Bondi-type systems near space-like infinity and the calculation of the NP-constants

Helmut Friedrich
Albert-Einstein-Institut
Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 5
14476 Golm, Germany

János Kánnár
MTA KFKI
Részecske és Magfizikai Kutatóintézet
Budapest, Pf. 49,
1525, Hungary

Abstract

We relate Bondi systems near space-like infinity to another type of gauge conditions. While the former are based on null infinity, the latter are defined in terms of Einstein propagation, the conformal structure, and data on some Cauchy hypersurface. For a certain class of time symmetric space-times we study an expansion which allows us to determine the behavior of various fields arising in Bondi systems in the region of space-time where null infinity touches space-like infinity. The coefficients of these expansions can be read off from the initial data. We obtain in particular expressions for the constants discovered by Newman and Penrose (NP-constants) in terms of the initial data. For this purpose we calculate a certain expansion introduced in [4] up to 3rd order.

1 Introduction

Most studies of gravitational fields near null infinity are based on the use of "Bondi-type" coordinates. In the first investigations of the behavior of the field near null infinity (cf. [1], [11], [22]) Bondi-type coordinates played a crucial role in the specification of the fall-off behavior of the field. The characterization of the asymptotic behavior of gravitational fields near null infinity in terms of the conformal geometry subsequently suggested by Penrose ([16], [17]) does not require the use of such a specific class of coordinates. Nevertheless, Bondi-type coordinates are usually also employed in this context because they allow us to exploit in a convenient way certain features of the null cone structure. If the gravitational field is, however, to be analyzed in detail in the region where future and past null infinity $\mathcal{J}^{\pm}$ "touch" space-like infinity, and if this is to be done such that $\mathcal{J}^{-}$ and $\mathcal{J}^{+}$ are treated on an equal footing, Bondi-type coordinates are not particularly helpful. Already in the simplest non-trivial case, that of the Schwarzschild solution, the use of double null coordinates leads to difficulties.

In [4] an initial value problem for the conformal vacuum field equations has been formulated which is designed to analyze near space-like and null infinity the Einstein propagation of asymptotically flat data on a Cauchy hypersurface $\tilde{S}$ in a finite picture. In this setting, which is based on certain conformally invariant structures, space-like infinity is represented by a cylinder $I \simeq]-1,1[\times S^2$ such that the sets $\mathcal{J}^{\pm} \simeq \mathbb{R} \times S^2$, representing future resp. past null infinity, "touch" the cylinder at its two boundary component $I^{\pm} = \{\pm 1\} \times S^2$. Though the underlying facts about the evolution equations which have been used here hold for much more general situations, the picture has been analyzed so far under certain simplifying assumptions on the initial data. The data are assumed to be time-symmetric and the conformal structure, which then represents the free datum, is assumed to extend smoothly through space-like infinity such that the latter is represented by a point i in an extended manifold $S = \tilde{S} \cup \{i\}$. The cylinder I is obtained by blowing up the point i to a sphere $I^0 \simeq \{0\} \times S^2$ and by smoothly extending the solution in a particular geometric gauge.

It can be seen already under these assumptions on the data that the new picture allows us to relate near $I^{\pm}$ properties of the data on $\tilde{S}$, which touches I at I^0, to properties of the field on null infinity by solving a hierarchy of differential equations on I. These equations have been used in [4] to derive certain "asymptotic regularity conditions" for the initial data whose imposition prevents a certain class of logarithmic singularities of the field at the sets $I^{\pm}$ from arising. However, it still has to be shown that the asymptotic regularity conditions ensure a time evolution of the data which extends near space-like infinity smoothly to null infinity.

In the present article we analyze the consistency of the early investigations of fields near null infinity with the picture developed in [4] and we demonstrate to some extent the efficiency of the latter in calculating near space-like infinity quantities on null infinity from the given data. For this purpose we make two different types of assumptions. On the one hand we shall consider space-times arising from time symmetric vacuum data as described above which satisfy the asymptotic regularity conditions. Our calculations of fields on the cylinder I rely only on these assumptions. On the other hand we shall assume that these data develop into solutions which admit a smooth conformal structure at null infinity and that the gauge conditions proposed in [4] extend in a smooth and regular way to $\mathcal{J}^{\pm}$. We expect that our analysis will contribute information on the solution process which in the end will allow us to remove the second type of assumptions and to show that the existence of the smooth evolution can be derived solely from assumptions on the initial data.

The present article can be divided into three different, though related, parts.

– In [4] an expansion of the field near space-like infinity in terms of a "radial" coordinate ρ, which vanishes on the cylinder I representing space-like infinity, has been introduced. We calculate the coefficients of this expansion to third order. This calculation is not only of interest because it allows us to study the NP-constants, which will be discussed below, but also because it provides some information on the smoothness of the evolution near null infinity for fields arising from data subject only to our first type of assumptions. Though the asymptotic regularity conditions referred to above exclude certain types of logarithmic singularities in the evolution near I, there exists another potential source of singularities. To show that in fact no further singularities can arise at any order, it is clearly of interest to understand the situation for the first few orders of the expansion. The potential singularities should show up for the first time at the order of our calculation. Our calculations show that at this order they are in fact excluded by the asymptotic regularity conditions.

We note that our expansion of the field near space-like and null infinity, which we carry out in terms of the conformally rescaled fields and associated gauge conditions, can be translated into an expansion of the field near space-like infinity in terms of the "physical" field and suitable coordinates. We shall not carry out such a translation because the main point of our consideration is the fact that we can relate quantities on null infinity to the data on $\tilde{S}$.

− Bondi-type coordinates and certain related frame fields (cf. the definition of the "NP-gauge" below) are based on the structure of null infinity. The gauge conditions in [4] (cf. the definition of the "F-gauge" below) are based on Cauchy data, the Einstein equations, and certain properties of conformal structures. We discuss in general terms how to construct near null infinity the transformation from the F-gauge into the NP-gauge. Using the expansion referred to above we then obtain expansions near I^+ of various quantities given in the NP-gauge in terms of the coordinates arising in the F-gauge and coefficients which are given directly in terms of the initial data on S. We note that these expansions imply expansions of quantities of physical interest on null infinity such as the Bondi-energy-momentum, the angular momentum (cf. [19] for various suggestions), the radiation field, etc. in terms of the coordinate ρ on null infinity, which vanishes at I^+, and coefficients derived from the initial data.

Since we need for our considerations quite detailed information on the structure of the initial data near space-like infinity, our explicit calculation are done only for time-symmetric data. However, many of our considerations apply also to more general situations and as soon as sufficient information on data with non-vanishing extrinsic curvature becomes available (cf. [2]), we shall be able to derive by similar calculations relations between fields on $\mathcal{J}^-$ and $\mathcal{J}^+$. These relations will contain non-trivial information on the evolution process.

− As a specific application of this discussion we reconsider the constants which have been associated by Newman and Penrose with asymptotically simple space-times (cf. [12], [14]). The NP-constants are given by certain integrals over spherical cuts of null infinity and have been shown to be absolutely conserved in the sense of being independent of the choice of cut. We derive for them expressions in terms of the initial data on $\tilde{S}$. Such expressions have been given already in the static case in [14]. We derive analogous expressions for a much more general class of space-times arising from time-symmetric initial data. For these data the time evolution of the field is in general not known explicitly as it is the case in the presence of a time-like Killing vector field. The fact that we can nevertheless obtain expressions in terms of the data illustrates to some extent the efficiency of the new picture. Though various authors (cf. [8], [20], [21]) discuss these constants from different points of view, no consensus has been found concerning their geometrical/physical significance. Whether our discussion will help clarify the meaning of the NP-constants remains to be seen. One of our main reasons for looking at them is the expectation that they may play a role in the construction of space-times. In numerical calculations they may certainly provide a check on the numerical accuracy.

2 Relating different gauge conditions near null infinity

We begin by giving an outline of the *finite, regular initial value problem near space-like infinity.* This has been introduced in the article [4], to which we refer for more details. It involves a gauge which we refer to as the *F-gauge.* We then recall the *NP-gauge*, employed in [14], to discuss the gravitational field near null infinity. Finally, we discuss how the NP-gauge is related to the F-gauge.

2.1 The regular finite initial value problem near space-like infinity

We want to discuss asymptotically flat solutions $(\tilde{M}, \tilde{g})$ to Einstein's field equations $\tilde{R}_{\mu\nu} = 0$ in a neighborhood $\tilde{M}_a$ of space-like infinity which covers parts of future and past null infinity. The solutions arise from asymptotically flat data on a smooth space-like Cauchy hypersurface

$\tilde{S} \subset \tilde{M}$ which are such that the intrinsic conformal structure on $\tilde{S}$ admits an extension with a certain smoothness to a smooth compact manifold S obtained from $\tilde{S}$ by adjoining a point i which represents space-like infinity, $S = \tilde{S} \cup \{i\}$. We assume that the solution, i.e. the evolution in time of these data, possesses a smooth conformal extension (M, g, Θ) such that we can write $M = \tilde{M} \cup \mathcal{J}^- \cup \mathcal{J}^+$, where $\mathcal{J}^\pm \simeq \mathbb{R} \times S^2$ represent future respectively past null infinity and Θ denotes a smooth "conformal factor" on M such that $\Theta > 0$ and $g = \Theta^2 \tilde{g}$ on $\tilde{M}$ while $\Theta = 0$, $d\Theta \neq 0$ on $\mathcal{J}^\pm$.

To analyze in detail the consequences of the field equations in a neighborhood of space-like infinity which covers parts of $\mathcal{J}^\pm$, the situation above has been discussed in the [4] in terms of a certain principal fiber bundle $M'_a \to M_a$ with projection π, 4−dimensional base space M_a, and bundle space M'_a which is a 5−dimensional manifold with boundary and edges. To describe this setting further we need to introduce some notation.

We employ the two-components spinor and space-spinor formalisms as used in [4] where ϵ_{ab}, ϵ^{ab} are the antisymmetric spinors with $\epsilon_{01} = 1$, $\epsilon^{01} = 1$. We set $\tau^{aa'} = \epsilon_0{}^a \, \bar{\epsilon}_{0'}{}^{a'} + \epsilon_1{}^a \, \bar{\epsilon}_{1'}{}^{a'}$. By $SU(2)$ will be denoted the group of 2×2 matrices $t = (t^a{}_b)$ satisfying

$$\epsilon_{ac} \, t^a{}_b \, t^c{}_d = \epsilon_{bd}, \qquad \tau_{ac} \, t^a{}_b \, t^c{}_d = \tau_{bd},$$

and by $U(1)$ its subgroup of diagonal matrices. A basis of the Lie-algebra of $SU(2)$ is then given by the 2×2 matrices

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; u_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; u_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad (2.1)$$

of which u_3 generates $U(1)$.

In the following will be described in detail the regular finite initial value problem at space-like infinity formulated in [4]. Though we shall remark in passing on the construction of the manifold M'_a and the underlying gauge conditions, we refer for the full details to the original article. The manifold M'_a is given by

$$M'_a = \{(\tau, \rho, t) \in \mathbb{R} \times \mathbb{R} \times SU(2) | \; 0 \leq \rho < a, \, -\frac{\omega}{\rho} \leq \tau \leq \frac{\omega}{\rho}\},$$

where a is a positive real number and $\omega = \omega(\rho, t)$ a smooth non-negative function, given below, such that $\frac{\omega}{\rho}$ extends to a smooth positive function with $\frac{\omega}{\rho} \to 1$ as $\rho \to 0$. By ρ and τ will also be denoted the projections of M'_a onto the first respectively second component of $\mathbb{R} \times \mathbb{R} \times SU(2)$. Then any coordinate system on $SU(2)$ will define together with the functions ρ and τ a coordinate system on M'_a. There will, however, arise no need for us to introduce coordinates on $SU(2)$. We denote the projection onto the third component of $\mathbb{R} \times \mathbb{R} \times SU(2)$ by t and regard the $SU(2)$-valued function t as a "coordinate" on M'_a.

The natural action on the right of $U(1)$ on $SU(2)$ induces a smooth action of $U(1)$ on M'_a. The quotient $M'_a/U(1)$ under this action will be denoted by M_a and the induced projection of M'_a onto M_a by π. We shall write $N = \pi(N')$ for any subset N' of M'_a. The following subsets of M'_a will be important for us:

$$\mathcal{J}'^{\pm} = \{\tau = \pm\frac{\omega}{\rho}, \rho > 0\} \simeq \mathbb{R} \times S^3,$$

$$I' = \{|\tau| < 1, \rho = 0\} \simeq \mathbb{R} \times S^3, \qquad I'^{\pm} = \{\tau = \pm 1, \, \rho = 0\} \simeq S^3,$$

$$C' = \{\tau = 0\}, \qquad I'^{0} = \{\tau = 0, \, \rho = 0\} = C' \cap I' \simeq S^3.$$

Because they cover only a part of null infinity close to space-like infinity, we should have denoted the first sets more precisely by $\mathcal{J}'^{\pm}_a$ but we dropped the subscript a for convenience. By definition the part of the physical manifold $\tilde{M}$ which is covered by M_a is given by $\tilde{M}_a = M_a \setminus (\mathcal{J}^- \cup \mathcal{J}^+ \cup I \cup$

$I^- \cup I^+$), the sets $\mathcal{J}^\pm$ represent *future* resp. *past null infinity* while the set I represents *space-like infinity* for $\tilde{M}_a$ and the metric induced on it by $\tilde{g}$. Thus $\tilde{M}_a$ covers a neighborhood of space-like and null infinity in $\tilde{M}$. The edges $I^\pm \simeq S^2$ of M_a at which future resp. past null infinity touches space-like infinity will play an important role in the following. We shall refer to the set C as the *initial hypersurface* since by definition $C \cap \tilde{M}_a = C \setminus I^0 = \tilde{S} \cap \tilde{M}_a$. There exists a neighborhood B_a of i in S and smooth surjective map $\pi' : C \to B_a$ which is injective on $C \setminus I^0$ and which maps I^0 onto i.

As described in [4], the manifold M'_a is obtained essentially by lifting M_a into the bundle of normalized (with respect to ϵ_{ab}) spin frames. The set $I'^0 \simeq SU(2)$ corresponds to the set of normalized (with respect to ϵ_{ab} and τ_{ab}) spin frames at the point i. With each such spin frame we associate a unit tangent vector of S at i. With this vector we associate in turn a curve through i in B_a and extend the spin frame along this curve by a certain transport process. Thus we obtain spin frames at each point of $B_a \setminus \{i\}$. These frames are transported off $B_a \setminus \{i\} \simeq C \setminus I^0$ into the space-time M_a by a certain propagation law along conformal geodesics orthogonal to C. The latter are given in our description of M'_a by the curves $\rho = const.$, $t = const.$ with τ a natural parameter along them. Since for given unit tangent vector at i the spin frame defining it is determined up to a phase factor, the spin frames at points of $M_a \setminus (I \cup I^- \cup I^+)$ are also given up to multiplications by phase factors, which corresponds to the action of the group $U(1)$. The transport laws as well as further details of the gauge conditions are encoded in the form of the data and certain properties of the unknowns for the reduced equations.

Since it turns out to be most convenient, we will carry out all our calculations on the manifold M'_a and use for the subsets of M'_a introduced above the same names as for their images under π.

We denote by Z_{u_i} the vector field generated by u_i and the obvious action of $SU(2)$ on M'_a and define complex vector fields $X_+ = -(Z_{u_2} + iZ_{u_1})$, $X_- = -(Z_{u_2} - iZ_{u_1})$, $X = -2\,i\,Z_{u_3}$ which satisfy the commutation relations

$$[X,\, X_+] = 2\,X_+, \quad [X,\, X_-] = -2\,X_-, \quad [X_+,\, X_-] = -X. \tag{2.2}$$

The conformal field equations, in the form used in [4], are given in a particular (conformal, coordinate, and frame) gauge which is explained, together with the equations, most naturally in the context of *normal conformal Cartan connections* (cf. [3]). Again, we shall not go through the complete argument but just describe the unknowns and equations. To obtain the equations on M'_a, we use the fact that the solder and the connection forms on the bundle of spin frames induce corresponding forms $\sigma^{aa'}$, $\omega^a{}_b$ on $M'_a \setminus I'$ which extend smoothly to M'_a. The metric $\epsilon_{ab}\,\bar{\epsilon}_{a'b'}\,\sigma^{aa'}\,\sigma^{bb'}$ on M'_a is degenerate because $<\sigma^{aa'}, X> = 0$ (the angle brackets denoting the dual pairing), but it descends to the Lorentz metric g on $\pi(M'_a \setminus I')$.

The equations are written as equations for the "vector"-valued unknown

$$u = (c^0{}_{ab},\, c^1{}_{ab},\, c^\pm{}_{ab},\, \chi_{(ab)cd},\, \xi_{abcd},\, f_{ab},\, \Theta_{(ab)cd},\, \Theta_g{}^g{}_{ab},\, \phi_{abcd}),$$

whose components have the following meaning. We consider the smooth vector fields

$$c_{aa'} = c^0{}_{aa'}\,\partial_\tau + c^1{}_{aa'}\partial_\rho + c^+{}_{aa'}\,X_+ + c^-{}_{aa'}\,X_-$$

which satisfy $<\sigma^{aa'}, c_{bb'}> = \epsilon_b{}^a\,\bar{\epsilon}_{b'}{}^{a'}$ on $M'_a \setminus I'$. All fields are written in space spinor notation based on the vector field $\sqrt{2}\,\partial_\tau = \tau^{aa'}\,c_{aa'}$. Since $\tau^{aa'}c_{aa'}$ is invariant under the action of $U(1)$ it descends to a vector field on $\pi(M'_a \setminus I')$ which is time-like, has norm $\tau_{aa'}\,\tau^{aa'} = 2$, and is orthogonal to $\tilde{S}$. We have

$$c_{aa'} = \frac{1}{\sqrt{2}}\,\tau_{aa'}\,\partial_\tau - \tau^b{}_{a'}\,c_{ab} \tag{2.3}$$

with $c_{ab} \equiv \tau_{(a}{}^{b'}\,c_{b)b'} = c^0{}_{ab}\,\partial_\tau + c^1{}_{ab}\partial_\rho + c^+{}_{ab}X_+ + c^-{}_{ab}X_-$. The connection defines connection coefficients $\Gamma_{abcd} = \tau_b{}^{a'}\Gamma_{aa'cd} = \tau_b{}^{a'} < \omega_{cd}, c_{aa'} >$ which can be decomposed in the form

$$\Gamma_{abcd} = \frac{1}{\sqrt{2}}\,(\xi_{abcd} - \chi_{abcd}) = \frac{1}{\sqrt{2}}\,(\xi_{abcd} - \chi_{(ab)cd}) - \frac{1}{2}\,\epsilon_{ab}\,f_{cd},$$

with fields satisfying $\chi_{abcd} = \chi_{ab(cd)}$, $\xi_{abcd} = \xi_{(ab)(cd)}$, $f_{ab} = f_{(ab)}$. The curvature is represented by the rescaled conformal Weyl spinor field $\phi_{abcd} = \phi_{(abcd)}$ and by a spinor field $\Theta_{abcd} = \Theta_{ab(cd)}$ which is the Ricci spinor field of a certain Weyl connection for $\tilde{g}$.

The pull back $\pi^* \Theta$, again referred to as the conformal factor and denoted by Θ, extends smoothly to M'_a and is known in our gauge explicitly. It is given by

$$\Theta = \frac{\Omega}{\rho}\left(1 - \tau^2 \frac{\rho^2}{\omega^2}\right), \tag{2.4}$$

and appears, together with the 1-form

$$d_{ab} = 2\,\rho\,\frac{U\,x_{ab} - \rho\,D_{ab}U - \rho^2\,D_{ab}W}{(U + \rho\,W)^3},$$

(with x_{ab} as given in appendix [A.2]) which characterizes in a certain way the difference between the Levi-Civita connection of g and the Weyl connection referred to above, as coefficient in the conformal field equations. We have set here

$$\begin{aligned} \Omega &= \tfrac{\rho^2}{(U+\rho\,W)^2}, \\ \omega \equiv 2\,\Omega\,(-D_{ab}\,\Omega\,D^{ab}\,\Omega)^{-\frac{1}{2}} &= \rho\,(U + \rho\,W)\left\{U^2 + 2\,\rho\,U\,x^{ab}D_{ab}U - \rho^2\,D^{ab}U\,D_{ab}U \right. \\ &\quad \left. +2\,\rho^2\,U\,x^{ab}D_{ab}W - 2\,\rho^3\,D^{ab}U\,D_{ab}W - \rho^4\,D^{ab}W\,D_{ab}W\right\}^{-\frac{1}{2}}, \end{aligned} \tag{2.5}$$

where the smooth functions $U = U(\rho,t)$, $W = W(\rho,t)$, which satisfy $U = 1$ and $W = \frac{1}{2}m_{ADM}$ on I^0, are given as part of the initial data on the initial hypersurface C', on which D_{ab} is the intrinsic covariant derivative. Note that the fields Ω, ω, d_{ab} do not depend on τ. The conformal factor satisfies the relations (cf. [3])

$$\begin{aligned} &\Theta > 0 \quad \text{on} \quad M'_a, \quad \{\Theta = 0\} = \mathcal{J}'^- \cup I'^- \cup I' \cup I'^+ \cup \mathcal{J}'^+, \\ &c_{aa'}(\Theta) \neq 0 \quad \text{and} \quad \epsilon^{ab}\,\bar{\epsilon}^{a'b'}\,c_{aa'}(\Theta)\,c_{bb'}(\Theta) = 0 \quad \text{on} \quad \mathcal{J}'^{\pm}. \end{aligned} \tag{2.6}$$

In the following we shall refer to the coordinates τ, ρ, t, the frame $\{c_{aa'}\}$, and the conformal gauge defined by (2.4) as the *F-gauge*.

2.1.1 The conformal evolution equations

We recall here a few general features of the conformal field equations and refer again to [4] for more details. The conformal field equations imply on M'_a evolution equations of the form

$$\{A^0\,\partial_\tau + A^1\,\partial_\rho + A^+\,X_+ + A^-\,X_-\}\,u = C\,u, \tag{2.7}$$

where A^0, A^1, $A^\pm$, C denote matrix-valued functions which depend on u and the coordinates. The system is, for u close to the data given below and for the coordinates taking values on M'_a near C', symmetric hyperbolic. Writing $u = (v, \phi)$ with

$$v = (c^0{}_{ab},\, c^1{}_{ab},\, c^\pm{}_{ab},\, \chi_{(ab)cd},\, \xi_{abcd},\, f_{ab},\, \Theta_{(ab)cd},\, \Theta_g{}^g{}_{ab}), \qquad \phi = (\phi_{abcd}), \tag{2.8}$$

the evolution equations for v are obtained, with our assumptions on the gauge, from the structural equations of the normal conformal Cartan connection associated with g. They read explicitly

$$\partial_\tau c^0{}_{ab} = -\chi_{(ab)}{}^{ef}\,c^0{}_{ef} - f_{ab}, \tag{2.9}$$

$$\partial_\tau c^\alpha{}_{ab} = -\chi_{(ab)}{}^{ef}\,c^\alpha{}_{ef}, \quad \alpha = 1, +, -, \tag{2.10}$$

$$\partial_\tau \xi_{abcd} = -\chi_{(ab)}{}^{ef}\,\xi_{efcd} + \frac{1}{\sqrt{2}}\,(\epsilon_{ac}\,\chi_{(bd)ef} + \epsilon_{bd}\,\chi_{(ac)ef})\,f^{ef} \tag{2.11}$$

$$-\sqrt{2}\,\chi_{(ab)(c}{}^{e}\,f_{d)e} - \frac{1}{2}\,(\epsilon_{ac}\,\Theta_{f}{}^{f}{}_{bd} + \epsilon_{bd}\,\Theta_{f}{}^{f}{}_{ac}) - i\,\Theta\,\mu_{abcd},$$

$$\partial_\tau f_{ab} = -\chi_{(ab)}{}^{ef}\,f_{ef} + \frac{1}{\sqrt{2}}\,\Theta_{f}{}^{f}{}_{ab}, \tag{2.12}$$

$$\partial_\tau \chi_{(ab)cd} = -\chi_{(ab)}{}^{ef}\,\chi_{efcd} - \Theta_{(cd)ab} + \Theta\,\eta_{abcd}, \tag{2.13}$$

$$\partial_\tau \Theta_{(ab)cd} = -\chi_{(cd)}{}^{ef}\,\Theta_{(ab)ef} - \partial_\tau\Theta\,\eta_{abcd} + i\,\sqrt{2}\,d^{e}{}_{(a}\mu_{b)cde}, \tag{2.14}$$

$$\partial_\tau \Theta_{g}{}^{g}{}_{ab} = -\chi_{(ab)}{}^{ef}\,\Theta_{g}{}^{g}{}_{ef} + \sqrt{2}\,d^{ef}\,\eta_{abef}, \tag{2.15}$$

where $\eta_{abcd} = \frac{1}{2}\,(\phi_{abcd} + \phi^{+}_{abcd})$ and $\mu_{abcd} = -\frac{i}{2}\,(\phi_{abcd} - \phi^{+}_{abcd})$, with $\tau_a{}^{a'}\tau_b{}^{b'}\tau_c{}^{c'}\tau_d{}^{d'}\bar{\phi}_{a'b'c'd'} = \phi^{+}_{abcd}$, denote the electric and the magnetic part of ϕ_{abcd} respectively. These equations are of the form

$$\partial_\tau\,v = K(v) + Q(v,v) + L(\phi), \tag{2.16}$$

with a linear function K and a quadratic function Q of v, both with constant coefficients, and a linear function L of ϕ with coefficients which depend on the coordinates. We have $L = 0$ on I'. The evolution equations for ϕ, derived from the Bianchi identities, are genuine partial differential equations. They will be considered in more detail below.

2.1.2 The initial data

Consequences of the finite regular initial value problem have been worked out so far for Cauchy data which are time symmetric and admit a smooth extension through space-like infinity. In fact, it has been assumed in [4], as will be done in the following, that the conformal structure is analytic near space-like infinity. We note that this condition is imposed only for convenience and could be relaxed. The free Cauchy data on $\tilde{S}$ are then given by the conformal structure of a smooth metric h on S which is analytic in some $h-$normal coordinates near i.

We assume h to be given near i in a certain conformal gauge, the *cn-gauge* (cf. [4]). This reduces the freedom of performing conformal rescalings $h \to \theta^2\,h$ to the choice of the 4 real parameters $\theta(i)$, $\theta_{,a}(i)$, the value of θ in a neighborhood of i then being determined by the conformal gauge. We assume that B_a is a convex $h-$normal neighborhood of i and that ρ descends to a radial normal coordinate on B_a.

The metric $\tilde{h}$ induced by $\tilde{g}$ on $\tilde{S}$ is related to h by a rescaling $\tilde{h} = \Omega^{-2}\,h$, where the conformal factor Ω satisfies $\rho\,\Omega^{-\frac{1}{2}} \to 1$ as $\rho \to 0$ and the Lichnerowicz (Yamabe) equation

$$(D_\alpha D^\alpha - \frac{1}{8}\,r)(\Omega^{-\frac{1}{2}}) = 0. \tag{2.17}$$

Here D denotes the covariant derivative and r the Ricci scalar of h. The form (2.5) of Ω in terms of the functions U and W is a consequence of this equation and the required asymptotic behavior of Ω, which ensures that $\tilde{h}$ is asymptotically flat.

The initial data on C' for the conformal field equations are derived from h and Ω. They are given by

$$c^{0}{}_{ab} = 0, \quad c^{1}{}_{ab} = \rho\,x_{ab}, \quad c^{+}{}_{ab} = z_{ab} + \rho\,\check{c}^{+}{}_{ab}, \quad c^{-}{}_{ab} = y_{ab} + \rho\,\check{c}^{-}{}_{ab},$$
$$\chi_{(ab)cd} = 0, \;\; \xi_{abcd} = \sqrt{2}\,\rho\,\check{\gamma}_{abcd}, \;\; f_{ab} = x_{ab}, \tag{2.18}$$
$$\Theta_{abcd} = -\frac{\rho^2}{\Omega}\,D_{(ab}\,D_{cd)}\Omega + \frac{1}{12}\,\rho^2\,r\,h_{abcd}, \;\; \phi_{abcd} = \frac{\rho^3}{\Omega^2}\,(D_{(ab}\,D_{cd)}\Omega + \Omega\,s_{abcd}),$$

with x_{ab}, y_{ab}, z_{ab}, and the expression h_{abcd} of the metric h in space spinor notation as given in appendix [A.2], and $s_{abcd} = s_{(abcd)}$ the trace free part of the Ricci tensor of h.

In chapter [4.1] we shall discuss how the coefficients $\check{c}^{\pm}{}_{ab}$, $\check{\gamma}_{abcd}$ defining the frame and the connection coefficients are determined on C' by the (3−dimensional) structure equations from r and s_{abcd}. The observation (cf. [4]) that the data above extend smoothly to $I'^{0} \subset C'$ is most important for our construction.

2.1.3 The transport equations on I

At first sight it may appear that the initial data on $\tilde{S}$, thus in particular on C', should be complemented by boundary data on I' for the solutions of equations (2.7) to be uniquely determined. However, it turns out that for any smooth solution to the evolution equations on M'_a which coincides on C' with the initial data above, we have the important relation

$$A^1 = 0 \quad \text{on} \quad I'. \tag{2.19}$$

As a consequence, equations (2.7) reduce to a symmetric hyperbolic system of the form $\{A^0\,\partial_\tau + A^+\,X_+ + A^-\,X_-\}\,u = C\,u$ on I' which allows us to determine the unknown u on I' uniquely in terms of the value of u on I'^{0}. Thus we find, as was to be expected, that any smooth solution of (2.7) on M'_a taking on C' our initial data is determined uniquely by its data on $\tilde{S}$.

More generally, by applying repeatedly the derivative operator ∂_ρ to the evolution equations, restricting to I', and observing (2.19), we obtain symmetric hyperbolic transport equations

$$\{A^0\,\partial_\tau + A^+\,X_+ + A^-\,X_-\}\,u^p = C_p\,u^p + g_p \quad \text{on} \quad I', \quad p = 0, 1, 2, \ldots, \tag{2.20}$$

for the quantities $u^p = (\partial^p_\rho\,u)|_{I'}$. Here the matrix-valued function C_p and the vector valued function g_p depend on p and the quantities $u^0, \ldots, u^{p-1}$, but the matrices A^0, $A^{\pm}$ are universal in the sense that they depend neither on p nor on the initial data. We shall employ the notation above more generally, such that applying it to the fields s_{abcd} and r on the Cauchy hypersurface we have $s^p_{abcd} = (\partial^p_\rho\,s_{abcd})|_{I'^0}$ and $r^p = (\partial^p_\rho\,r)|_{I'^0}$, respectively.

To integrate the transport equations (2.20) on I', we expand all fields in terms of the matrix elements of unitary representations of $SU(2)$ which are given, in terms of the matrix elements $(t^a{}_b)_{a,b=0,1}$ of the 2−dimensional standard representation of $t \in SU(2)$, by the complex-valued functions

$$\begin{gathered} SU(2,C) \ni t \to T_m{}^j{}_k(t) = \binom{m}{j}^{\frac{1}{2}}\binom{m}{k}^{\frac{1}{2}}\,t^{(b_1}{}_{(a_1}\ldots t^{b_m)_j}{}_{a_m)_k}, \quad T_0{}^0{}_0(t) = 1, \\ j, k = 0, \ldots, m, \quad m = 1, 2, 3, \ldots \end{gathered} \tag{2.21}$$

Here, as in the following, setting a string of indices into brackets with a lower index k is meant to indicate that the indices are symmetrized and then k of them are set equal to 1 while the remaining ones are set equal to 0. The functions $\sqrt{m+1}\,T_m{}^j{}_k(t)$ form a complete orthonormal set in the Hilbert space $L^2(\mu, SU(2))$ where μ denotes the normalized Haar-measure on $SU(2)$. Under complex conjugation we have

$$\overline{T_m{}^j{}_k(t)} = (-1)^{j+k}\,T_m{}^{m-j}{}_{m-k}(t), \quad t \in SU(2),$$

and, for $0 \le k, j \le m$, $m = 0, 1, 2, \ldots$, we have with $\beta_{m,j} = \{j\,(m-j+1)\}^{\frac{1}{2}}$

$$X\,T_m{}^k{}_j = (m-2j)\,T_m{}^k{}_j, \;\; X_+\,T_m{}^k{}_j = \beta_{m,j}\,T_m{}^k{}_{j-1}, \;\; X_-\,T_m{}^k{}_j = -\beta_{m,j+1}\,T_m{}^k{}_{j+1}. \tag{2.22}$$

A function f satisfying a relation $Xf = 2sf$ with an integer or half integer number s, is said to have spin weight s. We note the spin raising (lowering) property of the action of $X_{\pm}$ on such functions implied by (2.2), i.e. $X\,X_{\pm}\,f = 2\,(s \pm 1)\,X_{\pm}\,f$. By construction of the manifold M'_a any function occuring in our formalism has a well defined spin weight. This leads to a simplification of the expansion in terms of the functions $T_m{}^k{}_j$. The general form of these expansions has been discussed in detail in [4] and will be assumed here without further explanation.

The quantities u^0, u^1, u^2 have been determined in [4]. They are given here (with a correction and a useful change of notation) at the beginning of chapter [4.1]. The functions u^3 will be calculated in chapter [4.1]. The quantities u^p, $p = 2, 3, \ldots$ have been shown (cf. [4]) to develop a certain type of logarithmic singularity on the sets $I'^{\pm}$ unless the free datum h on S satisfies the *asymptotic regularity condition*

$$D_{(a_q b_q} \ldots D_{a_1 b_1}\, b_{abcd)}(i) = 0, \tag{2.23}$$

for $q = 0, 1, 2, \ldots$, where the spinor field $b_{abcd} = b_{(abcd)}$ represents the Cotton tensor of h. The values of the functions u^p, $p \leq 3$, which will be given below, have been calculated on I' under the assumption that (2.23) is satisfied for $q \leq 1$. The analysis of the quantities u^p, to the extent to which it has been carried out in [4], indicates another potential source for a singular behavior of the fields u^p, $p \geq 3$, at $I'^{\pm}$. This will be discussed further in chapter [4.1].

2.2 The NP-gauge

For simplicity we restrict our discussions now to the future of $\tilde{S}$ in M, we refer to future null infinity simply as to null infinity and we denote it by $\mathcal{J}$. In the following we shall describe a certain class of gauge conditions on (M, g) near null infinity, referred to as the *NP-gauge*, which comprise certain requirements on the conformal gauge, certain coordinates, and a certain orthonormal frame field. Though this gauge is known, our description will be quite detailed, because we will have to refer to it later. The Levi-Civita connection induced by the conformal metric g will be denoted by ∇.

Suppose $\{E^{\circ}_{aa'}\}$ is a smooth frame field, satisfying $g(E^{\circ}_{aa'}, E^{\circ}_{bb'}) = \epsilon_{ab}\, \bar{\epsilon}_{a'b'}$, which is defined in a neighborhood of null infinity. We call it an “adapted frame”, if it satisfies the following conditions. The vector field $E^{\circ}_{11'}$ is tangent to and parallel propagated along null infinity. On the neighborhood on which the frame is given there is exists a smooth function u° which induces an affine parameter on the null generators of $\mathcal{J}$ such that $E^{\circ}_{11'}(u^{\circ}) = 1$, which is constant on null hypersurfaces transverse to $\mathcal{J}$, and which satisfies $E^{\circ\,\alpha}_{00'} = g^{\alpha\beta}\nabla_{\beta} u^{\circ}$. Thus $E^{\circ}_{00'}$ is tangent to the hypersurfaces $\{u^{\circ} = const.\}$ and geodesic. The fields $E^{\circ}_{11'}$, $E^{\circ}_{00'}$ as well as the fields $E^{\circ}_{01'}$, $E^{\circ}_{10'}$, which are necessarily tangent to the slices $\{u^{\circ} = const.\} \cap \mathcal{J}$, are parallelly propagated in the direction of $E^{\circ}_{00'}$.

In terms of its NP-spin-coefficients (note the slight difference of our notation with that of [11])

$$\Gamma^{\circ}_{aa'bc} = \frac{1}{2}\left\{ E^{\circ\,\alpha}_{aa'} E^{\circ\,\beta}_{b1'} \nabla_{\alpha} E^{\circ}_{c0'\beta} + E^{\circ\,\alpha}_{aa'} E^{\circ\,\beta}_{c1'} \nabla_{\alpha} E^{\circ}_{b0'\beta} \right\}, \tag{2.24}$$

an adapted frame is characterized by the properties

$$\begin{aligned} &\Gamma^{\circ}_{10'11} = 0, \qquad \Gamma^{\circ}_{11'11} = 0 \quad \text{on} \quad \mathcal{J}, \\ &\Gamma^{\circ}_{10'00} = \bar{\Gamma}^{\circ}_{01'0'0'}, \quad \Gamma^{\circ}_{11'00} = \bar{\Gamma}^{\circ}_{01'0'1'} + \Gamma^{\circ}_{01'01}, \quad \Gamma^{\circ}_{00'ab} = 0, \quad a, b = 0, 1, \qquad \text{near} \quad \mathcal{J}. \end{aligned} \tag{2.25}$$

The first of these conditions tells us that $\mathcal{J}$ is shear free. This well known fact follows from the equation for the trace free part $s_{\alpha\beta}$ of the Ricci tensor of the conformal vacuum metric g,

$$\Theta\, s_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \nabla_{\gamma}\nabla^{\gamma}\Theta - 2\nabla_{\alpha}\nabla_{\beta}\Theta. \tag{2.26}$$

Transvection with $E^{\circ\,\alpha}_{10'} E^{\circ\,\beta}_{10'}$ and restriction to $\mathcal{J}$ gives $\Gamma^{\circ}_{10'11}\, E^{\circ}_{00'}(\Theta) = 0$, while $E^{\circ}_{00'}(\Theta) \neq 0$ on $\mathcal{J}$. We shall combine now the construction of an adapted frame with the freedom to perform rescalings

$$g \rightarrow g^{\star} = \theta^2 g, \quad \Theta \rightarrow \Theta^{\star} = \theta\, \Theta \tag{2.27}$$

with some positive function θ, to obtain another adapted frame $\{E^{\bullet}_{aa'}\}$ for which we get further simplifications besides (2.25). We start with an adapted frame $\{E^{\circ}_{aa'}\}$ as described above. For arbitrary $\theta > 0$ and for arbitrary function $p > 0$ which is constant on the generators of $\mathcal{J}$ we set

$$E^{\bullet}_{11'} = \theta^{-2}\, p\, E^{\circ}_{11'} \quad \text{and} \quad u^{\bullet}(u^{\circ}) = \int_{u^{\circ}_*}^{u^{\circ}} \theta^2(u')\, p^{-1}(u')\, du' + u^{\bullet}_* \quad \text{on} \quad \mathcal{J}, \tag{2.28}$$

where the integration is performed along the generators of $\mathcal{J}$. Then $E^{\bullet}_{11'}$ will be parallelly propagated and $E^{\bullet}_{11'}(u^{\bullet}) = 1$ will hold. We assume that $u^{\circ} = u^{\circ}_{*}$ and $u^{\bullet} = u^{\bullet}_{*}$ on $\mathcal{C}$ and set

$$E^{\bullet}_{00'} = p^{-1}\, E^{\circ}_{00'}, \quad E^{\bullet}_{11'} = \theta^{-2}\, p\, E^{\circ}_{11'}, \quad E^{\bullet}_{01'} = \theta^{-1} E^{\circ}_{01'} \quad \text{on} \quad \mathcal{C}. \tag{2.29}$$

Since $\mathcal{C}$ is diffeomorphic to S^2 and thus carries (up to diffeomorphisms) precisely one Riemannian conformal structure, we can fix coordinates $x^3 = \vartheta$, $x^4 = \varphi$ as well as the function θ on $\mathcal{C}$ such that the metric $h^{\star}$ induced by $g^{\star}$ on $\mathcal{C}$ is given by the standard S^2-metric $h^{\star} = d\,\vartheta^2 + \sin^2\vartheta\, d\,\varphi^2$. Using the transformation laws $\Gamma^{\bullet}_{10'00} = p^{-1}\left[\Gamma^{\circ}_{10'00} - E^{\circ}_{00'}(\log\theta)\right]$ and $\Gamma^{\bullet}_{01'11} = p\,\theta^{-2}\left[\Gamma^{\circ}_{01'11} + E^{\circ}_{11'}(\log\theta)\right]$ on $\mathcal{C}$, we can achieve, by suitable choice of $d\,\theta$ and p,

$$\Gamma^{\bullet}_{10'00} = 0, \quad \Gamma^{\bullet}_{01'11} = 0, \quad E^{\bullet}_{00'}(\Theta^{\star}) = const. \neq 0 \quad \text{on} \quad \mathcal{C}. \tag{2.30}$$

The transformation $s^{\star}_{\alpha\beta} = -\frac{2}{\theta}\left\{(\nabla_\alpha\nabla_\beta\theta - \frac{2}{\theta}\nabla_\alpha\theta\nabla_\beta\theta) - \frac{1}{4}g_{\alpha\beta}(\nabla_\gamma\nabla^\gamma\theta - \frac{2}{\theta}\nabla_\gamma\theta\nabla^\gamma\theta)\right\} + s_{\alpha\beta}$ of the trace free part $s_{\alpha\beta}$ of the Ricci tensor under the rescaling (2.27) implies a transformation of $\Phi_{22} = \frac{1}{2}s_{\alpha\beta}E^{\circ\,\alpha}_{11'}\,E^{\circ\,\beta}_{11'}$ into $\Phi^{\star}_{22} = \frac{1}{2}s^{\star}_{\alpha\beta}E^{\bullet\,\alpha}_{11'}\,E^{\bullet\,\beta}_{11'}$, which yields, with the assumption that $\Phi^{\star}_{22} = 0$ on $\mathcal{J}$, on the generators of $\mathcal{J}$ the ODE

$$E^{\circ}_{11'}\big(E^{\circ}_{11'}(\theta)\big) - \frac{2}{\theta}\big(E^{\circ}_{11'}(\theta)\big)^2 - \theta\,\Phi_{22} = 0. \tag{2.31}$$

This equation can be rewritten as a linear ODE for θ^{-1} which can be solved on the generators of $\mathcal{J}$ with $\theta > 0$. Using the initial data θ, $E^{\circ}_{11'}(\theta)$ on $\mathcal{C}$ determined above, we solve for θ to obtain

$$\Phi^{\star}_{22} = 0, \quad \Gamma^{\bullet}_{01'11} = 0 \quad \text{on} \quad \mathcal{J}. \tag{2.32}$$

Here the second equation is a consequence of the first, the field equations, and (2.30). We assume in the following (2.28). We observe that the induced metric on the sections $\{u^{\bullet} = const.\}$ is given as a consequence everywhere on $\mathcal{J}$ by the S^2-standard metric.

Once θ and $E^{\bullet}_{11'}$ have been fixed on $\mathcal{J}$, the vector field $E^{\bullet}_{01'}$ (whence $E^{\bullet}_{10'}$) tangent to $\{u^{\bullet} = const.\}$ is determined up to rotations. We choose some smooth field $E^{\bullet}_{01'}$ on $\mathcal{J}$, solve the equation

$$E^{\bullet}_{11'}(c) = -i\, E^{\bullet\,\alpha}_{10'}\, E^{\bullet\,\beta}_{11'}\, \nabla^{\star}_{\beta}\, E^{\bullet}_{01'\alpha} \tag{2.33}$$

for the function c with initial value $c = 0$ on $\mathcal{C}$, and replace $E^{\bullet}_{01'}$ by $e^{i\,c}\, E^{\bullet}_{01'}$ to achieve

$$\Gamma^{\bullet}_{11'01} = 0 \quad \text{on} \quad \mathcal{J}. \tag{2.34}$$

Observing the simplifications above, we contract the analogue of (2.26) for $g^{\star}$ with $E^{\bullet\,\alpha}_{01'}E^{\bullet\,\beta}_{10'}$ to conclude that $\nabla^{\star}_{\alpha}\nabla^{\star\,\alpha}\Theta^{\star} = 0$ on $\mathcal{J}$. A further contraction with $E^{\bullet\,\alpha}_{00'}E^{\bullet\,\beta}_{11'}$ gives

$$E^{\bullet}_{11'}\big(E^{\bullet}_{00'}(\Theta^{\star})\big) = 0, \quad \text{i.e.} \quad E^{\bullet}_{00'}(\Theta^{\star}) = const. \quad \text{on} \quad \mathcal{J}, \tag{2.35}$$

while a contraction with $E^{\bullet\,\alpha}_{00'}E^{\bullet\,\beta}_{01'}$ yields now $E^{\bullet}_{01'}\big(E^{\bullet}_{00'}(\Theta^{\star})\big) = \Gamma^{\bullet}_{11'00}E^{\bullet}_{00'}(\Theta^{\star})$, which implies

$$\Gamma^{\bullet}_{11'00} = 0 \quad \text{on} \quad \mathcal{J}. \tag{2.36}$$

To fix also $d\,\theta$ on $\mathcal{J}$, we use the conformal transformation law for the Ricci scalar, i.e.

$$R[g^{\star}] = \frac{1}{\theta^2}R[g] + \frac{12}{\theta^2}\nabla^{\star}_{\alpha}\,\theta\nabla^{\star\,\alpha}\,\theta - \frac{6}{\theta}\nabla^{\star}_{\alpha}\nabla^{\star\,\alpha}\,\theta. \tag{2.37}$$

If we require that $R[g^{\star}] = 0$ along $\mathcal{J}$, this equation takes on the generators of the null hypersurface $\mathcal{J}$ the form

$$E^{\bullet}_{11'}\big(E^{\bullet}_{00'}(\theta)\big) - \frac{2}{\theta}E^{\bullet}_{11'}(\theta)E^{\bullet}_{00'}(\theta) = F^{\star}, \tag{2.38}$$

of a linear ODE for the unknown $E^{\bullet}_{00'}(\theta)$, where the right hand side

$$F^{\star} = \mathfrak{Re}\left\{E^{\bullet}_{01'}(E^{\bullet}_{10'}(\theta)) - 2\,\Gamma^{\bullet}_{01'01}E^{\bullet}_{10'}(\theta) - \frac{2}{\theta}E^{\bullet}_{01'}(\theta)E^{\bullet}_{10'}(\theta) + \frac{1}{12\,\theta}R[g]\right\}$$

is given in terms of quantities which have been determined already on $\mathcal{J}$. Using the initial value $E^{\bullet}_{00'}(\theta) = p^{-1}\theta\,\Gamma^{\circ}_{10'00}|_{\mathcal{C}}$, fixed on $\mathcal{C}$ by (2.30), we can integrate the equation to achieve

$$R[g^{\star}] = 0, \quad \Gamma^{\bullet}_{10'00} = 0 \quad \text{on} \quad \mathcal{J}, \tag{2.39}$$

where the second equation follows again from our previous results and the field equations.

We do not require conditions of higher order on the conformal gauge. Assuming a conformal gauge as described here, we shall refer to an adapted frame $\{E^{\bullet}_{aa'}\}$ satisfying the conditions above as to an *NP-frame*, and to a normalized spin frame $\varepsilon^{\bullet}_{a}{}^{A} \equiv \{o^{\bullet\,A}, \iota^{\bullet\,A}\}$ which implies a NP-frame as to a *NP-spin-frame*.

We extend the coordinates x^3, x^4 to $\mathcal{J}$ such that they are constant on the null generators of $\mathcal{J}$. As described above, we define null hypersurfaces $\{u^{\bullet} = const.\}$ transverse to $\mathcal{J}$ and we denote by $r^{\bullet}$ the affine parameter on the null generators of these hypersurfaces which satisfies $E^{\bullet}_{00'}(r^{\bullet}) = 1$ and, on $\mathcal{J}$, $r^{\bullet} = 0$. The coordinates x^3, x^4 are extended such that they are constant on the null generators of $\{u^{\bullet} = const.\}$. Thus we get a *Bondi-type system* $(u^{\bullet}, r^{\bullet}, x^3, x^4)$ in some neighborhood of null infinity. Occasionally we shall change from the coordinates ϑ, φ, to a complex stereo-graphical coordinate given by $\zeta = e^{i\varphi}ctg\frac{\vartheta}{2}$. We write the volume element and the volume form alternatively

$$ds^2 = -(d\vartheta^2 + sin^2\vartheta\, d\varphi^2) = -P(\zeta)^{-2}d\zeta\, d\bar{\zeta}, \qquad \epsilon = \sin\vartheta\, d\vartheta \wedge d\varphi = [2P(\zeta)]^{-2}d\zeta \wedge d\bar{\zeta},$$

where we set $P(\zeta) = \frac{1}{2}(1 + \zeta\bar{\zeta})$. We shall refer to the conditions on the conformal scaling, the frame field, and the coordinates as to the *NP-gauge*.

2.3 Relating the NP-gauge to the F-gauge

While the NP-gauge is hinged on null infinity, the F-gauge is based on a Cauchy hypersurface and these gauge conditions are in general completely different. In the following we will study the transformation which relates one to the other. It is important for this that the conformal factor Θ, whence $\mathcal{J}$, is known explicitly in the F-gauge.

The vector fields $\{c_{aa'}\}$ tangent to the 5-dimensional bundle space M'_a are not directly related to the NP-gauge on the subset $M_a \setminus I$ of M. Let $S^2 \supset U \ni p \xrightarrow{s} s(p) \in SU(2)$ be a smooth local section, defined on some open subset U of S^2, of the Hopf fibration $SU(2) \to SU(2)/U(1) \simeq S^2$. It induces a smooth section $U \times \mathbb{R} \times \mathbb{R} \ni (p, \tau, \rho) \xrightarrow{S} (s(p), \tau, \rho) \in M'_a$. We denote the image of S by M^*_a. The vector fields tangent to $s(U)$ which have projection identical to that of $X_{\pm}$ are of the form $X_{\pm} + a_{\pm}\,X$ with some smooth functions $a_{\pm}$ on $s(U)$, satisfying $a_- = -\bar{a}_+$. Because of (2.2) $a_{\pm}$ cannot vanish on open subsets of $s(U)$. Consequently, the tangent vector fields $c^*_{aa'}$ of M^*_a satisfying $\pi_*(c^*_{aa'}) = \pi_*(c_{aa'})$ are given on M^*_a by

$$c^*_{aa'} = c_{aa'} + (a_+\,c^+{}_{aa'} + a_-\,c^-{}_{aa'})\,X,$$

with functions $a_{\pm}$ which are independent of τ and ρ. The connection coefficients defined on M^*_a by the connection form $\omega^b{}_c$ and the vector fields $c^*_{aa'}$ are given by

$$\Gamma^*_{aa'}{}^b{}_c = \Gamma_{aa'}{}^b{}_c + (a_+\,c^+{}_{aa'} + a_-\,c^-{}_{aa'})\,(\epsilon_0{}^b\,\epsilon_c{}^0 - \epsilon_1{}^b\,\epsilon_c{}^1).$$

In the remaining part of this section we shall work on $\pi(M'_a)$ and denote the projection of the vector fields $c^*_{aa'}$, which define a smooth orthonormal frame field on $\pi(M^*_a \setminus I')$, and the pull-back of $\Gamma^*_{aa'}{}^b{}_c$ by S again by $c^*_{aa'}$ and $\Gamma^*_{aa'}{}^b{}_c$. Similarly, the projections of $\mathcal{J}' \cap M^*_a$ and $I'^+ \cap M^*_a$ will be denoted by $\mathcal{J}$ and I^+.

The frame field $\{c^*_{aa'}\}$, which is in general not adapted to null infinity, will now be related close to I^+ to an adapted frame $\{E^\circ_{aa'}\}$. On $\mathcal{J}$ the vector field $E^\circ_{11'}$ must be of the form

$$E^{\circ\,\alpha}_{11'} = f\nabla^\alpha\Theta, \tag{2.40}$$

where ∇ and Θ denote the Levi-Civita connection and the conformal factor associated with the F-gauge. The requirement $0 = E^{\circ\,\beta}_{11'}\nabla_\beta E^{\circ\,\alpha}_{11'} = f\,\nabla^\beta\,\Theta\,\nabla_\beta\,f\,\nabla^\alpha\Theta + f^2\nabla_\beta\,(\frac{1}{2}\,\nabla_\alpha\Theta\,\nabla^\alpha\Theta)$ that $E^{\circ\,\alpha}_{11'}$ be parallelly propagated, gives after contraction with a vector field Z transverse to $\mathcal{J}$ the ODE

$$\nabla^\alpha\Theta\nabla_\alpha(\log f) = -\frac{Z(\frac{1}{2}\nabla_\beta\Theta\nabla^\beta\Theta)}{Z(\Theta)} \tag{2.41}$$

for f on the generators of $\mathcal{J}$. To fix f, we set $f = f_0 = const. > 0$ on some section $\mathcal{C}$ of $\mathcal{J}$. The function u° satisfying $E^{\circ\,\alpha}_{11'}(u^\circ) = 1$ on $\mathcal{J}$ and $u^\circ = u^\circ_*$ on $\mathcal{C}$ can be now be determined.

Let $\lambda^a{}_b \in SL(2,C)$ satisfy

$$E^\circ_{aa'} = \lambda^b{}_a\bar{\lambda}^{b'}{}_{a'}\,c^*_{bb'}. \tag{2.42}$$

Rewriting (2.40) in the form $E^\circ_{11'} = f\,c^*_{bb'}(\Theta)\epsilon^{ab}\bar{\epsilon}^{a'b'}c^*_{aa'}$, we find the relations

$$\lambda^0{}_1\bar{\lambda}^{0'}{}_{1'} = f\,c^*_{11'}(\Theta), \quad \lambda^0{}_1\bar{\lambda}^{1'}{}_{1'} = -f\,c^*_{10'}(\Theta), \quad \lambda^1{}_1\bar{\lambda}^{1'}{}_{1'} = f\,c^*_{00'}(\Theta). \tag{2.43}$$

From (2.42) we obtain $\lambda^0{}_1 E^\circ_{01'} = \lambda^0{}_0 E^\circ_{11'} - \bar{\lambda}^{0'}{}_{1'}c^*_{10'} - \bar{\lambda}^{1'}{}_{1'}c^*_{11'}$. Applying this to the function u°, we get

$$\lambda^0{}_0 = \bar{\lambda}^{0'}{}_{1'}c^*_{10'}(u^\circ) + \bar{\lambda}^{1'}{}_{1'}c^*_{11'}(u^\circ). \tag{2.44}$$

Together with the condition $\det(\lambda^a{}_b) = 1$ the relations (2.43), (2.44) allow us to determine the matrix elements $\lambda^a{}_b$ on $\mathcal{J}$ up to replacements $\lambda^a{}_b \to \lambda^a{}_b\,\eta^b{}_c$ with $(\eta^a{}_b) = diag(e^{i\,\alpha}, e^{-i\,\alpha}) \in U(1)$. After making here an arbitrary choice, the adapted frame $\{E^\circ_{aa'}\}$ is determined uniquely near $\mathcal{J}$.

To determine an NP-frame $\{E^\bullet_{aa'}\}$ near $\mathcal{J}$, we need to find an appropriate rescaling (2.27) and a scaling factor p. We set

$$c^\star_{aa'} = \theta^{-1}\,c^*_{aa'}, \qquad E^\bullet_{aa'} = \Lambda^b{}_a\bar{\Lambda}^{b'}{}_{a'}\,c^\star_{bb'} \tag{2.45}$$

with $\Lambda^a{}_b \in SL(2,C)$. Assuming (2.28), we have $E^{\bullet\,\alpha}_{11'} = f^\star\,\nabla^{\star\,\alpha}\Theta^\star$ with

$$f^\star = \frac{f\,p}{\theta} \quad \text{and} \quad E^\bullet_{00'}(\Theta^\star) = \frac{1}{f^\star} \quad \text{on} \quad \mathcal{J}. \tag{2.46}$$

We choose now θ, $d\,\theta$, and coordinates x^3, x^4 such that the induced metric on $\mathcal{C}$ is given by the S^2-standard metric and, with p chosen such that $p = \theta$ on $\mathcal{C}$, conditions (2.30) are satisfied with $E^\bullet_{00'}(\Theta^\star) = f_0^{-1}$.

Following the procedure of the previous section, we can determine the conformal factor θ on $\mathcal{J}$ such that (2.32) is satisfied. The transformation $\Lambda^a{}_b$ can be determined in the same way as $\lambda^a{}_b$. Imposing condition (2.34), we determine $\Lambda^a{}_b$ up to $U(1)-$transformations on $\mathcal{C}$. Conditions (2.35), (2.36) will now be satisfied as well and we can determine $d\,\theta$ on $\mathcal{J}$ such that (2.39) holds. Extending the tetrad to a neighborhood of $\mathcal{J}$ such that it is parallelly propagated in the direction of $E^\bullet_{00'}$, we get the desired NP-frame.

In our later calculations we will need the quantities $E^\bullet_{00'}(\Lambda^a{}_b)$. Using our gauge condition $\Gamma^\bullet_{00'ab} = 0$ and the transformation laws for the connection coefficients,

$$\Gamma^\star_{aa'bc} = \frac{1}{\theta}\left\{\Gamma^*_{aa'bc} + \epsilon_{a(b}c^*_{c)a'}(\log\theta)\right\},$$

$$E^\bullet_{aa'}(\Lambda^b{}_c) = -\Lambda^f{}_a\,\bar{\Lambda}^{f'}{}_{a'}\,\Lambda^h{}_c\,\Gamma^\star_{ff'}{}^b{}_h + \Lambda^b{}_d\,\Gamma^\bullet_{aa'}{}^d{}_c,$$

where $\Gamma^{\star}_{aa'bc}$ denotes the connection coefficients with respect to $\nabla^{\star}$ and $\{c^{\star}_{aa'}\}$, we find

$$E^{\bullet}_{00'}(\Lambda^b{}_c) = -\Lambda^f{}_0\,\bar{\Lambda}^{f'}{}_{0'}\,\Lambda^h{}_c\,\Gamma^{\star}_{ff'}{}^b{}_h. \tag{2.47}$$

In the considerations above we had to fix various quantities by prescribing data on the section $\mathcal{C}$. When we shall determine later the expansion of a NP-frame near I^+, it will be natural to try pushing $\mathcal{C}$ to I^+. A priori it is not clear, however, whether this can be done in a continuous way. We shall see, that for certain quantities the limits to I^+ do exist, while others quantities can only described in terms of their growth behavior near I^+.

3 The NP-constants

In 1965 Newman and Penrose discovered certain non-trivial quantities, defined by certain integrals over a 2-dimensional cross-section of $\mathcal{J}^+$, which are absolutely conserved in the sense that their values do not depend on the choice of the section (cf. [12, 14]). The interpretation of these ten real *NP-constants* is still open. In the case where the space-time admits a smooth conformal extension containing a point i^+ ("future time-like infinity") whose past light cone represents $\mathcal{J}^+$, these constants are essentially given by the five complex components of the rescaled conformal Weyl spinor (cf. [14, 7]). However, these quantities do not allow us a simple interpretation either. More interesting is the case of stationary vacuum space-times. In this case the constants have been calculated and have been given in the form $(mass) \times (quadrupole\ moment) - (dipole\ moment)^2$ (cf. [14, 18]).

If the evolution of the field in time is not given explicitly as in the presence of a time-like Killing vector field, there appears to be no obvious way to calculate the NP-constants. It turns out, however, that under suitable assumptions on the asymptotic behavior of the field near space-like infinity the constants can be calculated by integrating the transport equations on I' to a sufficiently high order. In the following we shall derive a formula for the constants in terms of quantities which can be determined by solving the transport equations.

To explain the original formula (cf. [14]), which is given in the Bondi-Sachs-Newman-Penrose framework, let $(u, r, \vartheta, \varphi)$ denote Bondi-coordinates on the physical space-time, where r denotes an affine parameter along the generators of the null hypersurfaces $\{u = const.\}$ and the generators are labeled by the standard coordinates (ϑ, φ) on the two-sphere. The null frame $\{\tilde{E}_{aa'}\}$ as well as a corresponding spinor dyad $\{\tilde{o}^A, \tilde{\iota}^A\}$, both defined on the physical space-time, are normalized with respect to the physical metric $\tilde{g}$. They are adapted to the Bondi-coordinates such that $\tilde{E}_{00'} = \partial_r$.

We assume that the conformal space-time with metric $g^{\star} := r^{-2}\tilde{g}$ admits a smooth extension as $r \to \infty$ to a smooth Lorentz space with boundary $\mathcal{J}^+ = \{r^{\bullet} = 0\}$ and that the functions $u^{\bullet} := u$, $r^{\bullet} := r^{-1}$, ϑ, and φ extend such as to define a smooth system of Bondi-type coordinates near $\mathcal{J}^+$. Furthermore, we assume that the frame $\{E^{\bullet}_{aa'}\}$ and the spinor dyad $\{o^{\bullet\,A}, \iota^{\bullet\,A}\}$, defined by

$$\begin{aligned} E^{\bullet}_{aa'} &= r^{2-a-a'}\,\tilde{E}_{aa'} \\ o^{\bullet\,A} &= r\,\tilde{o}^A, \quad \iota^{\bullet\,A} = \tilde{\iota}^A, \end{aligned} \tag{3.1}$$

such that they are normalized with respect to $g^{\star}$, extend to smooth frame resp. dyad near $\mathcal{J}^+$. The results of Newman and Unti (cf. [15]) then imply that $\{E^{\bullet}_{aa'}\}$ defines in fact a NP-frame.

Under our assumptions the component $\psi_0 = \psi_{ABCD}\tilde{o}^A\tilde{o}^B\tilde{o}^C\tilde{o}^D$ of the conformal Weyl spinor has an expansion $\psi_0 = \psi_0^0 r^{-5} + \psi_0^1 r^{-6} + O(r^{-7})$ with coefficients ψ_0^p which are independent of r. In terms of the physical space-time the NP-constants are given with this notation by the integrals

$$G_m = \oint {}_2\bar{Y}_{2,m}\psi_0^1 \sin\vartheta\, d\vartheta\, d\varphi, \tag{3.2}$$

which are calculated for fixed value of u. The functions ${}_2Y_{2,m}$, $m = -2, -1, 0, 1, 2$, denote spin-2 spherical harmonics (cf. [9]) which are obtained from the standard spherical harmonics by

$${}_2Y_{2,m} = \frac{1}{2\sqrt{6}} E^{\bullet\,\alpha}_{01'} E^{\bullet\,\beta}_{01'} \delta_\alpha \delta_\beta Y_{2,m} = \frac{1}{2\sqrt{6}} \eth^2 Y_{2,m}. \tag{3.3}$$

Here δ and $\eth$ denote the standard covariant differential operator on the unit 2-sphere and the "edth"-operator, respectively. In evaluating (3.2), it will be important that the operator $\eth$ is defined with respect to the complex null vector field $E^{\bullet}_{01'}$ (cf. [10]).

We reexpress the constants in terms of the fields $g^{\star}$, $E^{\bullet}_{aa'}$, $o^{\bullet\,A}$, $\iota^{\bullet\,A}$ satisfying the NP-gauge, in particular (2.39). Using the component $\phi_0 = r\,\psi_{ABCD}o^{\bullet\,A}o^{\bullet\,B}o^{\bullet\,C}o^{\bullet\,D}$ of the rescaled conformal Weyl spinor, and performing the obvious lift to M', we obtain for the NP-constants the formula

$$G_m = -\frac{1}{2\,\pi}\oint {}_2\bar{Y}_{2,m}\,E^{\bullet}_{00'}(\phi_0)\,d\mathcal{S}\,d\,\alpha. \tag{3.4}$$

Here $d\mathcal{S} = \sin\vartheta\,d\vartheta\,d\varphi$ denotes the surface element on the cross-section $\{r^{\bullet},\,u^{\bullet} = const.\} \subset \mathcal{J}^+$ and α denotes a parameter on the fibers of the principal fiber bundle $M' \to M$. The second integration can be performed without changing the result because the integrand is independent of the variable α.

The values of these integrals are independent of the value of the constant defining the cross-section as well as of the choice of the Bondi-coordinate $u^{\bullet}$ itself. Thus they are invariant under supertranslations (cf. [14]).

We shall determine the NP-constants by integrating the transport equations on I'. Since these equations and their unknowns are given in the F-gauge, we express (3.4) in this gauge. Using (2.45), we obtain in the notation of the previous chapter

$$\begin{aligned} G_m = -\frac{1}{2\,\pi}\oint {}_2\bar{Y}_{2,m}\frac{1}{\theta^4}\Big\{&\Lambda^b{}_0\Lambda^c{}_0\Lambda^d{}_0\Lambda^e{}_0\big[\Lambda^a{}_0\bar{\Lambda}^{a'}{}_{0'}\,c^*_{aa'}(\phi_{bcde}) - 3\,\phi_{bcde}E^{\bullet}_{00'}(\theta)\big] \\ &+ 4\,\theta\,\Lambda^b{}_0\Lambda^c{}_0\Lambda^d{}_0E^{\bullet}_{00'}(\Lambda^e{}_0)\,\phi_{bcde}\Big\}\,d\mathcal{S}\,d\,\alpha. \end{aligned} \tag{3.5}$$

This is the expression for the NP-constants which will be used in the calculations of section [4.3].

4 Time symmetric space-times

In this section we will use the assumptions of the regular finite initial value problem near space-like infinity and thus restrict our considerations to time symmetric space-times. We begin by solving the third order transport equations on I'. This calculation is of interest for two quite different reasons. First of all, it will give us a first insight into the potential source of singular behavior of the quantities u^p pointed out in section [2.1.3]. Further, besides giving information on this question of principle, the calculation will allow us to analyze the relation between the NP-constants and the initial data for asymptotically flat solutions. Under our assumptions, we will be able to evaluate the integral (3.5) in terms of quantities derived from the initial data.

4.1 Solving the third-order transport equation

The solutions u^p of equations (2.20) have been given in [4] for $p \le 2$. Since they will be used in the following calculations we reproduce them here, in a notation, though, which is more convenient for a systematic discussion of the higher order expansion coefficients. We also take the opportunity to correct a misprint in [4].

The solution u^0 of the transport equations (2.20) has the form

$$\begin{aligned} &(c^0{}_{ab})^0 = -\tau x_{ab}, \quad (c^1{}_{ab})^0 = 0, \quad (c^+{}_{ab})^0 = z_{ab}, \quad (c^-{}_{ab})^0 = y_{ab}, \quad \xi^0_{abcd} = 0, \\ &\chi^0_{(ab)cd} = 0, \quad f^0_{ab} = x_{ab}, \quad (\Theta_g{}^g{}_{ab})^0 = 0, \quad \Theta^0_{(ab)cd} = 0, \quad \phi^0_{abcd} = -6\,m\,\varepsilon^2_{abcd}, \end{aligned} \tag{4.1}$$

where $m = m_{ADM}$ denotes the ADM-mass of the initial data set. The spinors appearing on the right hand side of these and the following formulae are listed in (A.10) of appendix [A.3]. The solution u^1 is given by

$$
\begin{aligned}
&({c^0}_{ab})^1 = c^{01}(\tau)\,x_{ab}, \quad ({c^1}_{ab})^1 = x_{ab}, \quad ({c^+}_{ab})^1 = c^{\pm 1}(\tau) z_{ab},\\
&({c^-}_{ab})^1 = c^{\pm 1}(\tau) y_{ab}, \quad \xi^1_{abcd} = S^1(\tau)(\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}), \quad \chi^1_{(ab)cd} = K^1(\tau)\varepsilon^2_{abcd},\\
&f^1_{ab} = F^1(\tau)x_{ab}, \quad ({\Theta_g}^g{}_{ab})^1 = t^1(\tau)x_{ab}, \quad {\Theta^1}_{(ab)cd} = T^1(\tau)\varepsilon^2_{abcd},\\
&\phi^1_{abcd} = \phi^1_1(\tau)X_+W_1\varepsilon^1_{abcd} + [\phi^1_2(\tau) + \phi^1_3(\tau)W_1]\varepsilon^2_{abcd} - \phi^1_1(-\tau)X_-W_1\varepsilon^3_{abcd},
\end{aligned}
\tag{4.2}
$$

while u^2 takes the form

$$
\begin{aligned}
({c^0}_{ab})^2 &= [c^{02}_1(\tau) + c^{02}_2(\tau)W_1]x_{ab} + c^{02}_3(\tau)[X_-W_1y_{ab} + X_+W_1z_{ab}],\\
({c^1}_{ab})^2 &= c^{12}(\tau)x_{ab},\\
({c^+}_{ab})^2 &= [c^{\pm 2}_1(\tau) + c^{\pm 2}_2(\tau)W_1]z_{ab} + c^{\pm 2}_3(\tau)X_-W_1x_{ab},\\
({c^-}_{ab})^2 &= [c^{\pm 2}_1(\tau) + c^{\pm 2}_2(\tau)W_1]y_{ab} + c^{\pm 2}_3(\tau)X_+W_1x_{ab},\\
\xi^2_{abcd} &= [S^2_1(\tau) + S^2_2(\tau)W_1](\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}) + S^2_3(\tau)(\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac})X_-W_1\\
&\quad + S^2_3(\tau)(\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac})X_+W_1 + S^2_4(\tau)(\varepsilon^1_{abcd}X_+W_1 + \varepsilon^3_{abcd}X_-W_1),\\
\chi^2_{(ab)cd} &= [K^2_1(\tau) + K^2_2(\tau)W_1]\varepsilon^2_{abcd} + K^2_3(\tau)h_{abcd} + K^2_4(\tau)(\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac})X_-W_1\\
&\quad - K^2_4(\tau)(\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac})X_+W_1 + K^2_5(\tau)(\varepsilon^1_{abcd}X_+W_1 - \varepsilon^3_{abcd}X_-W_1),\\
f^2_{ab} &= [F^2_1(\tau) + F^2_2(\tau)W_1]x_{ab} + F^2_3(\tau)(X_-W_1y_{ab} + X_+W_1z_{ab}),\\
({\Theta_g}^g{}_{ab})^2 &= [t^2_1(\tau) + t^2_2(\tau)W_1]x_{ab} + t^2_3(\tau)(X_-W_1y_{ab} + X_+W_1z_{ab}),\\
\Theta^2_{(ab)cd} &= [T^2_1(\tau) + T^2_2(\tau)W_1]\varepsilon^2_{abcd} + T^2_3(\tau)h_{abcd} + T^2_4(\tau)(\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac})X_-W_1\\
&\quad - T^2_4(\tau)(\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac})X_+W_1 + T^2_5(\tau)(\varepsilon^1_{abcd}X_+W_1 - \varepsilon^3_{abcd}X_-W_1),\\
\phi^2_{abcd} &= \phi^2_1(\tau)X_+X_+W_2\varepsilon^0_{abcd} + [\phi^2_2(\tau)X_+W_1 + \phi^2_3(\tau)X_+W_2]\varepsilon^1_{abcd}\\
&\quad + [\phi^2_4(\tau) + \phi^2_5(\tau)W_1 + \phi^2_6(\tau)W_2]\varepsilon^2_{abcd} - [\phi^2_2(-\tau)X_-W_1 + \phi^2_3(-\tau)X_-W_2]\varepsilon^3_{abcd}\\
&\quad + \phi^2_1(-\tau)X_-X_-W_2\varepsilon^4_{abcd}.
\end{aligned}
\tag{4.3}
$$

The τ-dependent functions in these expressions are polynomials which are given in appendix [A.3].

The calculation of u^3 is facilitated by the following properties of the transport equations (2.20). For $p \geq 1$ they are of the form

$$\partial_\tau v^p = L_p\, v^p + l_p, \quad B^\alpha \partial_\alpha \phi^p = M_p \phi^p, \tag{4.4}$$

where, using the notation (2.8), we set $v^p = (\partial^p_\rho v)|_{I'}$, $\phi^p = (\partial^p_\rho \phi)|_{I'}$ and denote by L_p and l_p a matrix- resp. vector-valued function of the quantities $u^0, \ldots, u^{p-1}$, while M_p denotes a matrix-valued function which depends on the variables $u^0, \ldots, u^{p-1}$, v^p. The matrices B^α neither depend on p nor on the initial data. Thus, given the quantities u^q, $q \leq p-1$, we can integrate the first of equations (4.4), which is an ODE. To integrate the second equation, we expand the quantities u^p in terms of the functions ${T_m}^k{}_j$ given in (2.21) and use (2.22) to reduce the integration to that of a system of ODE's.

To determine the initial data for u^3 on ${I'}^0$, we have to expand the unknowns (2.18) in terms of ρ. Instead of prescribing the conformal metric h on the initial slice, which represents the free datum, we shall prescribe, in a fashion consistent with the 3-dimensional Bianchi identities, certain curvature quantities and use the 3-dimensional structure equations and the Yamabe equation to determine the remaining quantities.

The conformal factor, which appears in the expressions (2.18), is given in (2.5) in terms of the functions U and W. The function U, which is determined locally by h near space-like infinity, is given, by a procedure explained in [4], in the form

$$U = \sum_{p=0}^{\infty} U_p\, \rho^{2p}, \tag{4.5}$$

with ρ-dependent coefficients U_p. As shown in [4], the Taylor expansion of U in terms of ρ has in our gauge the form

$$U = 1 + \sum_{k=4}^{\infty} \frac{1}{k!} \hat{U}_k \rho^k. \tag{4.6}$$

For our calculations we shall need the coefficient $\hat{U}_4$, which will be determined later in this chapter.

The function W, which contains global information on the free initial data, is determined by solving the Yamabe equation on the initial hypersurface. We shall consider here a larger class of functions which are subject to the Yamabe equation only in a small neighborhood of space-like infinity. The coefficients in the Taylor expansion $W = W_0 + W_1\,\rho + \frac{1}{2} W_2\,\rho^2 + \frac{1}{3!} W_3\,\rho^3 + O(\rho^4)$ have expansion (cf. [4])

$$W_i = \sum_{m=0}^{2i} \sum_{k=0}^{m} W_{i;m,k}\, T_m{}^k{}_{\frac{m}{2}}.$$

They are restricted by the requirement that the Yamabe equation $(h^{\alpha\beta} D_\alpha D_\beta - \frac{1}{8} r_h)[W] = 0$ holds near $\{\rho = 0\}$, which implies the simplification

$$W_i = \sum_{k=0}^{2} W_{i;2i,k}\, T_{2i}{}^k{}_i, \qquad i \leq 3. \tag{4.7}$$

We get for the conformal factor and the trace-free part of its second covariant derivative

$$\begin{aligned}
\Omega &= \rho^2 - m\,\rho^3 + \left[\tfrac{3}{4}\, m^2 - 2\, W_1\right] \rho^4 + \left[-\tfrac{1}{2} m^3 + 3\, m\, W_1 - W_2\right] \rho^5 \\
&\quad + \left[\tfrac{5}{16}\, m^4 - 3\, m^2 W_1 + 3 W_1^2 + \tfrac{3}{2}\, m\, W_2 - \tfrac{1}{3} W_3 - \tfrac{1}{12} \hat{U}_4\right] \rho^6 + O(\rho^7), \\
D_{(ab} D_{cd)} \Omega &= \left[-6\, m\, \varepsilon^2_{abcd}\right] \rho + \left[(12\, m^2 - 36\, W_1) \varepsilon^2_{abcd} - 12\, (\varepsilon^1_{abcd} X_+ - \varepsilon^3_{abcd} X_-) W_1\right] \rho^2 \\
&\quad + \left[(-15\, m^3 + 96\, m\, W_1 - 36\, W_2) \varepsilon^2_{abcd} + (\varepsilon^1_{abcd} X_+ - \varepsilon^3_{abcd} X_-)(24\, m\, W_1 - 8\, W_2)\right. \\
&\quad \left. - \tfrac{1}{2} (\varepsilon^0_{abcd} X_+ X_+ + \varepsilon^4_{abcd} X_- X_-) W_2\right] \rho^3 + \left[(156\, W_1^2 - 150\, m^2 W_1 + 15\, m^4 + 81\, m\, W_2\right. \\
&\quad - 20\, W_3 - 4\, \hat{U}_4 + \tfrac{1}{12} X_+ X_- \hat{U}_4 - 6\, X_+ W_1 X_- W_1) \varepsilon^2_{abcd} \\
&\quad + (\varepsilon^1_{abcd} X_+ - \varepsilon^3_{abcd} X_-)(30\, W_1^2 - 30\, m^2 W_1 + 15\, m\, W_2 - \tfrac{10}{3} W_3 - \tfrac{5}{6} \hat{U}_4) \\
&\quad \left. + \tfrac{1}{2} (\varepsilon^0_{abcd} X_+ X_+ + \varepsilon^4_{abcd} X_- X_-)(3\, W_1^2 + \tfrac{3}{2} m\, W_2 - \tfrac{1}{3} W_3 - \tfrac{1}{12} \hat{U}_4) - \tfrac{2}{3} x_{e(a} \gamma^3{}_{bc}{}^e{}_{d)}\right] \rho^4 \\
&\quad + O(\rho^5).
\end{aligned} \tag{4.8}$$

From this we obtain as initial data for u^3 on I'^0

$$\begin{aligned}
&(c^0{}_{ab})^3 = 0, \quad (c^1{}_{ab})^3 = 0, \quad (c^+{}_{ab})^3 = 0, \quad (c^-{}_{ab})^3 = 0, \\
&\xi^3_{abcd} = 0, \quad \chi^3_{(ab)cd} = 0, \quad f^3_{ab} = 0, \quad (\Theta_g{}^g{}_{ab})^3 = 0, \\
&\Theta^3_{(ab)cd} = 3\, X_+ X_+ W_2 \varepsilon^0_{abcd} + (-72\, m\, X_+ W_1 + 48\, X_+ W_2) \varepsilon^1_{abcd} \\
&\qquad + (27\, m^3 - 288\, m\, W_1 + 216\, W_2) \varepsilon^2_{abcd} \\
&\qquad + (72\, m\, X_- W_1 - 48\, X_- W_2) \varepsilon^3_{abcd} + 3\, X_- X_- W_2 \varepsilon^4_{abcd}, \\
&\phi^3_{abcd} = (\varepsilon^0_{abcd} X_+ X_+ + \varepsilon^4_{abcd} X_- X_-)(9 W_1^2 - \tfrac{3}{2} m W_2 - W_3 - \tfrac{1}{4} \hat{U}_4) \\
&\qquad + 4 (\varepsilon^1_{abcd} X_+ - \varepsilon^3_{abcd} X_-)(9 W_1^2 - \tfrac{3}{2} m W_2 - 5 W_3 - \tfrac{5}{4} \hat{U}_4) \\
&\qquad + 6\, \varepsilon^2_{abcd} (12\, W_1^2 - 3 m\, W_2 - 20 W_3 - 4 \hat{U}_4 + \tfrac{1}{12} X_+ X_- \hat{U}_4 - 6 X_+ W_1 X_- W_1) \\
&\qquad - 4 x_{e(a} \gamma^3{}_{bc}{}^e{}_{d)} + 3 s^2_{abcd},
\end{aligned} \tag{4.9}$$

where $\gamma_{abcd} = (2\,\rho)^{-1}(\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}) + \check{\gamma}_{abcd}$ denote the connection coefficients on C'.

We determine now how the functions $\hat{U}_4$, γ^3_{abcd}, and s^2_{abcd} are related to the free data on the initial hypersurface C'. As shown in [4], the structure equations on C', which relate the connection coefficients to the curvature, read

$$\frac{1}{\sqrt{2}}\left\{\partial_\rho\check{\gamma}_{00ab} + \frac{\sqrt{2}}{\rho}\left[\check{\gamma}_{0000}\,z_{ab} - \check{\gamma}_{0011}\,y_{ab} + \frac{1}{\sqrt{2}}\check{\gamma}_{00ab}\right]\right\} = \check{\gamma}_{0000}\check{\gamma}_{11ab} - \check{\gamma}_{0011}\check{\gamma}_{00ab} - \frac{1}{2}s_{ab00} - \frac{1}{6\sqrt{2}}\,r\,y_{ab},$$
$$\frac{1}{\sqrt{2}}\left\{\partial_\rho\check{\gamma}_{11ab} + \frac{\sqrt{2}}{\rho}\left[\check{\gamma}_{1100}\,y_{ab} - \check{\gamma}_{1111}\,y_{ab} + \frac{1}{\sqrt{2}}\check{\gamma}_{11ab}\right]\right\} = \check{\gamma}_{1100}\check{\gamma}_{11ab} - \check{\gamma}_{1111}\check{\gamma}_{00ab} + \frac{1}{2}s_{ab11} - \frac{1}{6\sqrt{2}}\,r\,z_{ab},$$

and the components of $\check{\gamma}_{abcd}$ have Taylor expansions

$$\check{\gamma}_{01ab} = 0, \quad \check{\gamma}_{00ab} = \frac{1}{3!}\check{\gamma}^3_{00ab}\,\rho^3 + O(\rho^4), \quad \check{\gamma}_{11ab} = \frac{1}{3!}\check{\gamma}^3_{11ab}\,\rho^3 + O(\rho^4).$$

From this we get

$$\check{\gamma}^3_{0001} = -\frac{3}{4\sqrt{2}}s^2_{0001}, \qquad \check{\gamma}^3_{1101} = \frac{3}{4\sqrt{2}}s^2_{0111}, \qquad \check{\gamma}^3_{0000} = -\frac{3}{5\sqrt{2}}s^2_{0000},$$
$$\check{\gamma}^3_{1100} = \frac{3}{5\sqrt{2}}s^2_{0011} - \frac{1}{10\sqrt{2}}r^2, \quad \check{\gamma}^3_{0011} = -\frac{3}{5\sqrt{2}}s^2_{0011} + \frac{1}{10\sqrt{2}}r^2, \quad \check{\gamma}^3_{1111} = \frac{3}{5\sqrt{2}}s^2_{1111},$$

and obtain thus for the quantity $F_{abcd} = -4\,x_{e(a}\gamma^3{}_{bc}{}^e{}_{d)} + 3\,s^2_{abcd}$ the concise expressions

$$F_0 = \frac{9}{5}\,s^2_0, \quad F_1 = 3\,s^2_1, \quad F_2 = \frac{17}{5}\,s^2_2 - \frac{1}{15}r^2, \quad F_3 = 3\,s^2_3, \quad F_4 = \frac{9}{5}\,s^2_4, \tag{4.10}$$

where we set $F_i = F_{(abcd)_i}$, $s_i = s_{(abcd)_i}$, using the notation introduced in (2.21).

In the cn-gauge the curvature vanishes at zeroth and first order at space-like infinity. At second order this is in general not true and the prescription of the free data on S in terms of curvature quantities has to be consistent with the cn-gauge, the Bianchi identity, and the regularity condition (2.23) for $q = 1$. The content of the cn-gauge is expressed in second order in the curvature by the conditions

$$D_{ab}D^{ab}\,r = 0, \quad D_{ab}D^{ab}\,s_{cdef} = -\frac{5}{4}\,D_{cd}\,D_{ef}\,r, \quad D_{(ab}\,D_{cd}\,s_{efgh)} = 0 \quad \text{at} \quad i.$$

It follows that the spinor

$$t_{abcd\,efgh} = D_{ab}\,D_{cd}\,s_{efgh} - \frac{1}{3}\,h_{abcd}\,\Delta_h\,s_{efgh},$$

where Δ_h denotes the Laplacian corresponding to the metric h, is symmetric in the first and the last four indices separately. Using the Bianchi identity

$$D^{ab}\,s_{abcd} = \frac{1}{6}\,D_{cd}\,r,$$

we thus get

$$\frac{1}{6}\,D_{ab}\,D_{cd}\,r - \frac{1}{3}\,\Delta_h\,s_{abcd} = t^{ef}{}_{ab\,cdef} = t_a{}^e{}_b{}^f{}_{cdef} = D_a{}^e\,D_b{}^f\,s_{cdef} + \frac{1}{6}\,\Delta_h\,s_{abcd},$$

whence

$$D_a{}^e\,D_b{}^f\,s_{cdef} = \frac{19}{24}\,D_{ab}\,D_{cd}\,r.$$

No further conditions are implied at i on the Ricci scalar r at this order. Finally, we get from (2.23) for $q = 1$

$$D^h{}_{(a}\,D_{bc}\,s_{def)h} = 0 \quad \text{at} \quad i.$$

The relations above imply that the expansion of $t_{abcd\,efgh}$ in terms of symmetric spinors and ϵ_{ab}'s can be expressed completely in terms of symmetrized twofold contractions of this spinor, which in turn can all be expressed in terms of the symmetric spinor $D_{ab}\, D_{cd}\, r$. Working out this expansion we get

$$D^{ab}\, D^{cd}\, s_{efgh} = h^{(ab}{}_{(ef}\, D^{cd)}{}_{gh)}\, r - \frac{5}{15}\, h^{abcd}\, D_{ef}\, D_{gh}\, r \quad \text{at} \quad i, \tag{4.11}$$

in our gauge. Going through the procedure described in section (3.5) of [4] we get $s_{(abcd)_j} = s^2_j\, \rho^2 + O(\rho^3)$ and $r = r^2\, \rho^2 + O(\rho^3)$ with

$$s^2_j = \tfrac{3^{|2-j|}}{12} \sum_{k=0}^{4} R^*_k \binom{4}{j}^{-\frac{1}{2}} T_4{}^k{}_j, \qquad r^2 = \tfrac{2}{\sqrt{6}} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_2, \tag{4.12}$$

where we set $R^*_k = \frac{1}{2} \binom{4}{k}^{\frac{1}{2}} D_{(ab}\, D_{cd)_k}\, r^*$, with the star indicating that the quantities are given in our gauge at i. The 5 real numbers R^*_k contain precisely the information on the metric h which can at this order be freely specified in the cn-gauge.

We note that the Cotton spinor is then given at i by

$$D_{ab}\, b_{cdef} = -\frac{5}{8} \left\{ \epsilon_{a(b}\, D_{cd}\, D_{ef)}\, r + \epsilon_{b(a}\, D_{cd}\, D_{ef)}\, r \right\},$$

and the deviation of h from conformal flatness at i is encoded at this order in the symmetric spinor $D_{ab}\, D_{cd}\, r(i)$.

From (4.10), (4.12) we obtain

$$F_0 = \tfrac{27}{20} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_0, \quad F_1 = \tfrac{3}{8} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_1, \quad F_2 = \tfrac{3}{20\sqrt{6}} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_2,$$
$$F_3 = \tfrac{3}{8} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_3, \quad F_4 = \tfrac{27}{20} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_4.$$

Finally, we will calculate the coefficient $\hat{U}_4$ in the Taylor series (4.6). Only the coefficients U_0, U_1 and U_2 of the expansion (4.5) contribute to $\hat{U}_4$. These functions have the following expansions (cf. [4] for the defining integrals).

$$U_0 = exp\left\{ \frac{1}{4} \int_0^\rho \left(\Delta\rho'^2 + 6 \right) \frac{d\rho'}{\rho'} \right\} = 1 + \frac{1}{4!} [\sqrt{2}\, \gamma^3_{1100}]\, \rho^4 + O(\rho^5), \tag{4.13}$$

where we used the expansion

$$\Delta\rho^2 = -6 + \frac{2\sqrt{2}}{3} \gamma^3_{1100}\, \rho^4 + O(\rho^5).$$

Further we have, with L denoting the Yamabe operator,

$$U_1 = \frac{U_0}{2\,\rho} \int_0^\rho \frac{L[U_0]}{U_0} d\rho' = \frac{1}{2} \left[-\frac{7\sqrt{2}}{36} \gamma^3_{1100} - \frac{1}{48}\, r^2 \right] \rho^2 + O(\rho^3). \tag{4.14}$$

Finally, observing (4.12), we obtain

$$U_2 = -\frac{U_0}{2\,\rho^2} \int_0^\rho \frac{L[U_1]\,\rho'}{U_0} d\rho' = O(\rho).$$

Collecting results, we arrive at the expansion

$$U = 1 + \frac{1}{4!} \left[-\frac{4\sqrt{2}}{3}\, \gamma^3_{1100} - \frac{1}{4}\, r^2 \right] \rho^4 + O(\rho^5) = 1 + \frac{1}{4!} \left[-\frac{3}{10\sqrt{6}} \sum_{k=0}^{4} R^*_k\, T_4{}^k{}_2 \right] \rho^4 + O(\rho^5). \tag{4.15}$$

Since the initial datum for the conformal Weyl spinor is a non-linear function of the basic quantities and the transport equations are quadratic in the unknowns, we have to make use of the Clebsch-Gordan expansions of products like $T_{2\,m}^{\ k} T_{2\,n}^{\ l}$. These are readily calculated by using the definition (2.21). For the quantities relevant in our calculation we thus obtain

$$\begin{aligned} X_- W_1 X_+ W_1 &= -\sum_{k=0}^{4} a_k T_{4\,2}^{\ k} + 2\,b, & W_1^2 &= \sum_{k=0}^{4} a_k T_{4\,2}^{\ k} + b, \\ W_1 X_- W_1 &= -\tfrac{\sqrt{6}}{2}\sum_{k=0}^{4} a_k T_{4\,3}^{\ k}, & W_1 X_+ W_1 &= \tfrac{\sqrt{6}}{2}\sum_{k=0}^{4} a_k T_{4\,1}^{\ k}, \\ (X_- W_1)^2 &= \sqrt{6}\sum_{k=0}^{4} a_k T_{4\,4}^{\ k}, & (X_+ W_1)^2 &= \sqrt{6}\sum_{k=0}^{4} a_k T_{4\,0}^{\ k}, \end{aligned} \tag{4.16}$$

with coefficients

$$\begin{aligned} a_0 &= \tfrac{2}{\sqrt{6}} W_{1;2,0}^2, & a_1 &= \tfrac{2}{\sqrt{3}} W_{1;2,0} W_{1;2,1}, & a_2 &= \tfrac{2}{3}(W_{1;2,0} W_{1;2,2} + W_{1;2,1}^2), \\ a_3 &= \tfrac{2}{\sqrt{3}} W_{1;2,2} W_{1;2,1}, & a_4 &= \tfrac{2}{\sqrt{6}} W_{1;2,2}^2, & b &= -\tfrac{2}{3}(W_{1;2,0} W_{1;2,2} - \tfrac{1}{2} W_{1;2,1}^2). \end{aligned} \tag{4.17}$$

It was shown in [4] that the quantity ϕ_i^3 has an expansion of the form

$$\phi_i^3 = \sum_{m=|4-2i|}^{q} \sum_{k=0}^{m} \phi_{i;m,k}^3 T_{m\ \frac{m}{2}-2+i}^{\ k}. \tag{4.18}$$

Using the results above in the last equation of (4.9), this expansion reduces to

$$\begin{aligned} &\phi_{i;m,k}^3 = 0, \quad \text{for} \quad i = \{0,\ldots,4\} \quad \text{and} \quad m \geq 8, \\ &\phi_{0;6,k}^3 = -2\sqrt{30}\, W_{3;6,k}, \qquad \phi_{1;6,k}^3 = -10\sqrt{3}\, W_{3;6,k}, \quad \phi_{2;6,k}^3 = -20\, W_{3;6,k}, \\ &\phi_{3;6,k}^3 = -10\sqrt{3}\, W_{3;6,k}, \qquad \phi_{4;6,k}^3 = -2\sqrt{30}\, W_{3;6,k}, \\ &\phi_{0;4,k}^3 = 18\sqrt{6}\, a_k - 3\sqrt{6}\, m\, W_{2;4,k} + \tfrac{3}{2} R_k^*, \quad \phi_{1;4,k}^3 = 9\sqrt{6}\, a_k - \tfrac{3}{2}\sqrt{6}\, m\, W_{2;4,k} + \tfrac{3}{4} R_k^*, \\ &\phi_{2;4,k}^3 = 18\, a_k - 3\, m\, W_{2;4,k} + \tfrac{3}{2\sqrt{6}} R_k^*, \qquad \phi_{3;4,k}^3 = 9\sqrt{6}\, a_k - \tfrac{3}{2}\sqrt{6}\, m\, W_{2;4,k} + \tfrac{3}{4} R_k^*, \\ &\phi_{4;4,k}^3 = 18\sqrt{6}\, a_k - 3\sqrt{6}\, m\, W_{2;4,k} + \tfrac{3}{2} R_k^*, \\ &\phi_{i;2,k}^3 = 0 \ \text{ for } \ i = \{1,2,3\}, \qquad \phi_{2;0,0}^3 = 0. \end{aligned} \tag{4.19}$$

Given these data on I'^{0}, we are in the position to solve the transport equations on I'. The first of the systems (4.4) can be integrated step by step with the result

$$
\begin{aligned}
(c^0{}_{ab})^3 &= [c_1^{03}(\tau) + c_2^{03}(\tau)W_1 + c_3^{03}(\tau)W_2]x_{ab} + [c_4^{03}(\tau)X_+W_1 + c_5^{03}(\tau)X_+W_2]z_{ab} \\
&\quad + [c_4^{03}(\tau)X_-W_1 + c_5^{03}(\tau)X_-W_2]y_{ab}, \\
(c^1{}_{ab})^3 &= [c_1^{13}(\tau) + c_2^{13}(\tau)W_1]x_{ab} + c_3^{13}(\tau)[X_+W_1z_{ab} + X_-W_1y_{ab}], \\
(c^+{}_{ab})^3 &= [c_1^{\pm 3}(\tau)X_-W_1 + c_2^{\pm 3}(\tau)X_-W_2]x_{ab} + [c_3^{\pm 3}(\tau) + c_4^{\pm 3}(\tau)W_1 + c_5^{\pm 3}(\tau)W_2]z_{ab} \\
&\quad + c_6^{\pm 3}(\tau)X_-X_-W_2y_{ab}, \\
(c^-{}_{ab})^3 &= [c_1^{\pm 3}(\tau)X_+W_1 + c_2^{\pm 3}(\tau)X_+W_2]x_{ab} + [c_3^{\pm 3}(\tau) + c_4^{\pm 3}(\tau)W_1 + c_5^{\pm 3}(\tau)W_2]y_{ab} \\
&\quad + c_6^{\pm 3}(\tau)X_+X_+W_2z_{ab}, \\
\xi^3_{abcd} &= S_1^3(\tau)X_+X_+W_2\varepsilon^0_{abcd} + [S_2^3(\tau)X_+W_1 + S_3^3(\tau)X_+W_2]\varepsilon^1_{abcd} \\
&\quad + [S_2^3(\tau)X_-W_1 + S_3^3(\tau)X_-W_2]\varepsilon^3_{abcd} - S_1^3(\tau)X_-X_-W_2\varepsilon^4_{abcd} \\
&\quad + [S_4^3(\tau) + S_5^3(\tau)W_1 + S_6^3(\tau)W_2](\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}) \\
&\quad + [S_7^3(\tau)X_+W_1 + S_8^3(\tau)X_+W_2](\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac}) \\
&\quad + [S_7^3(\tau)X_-W_1 + S_8^3(\tau)X_-W_2](\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac}), \\
\chi^3_{(ab)cd} &= K_1^3(\tau)X_+X_+W_2\varepsilon^0_{abcd} + [K_2^3(\tau)X_+W_1 + K_3^3(\tau)X_+W_2]\varepsilon^1_{abcd} \\
&\quad + [K_4^3(\tau) + K_5^3(\tau)W_1 + K_6^3(\tau)W_2]\varepsilon^2_{abcd} - [K_2^3(\tau)X_-W_1 + K_3^3(\tau)X_-W_2]\varepsilon^3_{abcd} \\
&\quad + K_1^3(\tau)X_-X_-W_2\varepsilon^4_{abcd} + [K_7^3(\tau) + K_8^3(\tau)W_1]h_{abcd} \\
&\quad + [K_9^3(\tau)X_-W_1 + K_{10}^3(\tau)X_-W_2](\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac}) \\
&\quad - [K_9^3(\tau)X_+W_1 + K_{10}^3(\tau)X_+W_2](\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac}), \\
f^3_{ab} &= [F_1^3(\tau) + F_2^3(\tau)W_1 + F_3^3(\tau)W_2]x_{ab} + [F_4^3(\tau)X_-W_1 + F_5^3(\tau)X_-W_2]y_{ab} \\
&\quad + [F_4^3(\tau)X_+W_1 + F_5^3(\tau)X_+W_2]z_{ab}, \\
(\Theta_g{}^g{}_{ab})^3 &= [t_1^3(\tau) + t_2^3(\tau)W_1 + t_3^3(\tau)W_2]x_{ab} + [t_4^3(\tau)X_-W_1 + t_5^3(\tau)X_-W_2]y_{ab} \\
&\quad + [t_4^3(\tau)X_+W_1 + t_5^3(\tau)X_+W_2]z_{ab}, \\
\Theta^3{}_{(ab)cd} &= T_1^3(\tau)X_+X_+W_2\varepsilon^0_{abcd} + [T_2^3(\tau)X_+W_1 + T_3^3(\tau)X_+W_2]\varepsilon^1_{abcd} \\
&\quad + [T_4^3(\tau) + T_5^3(\tau)W_1 + T_6^3(\tau)W_2]\varepsilon^2_{abcd} - [T_2^3(\tau)X_-W_1 + T_3^3(\tau)X_-W_2]\varepsilon^3_{abcd} \\
&\quad + T_1^3(\tau)X_-X_-W_2\varepsilon^4_{abcd} + [T_7^3(\tau) + T_8^3(\tau)W_1]h_{abcd} \\
&\quad + [T_9^3(\tau)X_-W_1 + T_{10}^3(\tau)X_-W_2](\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac}) \\
&\quad - [T_9^3(\tau)X_+W_1 + T_{10}^3(\tau)X_+W_2](\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac}).
\end{aligned}
\tag{4.20}
$$

The τ-dependent functions in these expressions are given in appendix [A.3].

We now turn to the second of the transport equations (4.4), which is a partial differential equation. The system for the expansion coefficients ϕ_i^3 of the rescaled conformal Weyl spinor on I' has the form

$$
\begin{aligned}
(1+\tau)\partial_\tau\phi_0^3 + X_+\phi_1^3 - \phi_0^3 &= R_0, \\
\partial_\tau\phi_1^3 + \tfrac{1}{2}X_-\phi_0^3 + \tfrac{1}{2}X_+\phi_2^3 + \phi_1^3 &= R_1, \\
\partial_\tau\phi_2^3 + \tfrac{1}{2}X_-\phi_1^3 + \tfrac{1}{2}X_+\phi_3^3 &= R_2, \\
\partial_\tau\phi_3^3 + \tfrac{1}{2}X_-\phi_2^3 + \tfrac{1}{2}X_+\phi_4^3 - \phi_3^3 &= R_3, \\
(1-\tau)\partial_\tau\phi_4^3 + X_-\phi_3^3 + \phi_4^3 &= R_4,
\end{aligned}
\tag{4.21}
$$

where the right hand sides are given by

$$\begin{aligned}
R_0 &= A_1(\tau)X_+X_+W_2 + A_2(\tau)(X_+W_1)^2,\\
R_1 &= B_1(\tau)X_+W_1 + B_2(\tau)W_1X_+W_1 + B_3(\tau)X_+W_2,\\
R_2 &= C_1(\tau) + C_2(\tau)W_1 + C_3(\tau)(W_1)^2 + C_4(\tau)W_2 + C_5(\tau)X_+W_1X_-W_1,\\
R_3 &= B_1(-\tau)X_-W_1 + B_2(-\tau)W_1X_-W_1 + B_3(-\tau)X_-W_2,\\
R_4 &= -A_1(-\tau)X_-X_-W_2 - A_2(-\tau)(X_-W_1)^2,
\end{aligned} \tag{4.22}$$

with τ-dependent functions $A_i(\tau)$, $B_j(\tau)$, $C_k(\tau)$ which are listed in appendix [A.3]. These functions have been calculated from the lower order expansion coefficients (4.1)-(4.3) and from (4.20). The symmetry inherent in these expressions reflects the time-symmetry of the underlying space-time.

Using the expansion (4.18) and corresponding expansions of the terms above, we decompose (4.21) into the following equations. For $m \geq 6$ the coefficients $\phi^3_{i;m,k}$, $k = 0, \ldots, m$, satisfy the homogeneous system

$$\begin{aligned}
(1+\tau)\partial_\tau\phi^3_{0;m,k} - \phi^3_{0;m,k} + \sqrt{(\tfrac{m}{2}-1)(\tfrac{m}{2}+2)}\,\phi^3_{1;m,k} &= 0,\\
\partial_\tau\phi^3_{1;m,k} + \phi^3_{1;m,k} - \tfrac{1}{2}\sqrt{(\tfrac{m}{2}-1)(\tfrac{m}{2}+2)}\,\phi^3_{0;m,k} + \tfrac{1}{2}\sqrt{\tfrac{m}{2}(\tfrac{m}{2}+1)}\,\phi^3_{2;m,k} &= 0,\\
\partial_\tau\phi^3_{2;m,k} - \tfrac{1}{2}\sqrt{\tfrac{m}{2}(\tfrac{m}{2}+1)}\,\phi^3_{1;m,k} + \tfrac{1}{2}\sqrt{\tfrac{m}{2}(\tfrac{m}{2}+1)}\,\phi^3_{3;m,k} &= 0,\\
\partial_\tau\phi^3_{3;m,k} - \phi^3_{3;m,k} - \tfrac{1}{2}\sqrt{(\tfrac{m}{2}+1)\tfrac{m}{2}}\,\phi^3_{2;m,k} + \tfrac{1}{2}\sqrt{(\tfrac{m}{2}+2)(\tfrac{m}{2}-1)}\,\phi^3_{4;m,k} &= 0,\\
(1-\tau)\partial_\tau\phi^3_{4;m,k} + \phi^3_{4;m,k} - \sqrt{(\tfrac{m}{2}+2)(\tfrac{m}{2}-1)}\,\phi^3_{3;m,k} &= 0.
\end{aligned} \tag{4.23}$$

The coefficients $\phi^3_{i;4,k}$, $k = 0, \ldots, 4$, solve

$$\begin{aligned}
(1+\tau)\partial_\tau\phi^3_{0;4,k} - \phi^3_{0;4,k} + 2\,\phi^3_{1;4,k} &= 2\sqrt{6}\,A_1(\tau)\,W_{2;4,k} + \sqrt{6}\,A_2(\tau)\,a_k,\\
\partial_\tau\phi^3_{1;4,k} + \phi^3_{1;4,k} - \phi^3_{0;4,k} + \tfrac{1}{2}\sqrt{6}\,\phi^3_{2;4,k} &= \tfrac{1}{2}\sqrt{6}\,B_2(\tau)\,a_k + \sqrt{6}\,B_3(\tau)\,W_{2;4,k},\\
\partial_\tau\phi^3_{2;4,k} - \tfrac{1}{2}\sqrt{6}\,\phi^3_{1;4,k} + \tfrac{1}{2}\sqrt{6}\,\phi^3_{3;4,k} &= [C_3(\tau) - C_5(\tau)]\,a_k + C_4(\tau)W_{2;4,k},\\
\partial_\tau\phi^3_{3;4,k} + \phi^3_{3;4,k} + \phi^3_{4;4,k} - \tfrac{1}{2}\sqrt{6}\,\phi^3_{2;4,k} &= -\tfrac{1}{2}\sqrt{6}\,B_2(-\tau)\,a_k - \sqrt{6}\,B_3(-\tau)\,W_{2;4,k},\\
(1-\tau)\partial_\tau\phi^3_{4;4,k} + \phi^3_{4;4,k} - 2\,\phi^3_{3;4,k} &= -2\sqrt{6}\,A_1(-\tau)\,W_{2;4,k} - \sqrt{6}\,A_2(-\tau)\,a_k,
\end{aligned} \tag{4.24}$$

with the coefficients a_k defined in (4.17). The functions $\phi^3_{i;2,k}$, $k = 0, 1, 2$, satisfy

$$\begin{aligned}
\partial_\tau\phi^3_{1;2,k} + \phi^3_{1;2,k} + \tfrac{1}{\sqrt{2}}\,\phi^3_{2;2,k} &= \sqrt{2}\,B_1(\tau)\,W_{1;2,k},\\
\partial_\tau\phi^3_{2;2,k} - \tfrac{1}{\sqrt{2}}\,\phi^3_{1;2,k} + \tfrac{1}{\sqrt{2}}\,\phi^3_{3;2,k} &= C_2(\tau)\,W_{1;2,k},\\
\partial_\tau\phi^3_{3;2,k} - \phi^3_{3;2,k} - \tfrac{1}{\sqrt{2}}\,\phi^3_{2;2,k} &= -\sqrt{2}\,B_1(-\tau)\,W_{1;2,k},
\end{aligned} \tag{4.25}$$

while $\phi^3_{2;0,0}$ is subject to

$$\partial_\tau\phi^3_{2;0,0} = C_1(\tau) + [C_3(\tau) + 2\,C_5(\tau)]\,b, \tag{4.26}$$

with b as defined in (4.17).

These ordinary differential systems have to be integrated for the initial data (4.19) at $\tau = 0$. Since the equations are already quite complicated, we used the program MapleV.4 for this purpose. Synthesizing the result of these integrations according to (4.18), we obtain the following concise expressions for ϕ^3_i on I'.

$$
\begin{aligned}
\phi^3_0 =& -(1+\tau)(1-\tau)^5 X_+X_+W_3 + \tfrac{1}{12} f_0(\tau)\, m\, X_+X_+W_2 \\
& + \tfrac{1}{6} g_0(\tau)(X_+W_1)^2 + \tfrac{1}{4} h_0(\tau) X_+X_+ r^2, \\
\phi^3_1 =& -5\,(1+\tau)^2(1-\tau)^4 X_+W_3 + \tfrac{1}{6} f_1(\tau)\, m\, X_+W_2 \\
& + \tfrac{1}{3} g_1(\tau) W_1 X_+ W_1 + \tfrac{1}{2} h_1(\tau) X_+ r^2 + \tfrac{1}{2} k_1(\tau)\, m^2 X_+W_1, \\
\phi^3_2 =& -20\,(1+\tau)^3(1-\tau)^3 W_3 + f_2(\tau)\, m\, W_2 \\
& + g_2(\tau)(W_1)^2 + 3\, h_2(\tau) r^2 + k_2(\tau)\, m^2 W_1 \\
& + p(\tau)\, m^4 + [q(\tau) - g_2(\tau)]\, b, \\
\phi^3_3 =& \, 5\,(1+\tau)^4(1-\tau)^2 X_-W_3 - \tfrac{1}{6} f_1(-\tau)\, m\, X_-W_2 \\
& - \tfrac{1}{3} g_1(-\tau) W_1 X_- W_1 - \tfrac{1}{2} h_1(-\tau) X_- r^2 - \tfrac{1}{2} k_1(-\tau)\, m^2 X_-W_1, \\
\phi^3_4 =& -(1+\tau)^5(1-\tau) X_-X_-W_3 + \tfrac{1}{12} f_0(-\tau)\, m\, X_-X_-W_2 \\
& + \tfrac{1}{6} g_0(-\tau)(X_-W_1)^2 + \tfrac{1}{4} h_0(-\tau) X_-X_- r^2,
\end{aligned}
\tag{4.27}
$$

with τ-dependent functions which can be found in appendix [A.3]. All the functions ϕ^3_i have polynomial dependence on τ.

The most interesting feature of this solution is its smoothness at $\tau = \pm 1$, which, in view of the singular behavior of equations (4.23), (4.24) at these points, was not to be expected from the beginning. To explain its significance we indicate the argument which led to the asymptotic regularity condition (2.23). The Bianchi equations, which were used to obtain the evolution equations for the rescaled conformal Weyl spinor and, consequently, the second of the transport equations (4.4), form an overdetermined system. Thus there are further equations, to which we refer as to the constraints. In the present case the constraints take the form

$$
\begin{aligned}
\tau \partial_\tau \phi^3_1 + \tfrac{1}{2}(X_+\phi^3_2 - X_-\phi^3_0) - 3\phi^3_1 &= S_1, \\
\tau \partial_\tau \phi^3_2 + \tfrac{1}{2}(X_+\phi^3_3 - X_-\phi^3_1) - 3\phi^3_2 &= S_2, \\
\tau \partial_\tau \phi^3_3 + \tfrac{1}{2}(X_+\phi^3_4 - X_-\phi^3_2) - 3\phi^3_3 &= S_3,
\end{aligned}
\tag{4.28}
$$

where

$$
\begin{aligned}
S_1 &= F_1(\tau) X_+W_1 + F_2(\tau) W_1 X_+ W_1 + F_3(\tau) X_+W_2, \\
S_2 &= G_1(\tau) + G_2(\tau) W_1 + G_3(\tau)(W_1)^2 + G_4(\tau) W_2 + G_5(\tau) X_-W_1 X_+W_1, \\
S_3 &= -F_1(-\tau) X_-W_1 - F_2(-\tau) W_1 X_- W_1 - F_3(-\tau) X_-W_2,
\end{aligned}
\tag{4.29}
$$

with functions which are given in appendix [A.3]. As before, we obtain equations for the coefficients in the expansion (4.18). Together with (4.23), (4.24) these equations imply the systems

$$
\begin{aligned}
(1+\tau)(5\,\tau^2+3)\partial_\tau \phi^3_{0;6,k} + (5\,\tau^3 - 5\,\tau^2 + 5\,\tau + 7)\phi^3_{0;6,k} - 5\,(\tau-1)^3 \phi^3_{4;6,k} &= 0, \\
(1-\tau)(5\,\tau^2+3)\partial_\tau \phi^3_{4;6,k} + (5\,\tau^3 + 5\,\tau^2 + 5\,\tau - 7)\phi^3_{0;6,k} - 5\,(\tau+1)^3 \phi^3_{4;6,k} &= 0,
\end{aligned}
\tag{4.30}
$$

and

$$
\begin{aligned}
4(3+\tau^2)(1+\tau)\partial_\tau \phi^3_{0;4,k} - 2(1-\tau)^3 \phi^3_{0;4,k} + 2(1-\tau)^3 \phi^3_{4;4,k} =&\ T_1(\tau)\, a_k + T_2(\tau)\, W_{2;4,k}, \\
-4(3+\tau^2)(1-\tau)\partial_\tau \phi^3_{4;4,k} - 2(1+\tau)^3 \phi^3_{4;4,k} + 2(1+\tau)^3 \phi^3_{0;4,k} =&\ T_1(-\tau)\, a_k + T_2(-\tau)\, W_{2;4,k},
\end{aligned}
\tag{4.31}
$$

with functions T_1 and T_2 (given in appendix [A.3]) derived from the functions R_i and S_j.

It turns out that once these equations have been solved, the remaining expansion coefficients in (4.18) can be obtained either by purely algebraic operations or by solving ODE's which are regular for $\tau \in [-1, 1]$. This situation is the same for all orders $p \geq 3$ in (4.4). The solutions $y(\tau)$, with y denoting in the case above the column vector with entries given by the two unknowns of (4.30) resp. of (4.31), can then be given for $p \geq 3$ in the form (suppressing here all indices)

$$y(\tau) = X(\tau)\, X(0)^{-1}\, y_0 + X(\tau) \int_0^\tau X(\tau')^{-1}\, b(\tau')\, d\tau', \tag{4.32}$$

with $X(\tau)$ denoting a fundamental matrix of the system of ODE's under study. The vector-valued function $b(\tau)$ is built from solutions which are obtained by solving the equations of lower order. In [4] the equations (written there in a slightly different form) have been discussed in general and the fundamental matrices $X(\tau)$ have been derived. As in the case of (4.30), (4.31), there occur homogeneous as well as inhomogeneous systems for general $p \geq 3$. Thus for certain values of the indices (i.e. p and the indices which arise from expanding u^p in terms of the functions $T_m{}^i{}_j$) the functions $b(\tau)$ vanish and the solutions are of the form $y(\tau) = X(\tau)\, X(0)^{-1}\, y_0$. In these cases some of the entries of $X(\tau)$ have logarithmic singularities. The latter drop out of the final expression precisely if the asymptotic regularity conditions (2.23) are satisfied. In the remaining cases the entries of the matrices $X(\tau)$ are polynomials in τ but $\det(X) = c\, f(\tau)\, (1-\tau^2)^{p-2}$ with some constant $c \neq 0$ and some polynomial $f(\tau)$ satisfying $|f(\tau)| \geq 1$ for $|\tau| \leq 1$. Furthermore, the column vector $b(\tau)$ has poles. However, it has no logarithmic singularities if the solutions of the equations of lower order have no logarithmic singularities. Assuming condition (2.23), the remaining potential source of singularities of u^p, $p \geq 3$, at $|\tau| = \pm 1$ are the integrals on the right hand sides of the expressions (4.32). These have not been analyzed yet. To understand the general situation, it is clearly of interest to study the problem for the first few values of p. Remarkably, in the present case, $p = 3$, we find that the integrand in (4.32) has poles at $|\tau| = \pm 1$ and also outside the interval $[-1, 1]$, that the integral has poles and no logarithmic terms, but that the final solution is a polynomial in τ.

4.2 The detailed transformation formulae

In this section we will determine expansions for the conformal scale factor θ and the $SL(2, C)$-valued function $\Lambda^a{}_b$ which define the transformation from the F-gauge into the NP-gauge as described in section [2.3]. To calculate the NP-constants in terms of the initial data we shall determine the values of the integrals defining these quantities by taking their limits as $\rho \to 0$. The gauge in which these integrals are given is based on a section $\mathcal{C}$ of the generators of $\mathcal{J}^+$. We shall try to push this section to I^+. The usefulness of this procedure depends, of course, on the resulting form of the ODE's on $\mathcal{J}^+$ which were used in [2.3] to fix the F-gauge.

Near I^+ the hypersurface $\mathcal{J}^\pm$ can be given as the graph $\{\tau = \tau^s, \rho > 0\}$ of the function $\tau^s = \tau^s(\rho, t^a{}_b)$ which is given by

$$\tau^s = \frac{2\,\Omega}{\rho}[-D_{ab}\Omega D^{ab}\Omega]^{-\frac{1}{2}}. \tag{4.33}$$

Substituting the expansions (4.8) of Ω and those of the frame vectors into the expression above, we get the expansion

$$\tau^s = 1 + \frac{1}{2} m\,\rho + 2\,W_1 \rho^2 + O(\rho^3). \tag{4.34}$$

Setting in (2.41) $Z = \partial_\tau$, we obtain for the right hand side of this equation the expansion

$$\frac{Z(\frac{1}{2}\nabla_\beta\Theta\nabla^\beta\Theta)}{Z(\Theta)} = \frac{5}{3} m\,\rho^2 - \left(\frac{229}{63} m^2 - \frac{24}{5} W_1\right)\rho^3 + O(\rho^4). \tag{4.35}$$

Suppose $T = T^0\,\partial_\tau + T^1\,\partial_\rho + T^+\,X_+ + T^-\,X_-$ is a vector field defined near and tangent to $\mathcal{J}^+$. Denote by T^* the vector field which is induced by it on $\mathcal{J}^+$. If ρ and $t^a{}_b$ are used as coordinates on

$\mathcal{J}^+$, one finds for T^* the expression $T^* = T^1\,\partial_\rho + T^+\,X_+ + T^-\,X_-$. Applying this to the gradient of Θ on $\mathcal{J}^+$, we find that the left hand side of (2.41) is given by

$$\Big(\big\{-2\rho^2 + \tfrac{19}{3}m\,\rho^3 + O(\rho^4)\big\}\partial_\rho + \big\{\tfrac{36}{5}X_-W_1\,\rho^3 + O(\rho^4)\big\}X_+ + \big\{\tfrac{36}{5}X_+W_1\,\rho^3 + O(\rho^4)\big\}X_-\Big)(\log f).$$

Thus, dividing (2.41) on both sides by ρ^2, we get a differential equation of the form $T^*(\log f) = g$ on $\mathcal{J}^+$ with a vector field T^* and a function g which extend smoothly to I^+ such that $T^* = -2\,\partial_\rho + O(\rho)$ near I^+. For given datum f_0 on I^+ this equation has a unique smooth solution which can be expanded in terms of ρ. As shown in our general discussion, the value of f_0 has to be constant on $\mathcal{C}$ to fulfill the NP-gauge conditions. We choose $f_0 = -\frac{1}{2\sqrt{2}}$ on I^+ and find for the solution of (2.41) the expansion

$$f = -\frac{1}{2\sqrt{2}}\left\{1 + \frac{5}{6}m\,\rho + \left(\frac{191}{252}m^2 + \frac{6}{5}W_1\right)\rho^2 + O(\rho^3)\right\}. \tag{4.36}$$

To obtain the matrix elements $\lambda^a{}_b$ of (2.42) by using (2.43) we have to calculate the derivatives $c^*_{aa'}(\Theta)$ of the conformal factor. Using the expansion coefficients derived in [4.1], we get

$$\begin{aligned}
&c^*_{00'}(\Theta) = O(\rho^4),\\
&c^*_{01'}(\Theta) = \sqrt{2}\Big\{X_+W_1\rho^3 + O(\rho^4)\Big\},\quad c^*_{10'}(\Theta) = \sqrt{2}\Big\{X_-W_1\rho^3 + O(\rho^4)\Big\},\\
&c^*_{11'}(\Theta) = \sqrt{2}\Big\{-2\,\rho + 3\,m\rho^2 + (8\,W_1 - 3\,m^2)\rho^3 + O(\rho^4)\Big\}.
\end{aligned} \tag{4.37}$$

Substituting these expressions into the formulae (2.43) the matrix elements $\lambda^0{}_1$ and $\lambda^1{}_1$ can be calculated explicitly up to a $U(1)$ phase transformation. Since the choice of the latter is not important for the following we choose it suitably to obtain

$$\lambda^0{}_1 = \rho^{\frac{1}{2}}\Big\{1 - \tfrac{1}{3}m\rho + (-\tfrac{7}{5}W_1 + \tfrac{113}{252}m^2)\rho^2 + O(\rho^3)\Big\},\quad \lambda^1{}_1 = \rho^{\frac{5}{2}}\Big\{\tfrac{1}{2}X_+W_1 + O(\rho)\Big\}, \tag{4.38}$$

which allows us to determine also the expansion

$$\begin{aligned}
E^\circ_{11'} =& \sqrt{2}\Big\{\tfrac{1}{4}m\rho^2 + (-\tfrac{7}{12}m^2 + 2W_1)\rho^3 + O(\rho^4)\Big\}\partial_\tau\\
&+\sqrt{2}\Big\{\tfrac{1}{2}\rho^2 - \tfrac{7}{6}m\rho^3 + (\tfrac{577}{252}m^2 - \tfrac{31}{5}W_1)\rho^4 + O(\rho^5)\Big\}\partial_\rho\\
&+\sqrt{2}\Big\{-\tfrac{9}{5}X_-W_1\rho^3 + O(\rho^4)\Big\}X_+ + \sqrt{2}\Big\{-\tfrac{9}{5}X_+W_1\rho^3 + O(\rho^4)\Big\}X_-.
\end{aligned} \tag{4.39}$$

To solve the differential equation for the affine parameter on the generators of $\mathcal{J}^+$, we observe that already in the case of Minkowski space-time this parameter is a singular function of ρ, given by $u^\circ = -\sqrt{2}\rho^{-1} + u^\circ_*$. The inspection of the expansion (4.39) suggests to search for a solution of the form

$$u^\circ = w + \sqrt{2}\left(-\frac{1}{\rho} + \frac{7}{3}\,m\log\rho\right). \tag{4.40}$$

This ansatz does indeed lead to a smooth regular equation for w near I^+. It allows us to calculate the expansion

$$u^\circ = \sqrt{2}\left\{-\frac{1}{\rho} + \frac{7}{3}m\log\rho + u^\circ_* + \left(\frac{109}{126}m^2 + \frac{62}{5}W_1\right)\rho + O(\rho^2)\right\}, \tag{4.41}$$

where u°_* denotes an arbitrary constant initial datum on I^+. As described in chapter [2.3], the matrix elements $\lambda^0{}_0$ and $\lambda^1{}_0$ can now be determined. We obtain the expansions

$$\lambda^0{}_0 = \rho^{\frac{3}{2}}\Big\{\tfrac{77}{10}X_-W_1 + O(\rho)\Big\},\quad \lambda^1{}_0 = \rho^{-\frac{1}{2}}\Big\{-1 - \tfrac{1}{3}m\,\rho + O(\rho^2)\Big\}. \tag{4.42}$$

Knowing the matrix ${\lambda^a}_b$ on null infinity, we can calculate the limits of the NP-spin-coefficients $\Gamma^\circ_{01'11}$ and $\Gamma^\circ_{10'00}$ at I^+ as $\rho \to 0$. Substituting our expansions into the formula for the connection coefficients

$$\Gamma^\circ_{aa'bc} = {\lambda^f}_a {\bar{\lambda}^{f'}}_{a'} {\lambda^g}_b {\lambda^h}_c \Gamma^*_{ff'gh} - \epsilon_{gh} {\lambda^g}_b E^\circ_{aa'}({\lambda^h}_c), \tag{4.43}$$

we arrive at the expressions

$$\Gamma^\circ_{01'11}|_{I^+} = \lim_{\rho\to 0} \Gamma^\circ_{01'11} = 0, \quad \Gamma^\circ_{10'00}|_{I^+} = \lim_{\rho\to 0} \Gamma^\circ_{10'00} = \frac{11}{6\sqrt{2}}\, m. \tag{4.44}$$

The next step is to calculate the conformal scale factor θ by solving equation (2.31). To determine the Ricci spinor component $\Phi_{22} = \frac{1}{2} R_{\alpha\beta} E^{\circ\,\alpha}_{11'} E^{\circ\,\beta}_{11'}$, we have to determine the Ricci tensor $R_{\alpha\beta}$ of the metric g. The components of the tensor

$$\Theta_{\alpha\beta} := \frac{1}{2}\hat{R}_{(\alpha\beta)} - \frac{1}{12} g_{\alpha\beta}\hat{R} + \frac{1}{4}\hat{R}_{[\alpha\beta]} \tag{4.45}$$

in the frame $\{c^*_{aa'}\}$, where $\hat{R}_{\alpha\beta}$ resp. $\hat{R}$ denote the Ricci tensor and the curvature scalar induced by the Weyl connection $\hat{\nabla}$ with coefficients $\hat{\Gamma}_{\alpha\ \gamma}^{\ \beta} = \Gamma_{\alpha\ \gamma}^{\ \beta} + \delta^\beta_\alpha f_\gamma + \delta^\beta_\gamma f_\alpha - g_{\alpha\gamma} f^\beta$ (cf. [3]), are among the variables of the conformal field equations. Thus they are known to 3^{rd}-order in the ρ-coordinate. From the general transformation law

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - 2\nabla_{(\alpha} f_{\beta)} + 2 f_\alpha f_\beta - g_{\alpha\beta}(\nabla_\gamma f^\gamma + 2 f_\gamma f^\gamma) + 4\nabla_{[\alpha} f_{\beta]}, \tag{4.46}$$

we get the relation

$$\Theta_{\alpha\beta} = \frac{1}{2}\left(R_{\alpha\beta} - \frac{1}{6} g_{\alpha\beta} R\right) - \nabla_\beta f_\alpha + f_\alpha f_\beta - \frac{1}{2} g_{\alpha\beta} f_\gamma f^\gamma. \tag{4.47}$$

From this we derive the expression

$$\Phi_{22} = \Theta_{\alpha\beta} E^{\circ\,\alpha}_{11'} E^{\circ\,\beta}_{11'} + E^\circ_{11'}(E^{\circ\,\alpha}_{11'} f_\alpha) - (E^{\circ\,\alpha}_{11'} f_\alpha)^2. \tag{4.48}$$

Substituting here (4.39) and the expansion of the one-form f obtained from the solution of the field equations we get the expansion

$$\Phi_{22} = \frac{5}{6} m\rho^3 + \left(-\frac{167}{42} m^2 + \frac{18}{5} W_1\right)\rho^4 + O(\rho^5) \tag{4.49}$$

on $\mathcal{J}^+$.

On I^+ is induced in our gauge the standard S^2-metric. Therefore we solve equation (2.31) with the initial condition

$$\lim_{\rho\to 0} \theta = 1. \tag{4.50}$$

For the conformal scale factor we obtain then the expansion

$$\theta = 1 + \frac{5}{6} m\rho + \left(\frac{6}{5} W_1 + \frac{191}{252} m^2\right)\rho^2 + O(\rho^3). \tag{4.51}$$

By the choice of the initial value for the conformal factor the scale function p appearing in the gauge transformations is also fixed with

$$p \equiv 1 \quad \text{on} \quad \mathcal{J}^+. \tag{4.52}$$

In the conformal gauge characterized by the conformal factor $\Theta^\star := \theta\,\Theta$ the generators of null infinity are expansion free. Proceeding as indicated before, we construct the NP-frame $\{E^\bullet_{aa'}\}$.

Observing the expansions (2.42) and (2.45) of the null vectors $E^{\circ}_{11'}$ resp. $E^{\bullet}_{11'}$ and taking into account the properties of the conformal rescaling we get the relations

$$\Lambda^0{}_1 = \theta^{-\frac{1}{2}}\lambda^0{}_1 e^{ic}, \quad \Lambda^1{}_1 = \theta^{-\frac{1}{2}}\lambda^1{}_1 e^{ic}, \tag{4.53}$$

with function c, characterizing the phase freedom, which will be fixed later. Using (4.38) and (4.51) we get the expansions

$$\Lambda^0{}_1 = \rho^{\frac{1}{2}}\Big\{1 - \tfrac{3}{4}\,m\rho + \big(\tfrac{15}{32}\,m^2 - 2\,W_1\big)\,\rho^2 + O(\rho^3)\Big\}e^{ic}, \quad \Lambda^1{}_1 = \rho^{\frac{5}{2}}\Big\{\tfrac{1}{2}\,X_+W_1 + O(\rho)\Big\}e^{ic}, \tag{4.54}$$

from which we derive in turn the expansion

$$\begin{aligned} E^{\bullet}_{11'} =& \sqrt{2}\Big\{\tfrac{1}{4}m\rho^2 + (-m^2 + 2W_1)\rho^3 + O(\rho^4)\Big\}\partial_\tau \\ &+\sqrt{2}\Big\{\tfrac{1}{2}\rho^2 - 2\,m\rho^3 + (\tfrac{253}{56}m^2 - \tfrac{37}{5}W_1)\rho^4 + O(\rho^5)\Big\}\partial_\rho \\ &+\sqrt{2}\Big\{-\tfrac{9}{5}X_-W_1\rho^3 + O(\rho^4)\Big\}X_+ + \sqrt{2}\Big\{-\tfrac{9}{5}X_+W_1\rho^3 + O(\rho^4)\Big\}X_-, \end{aligned} \tag{4.55}$$

of the vector field $E^{\bullet}_{11'}$ tangent to the null generators of $\mathcal{J}^+$. Furthermore the new affine parameter has the form

$$u^{\bullet} = \sqrt{2}\left\{-\frac{1}{\rho} + 4\,m\log\rho + u^{\bullet}_* + \left(\frac{195}{28}\,m^2 + \frac{74}{5}\,W_1\right)\rho + O(\rho^2)\right\}, \tag{4.56}$$

with a free constant $u^{\bullet}_*$. Using the formula analogous to (2.44) we derive

$$\Lambda^0{}_0 = \rho^{\frac{1}{2}}\Big\{-\tfrac{101}{10}\,X_-W_1\rho + O(\rho^2)\Big\}e^{-ic}, \quad \Lambda^1{}_0 = \rho^{-\frac{1}{2}}\Big\{-1 - \tfrac{3}{4}\,m\,\rho + O(\rho^2)\Big\}e^{-ic}. \tag{4.57}$$

To determine of the phase factor $e^{\pm ic}$ we solve equation (2.33) along the generators of null infinity. Expanding the right hand side, we get

$$E^{\bullet}_{11'}(c) = 2\,\mathfrak{Im}\left\{\hat{\Lambda}^f{}_1\bar{\hat{\Lambda}}^{f'}{}_{1'}\hat{\Lambda}^g{}_1\hat{\Lambda}^h{}_0\Gamma^{\star}_{ff'gh} - \hat{\Lambda}^0{}_0E^{\bullet}_{11'}(\hat{\Lambda}^1{}_1) + \hat{\Lambda}^1{}_0E^{\bullet}_{11'}(\hat{\Lambda}^0{}_1)\right\}, \tag{4.58}$$

where $\hat{\Lambda}^a{}_b$ has been obtained from the matrix $\Lambda^a{}_b$ above by setting $c = 0$. Substituting the known data into the equation above, the solution c which is needed to satisfy the gauge condition $\Gamma^{\bullet}_{11'01}|_{\mathcal{J}} = 0$, is found to have an expansion

$$c = O(\rho^2), \tag{4.59}$$

which entails the expansions

$$e^{ic} = 1 + O(\rho^2), \quad E^{\bullet}_{11'}(e^{ic}) = O(\rho^3), \quad E^{\bullet}_{01'}(e^{ic}) = O(\rho^2). \tag{4.60}$$

The matrix elements $\Lambda^a{}_b$ are now determined on null infinity to the precision needed in our later calculations, but in the definition (3.5) of the NP-constants appear some of the transversal derivatives $E^{\bullet}_{00'}(\Lambda^a{}_b)$ of the matrix elements as well. Using the general formulae (2.47) we get the expansions

$$\begin{aligned} &E^{\bullet}_{00'}(\Lambda^0_0) = \sqrt{2}\rho^{\frac{1}{2}}\Big\{\tfrac{113}{40}X_-W_1 + O(\rho)\Big\}, && E^{\bullet}_{00'}(\Lambda^1_0) = \sqrt{2}\rho^{-\frac{3}{2}}\Big\{\tfrac{1}{4} + \tfrac{85}{48}m\rho + O(\rho^2)\Big\}, \\ &E^{\bullet}_{00'}(\Lambda^0_1) = \sqrt{2}\rho^{-\frac{1}{2}}\Big\{\tfrac{1}{4} + \tfrac{67}{48}\,m\,\rho + O(\rho^2)\Big\}, && E^{\bullet}_{00'}(\Lambda^1_1) = \sqrt{2}\rho^{\frac{3}{2}}\Big\{-\tfrac{47}{40}X_+W_1 + O(\rho)\Big\}, \end{aligned} \tag{4.61}$$

where we have taken the expressions (4.60) for the phase factor into account.

The transversal derivative of the conformal scale factor $E^{\bullet}_{00'}(\theta)$ is fixed on null infinity by the requirement $R[g^{\star}]|_{\mathcal{J}^+} = 0$. Thus it has to satisfy equation (2.38) with initial datum

$$E^{\bullet}_{00'}(\theta)|_{I^+} = \lim_{\rho\to 0}\,\theta\,p^{-1}\Gamma^{\circ}_{10'00} = \lim_{\rho\to 0}\Gamma^{\circ}_{10'00}. \tag{4.62}$$

Given the matrix ${\Lambda^a}_b$ and the conformal scale factor θ, all the terms appearing in equation (2.38) can be calculated in a straightforward way, with the exception of the curvature scalar $R[g]$, whose calculation requires some explanation. Contracting equation (4.47) we get the identity

$$R[g] = 6\,(\Theta_{aa'bb'} + \nabla_{aa'} f_{bb'} + f_{aa'} f_{bb'})\epsilon^{ab}\bar{\epsilon}^{a'b'}, \tag{4.63}$$

where

$$\nabla_{aa'} f_{bb'} = c^*_{aa'}(f_{bb'}) - (\Gamma^*_{aa'cb}\bar{\epsilon}_{b'c'} + \bar{\Gamma}^*_{aa'c'b'}\epsilon_{bc})f^{cc'}.$$

Expanding these quantities we get

$$\begin{aligned} R[g] &= \left(\tfrac{23}{3}\,m^2 - \tfrac{168}{5}W_1\right)\rho^2 + O(\rho^3),\\ F^\star &= \left(\tfrac{23}{36}\,m^2 - 4W_1 + \tfrac{6}{5}a_- X_+ W_1 - \tfrac{6}{5}a_+ X_- W_1\right)\rho^2 + O(\rho^3), \end{aligned} \tag{4.64}$$

which entail with (2.38) the expansion

$$E^\bullet_{00'}(\theta) = \sqrt{2}\left\{\frac{11}{12}\,m + \left(\frac{13}{6}\,m^2 - 4W_1 + \frac{6}{5}a_- X_+ W_1 - \frac{6}{5}a_+ X_- W_1\right)\rho + O(\rho^2)\right\}. \tag{4.65}$$

Given the expansion above, we can calculate expansions of various quantities of physical interest, such as the Bondi energy momentum, the angular momentum, and the radiation field on $\mathcal{J}^+$. Since the coefficients in these expansions are given directly in terms of the initial data on the Cauchy hypersurface S, the expansions contain information about the evolution of the field over an infinite range. As an example we will calculate below the NP-constants.

We close this section with a remark on the BMS group, the group of transformation between different Bondi-type systems. It was shown in [13] that for solutions for which the the condition $\lim_{u^\bullet \to -\infty} \Gamma^\bullet_{[e]\,01'00} = 0$ could be realized at space-like infinity, where the subscript "e" is to denote the electric part of the considered spin-coefficient, one can single out the inhomogeneous Lorentz group as the group of transformations preserving this condition. It turns out that under our assumptions, which include in particular the time-symmetry of the solution, the even stronger condition $\lim_{u^\bullet \to -\infty} \Gamma^\bullet_{01'00} = 0$ is satisfied. This means that for our solutions there is a natural way to single out the inhomogeneous Lorentz group as asymptotic symmetry group.

4.3 The NP-constants in time symmetric space-times

Using the formulae of the previous chapters we can express the NP-constants in terms of the initial data for the corresponding time symmetric solutions. All the quantities appearing in the integral (3.5) are known in terms of the initial data to the precision needed to perform the limit $\rho \to 0$.

We have to express the spin-2 spherical harmonics ${}_2\bar{Y}_{2,m}$ in terms of the functions ${T_m}^j{}_k$. By (3.3) the definition of the ð-operator is based on the choice of the complex null vector field $E^\bullet_{01'}$. In appendix [A.1] we have applied the standard choice and derived the relations between the operators X_+ and ð and between the spin-2 spherical harmonics ${}_2Y_{2,m}$ and the functions ${T_m}^j{}_k$. By this choice we should have $E^\bullet_{01'} = \frac{i}{\sqrt{2}}X_+$ on I^+. However, calculating the vector $E^\bullet_{01'}$ in the conventions used above, we get

$$E^\bullet_{01'}|_{I^+} = \frac{1}{\sqrt{2}}X_-. \tag{4.66}$$

There are two causes of the difference. We fixed the phase factor such as to simplify the calculations and the conventions used in the F-gauge and the NP-gauge are such that one has to swap the two spinors of the dyad to get from one to the other convention. The form (4.66) of $E^\bullet_{01'}$ corresponds to $-i\sqrt{2}\bar{m}$, if m denotes the standard complex null vector used in appendix [A.1]. This means that

(4.66) corresponds to the operator $-i\bar{ð}$ instead of ð discussed in the appendix. Observing this and (A.9) in (3.4) we obtain the formula

$$G_m = i^{2-m}(5\pi)^{\frac{1}{2}} \oint \bar{T}_4{}^{2-m}{}_4 E^{\bullet}_{00'}(\phi_0)\mu \quad \text{for} \quad m=-2,\ldots,2, \tag{4.67}$$

where $\mu = \frac{1}{4\pi^2} dS d\alpha$ is the Haar-measure on $SU(2)$.

To calculate (4.67) we expand the integrand in terms of ρ and take the limit as $\rho \to 0$. For this we have to determine for $E^{\bullet}_{00'}(\phi_0)$ only the terms of order O(1). In the limit only these terms give a contribution while the terms of order ρ^{-1} cancel each other. Using the explicit results of the previous chapters we arrive after some lengthy but straightforward calculations at the expression

$$\begin{gathered} G_m\big|_{I^+} = \lim_{\rho\to 0} G_m = i^{2-m}(10\pi)^{\frac{1}{2}} \\ \oint \bar{T}_4{}^{2-m}{}_4 \Big(-\frac{5}{32} X_- X_- r^2 + \frac{635}{8}\, m\, X_- X_- W_2 - \frac{1905}{2}\,(X_- W_1)^2 + \frac{16}{3} X_- X_- W_3 \Big)\mu. \end{gathered} \tag{4.68}$$

Expanding the functions in the brackets in terms of the functions $T_m{}^k{}_j$ and using the orthogonality relations satisfied by these functions we can perform the integration. All terms expect the last one give some contributions. Using the formulae (4.7), (4.12) and (4.16) we get the final expression

$$G_m\big|_{I^+} = \frac{i^{2-m}}{2}(15\pi)^{\frac{1}{2}} \left\{ 127\,(m\, W_{2;4,2-m} - 6\, a_{2-m}) - \frac{1}{2\sqrt{6}}\, R^*_{2-m} \right\}, \tag{4.69}$$

where the coefficients a_{2-m}, which are quadratic in $W_{1;2,k}$, are given by (4.17). We note that the structure of this more general expression is essentially the same as that of the expression obtained by Newman and Penrose in the case of static and stationary solutions.

5 Concluding remarks

We have seen that, under the assumptions explained above, certain fields which are given near space-like infinity in terms of Bondi-type systems can be expressed in a straightforward way in terms of the gauge conditions used in [4] and can thus be related directly to the structure of the Cauchy data which give rise to the space-times by Einstein evolution. The calculations involved are quite lengthy but taking into account that we relate quantities which are obtained by a non-linear evolution over an infinite domain of space-time to the data from which they arise, the overall structure of the argument is surprisingly simple.

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A Appendix

A.1 X_+ and the ð-operator

In this section we describe the relation between the operators ð, $\bar{ð}$ introduced in [13] and the operators X_+, X_-, X used in [4].

Consider on the group $SU(2)$, which is diffeomorphic to S^3, coordinates $\{x,\, y,\, \alpha\}$ such that outside a set of measure zero the general group element $t^a{}_b \in SU(2)$ is given by

$$t^a{}_b = \frac{1}{\sqrt{1+\zeta\bar{\zeta}}} \begin{pmatrix} e^{i\alpha} & ie^{-i\alpha}\zeta \\ ie^{i\alpha}\bar{\zeta} & e^{-i\alpha} \end{pmatrix}, \tag{A.1}$$

with $\zeta = x + iy$. Then α is a parameter and x and y are constant on the orbits of the the subgroup $U(1)$. The tangent vectors ∂_x, ∂_y, respectively ∂_α at the unit element coincide with the generators u_1, u_2, and u_3 of the Lie algebra of $SU(2)$. Writing $P = \frac{1}{2}(1 + \zeta\bar{\zeta})$, we get for the corresponding left invariant vector fields the expressions

$$\begin{aligned} Z_{u_1} &= P\cos(2\alpha)\partial_x + P\sin(2\alpha)\partial_y + \tfrac{1}{2}[x\sin(2\alpha) - y\cos(2\alpha)]\partial_\alpha, \\ Z_{u_2} &= -P\sin(2\alpha)\partial_x + P\cos(2\alpha)\partial_y + \tfrac{1}{2}[y\sin(2\alpha) + x\cos(2\alpha)]\partial_\alpha, \\ Z_{u_3} &= \tfrac{1}{2}\partial_\alpha, \end{aligned} \tag{A.2}$$

whence

$$\begin{aligned} X_+ &= -Z_{u_2} - iZ_{u_1} = e^{2i\alpha}\{-i\sqrt{2}(m - \tfrac{i}{2\sqrt{2}}\bar{\zeta}\partial_\alpha\}, \quad X = -2iZ_{u_3} = -i\partial_\alpha, \\ X_- &= -Z_{u_2} + iZ_{u_1} = e^{-2i\alpha}\{i\sqrt{2}(\bar{m} + \tfrac{i}{2\sqrt{2}}\zeta\partial_\alpha\}, \end{aligned} \tag{A.3}$$

where the vectors $m = \sqrt{2}P\partial_\zeta$ and $\bar{m} = \sqrt{2}P\partial_{\bar{\zeta}}$ define a complex dyad tangent to the surfaces $\{\alpha = const.\}$ which is null with respect to the standard S^2-metric $ds^2 = P^{-2}d\zeta\, d\bar{\zeta}$ on these surfaces.

We may identify $SU(2)$ with the spin frame bundle over the base manifold S^2 with structure group $U(1)$. The section $\{\alpha = 0\}$ can be identified with the base manifold (with a point omitted). Here we take the complex null frame $\{m, \bar{m}\}$ defined above, where a group element $u^a_{\ b} = diag(e^{i\alpha}, e^{-i\alpha}) \in U(1)$ acts as $u(\{m, \bar{m}\}) = \{e^{2i\alpha}m, e^{-2i\alpha}\bar{m}\}$. A function η on S^3 is said to have spin weight N, if it can be decomposed as $\eta|_{\zeta,\alpha} = e^{2Ni\alpha}\eta_0$, where the function η_0 is independent of the parameter α along the fibers. The ð-operator is defined by the complex null vector m and acts on a spin-N function as

$$ð\eta|_{\zeta,\alpha} = \sqrt{2}\{m(\eta_0) + N\eta_0\, \bar{m}^\gamma m^\beta \delta_\beta m_\gamma\}e^{2(N+1)i\alpha} = \sqrt{2}\{m(\eta_0) + \frac{1}{\sqrt{2}}N\bar{\zeta}\eta_0\}e^{2(N+1)i\alpha}, \tag{A.4}$$

where δ denotes the Levi-Civita differential operator induced by the standard S^2-metric. This means that $ð\eta$ has spin weight $N + 1$. (This treatment of the functions with spin weight and the ð operator is a bit different from the one which can be found in the literature (cf. [13, 9, 10]), where the expressions are evaluated on some cross-section of S^3.)

The horizontal lift of the vector m defined with respect to the Levi-Civita connection δ is given by

$$m_H|_{\zeta,\alpha} = m - \frac{i}{2\sqrt{2}}\hat{\zeta}\partial_\alpha. \tag{A.5}$$

This means that the ð-operator on S^3 is given by

$$ð|_{\zeta,\alpha} = \sqrt{2}\, e^{2i\alpha}m_H. \tag{A.6}$$

Comparing the formulae (A.3), (A.5) and (A.6) we get the relations

$$X_+ = -ið, \quad X_- = i\bar{ð}, \quad X = -[ð, \bar{ð}]. \tag{A.7}$$

The spherical harmonics $Y_{l,m}$ are defined as an orthogonal function system on the sphere S^2. They can be extended to S^3 as functions with zero spin weight, i.e. they became independent on the parameter along the fibers. This means that they can be expanded as $Y_{l,m} = \sum_{k,j} c_{kj} T_{2k}{}^j{}_k$ in terms of the functions $T_m{}^j{}_k$. The spherical harmonics satisfy the equation $ð\bar{ð}Y_{l,m} = -l(l+1)Y_{l,m}$, so using the relations (A.7) and (2.22) we arrive at the relation

$$Y_{l,m} = \sum_j c_j T_{2l}{}^j{}_l. \tag{A.8}$$

Taking into account the explicit coordinate expressions of the group elements one could determine the expansion coefficients c_j. Using the definition of the spin harmonics ${}_sY_{l,m}$ (cf. [9]) and equations (2.22), (A.7) and (A.8) one can also derive the relation between the functions ${}_sY_{l,m}$ and the functions $T_m{}^j{}_k$. We shall only need the transformation formulae

$$\begin{aligned} Y_{2,m} &= (-i)^{4-m}\left(\tfrac{5}{4\pi}\right)^{\frac{1}{2}} T_{4\ 2}^{2-m}, \\ {}_2Y_{2,m} &= (-i)^{2-m}\left(\tfrac{5}{4\pi}\right)^{\frac{1}{2}} T_{4\ 0}^{2-m}, \quad {}_{-2}Y_{2,m} = (-i)^{2-m}\left(\tfrac{5}{4\pi}\right)^{\frac{1}{2}} T_{4\ 4}^{2-m}. \end{aligned} \tag{A.9}$$

A.2 Some useful spinor identities

Here we describe irreducible decompositions of spinors with four unprimed indices in terms of the "primary spinors" ε^i_{abcd}, h_{abcd}, x_{ab}, y_{ab}, z_{ab} and ϵ_{ab}, where

$$\begin{aligned} x_{ab} &= \sqrt{2}\epsilon_{(a}{}^0\epsilon_{b)}{}^1, \qquad y_{ab} = -\tfrac{1}{\sqrt{2}}\epsilon_a{}^1\epsilon_b{}^1, \quad z_{ab} = \tfrac{1}{\sqrt{2}}\epsilon_a{}^0\epsilon_b{}^0, \\ \varepsilon^i_{abcd} &= \epsilon_{(a}{}^{(e}\epsilon_b{}^f\epsilon_c{}^g\epsilon_{d)}{}^{h)_i}, \quad h_{abcd} = -\epsilon_{a(c}\epsilon_{d)b}. \end{aligned} \tag{A.10}$$

It is well known that a spinor A_{abcd} satisfying $A_{abcd} = A_{(ab)(cd)} = -A_{cdab}$ can be decomposed in the form $A_{abcd} = \epsilon_{ac}A_{bd} + \epsilon_{bd}A_{ac}$ with $A_{ab} = \frac{1}{2}A_{afb}{}^f = A_{(ab)}$ and that a spinor S_{abcd} satisfying $S_{abcd} = S_{(ab)(cd)} = S_{cdab}$ can be written in the form $S_{abcd} = S_{(abcd)} + \frac{1}{3}h_{abcd}S$ with $S := S_{ef}{}^{ef}$. It follows from this that an arbitrary four index spinor with symmetries $X_{abcd} = X_{(ab)(cd)}$ can be expanded in terms of ε^i_{abcd}, $\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}$, $\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac}$, $\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac}$ and h_{abcd}.

The following relations were frequently used in the calculations:

$$y_{ab}x_{cd} = -\varepsilon^3_{abcd} - \frac{1}{2\sqrt{2}}(\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac}), \quad z_{ab}x_{cd} = \varepsilon^1_{abcd} + \frac{1}{2\sqrt{2}}(\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac});$$

$$x_{ab}x^{ab} = -1, \quad x_{ab}y^{ab} = 0, \quad x_{ab}z^{ab} = 0, \quad y_{ab}y^{ab} = 0, \quad y_{ab}z^{ab} = -\tfrac{1}{2}, \quad z_{ab}z^{ab} = 0;$$

$$\begin{aligned} x_a{}^f x_{bf} &= \tfrac{1}{2}\epsilon_{ab}, \quad y_a{}^f x_{bf} = \tfrac{1}{\sqrt{2}}y_{ab}, \quad z_a{}^f x_{bf} = -\tfrac{1}{\sqrt{2}}z_{ab}, \\ y_a{}^f y_{bf} &= 0, \quad y_a{}^f z_{bf} = -\tfrac{1}{2}\epsilon_a{}^1\epsilon_b{}^0, \quad z_a{}^f z_{bf} = 0; \end{aligned}$$

$$\begin{aligned} &\varepsilon^0_{abcd}x^{cd} = 0, && \varepsilon^0_{abcd}y^{cd} = -z_{ab}, && \varepsilon^0_{abcd}z^{cd} = 0, && \varepsilon^1_{abcd}x^{cd} = -\tfrac{1}{2}z_{ab}, \\ &\varepsilon^1_{abcd}y^{cd} = -\tfrac{1}{4}x_{ab}, && \varepsilon^1_{abcd}z^{cd} = 0, && \varepsilon^2_{abcd}x^{cd} = -\tfrac{1}{3}x_{ab}, && \varepsilon^2_{abcd}y^{cd} = \tfrac{1}{6}y_{ab}, \\ &\varepsilon^2_{abcd}z^{cd} = \tfrac{1}{6}z_{ab}, && \varepsilon^3_{abcd}x^{cd} = \tfrac{1}{2}y_{ab}, && \varepsilon^3_{abcd}y^{cd} = 0, && \varepsilon^3_{abcd}z^{cd} = \tfrac{1}{4}x_{ab}, \\ &\varepsilon^4_{abcd}x^{cd} = 0, && \varepsilon^4_{abcd}y^{cd} = 0, && \varepsilon^4_{abcd}z^{cd} = -y_{ab}; \end{aligned}$$

$$\begin{aligned} x_{(ab}x_{cd)} &= 2\,\varepsilon^2_{abcd}, \quad x_{(ab}y_{cd)} = -\varepsilon^3_{abcd}, \quad x_{(ab}z_{cd)} = \varepsilon^1_{abcd}, \\ y_{(ab}y_{cd)} &= \tfrac{1}{2}\varepsilon^4_{abcd}, \quad y_{(ab}z_{cd)} = -\tfrac{1}{2}\varepsilon^2_{abcd}, \quad z_{(ab}z_{cd)} = \tfrac{1}{2}\varepsilon^0_{abcd}; \end{aligned}$$

$$\begin{aligned} x_{(a}{}^f\varepsilon^0_{b)cdf} &= \tfrac{1}{\sqrt{2}}\varepsilon^0_{abcd}, \quad x_{(a}{}^f\varepsilon^1_{b)cdf} = \tfrac{1}{2\sqrt{2}}z_{ab}x_{cd}, \quad x_{(a}{}^f\varepsilon^2_{b)cdf} = \tfrac{1}{12}(\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}), \\ x_{(a}{}^f\varepsilon^3_{b)cdf} &= \tfrac{1}{2\sqrt{2}}y_{ab}x_{cd}, \quad x_{(a}{}^f\varepsilon^4_{b)cdf} = -\tfrac{1}{\sqrt{2}}\varepsilon^4_{abcd}, \quad h_{ab(c}{}^f x_{d)f} = \tfrac{1}{2}(\epsilon_{ac}x_{bd} + \epsilon_{bd}x_{ac}); \end{aligned}$$

$$y_{(d}{}^f\varepsilon^2_{c)abf} = -\tfrac{1}{2\sqrt{2}}\varepsilon^3_{abcd} + \tfrac{1}{24}(\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac}), \quad z_{(d}{}^f\varepsilon^2_{c)abf} = -\tfrac{1}{2\sqrt{2}}\varepsilon^1_{abcd} + \tfrac{1}{24}(\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac});$$

$$\varepsilon^2{}_{ab}{}^{ef}\varepsilon^1_{cdef} = -\tfrac{1}{12}\varepsilon^1_{abcd} + \tfrac{1}{8\sqrt{2}}(\epsilon_{ac}z_{bd} + \epsilon_{bd}z_{ac}), \quad \varepsilon^2{}_{ab}{}^{ef}\varepsilon^3_{cdef} = -\tfrac{1}{12}\varepsilon^3_{abcd} + \tfrac{1}{8\sqrt{2}}(\epsilon_{ac}y_{bd} + \epsilon_{bd}y_{ac});$$

$$\varepsilon^2_{abcd}\varepsilon^{2\,abcd} = \tfrac{1}{6}, \qquad \varepsilon^2_{ab}{}^{ef}\varepsilon^2_{cdef} = -\tfrac{1}{6}\varepsilon^2_{abcd} + \tfrac{1}{18}h_{abcd}.$$

A.3 The detailed expressions for u^p, $p = 0, \dots, 3$

The τ-dependent functions occuring in (4.2).

$$c^{01}(\tau) = m\,(\tfrac{4}{3}\,\tau^3 - \tfrac{1}{3}\,\tau^5),\quad c^{\pm 1}(\tau) = m\,(\tau^2 - \tfrac{1}{6}\,\tau^4),\quad S^1(\tau) = \sqrt{2}\,m\,(\tfrac{1}{2}\,\tau^2 - \tfrac{1}{4}\,\tau^4),$$
$$K^1(\tau) = m\,(-12\,\tau + 4\,\tau^3),\quad F^1(\tau) = \tfrac{1}{3}\,m\,\tau^4,\quad t^1(\tau) = \sqrt{2}\,4\,\tau\,m,$$
$$T^1(\tau) = 6\,m\,(1-\tau^2),\quad \phi^1_1(\tau) = -12\,(1-\tau)^2,\quad \phi^1_2(\tau) = -m^2\,(18\,\tau^2 - 3\,\tau^4),$$
$$\phi^1_3(\tau) = -36 + 36\,\tau^2.$$

The τ-dependent functions occuring in (4.3).

$$c^{02}_1(\tau) = m^2\,(-2\,\tau^3 - 3\,\tau^5 + \tfrac{8}{7}\,\tau^7 - \tfrac{1}{7}\,\tau^9),\qquad c^{02}_2(\tau) = 16\,\tau^3 - \tfrac{26}{5}\,\tau^5 + \tfrac{6}{5}\,\tau^7,$$
$$c^{02}_3(\tau) = 8\,\tau^3 - \tfrac{7}{5}\,\tau^5 - \tfrac{3}{5}\,\tau^7,\qquad c^{12}(\tau) = m\,(-4\,\tau^2 + \tfrac{2}{3}\tau^4),$$
$$c^{\pm 2}_1(\tau) = m^2\,(-2\,\tau^2 + 3\,\tau^4 - \tfrac{8}{9}\,\tau^6 + \tfrac{1}{14}\,\tau^8),\qquad c^{\pm 2}_2(\tau) = 12\,\tau^2 - 3\,\tau^4 + \tfrac{3}{5}\,\tau^6,$$
$$c^{\pm 2}_3(\tau) = -6\,\tau^2 - \tfrac{1}{2}\,\tau^4 + \tfrac{3}{10}\,\tau^6,\qquad S^2_1(\tau) = \sqrt{2}\,m^2\,(\tfrac{4}{3}\,\tau^4 - \tfrac{2}{9}\,\tau^6 - \tfrac{1}{28}\,\tau^8),$$
$$S^2_2(\tau) = \sqrt{2}\,(6\,\tau^2 - \tfrac{5}{2}\,\tau^4 + \tfrac{9}{10}\,\tau^6),\qquad S^2_3(\tau) = \sqrt{2}\,(-\tfrac{5}{4}\,\tau^4 + 3\,\tau^2 - \tfrac{9}{20}\,\tau^6),$$
$$S^2_4(\tau) = -36\,\tau^2 + 11\tau^4 + \tfrac{3}{5}\,\tau^6,\qquad K^2_1(\tau) = m^2\,(24\,\tau - 8\,\tau^3 + 4\,\tau^5 - \tfrac{4}{21}\,\tau^7),$$
$$K^2_2(\tau) = -144\,\tau + 72\,\tau^3 - \tfrac{108}{5}\,\tau^5,\qquad K^2_3(\tau) = m^2\,(-\tfrac{20}{3}\,\tau^3 + \tfrac{8}{3}\,\tau^5 - \tfrac{20}{63}\,\tau^7),$$
$$K^2_4(\tau) = -\sqrt{2}\,2\,\tau^3,\qquad K^2_5(\tau) = -48\,\tau + \tfrac{36}{5}\,\tau^5,$$
$$F^2_1(\tau) = m^2\,(-2\,\tau^2 + \tfrac{1}{3}\,\tau^4 - \tfrac{4}{9}\,\tau^6 + \tfrac{1}{7}\,\tau^8),\qquad F^2_2(\tau) = 2\,\tau^4 - \tfrac{6}{5}\,\tau^6,$$
$$F^2_3(\tau) = 3\,\tau^4 + \tfrac{3}{5}\,\tau^6,\qquad t^2_1(\tau) = \sqrt{2}\,m^2\,(-12\,\tau - \tfrac{8}{3}\,\tau^3 + \tfrac{4}{3}\,\tau^5),$$
$$t^2_2(\tau) = \sqrt{2}\,(48\,\tau - 16\,\tau^3),\qquad t^2_3(\tau) = \sqrt{2}\,(24\,\tau + 8\,\tau^3),$$
$$T^2_1(\tau) = m^2\,(-12 + 12\,\tau^2 - 10\,\tau^4 + \tfrac{2}{3}\,\tau^6),\qquad T^2_2(\tau) = 72 - 72\,\tau^2 + 36\,\tau^4,$$
$$T^2_3(\tau) = m^2\,(4\,\tau^2 - \tfrac{8}{3}\,\tau^4 + \tfrac{4}{9}\,\tau^6),\qquad T^2_4(\tau) = -\sqrt{2}\,6\,\tau^2,$$
$$T^2_5(\tau) = 24 - 12\,\tau^4,\qquad \phi^2_1(\tau) = -(-1+\tau)^4,$$
$$\phi^2_2(\tau) = 4\,m\,(\tfrac{37}{10}\,\tau^6 - \tfrac{41}{5}\,\tau^5 - \tfrac{41}{2}\,\tau^4 + 46\,\tau^3 - 18\,\tau^2),\qquad \phi^2_3(\tau) = 16\,(1+\tau)\,(-1+\tau)^3,$$
$$\phi^2_4(\tau) = 6\,(-\tfrac{8}{21}\,\tau^8 + \tfrac{14}{3}\,\tau^6 - 15\,\tau^4 + 6\,\tau^2)\,m^3,\qquad \phi^2_5(\tau) = 6\,m\,(-\tfrac{46}{5}\,\tau^6 + 62\,\tau^4 - 72\,\tau^2),$$
$$\phi^2_6(\tau) = -72\,(1+\tau)^2\,(-1+\tau)^2.$$

The τ-dependent functions occuring in (4.20).

$$
\begin{aligned}
c_1^{03}(\tau) &= (3\,\tau^3 + 18\,\tau^5 + \tfrac{283}{21}\,\tau^7 - \tfrac{1510}{189}\,\tau^9 + \tfrac{2972}{2079}\,\tau^{11} - \tfrac{74}{693}\,\tau^{13})\,m^3,\\
c_2^{03}(\tau) &= (-44\,\tau^3 - \tfrac{588}{5}\,\tau^5 + \tfrac{268}{7}\,\tau^7 - \tfrac{58}{7}\,\tau^9 + \tfrac{6}{5}\,\tau^{11})\,m,\\
c_3^{03}(\tau) &= 48\,\tau^3 - \tfrac{96}{5}\,\tau^5 + \tfrac{312}{35}\,\tau^7 - \tfrac{12}{7}\,\tau^9,\\
c_4^{03}(\tau) &= (-20\,\tau^3 - 6\,\tau^5 + \tfrac{439}{70}\,\tau^7 - \tfrac{573}{280}\,\tau^9 - \tfrac{1}{40}\,\tau^{11})\,m,\\
c_5^{03}(\tau) &= 16\,\tau^3 - 4\,\tau^5 - \tfrac{4}{7}\,\tau^7 + \tfrac{4}{7}\,\tau^9,\\
c_1^{13}(\tau) &= (12\,\tau^2 + 15\,\tau^4 - \tfrac{14}{3}\,\tau^6 + \tfrac{3}{7}\,\tau^8)\,m^2,\\
c_2^{13}(\tau) &= -72\,\tau^2 + 18\,\tau^4 - \tfrac{18}{5}\,\tau^6,\\
c_3^{13}(\tau) &= -36\,\tau^2 + 3\,\tau^4 + \tfrac{9}{5}\,\tau^6,\\
c_1^{\pm 3}(\tau) &= (18\,\tau^2 + 12\,\tau^4 - \tfrac{31}{5}\,\tau^6 + \tfrac{3}{2}\,\tau^8 - \tfrac{3}{40}\,\tau^{10})\,m,\\
c_2^{\pm 3}(\tau) &= -12\,\tau^2 + \tfrac{4}{5}\,\tau^6 - \tfrac{2}{7}\,\tau^8,\\
c_3^{\pm 3}(\tau) &= (\tfrac{9}{2}\,\tau^2 - \tfrac{33}{2}\,\tau^4 + \tfrac{50}{3}\,\tau^6 - \tfrac{515}{84}\,\tau^8 + \tfrac{25}{27}\,\tau^{10} - \tfrac{34}{693}\,\tau^{12})\,m^3,\\
c_4^{\pm 3}(\tau) &= (-48\,\tau^2 + 105\,\tau^4 - \tfrac{453}{10}\,\tau^6 + \tfrac{2847}{280}\,\tau^8 - \tfrac{7}{8}\,\tau^{10})\,m,\\
c_5^{\pm 3}(\tau) &= 36\,\tau^2 - 12\,\tau^4 + \tfrac{24}{5}\,\tau^6 - \tfrac{6}{7}\,\tau^8,\\
c_6^{\pm 3}(\tau) &= -3\,\tau^2 - 2\,\tau^4 + \tfrac{3}{5}\,\tau^6 + \tfrac{1}{14}\,\tau^8,\\
S_1^3(\tau) &= -9\,\tau^2 - 2\,\tau^4 + \tfrac{13}{5}\,\tau^6 + \tfrac{1}{14}\,\tau^8,\\
S_2^3(\tau) &= (108\,\tau^2 - 168\,\tau^4 + 86\,\tau^6 - \tfrac{39}{5}\,\tau^8 - \tfrac{3}{20}\,\tau^{10})\,m,\\
S_3^3(\tau) &= -72\,\tau^2 + 48\,\tau^4 - \tfrac{72}{5}\,\tau^6 - \tfrac{4}{7}\,\tau^8,\\
S_4^3(\tau) &= (-\tfrac{9}{4}\,\tau^2 - \tfrac{37}{4}\,\tau^4 + \tfrac{19}{2}\,\tau^6 - \tfrac{827}{168}\,\tau^8 + \tfrac{355}{378}\,\tau^{10} - \tfrac{6}{77}\,\tau^{12})\,\sqrt{2}\,m^3,\\
S_5^3(\tau) &= (6\,\tau^2 + \tfrac{69}{2}\,\tau^4 - \tfrac{333}{20}\,\tau^6 + \tfrac{1999}{560}\,\tau^8 + \tfrac{13}{80}\,\tau^{10})\,\sqrt{2}\,m,\\
S_6^3(\tau) &= (18\,\tau^2 - 6\,\tau^4 + \tfrac{24}{5}\,\tau^6 - \tfrac{9}{7}\,\tau^8)\,\sqrt{2},\\
S_7^3(\tau) &= (-3\,\tau^2 - \tfrac{33}{2}\,\tau^4 + \tfrac{177}{20}\,\tau^6 - \tfrac{379}{112}\,\tau^8 + \tfrac{1}{40}\,\tau^{10})\sqrt{2}\,m^3,\\
S_8^3(\tau) &= (6\,\tau^2 - 2\,\tau^4 + \tfrac{3}{7}\,\tau^8)\,\sqrt{2}\,m,\\
K_1^3(\tau) &= -6\,\tau - 8\,\tau^3 + \tfrac{18}{5}\,\tau^5 + \tfrac{4}{7}\,\tau^7,\\
K_2^3(\tau) &= (144\,\tau + 12\,\tau^3 - \tfrac{351}{5}\,\tau^5 + \tfrac{237}{5}\,\tau^7 - \tfrac{17}{4}\,\tau^9)\,m,\\
K_3^3(\tau) &= -96\,\tau + 16\,\tau^3 + \tfrac{72}{5}\,\tau^5 - \tfrac{64}{7}\,\tau^7,\\
K_4^3(\tau) &= (-54\,\tau + 12\,\tau^3 - 216\,\tau^5 + \tfrac{796}{7}\,\tau^7 - \tfrac{440}{21}\,\tau^9 + \tfrac{16}{11}\,\tau^{11})\,m^3,\\
K_5^3(\tau) &= (576\,\tau - 216\,\tau^3 + \tfrac{1962}{5}\,\tau^5 - \tfrac{714}{5}\,\tau^7 + \tfrac{23}{2}\,\tau^9)\,m,\\
K_6^3(\tau) &= -432\,\tau + 288\,\tau^3 - \tfrac{864}{5}\,\tau^5 + \tfrac{288}{7}\,\tau^7,\\
K_7^3(\tau) &= (40\,\tau^3 - 16\,\tau^5 + \tfrac{100}{21}\,\tau^7 - \tfrac{160}{189}\,\tau^9 + \tfrac{20}{693}\,\tau^{11})\,m^3,\\
K_8^3(\tau) &= (-240\,\tau^3 + \tfrac{582}{5}\,\tau^5 - \tfrac{218}{7}\,\tau^7 + \tfrac{23}{6}\,\tau^9)\,m,\\
K_9^3(\tau) &= (9\,\tau^3 - \tfrac{33}{20}\,\tau^5 - \tfrac{13}{20}\,\tau^7 + \tfrac{1}{80}\,\tau^9)\sqrt{2}\,m,\\
K_{10}^3(\tau) &= (-4\,\tau^3 + \tfrac{6}{5}\,\tau^5)\,\sqrt{2},
\end{aligned}
$$

$$
\begin{aligned}
F_1^3(\tau) &= (9\,\tau^2 + 2\,\tau^4 - \tfrac{7}{3}\,\tau^6 + \tfrac{26}{7}\,\tau^8 - \tfrac{20}{21}\,\tau^{10} + \tfrac{74}{693}\,\tau^{12})\,m^3,\\
F_2^3(\tau) &= (-60\,\tau^2 + 36\,\tau^4 - 12\,\tau^6 + \tfrac{106}{35}\,\tau^8 - \tfrac{6}{5}\,\tau^{10})\,m,\\
F_3^3(\tau) &= -\tfrac{24}{5}\,\tau^6 + \tfrac{12}{7}\,\tau^8,\\
F_4^3(\tau) &= (-12\,\tau^2 - 6\,\tau^4 + \tfrac{7}{2}\,\tau^6 + \tfrac{169}{56}\,\tau^8 + \tfrac{1}{40}\,\tau^{10})\,m,\\
F_5^3(\tau) &= 4\,\tau^4 - \tfrac{4}{5}\,\tau^6 - \tfrac{4}{7}\,\tau^8,\\
t_1^3(\tau) &= (36\,\tau + 20\,\tau^3 + 46\,\tau^5 - \tfrac{296}{21}\,\tau^7 + \tfrac{272}{189}\,\tau^9)\,\sqrt{2}\,m^3,\\
t_2^3(\tau) &= (-312\,\tau - 24\,\tau^3 - \tfrac{72}{5}\,\tau^5 - \tfrac{40}{7}\,\tau^7)\,\sqrt{2}\,m,\\
t_3^3(\tau) &= (144\,\tau - 96\,\tau^3 + \tfrac{144}{5}\,\tau^5)\,\sqrt{2},\\
t_4^3(\tau) &= (-96\,\tau - 12\,\tau^3 + \tfrac{294}{5}\,\tau^5 - \tfrac{86}{35}\,\tau^7)\,\sqrt{2}\,m,\\
t_5^3(\tau) &= (48\,\tau - \tfrac{48}{5}\,\tau^5)\,\sqrt{2},\\
T_1^3(\tau) &= 3 + 9\,\tau^2 - 3\,\tau^4 - \tau^6,\\
T_2^3(\tau) &= (-72 - 36\,\tau^2 + 81\,\tau^4 - \tfrac{423}{5}\,\tau^6 + \tfrac{33}{4}\,\tau^8)\,m,\\
T_3^3(\tau) &= 48 - 24\,\tau^2 + 16\,\tau^6,\\
T_4^3(\tau) &= (27 - 18\,\tau^2 + 180\,\tau^4 - 134\,\tau^6 + \tfrac{204}{7}\,\tau^8 - \tfrac{16}{7}\,\tau^{10})\,m^3,\\
T_5^3(\tau) &= (-288 + 216\,\tau^2 - 558\,\tau^4 + \tfrac{1326}{5}\,\tau^6 - \tfrac{243}{10}\,\tau^8)\,m,\\
T_6^3(\tau) &= 216 - 216\,\tau^2 + 216\,\tau^4 - 72\,\tau^6,\\
T_7^3(\tau) &= (-24\,\tau^2 + 16\,\tau^4 - \tfrac{20}{3}\,\tau^6 + \tfrac{32}{21}\,\tau^8 - \tfrac{4}{63}\,\tau^{10})\,m^3,\\
T_8^3(\tau) &= (144\,\tau^2 - 102\,\tau^4 + \tfrac{178}{5}\,\tau^6 - \tfrac{57}{10}\,\tau^8)\,m,\\
T_9^3(\tau) &= (27\,\tau^2 - \tfrac{81}{4}\,\tau^4 - \tfrac{11}{20}\,\tau^6 + \tfrac{9}{80}\,\tau^8)\sqrt{2}\,m,\\
T_{10}^3(\tau) &= (-12\,\tau^2 + 6\,\tau^4)\sqrt{2}.
\end{aligned}
$$

The τ-dependent functions occuring in (4.22).

$$
\begin{aligned}
A_1(\tau) &= \big(36\,\tau - 78\,\tau^2 + 82\,\tau^3 - \tfrac{97}{2}\,\tau^4 + \tfrac{6}{5}\,\tau^5 + \tfrac{169}{5}\,\tau^6 - \tfrac{208}{7}\,\tau^7 + \tfrac{54}{7}\,\tau^8\big)\,m,\\
A_2(\tau) &= -648\,\tau + 1728\,\tau^2 - 1692\,\tau^3 + 432\,\tau^4 + \tfrac{2592}{5}\,\tau^5 - \tfrac{2286}{5}\,\tau^6 + \tfrac{756}{5}\,\tau^7 - \tfrac{162}{5}\,\tau^8,\\
B_1(\tau) &= (108\,\tau - 234\,\tau^2 - 396\,\tau^3 + 1503\,\tau^4 - 579\,\tau^5 - \tfrac{14939}{20}\,\tau^6\\
&\quad + \tfrac{11682}{35}\,\tau^7 + \tfrac{40413}{560}\,\tau^8 - \tfrac{2591}{70}\,\tau^9 + \tfrac{177}{80}\,\tau^{10})m^2,\\
B_2(\tau) &= -648\,\tau + 1404\,\tau^2 - 540\,\tau^3 - 810\,\tau^4 + \tfrac{1404}{5}\,\tau^5 + \tfrac{1458}{5}\,\tau^6 - 108\,\tau^7 + \tfrac{108}{5}\,\tau^8,\\
B_3(\tau) &= (-72\,\tau + 168\,\tau^2 + 24\,\tau^3 - 274\,\tau^4 + 120\,\tau^5 + \tfrac{306}{5}\,\tau^6 - 32\,\tau^7 + \tfrac{6}{7}\,\tau^8)\,m,\\
C_1(\tau) &= (-27\,\tau + 342\,\tau^3 - 696\,\tau^5 + \tfrac{2598}{7}\,\tau^7 - \tfrac{4555}{63}\,\tau^9 + \tfrac{1079}{231}\,\tau^{11})\,m^4,\\
C_2(\tau) &= (504\,\tau - 3492\,\tau^3 + \tfrac{17607}{5}\,\tau^5 - \tfrac{41289}{35}\,\tau^7 + \tfrac{16559}{140}\,\tau^9)\,m^2,\\
C_3(\tau) &= -1296\,\tau + 2376\,\tau^3 - \tfrac{4752}{5}\,\tau^5 + 216\,\tau^7,\\
C_4(\tau) &= (-432\,\tau + 792\,\tau^3 - \tfrac{3072}{5}\,\tau^5 + \tfrac{816}{7}\,\tau^7)\,m,\\
C_5(\tau) &= -216\,\tau + 108\,\tau^3 + \tfrac{648}{5}\,\tau^5 + \tfrac{324}{5}\,\tau^7.
\end{aligned}
$$

The τ-dependent functions occuring in (4.27).

$$f_0(\tau) = -18 + 216\,\tau^2 - 240\,\tau^3 + 18\,\tau^4 - 48\,\tau^5 + 204\,\tau^6 - 144\,\tau^7 + 30\,\tau^8,$$
$$f_1(\tau) = -9 - 216\,\tau^2 + 696\,\tau^3 - 198\,\tau^4 - \tfrac{2544}{5}\,\tau^5 + \tfrac{984}{5}\,\tau^6 + \tfrac{936}{7}\,\tau^7 - \tfrac{411}{7}\,\tau^8,$$
$$f_2(\tau) = -3 - 216\,\tau^2 + 372\,\tau^4 - \tfrac{936}{5}\,\tau^6 + \tfrac{219}{7}\,\tau^8,$$
$$g_0(\tau) = 108 - 1944\,\tau^2 + 4752\,\tau^3 - 5724\,\tau^4 + \tfrac{19008}{5}\,\tau^5 - \tfrac{6264}{5}\,\tau^6 + \tfrac{864}{5}\,\tau^7 - \tfrac{108}{5}\,\tau^8,$$
$$g_1(\tau) = 54 - 972\,\tau^2 + 1620\,\tau^3 + 378\,\tau^4 - \tfrac{11448}{5}\,\tau^5 + \tfrac{5778}{5}\,\tau^6 + \tfrac{108}{5}\,\tau^7 - \tfrac{108}{5}\,\tau^8,$$
$$g_2(\tau) = 18 - 540\,\tau^2 + 972\,\tau^4 - \tfrac{2808}{5}\,\tau^6 + \tfrac{108}{5}\,\tau^8,$$
$$h_0(\tau) = \tfrac{3}{2}, \quad h_1(\tau) = \tfrac{3}{4}, \quad h_2(\tau) = \tfrac{1}{4},$$
$$k_1(\tau) = 108\,\tau^2 - 276\,\tau^3 - 129\,\tau^4 + \tfrac{4077}{5}\,\tau^5 - \tfrac{3289}{10}\,\tau^6 - \tfrac{9439}{35}\,\tau^7 + \tfrac{32803}{280}\,\tau^8 + \tfrac{463}{20}\,\tau^9 - \tfrac{2721}{280}\,\tau^{10},$$
$$k_2(\tau) = 252\,\tau^2 - 942\,\tau^4 + \tfrac{3614}{5}\,\tau^6 - \tfrac{6341}{35}\,\tau^8 + \tfrac{99}{7}\,\tau^{10},$$
$$p(\tau) = -\tfrac{27}{2}\,\tau^2 + \tfrac{171}{2}\,\tau^4 - 116\,\tau^6 + \tfrac{1299}{28}\,\tau^8 - \tfrac{911}{126}\,\tau^{10} + \tfrac{1079}{2772}\,\tau^{12},$$
$$q(\tau) = \tfrac{216}{5}\,\tau^8 - \tfrac{576}{5}\,\tau^6 + 648\,\tau^4 - 864\,\tau^2.$$

The τ-dependent functions occuring in (4.29).

$$F_1(\tau) = (72\,\tau^2 - \tfrac{1071}{2}\,\tau^4 + \tfrac{4077}{5}\,\tau^5 - \tfrac{2639}{20}\,\tau^6 - \tfrac{18878}{35}\,\tau^7 + \tfrac{113287}{560}\,\tau^8 + \tfrac{1389}{20}\,\tau^9 - \tfrac{15087}{560}\,\tau^{10})\,m^2,$$
$$F_2(\tau) = -864\,\tau^2 + 1584\,\tau^3 - 810\,\tau^4 - \tfrac{1296}{5}\,\tau^5 + \tfrac{882}{5}\,\tau^6 + \tfrac{432}{5}\,\tau^7 - \tfrac{108}{5}\,\tau^8,$$
$$F_3(\tau) = (-36\,\tau^2 - 40\,\tau^3 + 156\,\tau^4 - \tfrac{888}{5}\,\tau^5 + \tfrac{194}{5}\,\tau^6 + \tfrac{456}{7}\,\tau^7 - \tfrac{198}{7}\,\tau^8)\,m,$$
$$G_1(\tau) = (\tfrac{27}{2}\,\tau^2 + \tfrac{171}{2}\,\tau^4 - 348\,\tau^6 + \tfrac{6495}{28}\,\tau^8 - \tfrac{911}{18}\,\tau^{10} + \tfrac{1079}{308}\,\tau^{12})\,m^4,$$
$$G_2(\tau) = (-144\,\tau^2 - 1071\,\tau^4 + \tfrac{3679}{2}\,\tau^6 - \tfrac{220837}{280}\,\tau^8 + \tfrac{24999}{280}\,\tau^{10})\,m^2,$$
$$G_3(\tau) = 1116\,\tau^4 - 468\,\tau^6 + \tfrac{648}{5}\,\tau^8,$$
$$G_4(\tau) = 174\,m\,\tau^4 - \tfrac{1824}{5}\,m\,\tau^6 + \tfrac{684}{7}\,m\,\tau^8,$$
$$G_5(\tau) = 432\,\tau^2 - 234\,\tau^4 + \tfrac{306}{5}\,\tau^6 + \tfrac{216}{5}\,\tau^8.$$

The τ-dependent functions occuring in (4.31).

$$T_1(\tau) = -46656\,\tau + 124416\,\tau^2 - 138240\,\tau^3 + 51840\,\tau^4 + 31104\,\tau^5 - \tfrac{133056}{5}\,\tau^6 + 12096\tau^7 - \tfrac{50112}{5}\,\tau^8 + \tfrac{10368}{5}\,\tau^9,$$
$$T_2(\tau) = (5184\,\tau - 3456\,\tau^2 - 5088\,m\,\tau^3 - 6048\,\tau^4 + 12288\,\tau^5 + 384\,\tau^6 - 4128\,\tau^7 + 1824\,\tau^8 - 1344\,\tau^9 + 384\,\tau^{10})\,m.$$

References

[1] Bondi, H.,van der Burg, M. G. J. and Metzner, A. W. K. 1962 gravitational waves in general relativity, VII. Waves from isolated systems.. *Proc. Roy. Soc. Lond.* **A 269**, 21-52.

[2] Dain, S. In preparation.

[3] Friedrich, H. 1995 Einstein equations and conformal structure: existence of anti-de Sitter-type space-times. *J. Geom. Phys.* **17**, 125-184.

[4] Friedrich, H. 1998 Gravitational fields near space-like and null infinity. *J. Geom. Phys.* **24**, 83-163.

[5] Friedrich, H. 1998 Einstein equations and conformal structure. In: *The geometric universe: science, geometry and the work of Roger Penrose*, S. A. Hugget et al. (eds.), Oxford University Press, Oxford.

[6] Friedrich, H. 1998 Einstein's equation and geometric asymptotics. In: *Gravitation and Relativity: At the turn of the Millennium, Proceedings of the GR-15 Conference*, N. Dadhich, J. Narlikar (eds.), IUCAA, Pune, India.

[7] Friedrich, H. and Schmidt, B. G. 1987 Conformal geodesics in general relativity. *Proc. Roy. Soc. Lond.* **A 414**, 171-195.

[8] Goldberg, J. N. 1967 Invariant transformations and Newman-Penrose constants. *J. Math. Phys.* **8**, 2161-2166.

[9] Goldberg, J. N. et. al. 1967 Spin-s spherical harmonics and $\eth$. *J. Math. Phys.* **8**, 2155-2161.

[10] Ko, M., Newman, E. T. and Tod, K. P. 1977 $\mathcal{H}$-space and null infinity. In: *Asymptotic structure of space-time*, F. P. Esposito and L. Witten (eds.), Plenum Press, New York.

[11] Newman, E. & Penrose, R. 1962 An approach to gravitational radiation by a method of spin coefficients. *J. Math. Phys.* **3**, 566-578 & **4**, 988.

[12] Newman, E. T. and Penrose, R. 1965 10 exact gravitationally-conserved quantities. *Phys. Rev. Lett.* **15**, 231-233.

[13] Newman, E. T. and Penrose, R. 1966 Note on the Bondi-Metzner-Sachs group. *J. Math. Phys.* **7**, 863-870.

[14] Newman, E. T. and Penrose, R. 1968 New conservation laws for zero rest-mass fields in asymptotically flat space-time. Proc. Roy. Soc. Lond. **A 305**, 175-204.

[15] Newman, E. T. and Unti, T. W. J. 1962 Behavior of asymptotically flat empty spaces. *J. Math. Phys.* **3**, 891-901.

[16] Penrose, R. 1963 Asymptotic properties of fields and space-time. *Phys. Rev. Lett.*, **10**, 66-68.

[17] Penrose, R. 1965 Zero rest-mass fields including gravitation: asymptotic behavior.. *Proc. Roy. Soc. Lond.* **A 284**, 159-203.

[18] Penrose, R. 1967 Conserved quantities and conformal structure in general relativity. In: *Relativity and astrophysics*, J. Ehlers (ed.), Am. Math. Society.

[19] Penrose, R. 1982 Quasi-local mass and angular momentum in general relativity. *Proc. Roy. Soc. Lond.* **A381**, 53-63.

[20] Press, W. H. and Bardeen, J. M. 1971 Non-conservation of the Newman-Penrose conserved quantities. *Phys. Rev. Lett.* **27**, 1303-1306.

[21] Robinson, D. C. 1969 Conserved quantities of Newman and Penrose. *J. Math. Phys.* **10**, 1745-1753.

[22] Sachs, R. K. 1962 Gravitational waves in general relativity, VIII. Waves in asymptotically flat space-time. *Proc. Roy. Soc. Lond.* **A 270**, 103-126.