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Yang-Mills Integrals for Orthogonal, Symplectic and Exceptional Groups

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Abstract

We apply numerical and analytic techniques to the study of Yang-Mills integrals with orthogonal, symplectic and exceptional gauge symmetries. The main focus is on the supersymmetric integrals, which correspond essentially to the bulk part of the Witten index for susy quantum mechanical gauge theory. We evaluate these integrals for $D = 4$ and group rank up to three, using Monte Carlo methods. Our results are at variance with previous findings. We further compute the integrals with the deformation technique of Moore, Nekrasov and Shatashvili, which we adapt to the groups under study. Excellent agreement with all our numerical calculations is obtained. We also discuss the convergence properties of the purely bosonic integrals.

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I. INTRODUCTION

Recent attempts to describe D-branes through effective actions have revealed the existence of a new class of gauge-invariant matrix models. These models are related to (-1)-branes. They differ from the classic systems of random matrices, which have been extensively studied ever since Wigner's and Dyson's work in the 1950's. The models consist in matrix integrals of D non-linearly coupled matrices. They are obtained by complete dimensional reduction of D-dimensional Euclidean continuum (susy) Yang-Mills theory to zero dimensions, and we term them quite generally *Yang-Mills integrals*.

The supersymmetric $D = 4, 6, 10$ integrals with SU(N) symmetry have already found several important applications. They are relevant to the calculation of the Witten index of quantum mechanical gauge theory [1], [2], and to multi-instanton calculations [3], [4] of four-dimensional SU(∞) susy conformal gauge theory. The $D = 10$ integrals are furthermore the crucial ingredient in the so-called IKKT model [5], which possibly provides a non-perturbative definition of IIB superstring theory. Finally, it remains to be seen whether Yang-Mills integrals contain information on the full, unreduced field theory through the Eguchi-Kawai mechanism [6] as the size of the matrices gets large. Some very interesting recent considerations along these lines can be found in [7].

Yang-Mills integrals are ordinary, not functional integrals. Despite this tremendous simplification, no systematic analytic tools for their investigation are known to date. We have developed [8], [9], [10], [11] accurate and reliable Monte Carlo methods which allow to study the new matrix models as long as the dimension of the gauge group is not too large. We have found, e.g., that supersymmetry is generically *not* necessary for the existence of the integrals [9]. We also computed their asymptotic eigenvalue distributions, which we found to qualitatively differ between the susy and bosonic case as the size of the gauge group gets large [10]. For related, complementary studies see [12].

To date all existing studies have focused on the case where the gauge group is SU(N). In the present paper we generalize to the cases of all other semi-simple compact Lie groups of rank $r \leq 3$. These are, (besides the already known cases SU(2), SU(3), SU(4)) the groups SO(3), SO(4), SO(5), SO(6), SO(7), Sp(2), Sp(4), Sp(6), and G_2, for which we compute the susy $D = 4$ partition functions. We were motivated in part by a recent paper of Kac and Smilga [13] which presented conjectures about the values of the bulk part of the Witten index (and therefore for the corresponding integrals). Intriguingly, our results are at variance with their predictions in most cases, indicating that the index calculations for these groups are even more subtle than the corresponding considerations for SU(N), where the approach of [13] agrees with the known values.

Moore, Nekrasov and Shatashvili [14] recently employed sophisticated deformation techniques to evaluate the SU(N) susy bulk index for all N and $D = 4, 6, 10$. The method apparently leads to the correct result for all SU(N) [8], [9]. Below, we adapt the technique to the more general groups, and we find again excellent agreement with the Monte Carlo calculation. This further indicates that the deformation method is indeed reliable.

II. YANG-MILLS INTEGRALS FOR SEMI-SIMPLE COMPACT GAUGE GROUPS

For a general semi-simple compact Lie group G we define supersymmetric or bosonic Yang-Mills integrals as

$$\mathcal{Z}_{D,G}^{\mathcal{N}} := \int \prod_{A=1}^{\dim(G)} \left(\prod_{\mu=1}^{D} \frac{dX_\mu^A}{\sqrt{2\pi}} \right) \left(\prod_{\alpha=1}^{\mathcal{N}} d\Psi_\alpha^A \right) \exp\left[\frac{1}{4g^2} \mathrm{Tr}\,[X_\mu, X_\nu][X_\mu, X_\nu] + \frac{1}{2g^2} \mathrm{Tr}\,\Psi_\alpha [\Gamma^\mu_{\alpha\beta} X_\mu, \Psi_\beta] \right], \tag{1}$$

where $\dim(G)$ is the dimension of the Lie group and the D bosonic matrices $X_\mu = X_\mu^A T^A$ and the $\mathcal{N}$ fermionic matrices $\Psi_\alpha = \Psi_\alpha^A T_A$ are anti-hermitean and take values in the fundamental representation of the Lie algebra $\mathrm{Lie}(G)$, whose generators we denote by T^A. The integral eq.(1) depends on the gauge coupling constant in a trivial fashion, as we can immediately scale out g. Nevertheless, there is a natural convention for fixing g: For an orthogonal set of generators we should pick g according to their normalization: $\mathrm{Tr}\, T^A T^B = -g^2 \delta^{AB}$. This convention is imposed by the index calculations of the next section. In the bosonic case we simply drop the fermionic variables.

The supersymmetric integrals can formally be defined in $D = 3, 4, 6, 10$, which corresponds to $\mathcal{N} = 2, 4, 8, 16$ real supersymmetries. We are not aware of a mathematically rigorous investigation of their convergence properties. However, our numerical studies indicate that they are absolutely convergent in $D = 4, 6, 10$ (but not in $D = 3$) for all semi-simple compact gauge groups. The convergence properties of the bosonic ($\mathcal{N} = 0$) integrals are discussed in section 6. The variables Ψ_α^A are real Grassmann-valued and can be integrated out, leading to a bosonic integral with very special measure:

$$\mathcal{Z}_{D,G}^{\mathcal{N}} = \int \prod_{A=1}^{\dim(G)} \prod_{\mu=1}^{D} \frac{dX_\mu^A}{\sqrt{2\pi}} \exp\left[\frac{1}{4g^2} \mathrm{Tr}\,[X_\mu, X_\nu][X_\mu, X_\nu] \right] \mathcal{P}_{D,G}(X). \tag{2}$$

$\mathcal{P}_{D,G}(X)$ is a homogeneous Pfaffian polynomial of degree $\frac{1}{2}\mathcal{N}\, \dim(G)$ given by

$$\mathcal{P}_{D,G} = \mathrm{Pf}\mathcal{M}_{D,G} \qquad \text{with} \qquad (\mathcal{M}_{D,G})^{AB}_{\alpha\beta} = -i f^{ABC} \Gamma^\mu_{\alpha\beta} X_\mu^C, \tag{3}$$

where the structure constants [1] are defined through the real Lie algebra $\left[T^A, T^B\right] = f^{ABC} T^C$. Explicit expressions for the Gamma matrices $\Gamma^\mu_{\alpha\beta}$ and further details on $\mathcal{P}_{D,G}(X)$ may be found in [8].

III. GROUP VOLUMES AND BULK INDICES

It is well known that the susy Yang-Mills integrals eq.(2) naturally appear when one computes the Witten index of quantum mechanical gauge theory (i.e. the reduction of the

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[1] Note that in [8] we used hermitean generators but defined the structure constants through $\left[T^A, T^B\right] = i f^{ABC} T^C$, so eq.(3) remains valid.

field theory to one, as opposed to zero, dimension) by the heat kernel method. For the details of the method we refer to [1], [2]. Specifically, the integrals are related to the bulk part $\mathrm{ind}_0^D(G)$ of the index as

$$\mathrm{ind}_0^D(G) = \lim_{\beta\to 0}\mathrm{Tr}\,(-1)^F e^{-\beta H} = \frac{1}{\mathcal{F}_G}\mathcal{Z}^{\mathcal{N}}_{D,G}. \tag{4}$$

The total Witten index "$\mathrm{ind}^D(G)$" is then the sum of this bulk part and a boundary contribution "$\mathrm{ind}_1^D(G)$": $\mathrm{ind}^D(G) = \mathrm{ind}_0^D(G) + \mathrm{ind}_1^D(G)$. The constant $\mathcal{F}_G$ relating the bulk index ind_0^D and the Yang-Mills integral is independent of D and can be interpreted as

$$\mathcal{F}_G = \frac{1}{(2\pi)^{\frac{1}{2}\dim(G)}}\ \mathrm{Volume}\Big[\frac{G}{Z_G}\Big], \tag{5}$$

i.e. essentially the volume of the true gauge group, which turns out to be the quotient group G/Z_G, with Z_G the center group of G. In practice, great care has to be taken in using the relation eq.(5), as the volume depends on the choice of the local metric on the group manifold. For the present purposes we simply adapted our method for computing $\mathcal{F}_{\mathrm{SU}(N)}$ (see [9]) to the relevant gauge groups. An invariant average over the group allows to project onto gauge invariant states and to derive eq.(4) from the quantum mechanical path integral. In the ultralocal limit, the quantum mechanics of $D-1$ matrices turns into an integral over D matrices. Then, this integration is over the anti-hermitean generators of the group

$$\mathcal{D}U \to \frac{1}{\mathcal{F}_G}\prod_{A=1}^{\dim(G)}\frac{dX_D^A}{\sqrt{2\pi}}. \tag{6}$$

The normalized Haar measure $\mathcal{D}U$ on the group elements $U \in G$ simplifies significantly if we restrict attention to the Cartan subgroup of G. A beautiful result of Weyl allows to explicitly write down the restricted measure. If we parametrize the Cartan torus T by angles $-\pi \le \theta_1 \le \pi, \ldots, -\pi \le \theta_r \le \pi$, where $r=\mathrm{rank}(G)$, the measure reads

$$\mathcal{D}U \to \mathcal{D}T = \frac{1}{|W_G|}\left(\prod_{i=1}^{r}\frac{d\theta_i}{2\pi}\right)|\Delta_G|^2, \tag{7}$$

where $|W_G|$ is the order of the Weyl group W_G of G, and $|\Delta_G|^2$ the squared modulus of the Weyl denominator:

$$\Delta_G = \prod_{\alpha>0}\Big[e^{\frac{i}{2}(\theta,\alpha)} - e^{-\frac{i}{2}(\theta,\alpha)}\Big]. \tag{8}$$

Here the product is over the set of positive roots of the Lie algebra $\mathrm{Lie}(G)$. In the vicinity of the identity in G the angles θ_i are small and we can approximate the measure eq.(7) by

$$\frac{1}{|W_G|}\left(\prod_{i=1}^{r}\frac{d\theta_i}{2\pi}\right)\prod_{\alpha>0}\Big[\frac{1}{2}(\theta,\alpha) - \frac{1}{2}(\theta,\alpha)\Big]^2. \tag{9}$$

Now restricting the flat measure on $\mathrm{Lie}(G)$ on the right hand side of eq.(6) to the Cartan modes θ_i we get

$$\prod_{A=1}^{\dim(G)} \frac{dX_D^A}{\sqrt{2\pi}} \to \frac{\mathcal{F}_G}{(2\pi)^r} \frac{|Z_G|}{|W_G|} \Big(\prod_{i=1}^{r} d\theta_i \Big) \prod_{\alpha>0} \Big[\frac{1}{2}(\theta,\alpha) - \frac{1}{2}(\theta,\alpha) \Big]^2. \tag{10}$$

An important subtlety is that we needed to multiply the measure eq.(10) by an additional factor $|Z_G|$ of the order of the center group Z_G of G, as the averaging over the group manifold localizes on $|Z_G|$ points. Finally, the constant $\mathcal{F}_G$ in eq.(10) is fixed by noting that the flat measure on Lie(G) is normalized with respect to Gaussian integration

$$\int \Big(\prod_{A=1}^{\dim(G)} \frac{dX_D^A}{\sqrt{2\pi}} \Big) \exp \Big[-\frac{1}{2} \sum_A (X_D^A)^2 \Big] = 1. \tag{11}$$

The Gaussian integration of the right hand side of eq.(10) leads to Selberg-type integrals, see e.g. [15]. Explicit details on how to implement the above procedure for the groups under study can be found in the appendices. With our conventions for the normalization of the generators (SO(N): $\mathrm{Tr}\, T^A T^B = -2\delta^{AB}$ and Sp($2N$), G_2: $\mathrm{Tr}\, T^A T^B = -\frac{1}{2}\delta^{AB}$) one finds

$$\mathcal{F}_{\mathrm{SO}(N)} = \frac{1}{2C_N} \frac{\pi^{\frac{N}{2}}}{2^{\frac{N(N-5)}{4}} \prod_{j=1}^{N} \Gamma(j/2)}, \tag{12}$$

where $C_{2N} = 2$ and $C_{2N+1} = 1$, as well as

$$\mathcal{F}_{\mathrm{Sp}(2N)} = \frac{1}{2} \frac{2^{2N^2+\frac{N}{2}} \pi^{\frac{N}{2}}}{\prod_{j=1}^{N} \Gamma(2j)} \tag{13}$$

and finally

$$\mathcal{F}_{\mathrm{G}_2} = \frac{36864 \sqrt{3}\, \pi}{5}. \tag{14}$$

IV. DEFORMATION METHOD

In [14] it was suggested that the original susy Yang-Mills integrals eq.(1) may be vastly simplified by a deformation technique. It consists in adding a number of terms to the action which break the number of supersymmetries from $\mathcal{N} = 2, 4, 8, 16$ to $\mathcal{N} = 1$. Keeping one of the supersymmetries means that the partition function is "protected" and should not change under the deformation. This gives the correct result[2] for SU(N) and $D = 4, 6, 10$. The final outcome is a much simpler integral involving only a single Lie-algebra valued matrix. The remaining integral is still invariant under the gauge group, and one can therefore pass from the full algebra to the Cartan subalgebra degrees of freedom. This was derived in [14] in detail for SU(N) but should carry over immediately to other gauge groups. For $D = 4$ ($\mathcal{N} = 4$) one finds, in the notation of the previous section (here the product is over all roots α of Lie(G))

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[2]For unclear reasons it fails for $D = 3$.

$$\mathrm{ind}_0^{D=4}(G) = \frac{|Z_G|}{|W_G|}\,\frac{1}{E^r}\int\,\left(\prod_{i=1}^{r}\frac{dx_i}{2\pi i}\right)\,\prod_{\alpha}\frac{\left[\frac{1}{2}(x,\alpha)-\frac{1}{2}(x,\alpha)\right]}{\left[\frac{1}{2}(x,\alpha)-\frac{1}{2}(x,\alpha)-E\right]}. \tag{15}$$

This r-dimensional integral (r =rank(G)) is divergent. There are divergences due to the poles of the denominator of eq.(15), as well as at infinity, where the integrand tends to one. These divergences are present since the method starts from the partition sums eq.(1) with Minkowski signature, i.e. with the Wick-rotated versions of our integrals. The Minkowski integrals are divergent without a prescription. The poles are regulated by giving an imaginary part to the parameter E. The singularity at infinity is regulated by interpreting the integrals as *contour* integrals[3]. It would be interesting to complete the arguments by demonstrating that the Wick-rotation leads to precisely these prescriptions. Very encouraging signs for the consistency of this method are that the final result neither depends on the location of the parameter E nor on whether the contours are closed in the upper or the lower half plane (it is important though that all r contours are closed in the same way). For $D=6,10$ expressions very similar to eq.(15) can be found in [14].

We now present the explicit form of the contour integrals eq.(15) for the groups studied in the present work (see appendices for details)

$$\mathrm{ind}_0^{D=4}(\mathrm{SO}(2N+1)) = \frac{1}{2^N N!}\,\frac{1}{E^N}\,\oint\prod_{i=1}^{N}\frac{dx_i}{2\pi i}\,\prod_{i<j}^{N}\frac{(x_i^2-x_j^2)^2}{\left[(x_i-x_j)^2-E^2\right]\left[(x_i+x_j)^2-E^2\right]}\times$$

$$\times\prod_{i=1}^{N}\frac{x_i^2}{x_i^2-E^2} \tag{16}$$

$$\mathrm{ind}_0^{D=4}(\mathrm{SO}(2N)) = \frac{2}{2^{N-1} N!}\,\frac{1}{E^N}\,\oint\prod_{i=1}^{N}\frac{dx_i}{2\pi i}\,\prod_{i<j}^{N}\frac{(x_i^2-x_j^2)^2}{\left[(x_i-x_j)^2-E^2\right]\left[(x_i+x_j)^2-E^2\right]} \tag{17}$$

$$\mathrm{ind}_0^{D=4}(\mathrm{Sp}(2N)) = \frac{2}{2^N N!}\,\frac{1}{E^N}\,\oint\prod_{i=1}^{N}\frac{dx_i}{2\pi i}\,\prod_{i<j}^{N}\frac{(x_i^2-x_j^2)^2}{\left[(x_i-x_j)^2-E^2\right]\left[(x_i+x_j)^2-E^2\right]}\times$$

$$\times\prod_{i=1}^{N}\frac{x_i^2}{x_i^2-(\frac{E}{2})^2} \tag{18}$$

[3]This interpretation furthermore necessitates the inclusion of the factors of i in the measure of eq.(15) which would not be present in an ordinary integral over a real Lie algebra.

$$\mathrm{ind}_0^{D=4}(\mathrm{G}_2) = \frac{1}{12}\,\frac{1}{E^2}\oint \frac{dx_1}{2\pi i}\frac{dx_2}{2\pi i}\frac{(x_1-x_2)^2(x_1+x_2)^2x_1^2x_2^2}{\Big[(x_1-x_2)^2-E^2\Big]\Big[(x_1+x_2)^2-E^2\Big]\Big[x_1^2-E^2\Big]\Big[x_2^2-E^2\Big]}\times$$

$$\times\frac{(2x_1+x_2)^2(x_1+2x_2)^2}{\Big[(2x_1+x_2)^2-E^2\Big]\Big[(x_1+2x_2)^2-E^2\Big]} \qquad (19)$$

They are easily evaluated for low rank, and we present the results in table 1. We highlighted the cases which were not already indirectly known due to the standard low-rank isomorphisms so(3) = sp(2) = su(2), so(4) = su(2) $\oplus$ su(2), so(6) = su(4). Note, however, that these identities, as well as the final semi-simple Lie algebra isomorphism so(5) = sp(4), constitute non-trivial consistency checks on the expression eq.(15), as the precise form of the corresponding contour integrals is different in all these cases. It would be interesting to compute eqs. (16),(17),(18),(19) for arbitrary rank, as has been done in the case of SU(N) in [14]. We also checked that the analogous, more complicated $D = 6$ contour integrals lead to the same bulk indices, as one expects.

For the groups not related by an isomorphism to SU(N) the rational numbers in table 1 differ from the ones proposed in [13]. I would be important to understand why. We also do not see how the arguments of section 8 of [14], which seemed to furnish a shortcut explanation of the SU(N) results, could be adapted to reproduce the numbers highlighted in table 1.

We next turn to numerical verification of these proposed bulk indices.

Table 1: $D = 4$ and $D = 6$ bulk indices for the orthogonal, symplectic and exceptional groups of rank ≤ 3

Group	rank	$\mathrm{ind}_0^{D=4,6}$
SO(3)	1	1/4
SO(4)	2	1/16
SO(5)	2	**9/64**
SO(6)	3	1/16
SO(7)	3	**25/256**
Sp(2)	1	1/4
Sp(4)	2	**9/64**
Sp(6)	3	**51/512**
G_2	2	**151/864**

V. MONTE CARLO EVALUATION OF YANG-MILLS INTEGRALS

As in previous works [8,9], we evaluate the Yang-Mills integrals using Monte Carlo methods. Both the Pfaffian polynomial $\mathcal{P}_{D,G}$ and the action $\mathcal{S} = -\frac{1}{4g^2}\mathrm{Tr}\,[X_\mu, X_\nu][X_\mu, X_\nu]$ in eq.(2) are homogeneous functions of the X_μ^A

$$X_\mu^A \to \alpha X_\mu^A \quad (\forall \mu\ ; A) \Rightarrow \begin{cases} \mathcal{P}(\{X_\mu^A\}) \;\to\; \alpha^{\frac{1}{2}\mathcal{N}\dim(G)}\;\mathcal{P}(\{X_\mu^A\}) \\ \\ \mathcal{S}(\{X_\mu^A\}) \;\to\; \alpha^4\;\mathcal{S}(\{X_\mu^A\}) \end{cases}. \qquad (20)$$

We introduce polar coordinates $(X_1^1, \ldots, X_{\dim(G)}^D) = (\Omega_d, R)$, with $d = D\dim(G)$ the total dimension of the integral. As an example, $\mathcal{P}(\Omega, 1)$ and $\mathcal{S}(\Omega_d, 1)$ denote the value of the Pfaffian polynomial and the action, respectively, for a configuration $(X_1^1, \ldots, X_{\dim(G)}^D)$ on the surface of the $d-$dimensional unit hyper-sphere, with polar coordinates $(\Omega_d, R = 1)$).

Eq.(20) allows us to to perform the R-integration analytically for each value of Ω_d, and to express the Yang-Mills integral as an expectation value over these angular variables:

$$\mathcal{Z}_{D,G}^{\mathcal{N}} = \frac{\int \mathcal{D}\Omega_d \; z_G(\Omega_d)}{\int \mathcal{D}\Omega_d}, \tag{21}$$

with

$$z_G(\Omega_d) = 2^{-d/2-1} \frac{\Gamma\Big(\frac{d}{4} + \frac{\mathcal{N}}{8}\dim(G)\Big)}{\Gamma\Big(\frac{d}{2}\Big)} \times \frac{\mathcal{P}(\Omega, 1)}{\Big[\mathcal{S}(\Omega_d, 1)\Big]^{\frac{d}{4}+\frac{\mathcal{N}}{8}\dim(G)}}. \tag{22}$$

As discussed previously, the integrand $z_G(\Omega_d)$ is too singular to be obtained by direct sampling of random points on the surface of the unit hyper sphere. Slightly modifying our procedure from [9], we therefore write

$$\mathcal{Z}_{D,G}^{\mathcal{N}} = \left[\frac{\int \mathcal{D}\Omega_d \; z \times z^{\alpha_1 - 1}}{\int \mathcal{D}\Omega_d \; z}\right]^{-1} \left[\frac{\int \mathcal{D}\Omega_d \; z^{\alpha_1} \times z^{\alpha_2 - \alpha_1}}{\int \mathcal{D}\Omega_d \; z^{\alpha_1}}\right]^{-1} \left[\frac{\int \mathcal{D}\Omega_d \; z^{\alpha_2} \times z^{-\alpha_2}}{\int \mathcal{D}\Omega_d \; z^{\alpha_2}}\right]^{-1}. \tag{23}$$

Each of the terms [ ] in eq.(23) is computed in a separate run. For example, the second quotient in eq.(23):

$$\frac{\int \mathcal{D}\Omega_d \; z^{\alpha_1} \times z^{\alpha_2 - \alpha_1}}{\int \mathcal{D}\Omega_d \; z^{\alpha_1}} = \Big\langle z^{\alpha_2 - \alpha_1} \Big\rangle_{\alpha_1}, \tag{24}$$

is simply the average of $z^{\alpha_2 - \alpha_1}$ for points Ω_d on the unit hyper-sphere distributed according to the probability distribution $\pi(\Omega_d) \sim z^{\alpha_1}(\Omega_d)$ (As it stands, eq.(24) is immediately applicable to $D = 4$, where the integrand z is positive semi-definite [8,7]. In the general case, we have to sample with $|z^{\alpha_1}(\Omega_d)|$ (cf. [8])).

We sample angular variables Ω_d according to z^{α_1} with a Metropolis Markov-chain method, which we now explain: At each iteration of the procedure, two distinct indices (A_1, μ_1) and (A_2, μ_2), and an angle $0 < \phi < 2\pi$ are chosen randomly. An unbiased trial move $\Omega_d \to \Omega_d'$ is then constructed by modifying solely the coordinates $X_{\mu_1}^{A_1}$ and $X_{\mu_2}^{A_2}$:

$$\begin{bmatrix} X_{\mu_1}^{A_1} \\ \\ X_{\mu_2}^{A_2} \end{bmatrix} \to \begin{bmatrix} X_{\mu_1}^{A_1} \\ \\ X_{\mu_2}^{A_2} \end{bmatrix}' = \sqrt{(X_{\mu_1}^{A_1})^2 + (X_{\mu_2}^{A_2})^2} \begin{bmatrix} \sin(\phi) \\ \\ \cos(\phi) \end{bmatrix}, \tag{25}$$

all other elements of $\{X_\mu^A\}$ remaining unchanged. The trial move eq.(25) preserves the norm R of the vector $\{X_\mu^A\}$, i. e. keeps the configuration on the surface of the unit sphere. Furthermore, it is unbiased (the probability to propose $\Omega_d \to \Omega_d'$ is the same as for the reverse move).

Finally, the move (for the example in eq. (24)) is accepted according to the Metropolis acceptance probability

$$P(\Omega \to \Omega') = \min\left(1, \frac{z^{\alpha_1}(\Omega')}{z^{\alpha_1}(\Omega)}\right). \tag{26}$$

Empirically, we found the values $\alpha_1 = 0.95, \alpha_2 = 0.6$ to be appropriate. Each of the averages in eq.(23) was computed within between a few hours and more than a thousand hours of computer time (on a work station array), corresponding to a maximum of 5×10^9 samples. Results are presented in the table below.

Table 2: Direct evaluation of Yang-Mills integrals

Group G	Monte Carlo result $\mathcal{Z}_{D=4,G}^{\mathcal{N}=4}$	Exact
SO(3)	1.255 ± 0.003	1.2533...
SO(4)	0.197 ± 0.004	0.1963...
SO(5)	0.589 ± 0.004	0.589...
SO(6)	0.0407 ± 0.0007	0.04101...
SO(7)	0.0169 ± 0.0003	0.01708...
Sp(2)	1.253 ± 0.001	1.2533...
Sp(4)	18.65 ± 0.2	18.849...
Sp(6)	279.2 ± 9.7	285.59...
G_2	6943 ± 120	7011.4...

Dividing the Monte Carlo results for $Z_{D=4,G}^{\mathcal{N}=4}$ by the corresponding group volume factors (cf. eqs (12), (13), and (14)) we arrive at our numerical predictions for the bulk indices $\text{ind}_0^{D=4}(G)$, which we compare below to the proposed analytical values.

Table 3: Monte Carlo results for the $D = 4$ bulk index

Group G	Monte Carlo $\text{ind}_0^{D=4}(G)$	Exact (Table 1)
SO(3)	0.2503 ± 0.0006	0.25 (1/4)
SO(4)	0.0627 ± 0.0013	0.0625 (1/16)
SO(5)	0.1406 ± 0.001	0.1406 (**9/64**)
SO(6)	0.0620 ± 0.001	0.0625 (1/16)
SO(7)	0.0966 ± 0.0017	0.0976 (**25/256**)
Sp(2)	0.2500 ± 0.0002	0.25 (1/4)
Sp(4)	0.139 ± 0.0015	0.1406 (**9/64**)
Sp(6)	0.0973 ± 0.003	0.0996 (**51/512**)
G_2	0.173 ± 0.003	0.1747 (**151/864**)

Agreement between the Monte Carlo results and theory is excellent, both in cases where rigorous results are known (SO(3), SO(4), Sp(2)) and where the deformation technique was applied. Among the latter cases, we again indicate in bold type *new* values, which had not been obtained before.

VI. BOSONIC CONVERGENCE FOR ORTHOGONAL, SYMPLECTIC AND EXCEPTIONAL INTEGRALS

Our qualitative Monte Carlo method (cf. [9], [10], [11]) allows us to determine the convergence properties of the bosonic Yang-Mills integrals $Z_{D,G}^{\mathcal{N}=0}$. We have found the following:

$$\left.\begin{array}{l} \mathcal{Z}_{D,\mathrm{SO}(N=3,4)}^{\mathcal{N}=0} \\ \\ \mathcal{Z}_{D,\mathrm{Sp}(2)}^{\mathcal{N}=0} \end{array}\right\} < \infty \quad \text{for} \quad D \geq 5$$
$$\left.\begin{array}{l} \mathcal{Z}_{D,\mathrm{SO}(N\geq 5)}^{\mathcal{N}=0} \\ \\ \mathcal{Z}_{D,\mathrm{Sp}(4,6,\dots)}^{\mathcal{N}=0} \\ \\ \mathcal{Z}_{D,\mathrm{G}_2}^{\mathcal{N}=0} \end{array}\right\} < \infty \quad \text{for} \quad D \geq 3 \tag{27}$$

All other bosonic integrals diverge. We thus obtain conditions which are fully consistent with the group isomorphisms discussed in the appendix. Let us note that we also performed the same qualitative computations for the susy integrals, as an important check of the convergence of the underlying Markov chains during the simulation.

VII. CONCLUSIONS AND OUTLOOK

In this paper we provided further evidence that Yang-Mills integrals encode surprisingly rich and subtle structures, which may prove to have important bearings on gauge and string theory.

The chief result of the present paper was to demonstrate that Yang-Mills integrals, as well as the methods to study them, can be naturally generalized from the previously studied special unitary symmetries to other gauge symmetries. We numerically evaluated the partition functions for all semi-simple gauge groups of rank $r \leq 3$ and compared the results to conjectured exact values, which were obtained by a generalization of certain contour integrals derived from a supersymmetric deformation procedure. The connection between the Yang-Mills integrals and the bulk indices is provided by the group volumes, that we computed explicitly. We provided details on these very subtle calculations. Agreement between the approaches is perfect within the tight error margins left by our Monte Carlo technique.

It would be very interesting to gain a simpler understanding of the rational numbers collected in table 1, although it is already evident that the bulk indices of the groups in question are more complicated than those of the special unitary case. In particular it would be nice to find general formulas for arbitrary rank.

In the present paper we have focused on $D = 4$ since this is the case where our numerical approach is most accurate. One should clearly study the dimensions $D = 6$ and, especially, $D = 10$ as well. It is straightforward, if more involved, to work out the predictions of the deformation method for these cases, at least for low rank gauge groups. In [13] exact values for the *total* (bulk plus boundary) Witten index $\mathrm{ind}^D(G)$ were proposed. In $D = 4, 6$ one should have $\text{ind}^{D=4,6}(G) = 0$ while for $D = 10$ it is argued to be a positive integer which, for groups other than $\text{SU}(N)$, can be larger than one. It is important to check the arguments by computing the index from the path integral. The results in the present paper indicate that the bulk contributions $\text{ind}_0^D(G)$ are correctly reproduced by the defomation method; however, we still lack a reliable method for computing the boundary terms $\text{ind}_1^D(G)$.

Since the deformation method of [14] successfully reproduces the partition functions, it is natural to ask whether it can be extended to calculate correlation functions of the ensembles eq.(1), such as the quantities studied numerically (so far only for $\text{SU}(N)$) in [10], [11], [7]. This would likely lead to new insights both in string theory [5] and gauge theory [6], [7].

Numerically, it might be interesting to compare $\text{SU}(N)$, $\text{SO}(N)$ and $\text{Sp}(2N)$ for large values of N, as in the standard large $N \to \infty$ limit of 't Hooft these groups are expected to lead to identical results.

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APPENDIX A: DETAILS AND CONVENTIONS FOR $\text{SO}(N)$

The Lie algebra $\text{so}(N)$ has $\frac{1}{2}N(N-1)$ generators which we choose to be the following standard anti-symmetric matrices T^{pq} (i.e. the index A becomes a double index pq)

$$(T^{pq})_{jk} = \delta_j^p \delta_k^q - \delta_k^p \delta_j^q$$

where $p < q$ ($p, q = 1, 2, \ldots, N$). The Lie algebra reads then

$$[T^{pq}, T^{rs}] = \delta^{qr} T^{ps} - \delta^{qs} T^{pr} - \delta^{pr} T^{qs} + \delta^{ps} T^{qr} = \sum_{t<u} f^{pq,rs,tu} T^{tu}.$$

In this basis the generators are normalized as $\text{Tr}\, T^{pq} T^{rs} = -2\delta^{pq,rs}$ and the structure constants are given through $f^{pq,rs,tu} = -\frac{1}{2}\text{Tr}\left(T^{pq}[T^{rs}, T^{tu}]\right)$. The gauge potentials are then $X_\mu = \sum_{p<q} X_\mu^{pq} T^{pq}$ and the $\text{SO}(N)$ Yang-Mills integrals in these conventions read

$$\mathcal{Z}_{D,\text{SO}(N)}^{\mathcal{N}} = \int \prod_{p<q}^{\frac{1}{2}N(N-1)} \prod_{\mu=1}^{D} \frac{dX_\mu^{pq}}{\sqrt{2\pi}} \exp\left[\frac{1}{8}\text{Tr}\,[X_\mu, X_\nu][X_\mu, X_\nu]\right] \mathcal{P}_{D,N}(X), \tag{A1}$$

where $\mathcal{P}_{D,N}$ is the Pfaffian as defined in eq.(3). These conventions are such that, in view of the isomorphism $\text{so}(3) = \text{su}(2)$ we have for all $\mathcal{N}$ (i.e. $\mathcal{N} = 0, 4, 8, 16$)

$$\mathcal{Z}_{D,\text{SO}(3)}^{\mathcal{N}} = \mathcal{Z}_{D,\text{SU}(2)}^{\mathcal{N}} = \mathcal{Z}_{D,\text{Sp}(2)}^{\mathcal{N}}, \tag{A2}$$

where the sympletic case is discussed in appendix B. One checks that in these normalizations the isomorphism $\text{so}(4) = \text{so}(3) \oplus \text{so}(3)$ results in

$$\mathcal{Z}^{\mathcal{N}}_{D,\mathrm{SO}(4)} = 2^{-\frac{3}{2}(D-\frac{1}{2}\mathcal{N})}\Big(\mathcal{Z}^{\mathcal{N}}_{D,\mathrm{SO}(3)}\Big)^2, \tag{A3}$$

while the isomorphism so(6) = su(4) leads to

$$\mathcal{Z}^{\mathcal{N}}_{D,\mathrm{SO}(6)} = 2^{-\frac{15}{4}(D-\frac{1}{2}\mathcal{N})}\mathcal{Z}^{\mathcal{N}}_{D,\mathrm{SU}(4)}, \tag{A4}$$

where the SU(4) integral is defined as in [8]. Finally, the isomorphism so(5) = sp(4) translates into

$$\mathcal{Z}^{\mathcal{N}}_{D,\mathrm{SO}(5)} = 2^{-\frac{5}{2}(D-\frac{1}{2}\mathcal{N})}\mathcal{Z}^{\mathcal{N}}_{D,\mathrm{Sp}(4)}, \tag{A5}$$

where the Sp(4) integral is defined in appendix B. One verifies that these isomorphisms are in perfect agreement with the results of the present paper as well as with [9].

We next provide the details necessary for verifying the group volume factor $\mathcal{F}_{\mathrm{SO}(N)}$ of eq.(12). The natural Cartan subalgebra is spanned by the generators $T^{12}, T^{34}, \ldots$. The corresponding maximal compact tori are given for SO$(2N+1)$ by the $(2N+1)\times(2N+1)$ matrix

$$T = \begin{pmatrix} \mathrm{rot}\theta_1 & & & & \\ & \mathrm{rot}\theta_2 & & & \\ & & \ddots & & \\ & & & \mathrm{rot}\theta_N & \\ & & & & 1 \end{pmatrix} \tag{A6}$$

while for SO$(2N)$ one has the $2N\times 2N$ matrix

$$T = \begin{pmatrix} \mathrm{rot}\theta_1 & & & \\ & \mathrm{rot}\theta_2 & & \\ & & \ddots & \\ & & & \mathrm{rot}\theta_N \end{pmatrix} \tag{A7}$$

Here rotθ_i are the 2×2 rotation matrices

$$\mathrm{rot}\theta_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} \tag{A8}$$

and matrix elements with no entries are zero. The corresponding reduced, normalized Haar measure on SO$(2N+1)$ (i.e. eq.(7)) reads

$$\mathcal{D}T = \frac{2^{2N^2}}{2^N N!}\prod_{i=1}^{N}\frac{d\theta_i}{2\pi}\prod_{i<j}^{N}\sin^2\Big(\frac{\theta_i-\theta_j}{2}\Big)\sin^2\Big(\frac{\theta_i+\theta_j}{2}\Big)\prod_{i=1}^{N}\sin^2\Big(\frac{\theta_i}{2}\Big) \tag{A9}$$

while for SO$(2N)$ one has

$$\mathcal{D}T = \frac{2^{2N(N-1)}}{2^{N-1}N!}\prod_{i=1}^{N}\frac{d\theta_i}{2\pi}\prod_{i<j}^{N}\sin^2\Big(\frac{\theta_i-\theta_j}{2}\Big)\sin^2\Big(\frac{\theta_i+\theta_j}{2}\Big). \tag{A10}$$

Eqs.(A9),(A10) may also be used to work out the detailed form of the contour integrals eqs.(16),(17) of section 4: One simply expands the Haar measure around $\theta_i \sim 0$.

Finally we recall the center groups of SO(N): One has $Z_{\mathrm{SO}(2N+1)} = \{\mathbb{1}\}$ and $Z_{\mathrm{SO}(2N)} = \{\mathbb{1}, -\mathbb{1}\}$ and therefore $|Z_{\mathrm{SO}(2N+1)}| = 1$ and $|Z_{\mathrm{SO}(2N)}| = 2$.

APPENDIX B: DETAILS AND CONVENTIONS FOR SP($2N$)

The Lie algebra sp($2N$) has $2N^2+N$ generators which we choose as follows. Define the $N\times N$ matrices E^{pq} ($p,q,j,k=1,2,\dots,N$)

$$(E^{pq})_{jk} = \delta^p_j \delta^q_k$$

Then we define the $N\times N$ matrix generators $T^{a,pq}$ for $p<q$ by

$$T^{0,pq} = \frac{1}{2\sqrt{2}}\begin{pmatrix} E^{pq}-E^{qp} & 0 \\ 0 & E^{pq}-E^{qp}\end{pmatrix} \qquad T^{1,pq} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & i(E^{pq}+E^{qp}) \\ i(E^{pq}+E^{qp}) & 0\end{pmatrix}$$

$$T^{2,pq} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & (E^{pq}+E^{qp}) \\ -(E^{pq}+E^{qp}) & 0\end{pmatrix} \quad T^{3,pq} = \frac{1}{2\sqrt{2}}\begin{pmatrix} i(E^{pq}+E^{qp}) & 0 \\ 0 & -i(E^{pq}+E^{qp})\end{pmatrix}$$

and the remaining generators are ($p=1,\dots,N$)

$$T^{1,pp} = \frac{1}{2}\begin{pmatrix} 0 & iE^{pp} \\ iE^{pp} & 0\end{pmatrix}$$

$$T^{2,pp} = \frac{1}{2}\begin{pmatrix} 0 & E^{pp} \\ -E^{pp} & 0\end{pmatrix} \quad T^{3,pp} = \frac{1}{2}\begin{pmatrix} iE^{pp} & 0 \\ 0 & -iE^{pp}\end{pmatrix}$$

In this basis the generators are normalized as $\mathrm{Tr}\, T^A T^B = -\frac{1}{2}\delta^{AB}$ and the structure constants are given through $f^{ABC} = -2\mathrm{Tr}\, T^A[T^B,T^C]$ where A is the multi-index (a,pq). The Cartan subalgebra is spanned by the generators $T^{3,pp}$ with $p=1,\dots,N$. The corresponding maximal compact torus is given by the matrix

$$T = \begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_N} & & & \\ & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_N}\end{pmatrix} \tag{B1}$$

The corresponding normalized Haar measure reads

$$\mathcal{D}T = \frac{2^{2N^2}}{2^N N!}\prod_{i=1}^N \frac{d\theta_i}{2\pi}\prod_{i<j}^N \sin^2\left(\frac{\theta_i-\theta_j}{2}\right)\sin^2\left(\frac{\theta_i+\theta_j}{2}\right)\prod_{i=1}^N \sin^2\theta_i \tag{B2}$$

The center of Sp($2N$) is the group $Z_{\mathrm{Sp}(2N)} = \{\mathbb{1},-\mathbb{1}\}$ and thus $|Z_{\mathrm{Sp}(2N)}|=2$.

APPENDIX C: DETAILS AND CONVENTIONS FOR G_2

The Lie algebra G_2 has 14 generators. For the fundamental representation we choose them as the following explicit 7×7 matrices (see e.g. [16]). Define

$$(X^{p,q})_{jk} = \delta^p_j \delta^q_k - \delta^p_k \delta^q_j$$

$$(Y^{p,q})_{jk} = i(\delta^p_j \delta^q_k + \delta^p_k \delta^q_j)$$

Then

$$\begin{aligned}
T^1 &= (X^{1,2}+X^{3,4})\frac{1}{2\sqrt{6}} + (X^{5,6}+X^{6,7})\frac{1}{2\sqrt{3}}\\
T^2 &= (Y^{1,2}+Y^{3,4})\frac{1}{2\sqrt{6}} + (Y^{5,6}+Y^{6,7})\frac{1}{2\sqrt{3}}\\
T^3 &= (X^{1,7}-X^{4,5})\frac{1}{2\sqrt{2}}\\
T^4 &= (Y^{1,7}+Y^{4,5})\frac{1}{2\sqrt{2}}\\
T^5 &= (X^{1,6}+X^{4,6})\frac{1}{2\sqrt{3}} + (-X^{2,7}-X^{3,5})\frac{1}{2\sqrt{6}}\\
T^6 &= (Y^{1,6}-Y^{4,6})\frac{1}{2\sqrt{3}} + (-Y^{2,7}+Y^{3,5})\frac{1}{2\sqrt{6}}\\
T^7 &= (X^{1,3}+X^{2,4})\frac{1}{2\sqrt{2}}\\
T^8 &= (Y^{1,3}+Y^{2,4})\frac{1}{2\sqrt{2}}\\
T^9 &= (-X^{1,5}+X^{4,7})\frac{1}{2\sqrt{6}} + (X^{2,6}-X^{3,6})\frac{1}{2\sqrt{3}}\\
T^{10} &= (-Y^{1,5}-Y^{4,7})\frac{1}{2\sqrt{6}} + (Y^{2,6}+Y^{3,6})\frac{1}{2\sqrt{3}}\\
T^{11} &= (-X^{2,5}-X^{3,7})\frac{1}{2\sqrt{2}}\\
T^{12} &= (-Y^{2,5}+Y^{3,7})\frac{1}{2\sqrt{2}}
\end{aligned} \tag{C1}$$

Finally, the matrices T^{13} and T^{14} are diagonal matrices with elements:

$$T^{13} = \mathrm{diag}(\frac{i}{2\sqrt{6}}; \frac{-i}{2\sqrt{6}}; \frac{i}{2\sqrt{6}}; \frac{-i}{2\sqrt{6}}; \frac{i}{\sqrt{6}}; 0; \frac{-i}{\sqrt{6}}) \tag{C2}$$

$$T^{14} = \mathrm{diag}(\frac{i}{2\sqrt{2}}; \frac{i}{2\sqrt{2}}; \frac{-i}{2\sqrt{2}}; \frac{-i}{2\sqrt{2}}; 0; 0; 0) \tag{C3}$$

In this basis the generators are normalized as $\mathrm{Tr}\, T^A T^B = -\frac{1}{2}\delta^{AB}$ and the structure constants are given through $f^{ABC} = -2\mathrm{Tr}\, T^A[T^B, T^C]$. The Cartan subalgebra is spanned by the generators T^{13} and T^{14}. The corresponding maximal compact torus is given by the matrix

$$T = \begin{pmatrix} e^{i\theta_1} &&&&&& \\ & e^{-i\theta_2} &&&&& \\ && e^{i\theta_2} &&&& \\ &&& e^{-i\theta_1} &&& \\ &&&& e^{i(\theta_1+\theta_2)} && \\ &&&&& 1 & \\ &&&&&& e^{-i(\theta_1+\theta_2)} \end{pmatrix} \tag{C4}$$

The normalized Haar measure on G_2 with respect to this torus reads

$$\mathcal{D}T = \frac{2^{12}}{12}\,\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\sin^2\left(\frac{\theta_1}{2}\right)\sin^2\left(\frac{\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{2\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1+2\theta_2}{2}\right) \tag{C5}$$

The center of G_2 is trivial: $Z_{\mathrm{G}_2} = \{\mathbb{1}\}$ and thus $|Z_{\mathrm{G}_2}| = 1$.

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