{1}{\sin[\frac{1}{2}(\theta_i - \theta_{i-1} + i\beta)]} \\
&= \int \frac{\frac{d\theta_1}{2\pi}\ \frac{d\theta_N}{2\pi}\ \delta(N\frac{\theta_1+\theta_N}{2})}{\sinh\frac{1}{2}[\theta_1 - \theta_N - i(N-1)\beta]\ \sinh\frac{1}{2}[\theta_1 - \theta_N + i\beta]} \\
&= \frac{1}{2N\sinh N\frac{\beta}{2}}.
\end{aligned} \tag{2.20}$$

In the limit $\beta \to \infty$ our partition vanishes, which is not unexpected, since the Witten index of the $D = 4$ theory is zero. In the limit $\beta \to 0$ we recover the result for the completely reduced theory $\sim 1/N^2$, in agreement with [3]. A more careful analysis allows to reproduce also the numerical coefficient, in accordance with the conjecture made in [16]. In the limit $\beta \to 0$, the Ω-integral is saturated by the integration in the vicinity of the N central elements of $SU(N)$ which are parametrized by the element of the $su(N)$ Lie algebra [17]. After performing carefully the limit, one finds (see e.g. [15])

$$\mathcal{Z}_N(\beta) \to \frac{(g/\beta)^{\frac{1}{2}(N^2-1)}}{\beta\ \mathcal{F}_N}\ \mathcal{Z}_N^{(0)}\left(\frac{g}{\beta}\right)$$

where $\mathcal{Z}_N^{(0)}(g)$ is the partition function of the completely reduced theory, and one reproduces the result of [3]:

$$\mathcal{Z}_N^{(0)}(g) = \mathcal{F}_N \; g^{-\frac{1}{2}(N^2-1)} \; \frac{1}{N^2}. \tag{2.21}$$

(The numerical factor $\mathcal{F}_N$ depends on the way the integration measure is normalized. In the normalization used in [3] this factor is equal to one, but this is not the most natural choice from the point of view of applications to the D-brane physics.) In the particular case of the $SU(2)$ theory the results was obtaind by the direct calculation of the integrals [18].

Let us note that our partition function only formally coincides with that of the twisted matrix oscillator and, at least for finite N, there is an ambiguity related to the prescription for the contour integration. Witten's localization procedure leads to an integral over the *Lie algebra* and logically the integration with respect to θ's should be taken along the whole real axis. With this definition, only $(N-1)!$ terms in the expansion of the determinant will contribute to it. On the other hand, had we integrate in interval $[0, 2\pi]$, this would correspond to contour integration with respect to the eigenvalues of the *Lie group* element $\{t_k = e^{i\theta_k}\}_{k=1}^N$, where the contours circle the origin. In this case we would get contributions from all $N!$ terms in the expansion of the Cauchy determinant. The result would be given, instead of (2.20), by (see, for example, ref. [14])

$$\tilde{\mathcal{Z}}_N(\beta) = \frac{e^{-N^2\beta/2}}{(1-e^{-\beta})(1-e^{-2\beta})...(1-e^{-N\beta})}. \tag{2.22}$$

Unlike (2.20), the $\beta \to 0$ limit of this formula does not match the result of [3].

Which of the two formulas (2.20) or (2.22) is correct? Clearly the difference between them is due to a different treatment of the boundary conditions for the field ϕ in the formula (2.15). result considered then at hermitean oscillators). A happy resolution of this paradox would be that from the point of view of the application of Witten's localisation principle both formulas seem to be possible but the result depends on the boundary conditions and the contours of integration for the field $\phi(t)$ in the original action (2.5). However we feel that the question is rather subtle and more study is needed to clarify it. For example we cannot be sure that the supersymmetry of the original model isn't violated in one of two cases. On the other hand, the local BRST symmetry used for the calculations is certainly intact.

Now let us consider a slightly more ambitious problem, namely to calculate the generating functional of a set of BRST invariant operators made out of the gauge field ϕ. As mentioned before, such operators can be constructed as traces of the holonomy Ω in different representations, or, equivalently, as polynomials of the moments $\mathrm{Tr}(\Omega)^k$. We will add to the action the simplest possible source term

$$\lambda \mathrm{Tr}(\Omega^+ + \Omega). \tag{2.23}$$

Repeating the arguments, which led to (2.18), we find for the generating functional the following integral representation

$$\mathcal{Z}_N(\beta, \lambda) = \frac{1}{N!} \oint \prod_{k=1}^{N} \frac{d\theta_k}{2\pi} \; e^{N\lambda \cos\theta_k} \; \prod_{i \neq j} \frac{\sin[\frac{1}{2}(\theta_i - \theta_j)]}{\sin[\frac{1}{2}(\theta_i - \theta_j + i\beta)]} \tag{2.24}$$

where $\theta_i = \beta\phi_i$. If one follows the recipe of [3], the integration should be considered as a contour integration along the real axis, where the N integration variables are subjected

to the constraint $\theta_1 + ... + \theta_N = 0$. Then the result should be analytic as a function of β, which can therefore be given complex values. It is plausible that in the large N limit, which we are interested in, if the perturbation is sufficiently strong, the choice of the contours should be not important. The equivalence between the reduces SYM theory and the twisted matrix oscillator should takes place only in this limit.

It would be very interesting to understand what is the meaning, in terms of the original supersymmetric theory (2.1), of the deformation that leads to the partition function (2.23).

The reduction from 4 to 1 dimension of the original theory turns three of the components of the gauge field into Higgs fields (X_1, X_2 and $\bar{\phi}$). This makes the direct calculation of the partition function (which is related to the bulk part of the Witten index) more delicate, because of the absence of mass gap. By introducing the deformations (2.7), (2.8) and (2.13) we add an additional Higgs potential, thus breaking part of the supersymmetry. The effect of the source term, which we added to obtain the partition function (2.23), depends substantially on the way we have perturbed the theory. Indeed, it has positive ghost charge, and its effect would be zero, if the perturbation (2.13) of ghost charge -2 were not there to compensate it. This is also true in the completely reduced theory, discussed in ref. [5].

3. The partition function as a tau-function of the Toda hierarchy

Here we will show that our partition function with a source term $\lambda \mathrm{Tr}(\Omega^+ + \Omega)$ is a tau-function of discrete Toda chain. Let us rewrite the partition function of the model eq.(2.24) in the following form:

$$\mathcal{Z}_N(\beta, t) = \frac{1}{N!} \oint \prod_{j=1}^{N} \frac{dz_j}{2\pi} e^{U(z_j)} \frac{\Delta^2(z)}{\prod_{k,m}(e^{\beta/2} z_m - e^{-\beta/2} z_k)} \tag{3.1}$$

where $z_k = e^{i\theta_k}$, $U(z) = \sum_{n \neq 0} t_n z^n$, and $\Delta(z)$ is the Van-der-Monde determinant of z's. In our case $t_1 = t_{-1} = N\lambda$ and $t_n = 0$ for $n \neq \pm 1$, but most of the following conclusions are true for a general $U(z)$.

Let us now introduce the grand canonical partition function with the "charge" l:

$$\tilde{\tau}_l[t, \mu] = \sum_{N=1}^{\infty} e^{\mu N} e^{-lN\beta} \mathcal{Z}_N(\beta, t) \tag{3.2}$$

Due to the Cauchy identity the last equation can be rewritten in terms of a functional Fredholm determinant:

$$\begin{aligned} \tilde{\tau}_l[t, \mu] &= \sum_{N=1}^{\infty} e^{\mu N} \frac{e^{-lN\beta}}{N!} \oint \prod_{j=1}^{N} \frac{dz_j}{2\pi} e^{U(z_j)} \det_{k,m} \frac{1}{e^{\beta/2} z_m - e^{-\beta/2} z_k} \\ &= \mathrm{Det}(1 + e^{\mu - \beta l} \hat{K}), \end{aligned} \tag{3.3}$$

where the operator $\hat{K}$ is defined as

$$(\hat{K} f)(z) = \oint \frac{dz}{2\pi} \frac{e^{\frac{1}{2}[U(z) + U(z')]}}{e^{\beta/2} z - e^{-\beta/2} z'} f(z').$$

It is convenient to modify slightly the definition of the tau-function:

$$\tau_l[T,\mu] = \tilde{\tau}_l[t,\mu]\exp(-\sum_{n>0} n t_n t_{-n}), \tag{3.4}$$

where we introduced new couplings T_n by:

$$U(z) = \sum_{n\neq 0} z^n T_n (e^{-n\beta/2} - e^{n\beta/2}) \tag{3.5}$$

so that the old couplings are expressed through the new ones as:

$$t_n = T_n(e^{-n\beta/2} - e^{n\beta/2}). \tag{3.6}$$

We also note that

$$\tau_l[T,\mu] = \tau_0[T,\mu-\beta l] \equiv \tau[T,\mu-\beta l]. \tag{3.7}$$

Using, for example, the general construction of the paper [19] for our particular tau-function we conclude that it is a particular case of the tau-function of Toda hierarchy. It satisfies the Toda chain equations. Namely let us introduce a new function

$$e^{\Phi_l} = \frac{\tau_{l-1}}{\tau_l} = \frac{\tau[T,\mu-\beta(l-1)]}{\tau[T,\mu-\beta l]} \tag{3.8}$$

and the notations

$$ix_\pm = \sqrt{2}T_{\pm 1} = \pm(e^{\beta/2} - e^{-\beta/2})^{-1} t_{\pm 1}.$$

The first equation of the Toda hierarchy can be written as

$$\frac{\partial}{\partial x_+}\frac{\partial}{\partial x_-}\Phi_l + \frac{1}{2}\left(e^{\Phi_l - \Phi_{l+1}} - e^{\Phi_{l-1}-\Phi_l}\right) = 0 \tag{3.9}$$

Due to the symmetry $\theta \to -\theta$ of the measure our tau-function depends only on the variable $x = \sqrt{x_+ x_-}$. The corresponding reduced equation is:

$$\Phi_l'' + \frac{1}{x}\Phi_l' + \frac{1}{2}\left(e^{\Phi_l-\Phi_{l+1}} - e^{\Phi_{l-1}-\Phi_l}\right) = 0, \tag{3.10}$$

where the derivatives are taken with respect to x.

For the function $\psi_l = \Phi_l - \Phi_{l+1} = \log\frac{\tau_{l-1}\tau_{l+1}}{\tau_l^2}$ the Toda equation reads:

$$\psi_l'' + \frac{1}{x}\psi_l' + \frac{1}{2}\left(2e^{\psi_l} - e^{\psi_{l-1}} - e^{\psi_{l+1}}\right) = 0 \tag{3.11}$$

The tau-function, as well as $\Phi_l(0)$ and $\psi_l(0)$, can be determined for $x = 0$ using the methods of [14] (for $x = 0$ the tau-function is the grand canonical partition function of the matrix oscillator in the singlet representation of the $U(N)$ group, which is the same as the

partition function of N fermionic oscillators) and it can serve as a boundary condition for the Toda chain equation. For example, one finds from (2.22)

$$e^{\psi_l(0)} = \frac{1+e^{-\beta(\mu-l-3/2)}}{1+e^{-\beta(\mu-l-1/2)}}. \tag{3.12}$$

Let us also note that $\psi_l(x)$ is analytic in x^2 at the origin, which gives the second initial condition $\partial_x\psi_l|_{x=0}=0$.

Using these equations and the boundary conditions we expand the partition function (2.24) in powers of λ^2. In the first order:

$$\frac{1}{2N^2}\frac{\partial^2}{\partial\lambda^2}\log\mathcal{Z} = \frac{1-e^{-N\beta}}{1-e^{-\beta}} \tag{3.13}$$

This is the simplest correlation function $\langle \mathrm{tr}\Omega^+\mathrm{tr}\Omega\rangle$ of the holonomy wilson loop in our original model.

The large N limit of the initial partition function (2.24) or (3.1) can be studied in terms of a special scaling limit of these Toda equations (since $\mu\sim N$ in the legendre transform from canonical to microcanonical partition function), similar to the KP-hierarchy approach of a simpler zero-dimensional model of paper [5]. We leave this study to a future publication.

4. Saddle point equations in the large N limit

In this section we will investigate the large N limit, which is the most interesting from the point of view of applications. Since the potential $\lambda\cos\theta$ is symmetric, we assume that the saddle-point spectral density

$$\rho(\theta) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i)$$

is supported by the symmetric interval $[-a,\ a]$ with $0<a\le\pi$. The function $\rho(\theta)$ is determined by the saddle point equation

$$2\lambda\sin\theta = \fint_{-a}^{a}d\theta'\rho(\theta')\left(2\cot\frac{\theta-\theta'}{2}-\cot\frac{\theta-\theta'+i\beta}{2}-\cot\frac{\theta-\theta'-i\beta}{2}\right) \tag{4.1}$$

where we temporarily rescaled $\beta\epsilon\to\beta$. This equation is equivalent to a functional equation for the resolvent

$$W(\theta) = \frac{1}{2}\int_{-a}^{a}d\theta'\rho(\theta')\cot\frac{\theta-\theta'}{2}, \tag{4.2}$$

namely

$$\lambda\sin\theta = W(\theta+i0)+W(\theta-i0)-W(\theta+i\beta)-W(\theta-i\beta) \tag{4.3}$$

where $-a<\theta<a$, supplied with the normalization condition for the density

$$\oint_{\mathcal{C}}\frac{dz}{2\pi i}W(z) = 1 \tag{4.4}$$

where the contour of integration $\mathcal{C}$ circles interval $[-a,\ a]$.

It is easier to solve this equation for the function

$$\begin{aligned}\zeta(z) &= -2\cos z + 4\frac{\sinh\frac{\beta}{2}}{\lambda}\ \frac{W(z+i\beta/2)-W(z-i\beta/2)}{i} \\ &= -2\cos z - \frac{4\sinh^2\frac{\beta}{2}}{\lambda}\ \int_{-a}^{a}\frac{d\theta\rho(\theta)}{\cos(z-\theta)-\cosh\frac{\beta}{2}}\end{aligned} \tag{4.5}$$

which satisfies the simpler equation

$$\zeta(\theta+i\frac{\beta}{2}) = \zeta(\theta-i\frac{\beta}{2}) \qquad (\theta\in[-a,a]). \tag{4.6}$$

The solution can be formulated in terms of standard elliptic functions (see Appendix A for the derivation). We give it in the form which is convenient for the limit of small β (or, equivalently, finite β and $\epsilon\to 0$). The function $\zeta(z)$ will be given in a parametric form

$$\zeta=\zeta(v),\quad z=z(v)$$

where the parameter v belongs to the rectangle $-\frac{\pi}{2}<\mathrm{Re}v<\frac{\pi}{2},\ -\frac{\pi}{2}\tau<\mathrm{Im}v<\frac{\pi}{2}\tau$. The elliptic modulus q and the nome k^2

$$\begin{aligned}q &= e^{-\pi K/K'} = e^{i\pi\tau} \\ k &= \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4\end{aligned} \tag{4.7}$$

are given below as functions of β and λ.

The solution (in parametric form) is:

$$\zeta(v)\quad = \frac{\zeta_4 f^2(v)-\zeta_3 f^2(v_\infty)}{f^2(v)-f^2(v_\infty)}, \tag{4.8}$$

where $f(v)$ is a standard elliptic function

$$f(v) = \frac{2K'}{\pi}\mathrm{dn}\left(\frac{2K'}{\pi}v,\ k'\right) = 1+4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos(2nv), \tag{4.9}$$

and

$$\begin{aligned}z(v) &= i\frac{\beta}{\pi}v + i\ln\frac{\vartheta_1(v+v_\infty)}{\vartheta_1(v-v_\infty)} \\ &= i\frac{\beta}{\pi}v + i\ln\frac{\sin(v+v_\infty)}{\sin(v-v_\infty)} + 4i\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\ \frac{\sin 2nv_\infty\sin 2nv}{n}.\end{aligned} \tag{4.10}$$

The modulus τ is proportional to the ratio of v_∞ and β

$$\tau = 4i\frac{v_\infty}{\beta}, \qquad q = e^{-\frac{4\pi}{\beta}v_\infty}, \tag{4.11}$$

and is determined by

$$\frac{2\pi}{\lambda}\sinh(\beta/2) = \frac{\gamma}{2}E(k) - \frac{\zeta_4\zeta_5 + \zeta_1\zeta_3}{2\gamma}K(k). \tag{4.12}$$

The parameters $\zeta_1, ..., \zeta_5$ of the solution are expressed as functions of λ and v_∞ as follows:

$$\begin{aligned}
\zeta_4 - \zeta_3 &= -2e^{\frac{\beta}{\pi}v_\infty}\frac{f'(v_\infty)}{f(v_\infty)}\frac{\theta_1(2v_\infty)}{\theta_1'(0)}, \\
\zeta_4 + \zeta_3 &= -\frac{e^{\frac{\beta}{\pi}v_\infty}}{\theta_1'(0)}\left(\frac{f''(v_\infty)}{f'(v_\infty)} - \frac{f'(v_\infty)}{f(v_\infty)} + 2\theta_1'(2v_\infty) + \frac{\beta}{\pi}\theta_1(2v_\infty)\right)
\end{aligned} \tag{4.13}$$

$$\zeta_1 = \frac{\zeta_3\alpha^2 - \zeta_4 k^2}{\alpha^2 - k^2}, \quad \zeta_5 = \frac{\zeta_4 - \zeta_3\alpha^2}{1-\alpha^2}, \tag{4.14}$$

$$\alpha = \frac{f(v_\infty)}{1 + 2\sum_{n=1}^{\infty} q^{n^2}}, \quad \gamma = \frac{(\zeta_4 - \zeta_3)\alpha}{\sqrt{(1-\alpha^2)(\alpha^2 - k^2)}} \tag{4.15}$$

Finally, it is useful to know the value ζ_2 of the function $\zeta(z)$ at the branch point $z = i\frac{\beta}{2} + a$

$$\frac{\zeta_4 - \zeta_2}{\gamma} = \frac{\beta}{2\pi} + \frac{\beta}{v_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin 2nv_\infty. \tag{4.16}$$

5. Scaling limit

5.1. The resolvent in the scaling limit

Let us recall that the parameter β is the product of the physical time and the twisting parameter ϵ. Therefore the twisting is removed in the limit $\beta \to 0$. If $\beta \to 0$ with λ fixed, we reproduce the zero-dimensional case considered in [5]. In this section we will consider a nontrivial limit where both λ and β go to zero so that the ratio λ/β remains finite. In this limit, all observables depend only on the ratio λ/β, and this is why we call it "the scaling limit". Note that in the thermodynamical limit $N \to \infty$, the two limits $\lambda \to 0$ and $\epsilon \to 0$ will not commute.

In the scaling limit we have $\mathrm{Im}\tau >> 1$, $K \approx \ln\frac{4}{k'}$, $K' = \frac{\pi}{2}(1 + \frac{k'^2}{4})$ and, neglecting the exponentially small terms, we get

$$\begin{aligned}
\zeta &= \frac{\zeta_4 + \zeta_3}{2} - \frac{\zeta_{43}}{16q}\ \frac{1}{\sin(v+v_\infty)\sin(v-v_\infty)} \\
z &= i\frac{\beta}{\pi}v + i\ln\frac{\sin(v+v_\infty)}{\sin(v-v_\infty)}.
\end{aligned} \tag{5.1}$$

where $(\zeta_{ik} \equiv \zeta_i - \zeta_k)$

$$\begin{aligned}
\zeta_{43} &= 16qe^{\frac{\beta}{\pi}v_\infty}\sin^2 2v_\infty \\
\frac{\zeta_4 + \zeta_3}{2} &= -e^{\frac{\beta}{\pi}v_\infty}(2\cos 2v_\infty + \frac{\beta}{\pi}\sin 2v_\infty).
\end{aligned} \tag{5.2}$$

When the regularization is removed, i.e. in the limit $\beta \to 0$, a sensible limit is obtained when λ tends to zero linearly with β. The scaling coupling constant β/λ is obtained from (4.12) after substituting $E = 1, K = \frac{2\pi}{\beta} v_\infty$:

$$2\pi \frac{\beta}{\lambda} = 2 \sin 2v_\infty - 4v_\infty \cos 2v_\infty . \tag{5.3}$$

5.2. The free energy in the scaling limit

The derivative of the free energy

$$F(\lambda, \beta) = \lim \frac{1}{N^2} \ln \mathcal{Z}_N(\lambda, \beta)$$

is proportional to the first moment of the spectral density

$$F'_\lambda(\lambda, \beta) = \int_{-a}^{a} d\theta \rho(\theta) \cos\theta, \tag{5.4}$$

which can be evaluated by looking at expansion of $\zeta(z = \pi + iy)$, at $y \to \infty$,

$$\zeta(z) = e^y + \sum_{k=1}^{\infty} \zeta^{(k)} \ e^{-(2k+1)y}.$$

We have

$$F'_\lambda(\lambda, \beta) = \frac{\lambda}{8 \sinh^2 \frac{\beta}{2}} \ [\zeta^{(1)} - 1]. \tag{5.5}$$

The coefficient $\zeta^{(1)}$ is evaluated in the scaling limit $\beta \to 0$ in Appendix A. This allows us to write an explicit expression for the free energy in the limit $\beta \to 0$:

$$F'_\lambda(\lambda/\beta) = \frac{\lambda}{4\pi\beta}(4v_\infty - \sin 4v_\infty), \quad \pi \frac{\beta}{\lambda} = \sin 2v_\infty - 2v_\infty \cos 2v_\infty . \tag{5.6}$$

This expression for the free energy is universal in a certain sense: if one deforms the potential (2.23) to a more general one:

$$\lambda \mathrm{Tr} \sum_{n=-\infty}^{\infty} t_n \Omega^n, \tag{5.7}$$

then the scaling limit of the free energy will have the same form (5.6), where λ will be substituted by some function of the couplings $\tilde{\lambda}(g_1, g_2, \cdots)$. The universal form of the free energy can only change if we tune the couplings g_n to some multicritical point.

The corrections to the eq. (5.6) are of two kinds: power-like corrections and exponentially small terms of the type

$$q = e^{-4\pi v_\infty / \beta}.$$

In the limit $\lambda \to \infty$ we have $q = \exp - \left(\frac{24\pi^4}{\lambda\beta^2}\right)^{1/3}$. These terms are of course invisible compared with the power-like corrections but they imply the existence of essential singularity in the $\beta \to 0$.

If we return to the original notations in terms of β,ϵ and λ we conclude from (5.6) that $F(\lambda, \beta, \epsilon) = \epsilon\lambda f(\lambda/(\beta\epsilon))$. Hence the principal $\sim N^2$ correction to the free energy tends to zero in the limit $\epsilon \to 0$ (when we recover the original unperturbed reduced SYM theory). On the other hand, as we will see below from (5.6), there is no regular expansion in powers of ϵ in the weak coupling phase which signifies that there is an essential singularity at the origin of this coupling and, correspondingly, in the moduli space of our theory.

5.3. The Gross-Witten phase transition and the strong coupling phase

The matrix integral we are considering has qualitatively the same phase structure as the $U(\infty)$ gauge theory on a two-dimensional sphere. The weak coupling phase considered above, describes the range of couplings $\lambda > \lambda_c$ where

$$\lambda_c = \frac{1 - e^{-\beta}}{\epsilon} \tag{5.8}$$

is determined by from the condition $a = \pi$, i.e. that the two endpoints of the cut meet on the unit circle. (Eq. (5.8) follows from (5.3) with $v_\infty = \pi/2$; the length of the cut as a function of λ is given in Appendix A.) The singularity near this critical point is as usual of third order.

The strong coupling solution is obtained by expanding the spectral density and the kernel in a Fourier series

$$\rho(\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} c_k \cos(k\theta).$$

$$\cot \frac{\theta + i0}{2} + \cot \frac{\theta - i0}{2} - \cot \frac{\theta + i\beta}{2} - \cot \frac{\theta - i\beta}{2} = 4 \sum_{k=1}^{\infty} [1 - e^{-k\beta}] \sin(k\theta)$$

$$\fint_{-\pi}^{\pi} d\theta' \rho(\theta') \cot \frac{\theta - \theta'}{2} = \sum_{k=1}^{\infty} c_k \sin(k\theta)$$

It is therefore clear that only the c_0 term of the expansion of the spectral density has to be retained. One finds

$$\rho(\theta) = \frac{1}{2\pi} \left(1 + \frac{\lambda}{1 - e^{-\beta}} \cos\theta \right) \qquad \text{for} \qquad 0 < \lambda < \lambda_c = 1 - e^{-\beta}. \tag{5.9}$$

For the free energy we find then:

$$F'_\lambda = \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos\theta = \frac{1}{2} \, \frac{\lambda}{1 - e^{-\beta}} \qquad \text{for} \qquad 0 < \lambda < \lambda_c \tag{5.10}$$

In the scaling limit $\epsilon \to 0$ we obtain: $F'_\lambda = \frac{1}{2} \frac{\lambda}{\beta}$ for $\lambda < \beta$. At the critical point $\lambda_c = \beta$ we have $F'_\lambda|_{\lambda=\beta} = \frac{1}{2}$ and $F''_\lambda|_{\lambda=\beta} = \frac{1}{2\beta}$. A simple calculation using eq. (5.6) gives in the weak coupling phase $\lambda > \lambda_c$ the same values of first two derivatives of the free energy at the critical point. This means that we have, as usually, the 3-rd order Gross-Witten phase transition. Note that in the limit $\beta \to \infty$ our model reduces indeed to the one-plaquette model originally studied by Gross and Witten [20].

5.4. Reduction to the zero-dimensional theory: $\beta << \lambda$

In this limit the theory appears to be the zero-dimensional reduction of $CN = 1$ SYM studied in [5]. The integral (2.24) reduces (after the rescaling $\theta \to \beta\theta$) to a simpler integral:

$$\mathcal{Z}_N(\beta, \lambda) = \frac{2\pi N}{N!} \frac{\beta}{2\pi}^N e^{N\lambda} \int \prod_{k=1}^{N} \frac{d\theta_k}{2\pi} \, e^{-\frac{1}{2} N\lambda\beta^2\theta_k^2} \prod_{i \neq j} \frac{(\theta_i - \theta_j)}{(\theta_i - \theta_j + i)} \tag{5.11}$$

This model was studied in [6] and later in [5].
We find from (5.6) the following expansions in half-length of the cut $a = 2v_\infty$:

$$F'_\lambda = 1 - \frac{a^2}{10} + \frac{a^4}{4200} + \mathcal{O}(a^6) \tag{5.12}$$

$$\frac{\beta}{\lambda} = \frac{a^3}{3\pi}(1 - \frac{a^2}{10}) + \mathcal{O}(a^7) \tag{5.13}$$

This gives the following asymptotics for the free energy:

$$F(\beta, \lambda) = \lambda \left[1 - \frac{3}{10} \left(\frac{3\pi\beta}{\lambda} \right)^{2/3} - \frac{27}{1400} \left(\frac{3\pi\beta}{\lambda} \right)^{4/3} + O\left(\frac{\beta^2}{\lambda^2} \right) \right] \tag{5.14}$$

The first term of this expansion matches the asymptotics of big λ obtained in [5] for the integral (5.11) in the large N limit. The next terms are not supposed to match with [5] since we already used the scaling limit expression of the free energy with the finite compactification radius β.

6. Conclusions

Let us outline the main results of the paper:
1. We consider a topological sector of the one-dimensionally reduced $\mathcal{N} = 1$ SYM_4 theory on the light-cone time circle with a a special massive perturbation. Using Wittens non-abelian localisation principle [11], we represented the partition function with periodic b.c. in terms of a solvable matrix quantum mechanics (twisted matrix oscillator).
2. We find the integrability properties of this model relating it to the Toda hierarchy. The generating functional as a function of its parameters satisfies the Toda chain equation.
3. In the large N limit we find the exact solution of the model: the generating functional is parametrized in terms of elliptic functions. We find the Gross-Witten type phase transition and identify its location. The strong coupling solution is also found.
4. An interesting model corresponds to the analytical continuation $\beta \to i\beta$ (inverted matrix oscillator) being known to have the properties of the $c = 1$ non-critical strings.
5. In the scaling limit of vanishing perturbation we find a simple universal (with respect to certain deformations of parameters of the generating functional) expression for the free energy and Wilson loop correlators along the light-cone circle. Its strong compactification limit restores similar results for the completely reduced $\mathcal{N} = 1$ SYM_4 considered in [5].

Some remaining problems:
1. We need further understanding of the space time symmetries of the model and of the correlators corresponding to our generating functional.
2. The representations similar to the eq.(2.24) for the SYM_4 can be found also for the SYM_6 and SYM_{10} reduced to the light-cone time circle: we just have to take the corresponding eigenvalue integrals for the partition functions in the paper [5]and substitute there the rational functions by trigonometric ones. Unfortunately, we cannot apply the powerful methods used here to those models: we don't know any relation of them to the integrable hierarchies and we cannot solve exactly the large N saddle point equation. On the other hand, it seems to be possible to investigate this saddle point equation in the scaling limit similar to tha used in the present paper.

Another interesting question is whether two prescriptions for the contour integration with respect to the eigenvalues of ϕ coincide for the infinite N in some part of the phase space of parameters $\epsilon\beta$ and λ. It is clear that they give different answers in the strong

coupling phase $\lambda < \lambda_c(\beta)$ (since they are already different for $\lambda = 0$). As for the weak coupling phase $\lambda > \lambda_c(\beta)$ of our large N solution (which is not even analytical at $\lambda = 0$), it is possible that the saddle point approximation does not distinguish between two different prescriptions of integration over $\phi(t)$ (contour integration over the Cartan subalgebra, on the one hand, and integration over θ's in the finite interval $[0, 2\pi]$, on the other hand). This hypothesis is to be verified. A weaker version of it could be the coincidence of two prescriptions in the scaling limit (eq. (5.6)).

7. Acknowledgments

We are grateful to N. Nekrasov for many valuable advices and in particular for explaining to us the construction of refs. [9,10]. We also thank A.S.Schwarz, S.Shatashvili and A.Wainstein for useful discussions. One of us (J.H.) would like to thank J. L. Gervais for his kind invitation to LPT ENS in January 1999. This research is supported in part by European TMR contract ERBFMRXCT960012.

Appendix A. Solution of the saddle point equations

A.1. The function $z = z(\zeta)$ as an elliptic integral

It follows from the integral representation of $\zeta(z)$ that it is real when $z \in \mathbb{R}, i\mathbb{R}, i\mathbb{R}\pm\pi$, satisfies

$$\zeta(z) = \zeta(z + 2\pi) = \zeta(-z) = \overline{\zeta(\bar{z})} \tag{A.1}$$

and by (4.6) is also real along the interval $[\frac{i}{2}\beta - a, \frac{i}{2}\beta + a]$. Therefore this function defines a map of the half strip $0 < \mathrm{Re}z < \pi, \mathrm{Im}z > 0$ with a cut $[\frac{i}{2}\beta, \frac{i}{2}\beta + a]$, to the upper half plane $\mathrm{Im}\zeta > 0$ (Fig. 1). The inverse map $z = z(\zeta)$ is given by the Schwarz-Christoffel formula (see, e.g. [21]):

$$z = i \int_{\zeta_4}^{\zeta} \frac{dt\ (t - \zeta_2)}{Y(t)} \tag{A.2}$$

where

$$Y(t) = \sqrt{(t - \zeta_1)(t - \zeta_3)(t - \zeta_4)(t - \zeta_5)}\ . \tag{A.3}$$

By construction, the map (A.2) acts on the special points $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 < \zeta_5$ and ∞ as is shown in the two first coloumns of Table 1.

The values of ζ at the special points of the map are determined as functions of β and λ by the assymptotics of $\zeta(z)$ at infinity. The expansion of the function (4.5) at $z \to \infty$ contains only odd powers of e^{iz}. If we approach infinity as $z = \pi + iy,\ y \to \infty$, the asymptotics of $\zeta(z)$ is

$$\zeta_+(y) \equiv \zeta(\pi + iy) = e^y + \sum_{n=0}^{\infty} \zeta^{(2n+1)} e^{-(2n+1)y} \tag{A.4}$$

where

$$\zeta^{(1)} = \left(1 + \frac{8 \sinh^2 \frac{\beta}{2}}{\lambda} \int_{-a}^{a} d\theta \rho(\theta) \cos\theta\right), \tag{A.5}$$

etc.

A.2. Elliptic parametrization of the solution

The map (A.2) and the condition (4.4) can be expressed explicitly in terms of standard elliptic integrals (see [22], 256.02) with parameters

$$k=\sqrt{\frac{\zeta_{54}\zeta_{31}}{\zeta_{53}\zeta_{41}}},\quad k'=\sqrt{1-k^2}=\sqrt{\frac{\zeta_{43}\zeta_{51}}{\zeta_{53}\zeta_{41}}} \tag{A.6}$$

$$\gamma=\sqrt{\zeta_{53}\zeta_{41}},\quad \alpha^2=\frac{\zeta_{54}}{\zeta_{53}},\quad \nu=\arcsin\sqrt{\frac{\zeta_{41}}{\zeta_{51}}} \tag{A.7}$$

where the notation $\zeta_{ij}=\zeta_i-\zeta_j$ is used. Namely

$$\begin{aligned} z(\zeta)&=\frac{2\zeta_{43}}{\gamma}\left(\Pi(\varphi,\alpha^2,k)+\frac{\zeta_{32}}{\zeta_{43}}F(\varphi,k)\right)\\ \varphi&=\arcsin\frac{1}{\alpha}\sqrt{\frac{\zeta-\zeta_4}{\zeta-\zeta_3}}, \end{aligned} \tag{A.8}$$

and

$$\frac{2\pi}{\lambda}\sinh(\beta/2)=\frac{\gamma}{2}E(k)-\frac{\zeta_4\zeta_5+\zeta_1\zeta_3}{2\gamma}K(k). \tag{A.9}$$

It is convenient to introduce as a parameter the elliptic amplitude u related to the angle φ as $\mathrm{sn}u=\sin\varphi$

$$u\ =i\frac{\sqrt{\zeta_{53}\zeta_{41}}}{2}\int_{\zeta_4}^{\zeta}\frac{dt}{Y(t)}=F(\varphi,k),\quad \mathrm{sn}u=\sin\varphi=\frac{1}{\alpha}\sqrt{\frac{\zeta-\zeta_4}{\zeta-\zeta_3}}. \tag{A.10}$$

Then the function

$$\zeta(u)=\frac{\zeta_4-\zeta_3\alpha^2\mathrm{sn}^2u}{1-\alpha^2\mathrm{sn}^2u} \tag{A.11}$$

maps the upper ζ-half-plane is mapped to the rectangle $0\le \mathrm{Re}u\le K,\ 0\le \mathrm{Im}u\le K'$ with K and K' being the complete elliptic integrals associated with the moduli k and k'. The special points $\zeta=\zeta_1,\dots,\zeta_5$ and ∞ correspond to the points $u_1,\dots,u_5$ and u_∞ along the boundary of the rectangle as is shown in Table 1. Note that

$$\frac{1}{\alpha}=\mathrm{sn}(u_\infty).$$

ζ	z	u	v
$-\infty$	$+i\infty$	$u_\infty+i0$	$v_\infty+0$
ζ_1	$\frac{i}{2}\beta$	$K+iK'$	$\frac{\pi}{2}$
ζ_2	$\frac{i}{2}\beta+a$	u_2	v_2
ζ_3	$\frac{i}{2}\beta$	iK'	$\frac{\pi}{2}(1+\tau)$
ζ_4	0	0	$\frac{\pi}{2}\tau$
ζ_5	π	K	0
$+\infty$	$\pi+i\infty$	$u_\infty+i0$	$v_\infty-0$

Table 1. The values of z, ζ, u and v at the special points of the map.

The function $z(u)$ defined by the integral (A.2) reads, in the parametrization (A.10),

$$z(u) = \frac{2\zeta_{43}}{\sqrt{\zeta_{53}\zeta_{41}}} \left(\int_0^u \frac{du}{1-\alpha^2 \mathrm{sn}^2 u} + \frac{\zeta_{32}}{\zeta_{43}} u \right). \tag{A.12}$$

We will express the integral in (A.12) in terms of Jacobian elliptic functions. Since the point u_∞ is between u_5 and u_1), it has the form

$$u_\infty = K + i\xi, \quad 0 < \xi < K'. \tag{A.13}$$

We find, using eq. 433.01 of [22],

$$z(u) = i \ln \frac{H_1(u+i\xi)}{H_1(u-i\xi)} + \frac{u}{K} \left[\frac{2\zeta_{42}}{\sqrt{\zeta_{53}\zeta_{41}}} K - \pi \Lambda_0(\nu, k) \right] \tag{A.14}$$

where $H_1(u)$ is a standard Jacobian elliptic function and

$$\begin{aligned} \Lambda_0(\nu, k) &= \frac{2}{\pi}[(E-K)F(\nu, k') + KE(\nu, k')] \\ &= i\frac{H_1'(i\xi)}{H_1(i\xi)}, \qquad \left(\nu = \arcsin \sqrt{\frac{\zeta_{41}}{\zeta_{51}}} = \arcsin \frac{\sqrt{\alpha^2 - k^2}}{k'} \right) \end{aligned}$$

is known as the Heuman's Lambda function. The condition $z(iK') = z(K+iK')$ is satisfied only if the coefficient in front of the linear term in u is zero, hence the condition

$$\frac{2\zeta_{42}}{\sqrt{\zeta_{53}\zeta_{41}}} K = \pi \Lambda_0(\nu, k) \tag{A.15}$$

From $H_1(u+iK') = e^{\frac{-i\pi u}{K}} H_1(u-iK')$ we find

$$z(iK') = i\frac{\pi}{K}\xi$$

which allows to determine ξ:

$$\xi = \frac{K}{2\pi}\beta. \tag{A.16}$$

The final expression for $z(u)$ is therefore

$$z(u) = i \ln \frac{H_1(u + i\frac{K}{2\pi}\beta)}{H_1(u - i\frac{K}{2\pi}\beta)}. \tag{A.17}$$

A.3. The dual modulus

We are going to write our solution in a form, which will allow to perform painlessly the scaling limit $\beta \to 0$. In this limit $a_{43} \approx -i\beta\frac{d\zeta}{dz} \to 0$ and, according to (A.6), $k' \approx$

$4e^{-K} \to 0$. Therefore it is more convenient to expand the solution in the dual modular parameter

$$q = e^{-\pi K/K'} = e^{i\pi\tau} \tag{A.18}$$

and use the variable v

$$v = \frac{\pi}{2}\tau - i\frac{\pi/2}{K'}\ u$$

as a parameter. The elliptic nome is expressed as a function of q as

$$k = \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4 \tag{A.19}$$

The parameters corresponding to the special points of the map are given by the last coulumn of Table 1. The parameter of infinity is equal, by (A.13) and (A.16), to

$$v_\infty = \frac{K}{4K'}\beta = -\frac{i\tau}{4}\beta \qquad (0 < v_\infty < \frac{\pi}{2}). \tag{A.20}$$

The domains of the four variables z, ζ, u and v are depicted in Fig. 1.

We will write the solution as a function of the parameters $v_\infty = v_\infty(\lambda)$ and β. It will be written as a series in the expansion parameter q

$$q = e^{-\frac{4\pi}{\beta}v_\infty} = e^{i\pi\tau}, \qquad \tau = 4i\frac{v_\infty}{\beta}. \tag{A.21}$$

The expansion of the function $\zeta(u)$ is obtained by plugging in eq. (A.11) the representation of sn in terms of the dual modulus

$$\frac{1}{\mathrm{sn}u} = \mathrm{dn}\left(\frac{2K'}{\pi}v,\ k'\right) = \frac{\pi}{2K'}f(q) \tag{A.22}$$

$$f(q) = \left[1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos(2nv)\right]. \tag{A.23}$$

The function $z(v)$ reads, in terms of the standard elliptic functions associated with the dual modulus,

$$\begin{aligned} z(v) &= -\frac{4}{\pi\tau}v_\infty v + i\ln\frac{\vartheta_1(v+v_\infty)}{\vartheta_1(v-v_\infty)} \\ &= i\frac{\beta}{\pi}v + i\ln\frac{\sin(v+v_\infty)}{\sin(v-v_\infty)} + 4i\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\ \frac{\sin 2nv_\infty \sin 2nv}{n}\ . \end{aligned} \tag{A.24}$$

Finally, (A.15) expands as

$$\frac{4v_\infty}{\beta}\frac{\zeta_{42}}{\sqrt{\zeta_{53}\zeta_{41}}} = \frac{2}{\pi}v_\infty + \frac{\vartheta_4'(v_\infty)}{\vartheta_4(v_\infty)} = \frac{2}{\pi}v_\infty + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin 2nv_\infty\,. \tag{A.25}$$

In order to fix completely the solution, let us consider the vicinity of the point v_∞ and compare the explicit dependence $\zeta = \zeta(z)$ with the asymptotics (A.4)at $z \to \pi + i\infty$. The half-line

$$z = \pi + iy \quad (y > 0)$$

is parametrized by the interval $0 < v < v_\infty$. In the left vicinity of the point v_∞

$$v = v_\infty - \epsilon \quad (\epsilon > 0)$$

the functions $z = \pi + iy$ and ζ have the form

$$e^y = \frac{A}{\epsilon} + B + C\epsilon, \quad \zeta = \frac{P}{\epsilon} + Q + R\epsilon$$

with

$$P = -\zeta_{43}\frac{f(v_\infty)}{2f'(v_\infty)}, \quad Q = -P\left(2\frac{\zeta_4}{\zeta_{43}}\frac{f'(v_\infty)}{f(v_\infty)} - \frac{1}{2}\frac{f'(v_\infty)}{f(v_\infty)} - \frac{1}{2}\frac{f''(v_\infty)}{f'(v_\infty)}\right),$$

$$A = e^{\frac{\beta}{\pi}v_\infty}\frac{\theta_1(2v_\infty)}{\theta_1'(0)}, \quad B = -A\left(\frac{\beta}{\pi} + \frac{\theta_1'(2v_\infty)}{\theta_1(2v_\infty)}\right).$$

The leading asymptotics (A.4) of $\zeta(z)$ is achieved if $A = P$ and $B = Q$, which yields (4.13).

¿From (A.25) we get

$$\zeta_{42} = \sqrt{\zeta_{53}\zeta_{41}}\frac{\beta}{2\pi}\left(1 + 2\pi q\frac{\sin 2v_\infty}{v_\infty}\right) \tag{A.26}$$

(Note that the relation $2\zeta_2 = \zeta_1 + \zeta_3 + \zeta_4 + \zeta_5$ is satisfied.)

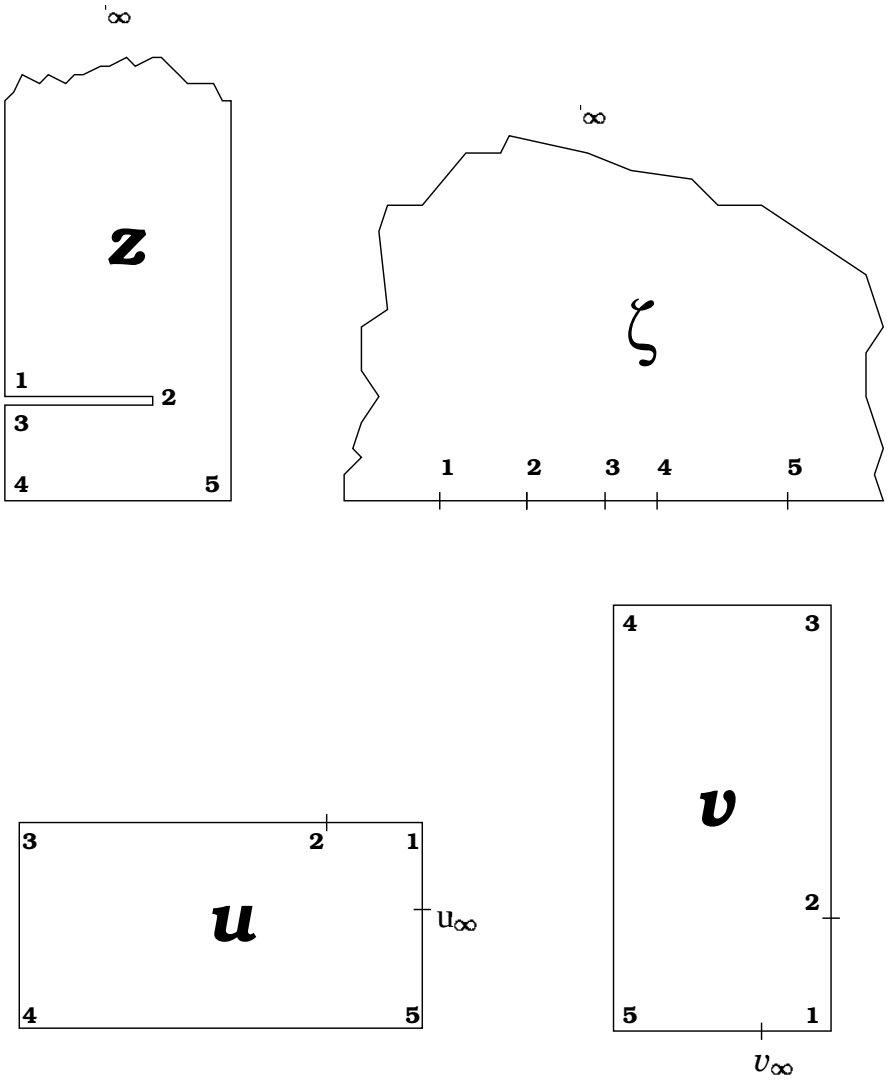

Fig.1: The domains of the variables z, ζ, u, v.

A.4. The limit of large τ (small β)

In this limit, which corresponds to the scaling limit discussed in Section 4, the v-rectangle can be replaced by an infinite half-strip and elliptic functions degenerate to trigonometric functions. After substituting $E = 1, K = \frac{2\pi}{\beta} v_\infty$ in the normalization condition (A.9), we get in this limit

$$\frac{4\pi}{\lambda} \sinh \frac{\beta}{2} e^{-\frac{\beta}{\pi} v_\infty} = 2 \sin 2v_\infty - 4v_\infty [\cos 2v_\infty - \frac{\beta}{2\pi} \sin 2v_\infty]. \tag{A.27}$$

The parameters of the solution are obtained from

$$P = \frac{\zeta_{43}}{16q} \frac{1}{\sin 2v_\infty}, \quad Q = \zeta_4 + P \cot 2v_\infty, \quad R = P \left(\frac{2}{3} + \cot^2 2v_\infty \right)$$

$$A = e^{\frac{\beta}{\pi}v_\infty}\sin 2v_\infty, \quad B = -A\left(\cot 2v_\infty + \frac{\beta}{\pi}\right), \quad C = A\left(-\frac{1}{3} + \frac{\beta}{\pi}\cot 2v_\infty + \frac{\beta^2}{2\pi^2}\right).$$

From $A = P$ we get

$$\zeta_{43} = 16qe^{\frac{\beta}{\pi}v_\infty}\sin^2 2v_\infty \tag{A.28}$$

and $B = Q$ implies

$$\frac{\zeta_3 + \zeta_4}{2} = -e^{\frac{\beta}{\pi}v_\infty}(2\cos 2v_\infty + \frac{\beta}{\pi}\sin 2v_\infty) = -\left(e^{\frac{\beta}{\pi}v_\infty}\sin 2v_\infty\right)'_{v_\infty}. \tag{A.29}$$

It is useful to note that

$$\frac{\zeta_{53} + \zeta_{54}}{\zeta_{41} + \zeta_{\zeta 1}} = \cot^2 v_\infty, \quad \frac{\zeta_{42} + \zeta_{32}}{\zeta_{41} + \zeta_{31}} = \frac{\beta}{2\pi}\cot v_\infty, \quad \frac{\zeta_{43}}{\zeta_{41} + \zeta_{31}} = 16q\cos^2 v_\infty.$$

Finally, the coefficient $\zeta^{(1)}$ is obtained as

$$\begin{aligned}\zeta^{(1)} &= A(R - C)\\ &= 1 + 2\beta^2\left(\frac{4v_\infty - \sin 4v_\infty}{4\pi\beta\epsilon} + \frac{4v_\infty^2 - \sin^2 2v_\infty - 2v_\infty\sin 4v_\infty}{8\pi^2} + \mathcal{O}(\beta)\right).\end{aligned} \tag{A.30}$$

A.5. The length of the cut

The branch point of the Riemann surface of $\zeta(z)$ is at $z_2 = z(v_2)$, where $\zeta(v_2) = \zeta_2$. Taking the limit of (A.26),

$$\zeta_{42} = \frac{\beta}{\pi}\sin 2v_\infty e^{\frac{\beta}{\pi}v_\infty} \tag{A.31}$$

we rewrite the solution (5.1) in the form

$$\begin{aligned} z &= i\frac{\beta}{\pi}v + i\ln\frac{\cot v + \cot v_\infty}{\cot v - \cot v_\infty}\\ \cot^2 v &= \frac{(\zeta_4 - \zeta)\cot v_\infty + \frac{2\pi}{\beta}\zeta_{42}}{(\zeta_4 - \zeta)\tan v_\infty - \frac{2\pi}{\beta}\zeta_{42}}.\end{aligned} \tag{A.32}$$

Putting $v = v_2$ in (A.32), we get

$$a = -\frac{\beta}{\pi}\delta_2 + i\ln\frac{\tanh\delta_2 - i\cot v_\infty}{\tanh\delta_2 + i\cot v_\infty},$$

$$\tanh^2\delta_2 = \frac{1 - \frac{\beta}{2\pi}\cot v_\infty}{1 + \frac{\beta}{2\pi}\tan v_\infty}$$

and finally

$$\cos(a + \frac{\beta}{\pi}\delta_2) = \cos 2v_\infty - \frac{\beta}{2\pi}\sin 2v_\infty \approx \cos 2v_\infty,$$

which allows us to evaluate a

$$a \approx 2v_\infty - \frac{\beta}{\pi}\delta_2, \quad \delta_2 \approx \ln\left(\frac{4\pi}{\beta}\sin 2v_\infty\right). \tag{A.33}$$

Appendix B. Direct scaling analysis of the equations on parameters of the large N solution

Six conditions on the length of the cut, a, and the 5 parameters of the map $z \mapsto \zeta(z)$ (which we denote here by $a_1, \ldots, a_n$ instead of $\zeta_1, \ldots, \zeta_n$) are

$$\begin{aligned}
&(1) \qquad a = \int_{a_2}^{a_3} dt \frac{t-a_2}{|Y(t)|} \\
&(2) \qquad 0 = \int_{a_1}^{a_3} dt \frac{t-a_2}{|Y(t)|} \qquad \left(\text{resp.} \quad \pi = \int_{a_4}^{a_5} dt \frac{t-a_2}{|Y(t)|}\right) \\
&(3) \qquad \tfrac{1}{2}\beta = \int_{a_3}^{a_4} dt \frac{t-a_2}{|Y(t)|} \\
&(4) \qquad a_1 + a_3 + a_4 + a_5 = 2a_2 \\
&(5) \qquad \ln a_5 = \int_{a_5}^{\infty} dt \left(\frac{t-a_2}{Y(t)} - \frac{1}{t}\right) \qquad (a_5 > 0), \\
&(6) \qquad \frac{4\pi}{\lambda}\sinh(\beta/2) = \int_{a_4}^{a_5} d\zeta \frac{\zeta(\zeta-a_2)}{|Y(\zeta)|}
\end{aligned} \tag{B.1}$$

with $Y(t)$ as in (A3) (where the a_i were denoted by ζ_i). While the first three conditions are implied by the geometry of the map ζ, conditions (4) and (5) follow when comparing the (from (4.5)) known asymptotics of ζ, e.g. for $z = \pi + iy$ ($y \to \infty$; cp. (A4)), to the one implied by the integral representation (A2) which says that

$$y = \ln\zeta_+ + \left(\int_{a_5}^{\infty} dt \left(\frac{t-a_2}{Y(t)} - \frac{1}{t}\right) - \ln a_5\right) - \frac{1}{\zeta_+}\left(\frac{1}{2}(a_1+a_3+a_4+a_5) - a_2\right) - \sum_{n=2}^{\infty}\frac{b_n}{\zeta_+^n} \tag{B.2}$$

with $b_2 = \frac{1}{16}(a_1^2 + a_3^2 + a_4^2 + a_5^2 - \prod_{2\neq i\neq j\neq 2} a_i a_j)$ already determining $\int_{-a}^{+a} d\theta\, \rho(\theta)\cos(\theta)$.

With

$$a_{ij} = a_i - a_j, \;\; \gamma = \sqrt{a_{53}a_{41}}, \qquad \nu = \arcsin\sqrt{\frac{a_{41}}{a_{51}}}k = \frac{\sqrt{a_{54}a_{31}}}{\gamma}, \;\; k' = \sqrt{1-k^2} = \frac{\sqrt{a_{43}a_{51}}}{\gamma}, \tag{B.3}$$

the conditions (2), (3), (5) and (6) read

$$\begin{aligned}
(2) \qquad K(k) &= \frac{a_{43}}{a_{42}}\Pi\left(\frac{a_{31}}{a_{41}},\, k\right) \\
&= \frac{a_{51}}{a_{52}}\Pi\left(-\frac{a_{31}}{a_{53}},\, k\right) \\
&= -\frac{a_{42}}{a_{32}}\Pi\left(\frac{a_{54}}{a_{53}},\, k\right) + \frac{\pi\gamma}{2a_{32}}
\end{aligned} \tag{B.4}$$

$$(3) \qquad \begin{aligned} \frac{1}{2}\beta &= \frac{2a_{31}}{\gamma}\Pi\left(\frac{a_{43}}{a_{41}}, k'\right) - \frac{2a_{21}}{\gamma}K(k') \\ &= -\frac{2a_{54}}{\gamma}\Pi\left(\frac{a_{43}}{a_{53}}, k'\right) + \frac{2a_{52}}{\gamma}K(k') \end{aligned} \tag{B.5}$$

$$(5) \qquad \frac{2a_{54}}{\gamma}\Pi\left(\nu,\ \frac{a_{43}}{a_{53}},\ k'\right) - \frac{2a_{52}}{\gamma}F(\nu, k') = \ln\left(\frac{a_{53}+a_{41}}{4}\right) \tag{B.6}$$

$$(6) \qquad \frac{2\pi}{\lambda}\sinh(\beta/2) = \frac{\gamma}{2}E(k) - \frac{a_4a_5 + a_1a_3}{2\gamma}K(k). \tag{B.7}$$

The scaling limit can be obtained by direct analysis of the equations (B.1) in the limit $\beta \to 0, a \to a_0 > 0$. Letting

$$a_{43} = u, a_{32} = v, a_{53} = w,$$

$$\tilde{u} = \frac{u}{a_{31}}, \tilde{v} = \frac{v}{a_{31}}, \tilde{w} = \frac{w}{a_{31}},$$

one has

$$(3) \qquad \beta = 2\int_0^1 ds \frac{\tilde{v} + \tilde{u}s}{\sqrt{s(1-s)(\tilde{w} - \tilde{u}s)(1+\tilde{u}s)}} \tag{B.8}$$

$$= \frac{2v}{wa_{31}}\int_0^1 ds \frac{1 + \frac{u}{v}s}{\sqrt{s(1-s)(1-\frac{u}{w}s)(1+\frac{u}{a_{31}}s)}}$$

$$(1) \qquad a = \tilde{v}\int_0^1 ds \frac{1-s}{\sqrt{s(s+\frac{\tilde{u}}{\tilde{v}})(\tilde{w}+\tilde{v}s)(1-\tilde{v}s)}} \tag{B.9}$$

$$= \frac{v}{wa_{31}}\int_0^1 ds \frac{1-s}{\sqrt{s(s+\frac{u}{v})(1+\frac{v}{w}s)(1-\frac{v}{a_{31}}s)}}$$

$$(2) \qquad \begin{aligned} \pi &= \frac{v}{\sqrt{wa_{31}}}\int_{u/w}^1 ds \frac{1+\frac{w}{v}s}{\sqrt{s(1-s)(s-u/w)(1+s\tilde{w})}} \\ &\approx \frac{2\tilde{v}}{\sqrt{\tilde{w}}}K(r) + \sqrt{w}\int_0^1 \frac{ds}{\sqrt{(1-s)(1+s\tilde{w})}} \end{aligned} \tag{B.10}$$

where $r^2 = (1+\tilde{u})^{-1}(1-\frac{u}{w}) \approx 1 - \tilde{u}(\frac{\tilde{w}+1}{\tilde{w}})$.

In order to have $\beta \to 0$ and a finite, we must have $u, v \to 0$,

$$\tilde{\epsilon} \equiv \frac{2\tilde{v}}{\sqrt{\tilde{w}}} \to 0. \tag{B.11}$$

If one wants to keep, according to (B.9), a finite in this limit, u/v must go to zero such that

$$-\tilde{\epsilon}\ln\frac{u}{v} = 2a_0 \tag{B.12}$$

finite, i.e. $u \to 0$ exponentially faster than $\tilde{v}/\sqrt{\tilde{w}}$ (and the $\approx$ sign in (B10) and thereafter, means that such terms are dropped).

One also finds that $\frac{u}{w} \to 0$ (even if $w \to 0$) as if not, the r.h.s. of (B.10) would go to zero. So $\beta \to 0$, $a \to a_0 > 0$ implies

$$u, v \to 0, \quad \frac{u}{v} \to 0, \quad \frac{u}{w} \to 0 \tag{B.13}$$

together with (B.11), (B.12), and

$$\beta \approx \tilde{\epsilon}\pi. \tag{B.14}$$

In order to extract more quantitative information from (B.10) consider the equivalent condition (B.4),

$$(\tilde{u} + \tilde{v})K(k) = \tilde{u}\Pi\left(\frac{1}{1+\tilde{u}}, k\right) \tag{B.15}$$

where

$$k^2 = \left(1 - \frac{u}{w}\right)\left(\frac{1}{1+\tilde{u}}\right) < \frac{1}{1+\tilde{u}}. \tag{B.16}$$

As $\tilde{u} \to 0$, $k'^2 = 1 - k^2 \approx \frac{u}{w} - \tilde{u} \to 0$ and we can use some standard expansions for the third elliptic integral appearing in (B.15), e.g. (412.01 of [BF])

$$\Pi\left(\frac{1}{1+\tilde{u}}, k\right) = K(k) + \frac{\pi}{2}\frac{1}{\sqrt{1+\tilde{u}}}\frac{1 - \Lambda_0(\theta, k)}{\sqrt{\frac{\tilde{u}}{1+\tilde{u}}\frac{1}{1+\tilde{u}}\frac{\tilde{u}}{\tilde{w}}}} \tag{B.17}$$

where

$$\sin\theta = \sqrt{\frac{\frac{\tilde{u}}{1+\tilde{u}}}{\frac{u}{w} + \frac{\tilde{u}}{1+\tilde{u}}}} \approx \sqrt{\frac{\tilde{w}}{\tilde{w}+1}} \tag{B.18}$$

and the first terms in the expansion of Heumann's Lambda function $\Lambda_0(\theta, k)$ (904.00 of [BF]) are

$$\Lambda_0(\theta, k) = \frac{2}{\pi}\left(E\theta - \frac{1}{4}(2K - E)k'^2(\theta - \sin\theta\cos\theta) + ...\right) \approx \frac{2}{\pi}\arcsin\sqrt{\frac{\tilde{w}}{\tilde{w}+1}}. \tag{B.19}$$

Inserting (B.18) into (B.15) and using

$$K(k) \approx -\frac{1}{2}\ln k'^2 + \ln 4 \quad \text{for} \quad k' \to 0$$

one finds:

$$\tilde{v}\left(-\frac{1}{2}\ln\left(\frac{u}{w} + \frac{\tilde{u}}{1+\tilde{u}}\right) + \ln 4\right) \approx \frac{\pi}{2}\sqrt{\tilde{w}}\left(1 - \frac{2}{\pi}\arcsin\sqrt{\frac{\tilde{w}}{1+\tilde{w}}}\right),$$

and, using (B.14)

$$\frac{u}{w} + \frac{\tilde{u}}{1+\tilde{u}} \approx 16 e^{-\frac{2\pi^2}{\beta}(1 - \frac{2}{\pi}\arcsin\sqrt{\frac{\tilde{w}}{\tilde{w}+1}})} \tag{B.20}$$

$$\tilde{u} \approx \frac{16\tilde{w}}{1+\tilde{w}} e^{-\frac{2\pi^2}{\beta}\left(1-\frac{2}{\pi}\arcsin\sqrt{\frac{\tilde{w}}{\tilde{w}+1}}\right)}. \tag{B.21}$$

Apart from converting $\ln \tilde{u}$ terms into $\tilde{w}, \beta$ dependencies, all other $\tilde{u}$-dependencies are dropped, due to this exponential decay (B.21). Eq. (B.1.5) can then be stated explicitly as an expression for a_{31} in terms of β and $\tilde{w}$ as follows:

$$\begin{aligned}
\ln a_5 &\approx \int_0^\infty ds \left(\frac{1}{\sqrt{(s+a_{31}+w)s}} - \frac{1}{s+a_3+w} \right) + \frac{v}{w} \int_1^\infty \frac{ds}{s\sqrt{(s-1)(\beta+\frac{1}{\tilde{w}})}} \\
&= \lim_{\Lambda\to\infty} \left(\ln\left(2\sqrt{s^2+sa_{51}}+2s+a_{51}\right)\Big|_0^\Lambda + \ln\frac{a_5}{\Lambda} \right) + \frac{v}{w}\sqrt{\tilde{w}}\left(\arcsin\left(\frac{1-\tilde{w}}{1+\tilde{w}}\right)+\frac{\pi}{2}\right) \\
&= \ln 4 + \ln\frac{a_5}{a_{51}} + \frac{\beta}{2\pi}\left(\arcsin\left(\frac{1-\tilde{w}}{1+\tilde{w}}\right)+\frac{\pi}{2}\right)
\end{aligned} \tag{B.22}$$

Hence

$$a_{31} \approx \frac{4}{1+\tilde{w}}\; e^{\frac{\beta}{4}\left[1+\frac{2}{\pi}\arcsin\left(\frac{1-\tilde{w}}{1+\tilde{w}}\right)\right]}. \tag{B.23}$$

The last equation needed to calculate the a_i as functions of $\beta \to 0$ and λ is (B.7), resp.

$$\frac{4\pi\sinh(\beta/2)}{\lambda} = \sqrt{w(a_{31}+u)}E(k) + \frac{2va_3}{\sqrt{wa_{31}}}K(k) \tag{B.24}$$

as $\gamma^2 = w(a_{31}+u)$ and (due to (B.1.4))

$$a_4a_5 + a_1a_3 = -2va_3. \tag{B.25}$$

(B.24) can be simplified substantially even without neglecting $\tilde{u}$-terms, by noting that (B.15), (B.17) imply

$$\frac{1}{\pi}\frac{2v}{\sqrt{wa_{31}}}K(k) = \sqrt{1+\tilde{u}}\;[1-\Lambda_0(\theta,k)]. \tag{B.26}$$

With

$$\frac{a_3}{2} = \frac{a_{31}}{4}(1-2\tilde{v}-\tilde{w}-\tilde{u}), \quad \tilde{v} = \frac{\beta}{2\pi}\sqrt{\tilde{w}} \tag{B.27}$$

one therefore gets

$$\begin{aligned}
\frac{2\sinh(\beta/2)}{\lambda} =& \frac{a_{31}}{4}\sqrt{1+\tilde{u}}\left(\frac{2}{\pi}\sqrt{\tilde{w}}E + (1-2\tilde{v}-\tilde{w}-\tilde{u})(1-\Lambda_0)\right) \\
\approx& \frac{a_{31}}{4}\left(\frac{2}{\pi}\sqrt{\tilde{w}} + \left(1-\frac{\beta}{\pi}\sqrt{\tilde{w}}-\tilde{w}\right)\left(1-\frac{2}{\pi}\arcsin\sqrt{\frac{\tilde{w}}{\tilde{w}+1}}\right)\right)
\end{aligned} \tag{B.28}$$

which is an (implicit) equation for $\tilde{w}$ as a function of β and λ, when inserting (B.23). For $\tilde{w} \to 0$ it reads

$$\frac{1-e^{-\beta}}{\lambda} \approx 1 - \frac{\beta}{\pi}\sqrt{\tilde{w}} - 2\tilde{w}. \tag{B.29}$$

The length of the cut is given by (cp. (B.1.1),(B9))

$$a \to a_0 = \pi \left(1 - \frac{2}{\pi} \arcsin \sqrt{\frac{\tilde{w}}{\tilde{w}+1}} \right) \tag{B.30}$$

The second of the final scaling eqs. (5.6) follows from (B.28) and (B.30) if we neglect all terms proportional to β or exponentially small terms and use $a = 2v_\infty$.

Finally note that the second line of (B.10), via $K(r) \approx \frac{1}{2} \ln \frac{16\tilde{w}}{\tilde{u}(\tilde{w}+1)}$ implies (B.21) (shortcutting the argument (B.15–21)), when using

$$\sqrt{\tilde{w}} \int_0^1 \frac{ds}{\sqrt{(1-s)(1+s\tilde{w})}} = \frac{\pi}{2} - \arcsin \frac{1-\tilde{w}}{1+\tilde{w}}$$

and

$$\begin{aligned} \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{1-\tilde{w}}{1+\tilde{w}} &= \frac{2}{\pi} \arcsin \frac{1}{\sqrt{1+\tilde{w}}} \\ &= 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{\tilde{w}}{\tilde{w}+1}}. \end{aligned}$$

Appendix C. Inverted oscillator: the point $\beta = i\pi$

An interesting analytical continuation of our model corresponds to the imaginary values of the generator of $O_\epsilon(2)$ symmetry of the original supersymmetric model. If we renormalise ϵ to one it is equivalent to the change $\beta \to i\beta$ in (2.24). The corresponding saddle point equation reads:

$$2\lambda \sin u = \fint_{-a}^{a} du' \rho(u') \left(2 \cot \frac{u-u'}{2} - \cot \frac{u-u'+\beta}{2} - \cot \frac{u-u'-\beta}{2} \right) \tag{C.1}$$

According to the arguments and results of the paper [14] the inverted twisted matrix oscillator describes the compactified $c = 1$ string, or, in other words, the compactified bosonic field coupled to the 2d quantum gravity. So at least the critical regime of $c = 1$ string with the typical inverse logarithmic dependence of the physical quantities on the cosmological coupling should show up at some point. Let us demonstrate it in the case which we can solve explicitly, namely for $\beta = i\pi$. The equation (C.1) in this case looks as:

$$\frac{\lambda}{2} \sin u = \fint_{-a}^{a} du' \frac{\rho(u')}{\sin(u-u')}. \tag{C.2}$$

The spectral density is

$$\rho(u) = \frac{\lambda}{2\pi} \sqrt{\sin^2 a - \sin^2 u}.$$

The normalization condition gives

$$1 = \int_{-a}^{a} du \rho(u) = \frac{\lambda}{2\pi} \int_{-a}^{a} du \sqrt{\sin^2 a - \sin^2 u} = \frac{2\lambda}{\pi} [E(\sin a) - \cos^2 a \; K(\sin a)].$$

or

$$E(k) - k'^2 K(k) = \frac{\pi}{\lambda}, \quad k = \sin a. \tag{C.3}$$

Consider the limit when the eigenvalues occupy almost the whole interval $[-\pi, \pi]$ allowed by the periodicity: $a \sim \pi, \;\; k^2 \simeq 1$. In terms of k' we have the following assymptotics:

$$E \simeq 1 + \frac{1}{2} k'^2 \log(4/k') \tag{C.4}$$

$$K \simeq \log(4/k') \tag{C.5}$$

By the use of (C.4)and (C.5)we obtain from (C.3):

$$k'^2 \simeq \frac{2}{\pi} \frac{(\lambda - \lambda_c)}{|\log(\lambda - \lambda_c)|} \tag{C.6}$$

for $\lambda \to \lambda_c = \pi$.

For the simplest physical quantity: the derivative of the free energy we obtain:

$$F'_\lambda = \int_{-a}^{a} du \rho(u) \cos u = \frac{\lambda k^2}{2} \tag{C.7}$$

from where we obtain the scaling asymptotics typical for the $c = 1$ noncritical string discovered in [23]:

$$F(\lambda) \simeq \frac{\pi^2}{4} - \frac{1}{4} \frac{(\lambda - \lambda_c)^2}{|\log(\lambda - \lambda_c)|} \tag{C.8}$$

The considered case $\beta = i\pi$ of the $c = 1$ matrix model corresponds to the Kosterlitz-Thouless phase transition point. It would be interesting to study the vicinity of this point by generalizing our solution to all imaginary β.

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