

HOLOGRAPHIC DUALITY, SUPERSYMMETRY, AND PAINLEVÉ EQUATION

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An example of the holographic correspondence between 2d, N=2 quantum field theories and classical 4d, N=2 supergravity theories is found. The constraints on the target space geometry of the 4d, N=2 non-linear sigma-models in N=2 supergravity background are interpreted as the renormalization flow equations in two dimensions. Our geometrical description of the renormalization flow is manifestly covariant under reparametrization of the 2d coupling constants. The proposed holography is described in terms of the (Weyl) anti-self-dual Einstein metrics, whose exact regular (Tod-Hitchin) solutions are governed by the Painlevé VI equation.

1 Introduction

The evidence for the holographic principle (see [1] for a review) goes far beyond the (superconformal) Maldacena conjecture. Certain Quantum Field Theories (QFT) apparently allow the dual description in terms of the effective gravity theories in higher dimensions. It is, however, not clear *where* does the holographic correspondence apply and *why* does it exist at all?

We give a new (different from the standard AdS/CFT) example of the holographic correspondence between 2d, N=2 supersymmetric QFT and classical 4d, N=2 supergravity theories, by identifying the geometrical constraints on the target space geometry of the 4d Non-Linear Sigma-Models (NLSM), in the N=2 supergravity background, with the RG flow equations in the 2d, N=2 QFT. For simplicity, we restrict ourselves to the four-dimensional NLSM target spaces, by considering a single NLSM hypermultiplet only. This limited class of the 4d, N=2 NLSM describes integrable deformations of the 2d, N=2 Superconformal Field Theories (SCFT). The exact RG flow is highly constrained by N=2 supersymmetry, being described by the effective regular (Tod-Hitchin) metrics governed by the exact solutions to the Painlevé VI equation [3, 4].

The N=2 scalar (hypermultiplet) couplings in the 2d, N=2 supergravity are

well-known to be described by the NLSM with the quaternionic-Kähler target spaces of negative scalar curvature [2]. The four-dimensional quaternionic-Kähler target space is (Weyl) Anti-Self-Dual (ASD) and Einstein, i.e.

$$W_{abcd}^+ = 0 \quad \text{and} \quad R_{ab} = \frac{1}{2}\Lambda g_{ab}, \quad \Lambda = -24\kappa^2, \quad (1)$$

where $W = W^- + W^+$ is the Weyl tensor, R_{ab} is the Ricci tensor, $a, b, c, d = 1, 2, 3, 4$, and κ is the gravitational coupling constant.

The R-symmetry of N=2 supersymmetry implies the $SU(2)$ isometry of the NLSM metric. The (non-degenerate) action of this isometry in four internal dimensions leads to the three-dimensional orbits that can be parametrized by the ‘radial’ coordinate (t) to be identified with the RG parameter of the dual QFT in 2d. The $SU(2)$ -invariant metrics are most conveniently described in the Bianchi IX formalism of general relativity, having manifest $SU(2)$ symmetry. Given a ‘radial’ coordinate r and ‘Euler angles’ (θ, ψ, ϕ) , the $SU(2)$ -covariant one-forms are $\sigma_1 = \frac{1}{2}(\sin \psi d\theta - \sin \theta \cos \psi d\phi)$, $\sigma_2 = -\frac{1}{2}(\cos \psi d\theta + \sin \theta \sin \psi d\phi)$ and $\sigma_3 = \frac{1}{2}(d\psi + \cos \theta d\phi)$, being subject to the relation $\sigma_i \wedge \sigma_j = \frac{1}{2}\varepsilon_{ijk}d\sigma_k$.

The simplest *symmetric* quaternionic space, which is relevant for our purposes, is given by the coset $SU(2, 1)/U(2)$ whose natural metric is dual to the standard Fubini-Study metric,

$$ds^2 = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{(1-r^2)^2}\sigma_2^2 + \frac{r^2}{(1-r^2)}(\sigma_1^2 + \sigma_3^2). \quad (2)$$

Note that the coefficient at σ_2^2 decays faster than the coefficients at σ_1^2 and σ_3^2 . The conformal structure associated with the metric (2) inside the unit ball in \mathbf{C}^2 survives on the two-dimensional subspace of the boundary of \mathbf{C}^2 that is annihilated by σ_2 .

The metric (2) can be identified with the Zamolodchikov metric of certain 2d, N=2 SCFT [5]. It is not difficult to identify the corresponding SCFT since the coset $SU(2, 1)/U(2)$ appears in the Kazama-Suzuki coset construction list (see ref. [6] for a review). In fact, there is even 2d, N=4 supersymmetry in the 2d SCFT associated with this coset [7].

2 Weyl self-duality and RG flow

The (Weyl) ASD equations can be put into an equivalent form of the first-order system of Ordinary Differential Equations (ODE) that are going to be identified with the RG flow equations in 2d QFT. We are thus led to a study of

the $SU(2)$ -invariant deformations of the metric (2) subject to the constraints (1). This well-defined mathematical problem was already addressed by Tod [3] and Nitchin [4]. A generic $SU(2)$ invariant metric in the Bianchi IX formalism reads

$$ds^2 = w_1 w_2 w_3 dt^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2 . \quad (3)$$

Being applied to eq. (3), the Weyl ASD conditions of eq. (1) give rise to the ODE system [3]

$$\dot{A}_1 = -A_2 A_3 + A_1 (A_2 + A_3) , \quad \text{and cyclic permutations} , \quad (4)$$

where the dot means differentiation with respect to t , while A_i , $i = 1, 2, 3$, are defined from the auxiliary ODE system

$$\dot{w}_1 = -w_2 w_3 + w_1 (A_2 + A_3) , \quad \text{and cyclic permutations} . \quad (5)$$

The metric (2) corresponds to the case when all A_i vanish. The Einstein condition in eq. (1) can be easilly satisfied by conformal rescaling of the (Weyl) ASD metric (see below). Having solved eq. (4), its solution can be substituted into eq. (5). To solve the last equations, it is convenient to change variables as [3]

$$w_1 = \frac{\Omega_1 \dot{x}}{\sqrt{x(1-x)}} , \quad w_2 = \frac{\Omega_2 \dot{x}}{\sqrt{x^2(1-x)}} , \quad w_3 = \frac{\Omega_3 \dot{x}}{\sqrt{x(1-x)^2}} , \quad (6)$$

where Ω_i are constrained by the algebraic relation

$$\Omega_2^2 + \Omega_3^2 - \Omega_1^2 = \frac{1}{4} . \quad (7)$$

Equation (5) then takes the form [3, 4]

$$\Omega'_1 = -\frac{\Omega_2 \Omega_3}{x(1-x)} , \quad \Omega'_2 = -\frac{\Omega_3 \Omega_1}{x} , \quad \Omega'_3 = -\frac{\Omega_1 \Omega_2}{1-x} , \quad (8)$$

where the prime denotes differentiation with respect to x . The constraint (7) is preserved under eq. (8), so that the transformation (6) is consistent. In terms of the new variables (x, Ω_i) , the Einstein condition of eq. (1) on the metric in terms of the new variables,

$$ds^2 = e^{2u} \left[\frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2} \right] , \quad (9)$$

amounts to the algebraic relation

$$96\kappa^2 e^{2u} = \frac{8x\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3(x(\Omega_1^2 + \Omega_2^2) - (1 - 4\Omega_3^2)(\Omega_2^2 - (1-x)\Omega_1^2))}{(x\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2 - (1-x)\Omega_1^2))^2}. \quad (10)$$

The ODE system (4), $\dot{A}_i = C_i^{jk} A_j A_k$, can be naturally interpreted as the RG flow equations in the dual 2d QFT originating from the 2d SCFT in its UV-fixed point. The Kähler nature of this 2d QFT implies that it should be N=2 supersymmetric. The universal coefficients C_i^{jk} can be identified with the (normalized) OPE coefficients of the 2d, N=2 SCFT.

The exact solutions to the ODE system (4) are known to be dictated by the particular Painlevé VI equation, whose parameters are all fixed by the quaternionic-Kähler property of the metric [3, 4],

$$y'' = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) (y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[\frac{1}{8} - \frac{x}{8y^2} + \frac{x-1}{8(y-1)^2} + \frac{3x(x-1)}{8(y-x)^2} \right], \quad (11)$$

where $y = y(x)$, and the primes denote differentiation with respect to x .

The equivalence between eqs. (4) and (11) is established via eq. (8) and the relations [4]

$$\begin{aligned} \Omega_1^2 &= \frac{(y-x)^2 y(y-1)}{x(1-x)} \left(v - \frac{1}{2(y-1)} \right) \left(v - \frac{1}{2y} \right), \\ \Omega_2^2 &= \frac{(y-x)y^2(y-1)}{x} \left(v - \frac{1}{2(y-x)} \right) \left(v - \frac{1}{2(y-1)} \right), \\ \Omega_3^2 &= \frac{(y-x)y(y-1)^2}{(1-x)} \left(v - \frac{1}{2y} \right) \left(v - \frac{1}{2(y-x)} \right), \end{aligned} \quad (12)$$

where the auxiliary variable v is defined by the auxiliary equation

$$y' = \frac{y(y-1)(y-x)}{x(x-1)} \left(2v - \frac{1}{2y} - \frac{1}{2(y-1)} + \frac{1}{2(y-x)} \right). \quad (13)$$

An exact solution to eq. (11), leading to a *complete* (regular) metric, is known to be unique, while it can be expressed in terms of the standard theta-functions $\vartheta_\alpha(z|\tau)$, $\alpha = 1, 2, 3, 4$. In order to write down the solution $y(x)$ explicitly, the theta-function arguments should be related by $z = \frac{1}{2}(\tau - k)$,

where k is considered to be an arbitrary (real and positive) parameter. Their relation to x is defined by $x = \vartheta_3^4(0)/\vartheta_4^4(0)$, where the value of z is explicitly indicated, as usual. The relevant exact solution to the Painlevé VI equation reads [4]

$$y(x) = \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3} \left[1 + \frac{\vartheta_3^4(0)}{\vartheta_4^4(0)} \right] + \frac{\vartheta_1'''(z)\vartheta_1(z) - 2\vartheta_1''(z)\vartheta_1'(z) + 2\pi i(\vartheta_1''(z)\vartheta_1(z) - \vartheta_1'^2(z))}{2\pi^2\vartheta_4^4(0)\vartheta_1(z)(\vartheta_1'(z) + \pi i\vartheta_1(z))}. \quad (14)$$

The parameter $k > 0$ describes the monodromy of the solution (14) around its essential singularities (branch points) $x = 0, 1, \infty$. This (non-abelian) monodromy is generated by the matrices (with the eigenvalues $\pm i$)

$$M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & i^{1-k} \\ i^{1+k} & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & i^{-k} \\ -i^k & 0 \end{pmatrix}. \quad (15)$$

The function (14) is meromorphic outside $x = 0, 1, \infty$, with the simple poles at $\bar{x}_1, \bar{x}_2, \dots$, where $\bar{x}_n \in (x_n, x_{n+1})$ and $x_n = x(ik/(2n-1))$ for each positive integer n . Accordingly, the metric is well-defined (complete) for $x \in (\bar{x}_n, x_{n+1}]$, i.e. in the unit ball with the origin at $x = x_{n+1}$ and the boundary at $x = \bar{x}_n$ [4]. Near the boundary the metric (9) has the asymptotical behaviour

$$ds^2 = \frac{dx^2}{(1-x)^2} + \frac{4}{(1-x)\cosh^2(\pi k/2)}\sigma_1^2 + \frac{16}{(1-x)^2\sinh^2(\pi k/2)\cosh^2(\pi k/2)}\sigma_2^2 + \frac{4}{(1-x)\sinh^2(\pi k/2)}\sigma_3^2 + \text{regular terms}. \quad (16)$$

It is clear from eq. (16) that the coefficient at σ_2^2 vanishes faster than the others, like in eq. (2), so that there is the natural conformal structure,

$$\sinh^2(\pi k/2)\sigma_1^2 + \cosh^2(\pi k/2)\sigma_3^2, \quad (17)$$

on the two-dimensional boundary annihilated by σ_2 . The only relevant parameter $\tanh^2(\pi k/2)$ in eq. (17) represents the central charge of the 2d CFT on the boundary. In the interior of the ball we have the spectral flow with the monotonically decreasing ‘effective’ central charge (called c -function), in accordance with the c -theorem [5]. The RG evolution ends at another (IR) fixed point where the solution (14) has a removable pole.

3 Conclusion

Local 4d, N=2 supersymmetry appears to be the sole source of the constraints (1) on the effective metric. The regular $SU(2)$ -invariant metric solutions are also unique, being parametrized by the CFT central charge describing the monodromy of the ‘master’ solution to the underlying Painlevé equation. Our geometrical description of the RG flow by eq. (1) is manifestly covariant with respect to arbitrary reparametrizations of the 2d QFT coupling constants. In our explicit example of the holographic correspondence, the RG flow in a 2d, N=2 QFT is described by the ODE system (4) whose coefficients are the universal (normalized) OPE coefficients of the underlying CFT at the UV-fixed point of the QFT. Unlike the 2d, N=2 supersymmetric RG flow solutions found by Cecotti and Vafa [8], our RG flow has an IR fixed point and, therefore, it can be interpreted as a domain-wall solution.

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