BLACK HOLE COLLISIONS: HOW FAR CAN PERTURBATION THEORY GO?

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The computation of gravitational radiation generated by the coalescence of inspiralling binary black holes is nowdays one of the main goals of numerical relativity. Perturbation theory has emerged as an ubiquitous tool for all those dynamical evolutions where the two black holes start close enough to each other, to be treated as single distorted black hole (close limit approximation), providing at the same time useful benchmarks for full numerical simulations. Here we summarize the most recent developments to study evolutions of perturbations around rotating (Kerr) black holes. The final aim is to generalize the close limit approximation to the most general case of two rotating black holes in orbit around each other, and thus provide reliable templates for the gravitational waveforms in this regime. For this reason it has become very important to know if these predictions can actually be trusted to larger separation parameters (even in the region where the holes have distinct event horizons). The only way to extend the range of validity of the linear approximation is to develop the theory of second order perturbations around a Kerr hole, by generalizing the Teukolsky formalism.

The prediction of accurate waveforms generated during the final orbital stage of binary black holes has become a worldwide research topic in general relativity during this decade. The main reason is that these catastrophic astrophysical events, considered one of the strongest sources of gravitational radiation in the universe, are potentially observable by LIGO, VIRGO and other interferometric detectors, now under construction. For its strong nonlinear features this black hole merger problem is only fully tractable by direct numerical integration (with supercomputers) of Einstein equations. Several difficulties remain to be solved in this approach such as the presence of early instabilities in the codes for numerical evolution of Einstein theory (see E.Seidel's contribution to these proceedings for a summary and references on this problem), and finding a new prescription for astrophysically realistic initial data representing orbiting black holes. Meanwhile, perturbation theory has shown not only to be the main approximation scheme for computation of gravitational radiation, but also a useful tool to provide benchmarks for full numerical simulations. The idea is that one can start a full numerical collision evolution with supercomputers and eventually have perturbation theory take over in case the numerical evolution crashes. From the theoretical point of view perhaps the more relevant contribution during the nineties in perturbative theory has been the "close limit approximation" [1]. It considers the final merger state of two black holes as described by a *single* perturbed one. This idea was applied to the head-on collision of two black holes and the emitted gravitational radiation was computed by means of the techniques used in first order perturbation theory around a Schwarzschild black hole. When the results of this computation have been compared with those of the full numerical integration of Einstein equations the agreement was so good that it was disturbing [2]. This encouraged the significant effort invested into the development of a second order Zerilli formalism of metric perturbations about the Schwarzschild background. The method was successfully implemented with particular emphasis on the comparison with the fully numerically generated results. In the case of two initially stationary black holes (Misner data) the agreement of the results is striking [3].

Second order perturbation theory confirmed the success of the close limit approximation with an impressive agreement in both waveforms and energy radiated against the full numerical simulations. There has been a tantamount success in the extension of these studies to the case of initially moving towards each other black holes [4], and for slowly rotating ones [5] (See Ref. [6] for a comprehensive review).

All the above close limit computations are based on the Zerilli [7] approach to metric perturbations of a Schwarzschild, i.e. nonrotating, black hole. This method uses the Regge-Wheeler [8] decomposition of the metric perturbations into multipoles (tensor harmonics). Einstein equations in the Regge-Wheeler gauge reduce to two single wave equations for the even and odd parity modes of the gravitational perturbations. There is, however, the strong belief that binary black holes in a realistic astrophysical scenario merge together into a single, highly rotating, black hole. There is also concrete observational evidence of accreting black holes [9] that places the rotation parameter as high as $a/M \simeq 0.95$. Finally, highly rotating black holes provide a new scenario to compare perturbative theory with full numerical integrations of Einstein equations.

The Regge-Wheeler-Zerilli techniques cannot be extended to study perturbations on a Kerr black hole background (see Ref. [5] for the slowly rotating case). In this case there is not a multipole decomposition of metric perturbations (in the time domain) and Einstein equations cannot be uncoupled into wave equations. A reformulation of the gravitational field equations due to Newman and Penrose [10], based on the Einstein equations and Bianchi identities projected along a null complex tetrad, $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$, allowed Teukolsky [11] to write down a single master wave equation for the perturbations of the Kerr metric in terms of the Weyl scalars ψ_4 or ψ_0. This formulation has several advantages: i) It is a first order gauge invariant description. ii) It does not rely on any multipole decomposition. iii) The Weyl scalars are objects defined in the full nonlinear theory. In addition, the Newman-Penrose formulation constitutes a simpler and more elegant framework to organize higher order perturbation schemes as we will see in the next section.

Since the seventies the Teukolsky equation for the first order perturbations around a rotating black hole has been Fourier transformed and integrated in the frequency domain for a variety of situations where initial data played no role (see Ref. [13] for a review). Very recently it was proved [14,15] that nothing is intrinsically wrong with the Teukolsky equation when sources extend to infinity and that a regularization method produces sensible results. In order to incorporate initial data and have a notable computational efficiency, concrete progress has been made recently to complete a computational framework that allows to integrate the Teukolsky equation in the *time domain*: First, an evolution code for integration of the Teukolsky wave equation is now available [16] and successfully tested [17]. Second, non conformally flat Cauchy data, compatible with Boyer-Lindquist slices of the Kerr geometry, began to be studied with a Kerr-Schild [18,19] or an axially symmetric [20,21] ansatz. Finally, an expression connecting ψ_4 to only Cauchy data,

$$\boxed{\psi_4 = \psi_4(h_{ij}, K_{ij}), \partial_t \psi_4 = \dot{\psi}_4(h_{ij}, K_{ij})},$$

has been worked out explicitly [22,17,23]. Note that those relashions hold for the full spacetime and thus to any order in the perturbations (see Lousto's proceeding contribution in this volume for a comprehensive review on this problems).

Assuming that we can solve for the first order perturbations problem, we decided to go one step forward in setting the formalism for the second order perturbations. As motivations for this work we can cite the spectacular results presented in Ref. [3] for the head-on collision and the hope to obtain similar agreement for the orbital binary black hole case in the close limit. Second order perturbations of the Kerr metric may even play a more important role in this case since we expect the perturbative parameter to be linear in the separation of the holes[24] while in the head on case it is quadratic in the separation[25]. The nonrotating limit of our approach will also provide an independent test and clarify some aspects of Ref. [3] results. High precision comparison with full numerical integration of Einstein equations using perturbative theory as benchmarks is also one of the main goals in this program. However, the main mission of second order perturbations is to provide error bars. It is well known that linearized perturbation theory does not provide, in itself, any indication on how good the perturbative approximation is. In fact, it is in general very difficult to estimate the errors involved in replacing an exact solution of the full Einstein equations with an approximate (perturbative) solution, i.e., to determine how small a perturbative parameter ε must be in order that the approximate solution have sufficient accuracy. Moreover, first order perturbation theory can be very sensitive to the choice of parametrization, i. e. different choices of the perturbative parameter can affect the accuracy of the linearized approximation [25]. The only reliable procedure to resolve the error and/or parameter arbitrariness is to carry out computations of the radiated waveforms and energy to second order in the expansion parameter. The ratio of second order corrections to the linear order results constitutes the only direct and systematically independent measure of the goodness of the perturbation results.

In Ref. [12] we extend to second (and higher) order the Teukolsky derivation of the equation that describes first order perturbations about a Kerr hole. To do so we consider the Newman-Penrose [10] formulation of the Bianchi identities and Einstein equations for the Kinnersley null tetrad (with l^μ and n^μ, m^μ along the principal null directions), make a perturbative expansion of it, and decouple the equation that describes the evolution of second (and higher) order perturbations. This equation takes the following form

$$\boxed{\widehat{\mathcal{T}}\psi^{(2)} = S[\psi^{(1)}, \partial_t\psi^{(1)}]}, \tag{1}$$

where $\psi \doteq (\rho^{(0)})^{-4}\psi_4$, $\widehat{\mathcal{T}}$ is the same (zeroth order) wave operator that applies to first order perturbations[26] that in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$, takes the following familiar form,

$$\begin{aligned}\widehat{\mathcal{T}} &= \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\vartheta\right]\partial_{tt} + \frac{4Mar}{\Delta}\partial_{t\varphi} - 4\left[r + ia\cos\vartheta - \frac{M(r^2-a^2)}{\Delta}\right]\partial_t \\ &- \Delta^2\partial_r\left(\Delta^{-1}\partial_r\right) - \frac{1}{\sin\vartheta}\partial_\vartheta\left(\sin\vartheta\partial_\vartheta\right) - \left[\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta}\right]\partial_{\varphi\varphi} \\ &+ 4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\vartheta}{\sin^2\vartheta}\right]\partial_\varphi + \left(4\cot^2\vartheta + 2\right),\end{aligned} \tag{2}$$

where M is the mass of the black hole, a its angular momentum per unit mass, $\Sigma \equiv r^2 + a^2\cos^2\vartheta$, and $\Delta \equiv r^2 - 2Mr + a^2$. The source term S is quadratic in the first order perturbations:

$$\begin{aligned}\mathcal{S} &= 2(\rho^{(0)})^{-4}\Sigma\Big\{\left[\left(\overline{\delta} + 3\alpha + \overline{\beta} + 4\pi - \overline{\tau}\right)^{(0)}(\delta + 4\beta - \tau)^{(1)} - \overline{d}_4^{(0)}(D + 4\epsilon - \rho)^{(1)}\right]\psi_4^{(1)} \\ &+ \left[(\Delta + 4\mu + \overline{\mu} + 3\gamma - \overline{\gamma})^{(0)}\left(\overline{\delta} + 4\pi + 2\alpha\right)^{(1)}\right]\psi_3^{(1)} \\ &- 3\left[\psi_2^{(1)}\Delta^{(0)}\lambda^{(1)} + \lambda^{(1)}\overline{d}_4^{(0)}\psi_2^{(1)}\right]\Big\},\end{aligned} \tag{3}$$

where all the NP spin-coefficients and directional derivatives can be expressed in terms of metric perturbations (see Ref.[12] for details).

As we show explicitly in Ref. [12], $\psi_4^{(2)}$ is neither invariant under first order coordinates transformations nor second order tetrad rotations. Thus, in order to integrate Eq. (1), one would have to evolve $\psi^{(2)}$ in a fixed gauge (and tetrad) and then compute physical quantities, like radiated energy and waveform, in an asymptotically flat gauge. This sort of approach was followed in Ref. [5] to study second order perturbations of a Schwarzschild black hole in the Regge-Wheeler gauge which is a unique gauge that allows to invert expressions in terms of generic perturbations and thus recover the gauge invariance. There is not a generalization of the Regge-Wheeler gauge when studying perturbations of a Kerr hole, essentially because one cannot perform a simple multipole decomposition of the metric. Instead, Chrzanowski [28] found two convenient gauges that allowed him to invert the metric perturbations in terms of the Weyl scalars ψ_4 or ψ_0. One can then use these metric perturbations to explicitly compute the source (3), in terms of $\psi_4^{(1)}$ or $\psi_0^{(1)}$ only, which is the object we directly obtain from the integration of the first order Teukolsky equation.

The energy and momenta radiated at infinity to second perturbative order can be computed using the standard methods of linearized gravity defined in asymptotically flat coordinates at future null infinity). For outgoing waves the total radiated energy per unit time ($u = t - r$) can thus be obtained from the Landau-Lifschitz pseudo tensor as

$$\frac{dE}{du} = \lim_{r\to\infty}\left\{\frac{r^2}{4\pi}\int_\Omega d\Omega \left|\int_{-\infty}^{u} d\tilde{u}\ \psi_4(\tilde{u},r,\vartheta,\varphi)\right|^2\right\}, \quad d\Omega = \sin\vartheta\ d\vartheta\ d\varphi, \tag{4}$$

where we can consider $\psi_4 = \psi_4^{(1)} + \psi_4^{(2)\ AF} + ...$

In the same way, one can also compute the total linear momentum radiated at infinity per unit time along cartesian-like coordinates as[29]

$$\frac{dP_\mu}{du} = -\lim_{r\to\infty}\left\{\frac{r^2}{4\pi}\int_\Omega d\Omega\ \tilde{l}_\mu \left|\int_{-\infty}^{u} d\tilde{u}\ \psi_4(\tilde{u},r,\vartheta,\varphi)\right|^2\right\}, \tag{5}$$
$$\tilde{l}_\mu = (1, -\sin\theta\cos\varphi, -\sin\theta\sin\varphi, -\cos\theta),$$

and the angular momentum carried away by the waves[30]

$$\frac{dJ_z}{du} = -\lim_{r\to\infty}\left\{\frac{r^2}{4\pi}\ Re\int_\Omega d\Omega\left(\partial_\varphi\int_{-\infty}^{u} d\tilde{u}\ \psi_4(\tilde{u},r,\vartheta,\varphi)\right)\left(\int_{-\infty}^{u} du'\int_{-\infty}^{u'} d\tilde{u}\ \overline{\psi}_4(\tilde{u},r,\vartheta,\varphi)\right)\right\}. \tag{6}$$

Higher than first order calculations are always characterized by an extraordinary complexity and a number of subtle, potentially confusing, gauge issues mainly due to the fact that a general second order gauge invariant formulation is not yet at hand in the literature. In general, gauge invariant quantities have an inherent physical meaning and they automatically lead to the simpler and direct interpretation of the results. In the Newman-Penrose formalism one has not only to look at gauge invariance (i. e. invariance under infinitesimal coordinates transformations), but also at invariance under tetrad rotations (see Ref.[12]). More specifically, the problem here is that the waveform $\psi_4^{(2)}$ in Eq. (1)) is neither first order coordinate gauge invariant nor tetrad invariant. It is only invariant under purely *second* order changes of coordinates, simply because ψ_4 vanishes on the background (Kerr metric). The question that arises therefore is whether $\psi_4^{(2)}$ can be unambiguously compared with, for instance, full numerical computations of the covariant ψ_4^{Num}. Invariant objects to describe second perturbations lead us to reliable physical answers without having to face gauge difficulties.

To handle this problem we give an explicit and general prescription for the construction of second order gauge and tetrad invariant objects representing outgoing radiation and thus explictly build up a coordinate and tetrad invariant quantity up to second order, $\psi_I^{(2)}$, which has the property of reducing to the linear part (in the second order perturbations of the metric) of $\psi_4^{(2)}$ in an asymptotically flat gauge at the "radiation zone", far from the sources. This property ensures us direct comparison with ψ_4^{Num} by constructing $\psi_4^{(1)} + \psi_{4\ I}^{(2)}$. To do so we impose the waveform $\psi_I^{(2)}$ to be invariant under a "combined" transformation of both the coordinates and the tetrad frame to first and second order. The resulting second order invariant waveform can then be built up out of the original $\psi_4^{(2)}$ plus corrections (quadratic in the first order quantities) that cancel out the gauge and tetrad dependence of $\psi_4^{(2)}$. The good of all this complex construction is that at the end ψ_I fulfills a single wave equation of the same form as Eq. (1)

$$\widehat{\mathcal{T}}[\psi^{(2)} + Q] = S + \widehat{\mathcal{W}}[Q] = \boxed{\widehat{\mathcal{W}}[\psi_I] = S_I}, \tag{7}$$

being $\psi^{(2)} \doteq (\rho^{(0)})^{-4}\psi_4^{(2)}$ and S_I a "corrected" source term build up out of (known to this level) first order perturbations.

A number of interesting conceptual and technical issues raised from this computation, like the appearance of nonlocalities in the definition of the gauge invariant waveform when we want to relate it to known first order objects and its non uniqueness. Seen in retrospective, our method of generating a gauge invariant object is like a machine that transforms any (first order) gauge into an asymptotically flat gauge, in particular, into the outgoing radiation gauge[28]. In fact,in this gauge we have $(\psi_4^{(2)})^{ORG} = (\psi_{4L}^{(2)})^{ORG} = (\psi_I^{(2)})^{ORG} = \psi_I^{(2)}$.

The spirit of this work has been to show that there exists a gauge invariant way to deal with second order perturbations in the more general case of a rotating black hole and to provide theoretical support to the numerical integration of the second order perturbation problem. In order to implement such integration of Eq. (7) (or equivalently of Eq. (1)) we proceed as follows: We assume that on an initial hypersurface we know the first and second order perturbed metric and extrinsic curvature. We then solve the first order problem, i.e. solve the standard Teukolsky equation for $\psi_4^{(1)}$ (and for $\psi_0^{(1)}$). Next we build up the perturbed metric coefficients in, for instance, the outgoing radiation gauge (ORG). The perturbed spin coefficients and the covariant basis are given by expressions in terms of metric perturbations [12]. Those are all the necessary elements to build up the effective source term appearing on the right hand side of our evolution equation. For the computation of the radiated energy and momentum one uses Eqs. (4) and (5). The advantage of this procedure is that we can now use the same (2+1)-dimensional code for evolving the first order perturbations [16] by adding a source term.

An important application of the formalism presented here (see also Ref.[12]) is to extend the numerical computations to the more interesting case of rotating black holes in orbit around each other. The numerical integration of Eq. (7) will be relevant not only for establishing the range of validity of the collision parameters in the close limit approximation, but (hopefully) to produce a more precise computation of the gravitational radiation. Direct comparison with the existing codes for numerical integration of the full nonlinear Einstein equations is possible.

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