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## Three-graviton scattering in M-theory

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### Abstract

The leading eikonal  $S$ -matrix for three-graviton scattering in  $d = 11$  supergravity and Matrix theory are shown to precisely agree. The result unifies the source-probe plus recoil approach of Okawa and Yoneya and relaxes the restriction imposed by those authors that all D-particle impact parameters and velocities are mutually perpendicular. Furthermore, the unified  $S$ -matrix approach facilitates a clean-cut study of M-theoretic  $\mathcal{R}^4$  curvature corrections to the low energy supergravity effective action. In particular, the leading  $\mathcal{R}^4$  correction to the three-graviton  $S$ -matrix is computed and compared to the corresponding next to leading order two-loop  $U(3)$  amplitude in Matrix theory. We find a clear disagreement of the two resulting tensor structures. © 1999 Elsevier Science B.V. All rights reserved.

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### 1. Introduction

According to current thinking, the various known string theories should be regarded as appropriate limits of a more fundamental eleven-dimensional theory, referred to as M-theory [1]. The cornerstone of our present understanding of M-theory is that its

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low energy effective action ought be  $d = 11$  supergravity [2]. It has been proposed, however, that the quantum degrees of freedom of light-cone M-theory are captured by a supersymmetric quantum mechanical  $U(N)$  Yang–Mills model, known as Matrix theory [3,4]. In practical terms this has meant that a large body of research has been devoted to comparing quantities computed via Matrix theory with those in  $d = 11$  supergravity. In particular, at the level of comparing phase shifts for eikonal scattering of gravitons [5–7] along with the complete tree level  $t$ -channel ( $2 \rightarrow 2$ ) graviton and antisymmetric tensor  $S$ -matrices [8,9], impressive agreement has been found. It has also been possible to successfully compare the conserved currents of the two models [10].

Nevertheless, it should be noted that computations to date have only managed to show the equivalence of one and two-loop computations in a relatively simple quantum mechanical model with what amounts to tree level supergravity. Therefore, the capability of the Matrix theory to uncover genuinely new physics seems somewhat limited. This is the main question we shall address in this paper, that is whether the model serves as a tool to study quantum corrections to the supergravity action. To this end it is clearly of central importance to determine the exact nature of the proposed correspondence.

The first issue is to identify the correspondence between the states of the two theories. Indeed, one of the motivations for the original conjecture [3] was the realization that the one-dimensional super Yang–Mills model possesses asymptotic excitations that behave as supergravitons of eleven-dimensional supergravity. This correspondence was refined in [11] where explicit asymptotic wave functions of gravitons, antisymmetric tensors and gravitini were found in the quantum mechanical model. Following the lines of [11], it has been possible to find a formalism to compute eikonal scattering amplitudes for these excitations in Matrix theory [8,9]. In this article we apply this method (which we often refer to as the Matrix theory LSZ formalism) to multi-particle scattering. In particular, we consider three-graviton amplitudes (studied already extensively within an eikonal phase shift framework by Okawa and Yoneya [7,12]).

The motivations for this computation are twofold. Firstly, given the agreement found in [7] for this process, such a calculation provides both a detailed test of our approach and at the same time verifies their work. Actually our formalism will provide not only a check of the results of [7,12], but also an extension and unification of them. In particular, we have been able to drop the restriction made by [7] that all D-particle velocities and impact parameters are mutually perpendicular. Furthermore, in a direct comparison of scattering amplitudes, there is no need to distinguish between recoil and non-recoil terms, as long as one sums over all the Feynman diagrams in the theory, including, in particular, the one-particle reducible graphs. Our result constitutes the complete agreement of  $t$ -channel three-particle spin independent  $S$ -matrices in Matrix theory and tree level supergravity.

The second motivation of our computation arises from the observation of [13] that the next to leading term in the two-loop effective action of Matrix theory, which is of order  $v^8/r^{18}$  (in relative velocities and distances between the supergravitons), has the correct scaling to match the first correction to graviton scattering induced by higher order  $\mathcal{R}^4$  curvature corrections [14] to  $d = 11$  supergravity. Although this observation origi-

nally concerned two-graviton scattering, we stress that two loops in the Matrix theory corresponds generically to three-particle interactions. Two-particle scattering arises then only as a sub-case in which the momenta of two of the three particles are identified. A genuine three-particle scattering computation involves a wide array of kinematical invariants and therefore allows a detailed comparison of the tensorial structures of amplitudes in the two theories. Thanks to the absence of lower order  $\mathcal{R}^2$  or  $\mathcal{R}^3$  couplings, the correction to the eikonal three-graviton scattering in  $d = 11$  supergravity induced by the M-theoretic  $\mathcal{R}^4$  term is easily computed via Feynman diagrams and takes a rather simple form, as we will see in what follows.

We should remark that this question has been studied before in the context of two-graviton scattering [15] where it was found that while the scaling dependence in  $v$ ,  $r$ , the Planck mass  $M$  and the compactified radius  $R$  is indeed correct, there is however a mismatch of factors of  $N$ . In principle one may be content with this mismatch but a number of questions remain open. In particular one might think that the simple introduction of the factors of  $N$  in what really amounts to an  $N = 2$  calculation in [15] is somewhat naive since it does not take into account bound state effects. This is reflected in the fact that we have no control over the Matrix theory LSZ procedure for two or three-particle scattering for arbitrary values of  $N$ , essentially because the ground-state Matrix theory wavefunction is still unknown<sup>5</sup>. From the viewpoint of the finite  $N$  matrix conjecture of Susskind [4], however, we are no longer subject to such a restriction. Is it then possible to find a stronger and more conclusive test? We believe that a detailed comparison of tensorial structures of the three-graviton amplitudes of the two models provides such a test. In addition the above-mentioned precise and complete agreement found for  $3 \rightarrow 3$  graviton scattering at leading order to be presented, gives one great confidence in our methods.

The results of our analysis show a definitive disagreement between the next to leading Matrix theory and quantum corrected supergravity amplitudes. We find different tensorial structures in the amplitudes of the two models, thus ruling out the proposed correspondence of  $v^8$  two-loop Matrix theory and  $\mathcal{R}^4$  corrected supergravity.

The outline of the paper is as follows. We present our supergravity computation of eikonal three-graviton scattering at leading order in subsection 2.1 and include the  $\mathcal{R}^4$  correction in subsection 2.2. In section three we turn to Matrix theory, where we compute the leading S-matrix contribution to three-particle scattering; in section four we expand this amplitude to obtain the next-to-leading  $v^8$  term. In the conclusions, we present the possible viewpoints explaining the mismatch we have found.

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<sup>5</sup> Progress towards understanding at least the asymptotics of the ground-state wavefunction may be found in [16,17].

## 2. Three-graviton scattering in $d = 11$ supergravity

### 2.1. Computation of the leading (tree-level) S-matrix

By definition M-theory at low energies is eleven-dimensional supergravity [2], whose bosonic sector is given by the action

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{2\kappa_{11}^2} \sqrt{-g} \mathcal{R} - \frac{1}{8} \sqrt{-g} (F_{MNPQ})^2 \\ & - \frac{\sqrt{3}}{12^3 \kappa_{11}^3} \varepsilon^{M_1 \dots M_{11}} F_{M_1 M_2 M_3 M_4} F_{M_5 M_6 M_7 M_8} C_{M_9 M_{10} M_{11}}, \end{aligned} \quad (1)$$

where  $F_{MNPQ} = 4 \partial_{[M} C_{NPQ]}$ ,  $g = \det g_{MN}$  and  $M = 0, \dots, 10$ .  $\kappa_{11}$  is the eleven-dimensional gravitational coupling constant. Perturbative quantum gravity may be studied by considering small fluctuations  $h_{MN}$  around the flat metric  $\eta_{MN}$ , i.e.  $g_{MN} = \eta_{MN} + \kappa_{11} h_{MN}$ . After employing the harmonic gauge  $\partial_N h^N_M - (1/2) \partial_M h^N_N = 0$ , one derives the graviton propagator

$$\langle h_{MN}(k) h_{PQ}(-k) \rangle = -\frac{i/2}{k^2 + i\epsilon} (\eta_{MP} \eta_{NQ} + \eta_{MQ} \eta_{NP} - \frac{2}{9} \eta_{MN} \eta_{PQ}). \quad (2)$$

We want to study three-graviton scattering at tree level. At this order, as can be easily seen from the supergravity action (1), the only contribution comes from the pure gravity sector, that is the Einstein–Hilbert term. In particular, in our computations we shall need the three-graviton and four-graviton vertices arising from its expansion. These are rather lengthy expressions and may be found in [18].

We consider now the elastic scattering process  $1+2+3 \rightarrow 1'+2'+3'$  of three gravitons into three gravitons and concentrate only on the terms in the amplitude proportional to

$$(h_1 \cdot h'_1) (h_2 \cdot h'_2) (h_3 \cdot h'_3), \quad (3)$$

$h_i$  being the external transverse graviton polarization tensors and  $(h_1 \cdot h'_1) \equiv h_1^{mn} h'^{mn}$ . The eleven dimensional momenta are conveniently parametrized in a light-cone frame  $M = (+, -, m)$  as

$$p_i = \left[ -\frac{1}{2} \left( \mathbf{v}_i - \frac{\mathbf{q}_i}{2} \right)^2, 1, \mathbf{v}_i - \frac{\mathbf{q}_i}{2} \right], \quad p'_i = \left[ -\frac{1}{2} \left( \mathbf{v}_i + \frac{\mathbf{q}_i}{2} \right)^2, 1, \mathbf{v}_i + \frac{\mathbf{q}_i}{2} \right], \quad (4)$$

where  $p_i^2 = 0 = p_i'^2$  and  $i = 1, 2, 3$ , using a vector notation for the  $SO(9)$  indices  $m = 1, \dots, 9$ . Note that we are considering only processes with zero compactified  $q$ -momentum transfer between in-going particles  $i$  and outgoing ones  $i'$ . Conservation of transverse momentum and energy implies

$$\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0, \quad \mathbf{v}_1 \cdot \mathbf{q}_1 + \mathbf{v}_2 \cdot \mathbf{q}_2 + \mathbf{v}_3 \cdot \mathbf{q}_3 = 0. \quad (5)$$

Moreover we will study the amplitude in an eikonal limit. To be precise this means we keep only terms with at least a double pole ( $1/(q_1^2 q_2^2)$  and permutations). Terms in

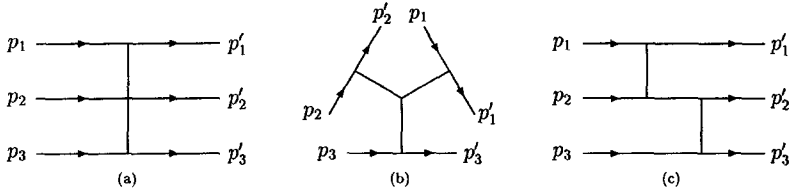


Fig. 1. The Einstein-Hilbert graphs, (a) V-type, (b) Y-type and (c) “re-scattering” graphs.

which this minimal pole structure is cancelled represent contact interactions and cannot be reliably computed in the eikonal Matrix theory framework we present here. At tree level there are then only the three types of diagrams of Fig. 1 up to permutations of the external legs.

The straightforward but tedious evaluation of these graphs was performed with the help of the computer algebra system *Form* [19]. There are three diagrams of V-type (a) yielding

$$A_V = 2 \frac{q_1^2 + q_2^2 + q_3^2}{q_1^2 q_2^2 q_3^2} v_{12}^2 v_{23}^2 v_{31}^2 + \mathcal{O}(v^5 q^{-3}), \tag{6}$$

where we suppress the terms of higher order in  $\mathbf{v}_i$  and lower order in  $\mathbf{q}_i$ . Similarly, there is only one Y-type graph (b) that can be written as follows:

$$A_Y = -\frac{1}{q_1^2 q_2^2 q_3^2} [(q_1^2 + q_2^2 + q_3^2) v_{12}^2 v_{23}^2 v_{31}^2 - \mathcal{T}^2] + \mathcal{O}(v^5 q^{-3}), \tag{7}$$

where

$$\mathcal{T} = (v_{23}^2 \mathbf{q}_2 \cdot \mathbf{v}_{12} + v_{31}^2 \mathbf{q}_3 \cdot \mathbf{v}_{23} + v_{12}^2 \mathbf{q}_1 \cdot \mathbf{v}_{31}) . \tag{8}$$

Notice that the combination  $\mathcal{T} \rightarrow \text{sgn}(\pi) \mathcal{T}$  under any permutation  $\pi$  of the labels 1, 2 and 3. In particular it is then invariant for cyclic permutations of the three labels. Finally we have the contributions of the six re-scattering graphs (c):

$$A_r = -\frac{1}{q_1^2 q_2^2 q_3^2} \left\{ (q_1^2 + q_2^2 + q_3^2) v_{12}^2 v_{23}^2 v_{31}^2 - \left[ \left( \frac{q_1^2 v_{12}^2 v_{31}^2}{\mathbf{q}_2 \cdot \mathbf{v}_{12}} \right) \mathcal{T} + \frac{1}{8} \left( \frac{q_1^2 v_{12}^2 v_{31}^2}{\mathbf{q}_2 \cdot \mathbf{v}_{12}} \right)^2 + \text{cyclic} \right] \right\} + \mathcal{O}(v^5 q^{-3}), \tag{9}$$

where cyclic indicates the two cyclic permutations of the labels 1, 2 and 3. Summing these three diagrams up one obtains the final result for the eikonal three-graviton amplitude:<sup>6</sup>

$$\mathcal{A}_{EH} = \frac{1}{q_1^2 q_2^2 q_3^2} \left\{ \mathcal{T}^2 + \left[ \left( \frac{q_1^2 v_{12}^2 v_{31}^2}{\mathbf{q}_2 \cdot \mathbf{v}_{12}} \right) \mathcal{T} + \frac{1}{8} \left( \frac{q_1^2 v_{12}^2 v_{31}^2}{\mathbf{q}_2 \cdot \mathbf{v}_{12}} \right)^2 + \text{cyclic} \right] \right\} + \mathcal{O}(v^5 q^{-3}). \tag{10}$$

<sup>6</sup> Throughout this paper, we discard the overall coefficients of complete amplitudes.

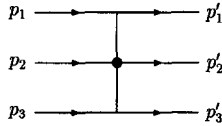


Fig. 2. The  $\mathcal{R}^4$  graph with the  $\mathcal{R}^4$  vertex inserted in the middle.

As discussed in the Introduction, we deliberately omitted the  $N$ -dependence in the formulae above, because we have complete control of our LSZ matrix theory procedure for  $N_i = 1$  only. Anyway they can be easily reintroduced with the net result that (10) takes an overall factor of  $N_1 N_2 N_3$ , where  $N_i$  is the  $p_-$  momentum of the graviton  $i$  and where we normalize each external leg with a factor of  $1/\sqrt{N_i}$ .

### 2.2. The $t_8 t_8 \mathcal{R}^4$ contribution

In this subsection we will compute the leading correction, in a small velocity and momentum transfer expansion, to the eikonal three-graviton scattering involving the higher derivative  $\mathcal{R}^4$  term.

It has been conjectured in [14] that the eleven dimensional supergravity action should contain a  $\mathcal{R}^4$  term, whose form in uncompactified eleven dimensions is

$$S_{\mathcal{R}^4} = \frac{\pi^2}{9 \cdot 2^7 \kappa_{11}^{2/3}} \int d^{11}x \sqrt{-g} t_8 t_8 \mathcal{R}^4, \tag{11}$$

where

$$t_8 t_8 \mathcal{R}^4 = t_8^{M_1 M_2 \dots M_8} t_8^{N_1 N_2 \dots N_8} \mathcal{R}_{M_1 M_2 N_1 N_2} \mathcal{R}_{M_3 M_4 N_3 N_4} \mathcal{R}_{M_5 M_6 N_5 N_6} \mathcal{R}_{M_7 M_8 N_7 N_8}. \tag{12}$$

The explicit form of the eight tensor  $t_8$  is given, e.g., in Ref. [20] for the ten-dimensional case. The tensor  $t_8$  entering (11), (12), is obtained by trivially extending the range of the indices to include the eleventh coordinate. From a supergravity point of view the (linearized) couplings in (11) arise as counter terms coming from a one-loop four-graviton scattering [21,22]. In this respect the coefficient in (11) would be UV divergent, but its finite value is fixed by requiring consistency with results obtained in IIA and IIB string theory [21]. Explicit computations have excluded the presence of one-loop counter terms of the form  $\mathcal{R}^2$  or  $\mathcal{R}^3$  in  $d = 11$  supergravity.<sup>7</sup> It is then not difficult to realize that the first leading contribution to the eikonal three-graviton scattering involving the couplings (11), is the unique graph shown in Fig. 2, that involves the linearized piece of each of the four Riemann tensors appearing in (12). Any other possible contribution,

<sup>7</sup> Strictly speaking what has been computed in the literature is the *background* effective action with background fields *on-shell* in which case the absence of  $\mathcal{R}^2$  and  $\mathcal{R}^3$  curvature corrections has been explicitly verified by Fradkin and Tseytlin [23]. However, it is an old result [24] that the  $S$ -matrix may be obtained from the on-shell background effective action by substitution of an iterative solution to the full field equations of the form  $g_{MN}^{\text{cl}} = g_{MN}^{\text{as}} + \dots$  where  $g_{MN}^{\text{as}}$  is an asymptotic field on mass shell depending physical polarizations (in particular, here we must take  $g_{MN}^{\text{as}}$  to be an asymptotic scattering solution in a flat background). In practice, this amounts to adding all possible trees to the effective vertices given by (11).

involving for instance Y-type or re-scattering-type graphs will be either sub-dominant in a small velocity and momentum transfer expansion or outside the eikonal kinematical regime. We then need to compute only one tree level graph with the insertion of the  $\mathcal{R}^4$  term as shown in Fig. 2 (up to permutations of the external legs). This can be most easily done by noticing that the linearized tensorial structure appearing in (12) is precisely the same as that obtained by computing four-graviton tree level scattering in a theory of pure gravity in any space-time dimension [18].<sup>8</sup> By using the results of [18] the computation of the graph in Fig. 2 is then greatly simplified. We find that the result for the part of the amplitude with the external polarizations contracted as in (3) and in the kinematical parameterization (4), can be written as follows (neglecting an overall coefficient):

$$\mathcal{A}_{\mathcal{R}^4} = \left\{ \frac{1}{q_1^2 q_3^2} [v_{12}^2 v_{23}^2 q_2^2 + \mathcal{T}(\mathbf{q}_1 \cdot \mathbf{v}_{12})]^2 + \text{cyclic} \right\}, \quad (13)$$

where  $\mathcal{T}$  was defined in (8). A clarification is now needed. The result (11), from which we computed the graph in Fig. 2 using the kinematics (4), applies strictly to eleven uncompactified space-time dimensions. However, the correspondence with Matrix theory at finite  $N$  requires a compactification on an almost time-like circle [4]. This means that we should have first compactified the theory on a spacelike circle and then performed a computation analogous to that reported in [21], e.g. a one-loop four-graviton scattering with two of them, according to Fig. 2, carrying equal and non-vanishing Kaluza Klein momentum. This would give the counter term of the form (11) which has the correct compactified radius  $R$  and Planck constant  $\kappa_{11}$  dependence to match the two-loop Matrix theory computation we consider in this article, but also terms with inappropriate  $R$  dependence, namely the analogs of the  $\zeta(3)/R^3$  found in [21] for the case of four-graviton scattering with all external legs carrying vanishing Kaluza–Klein momentum. To reach the discrete light cone kinematics of (4) one must take the limit  $R \rightarrow 0$ , so that such additional terms should, in principle, not be neglected. However, our philosophy is to study only those terms having the right dependence in the radius  $R$  and Planck constant  $\kappa_{11}$  to match two-loops in Matrix theory perturbation theory. In particular, (13) does not represent the complete eikonal, leading  $\mathcal{R}^4$  correction at the three-graviton scattering for  $d = 11$  supergravity on a circle, but only the terms that have a chance to be reproduced by a perturbative two-loop Matrix theory computation involving supergravitons. The  $N$ -dependence of (13), that we omitted, is easily computed to be globally of order  $N^5$ , in disagreement with the  $N^3$  dependence arising at two loops in Matrix theory. This reproduces indeed the disagreement found in [15].

<sup>8</sup> It should be noted that in this fashion one only obtains the *on-shell* vertex function. The key observation is that in the eikonal and spin-less limit (where one discards terms cancelling the double pole as well as contractions of momenta with polarizations) the two *a priori* off-shell legs entering the  $\mathcal{R}^4$  vertex are effectively put on-shell.

### 3. Scattering gravitons in Matrix theory

We now turn to the two-loop Matrix theory calculation, which has been carefully computed to leading order by Okawa and Yoneya [7]. We have reconsidered their computation and find results in accordance with theirs. Importantly, however, we rectify a hole in the original supergravity–Matrix theory agreement presented in [7]. In more detail, the technical assumption made by Okawa and Yoneya that all inner products of impact parameters  $\mathbf{b}_{ij}$  and relative velocities  $\mathbf{v}_{ij}$  vanish,  $\{\mathbf{b} \cdot \mathbf{v}\} = 0$ , can be shown to pose no restriction for one- and two-particle dynamics. But for three particles it constitutes a genuine restriction. We will show that this restriction may be dropped rather easily in our framework of comparing S-matrices.

#### 3.1. The setup

Let us summarize the Okawa–Yoneya result in our notation. The Euclidean Matrix theory action reads (setting the Yang–Mills coupling and compactified radius to unity)

$$S = \int dt \operatorname{tr} \left[ \frac{1}{2} (D_t X^m)^2 - \frac{1}{4} [X^m, X^n]^2 + \frac{1}{2} (\psi^T D_t \psi - \psi^T \gamma_m [X^m, \psi]) \right], \quad (14)$$

where  $D_t X^m = \partial_t X^m - i[A, X^m]$  and  $D_t \psi = \partial_t \psi - i[A, \psi]$ ;  $A$ ,  $X^m$  and  $\psi_\alpha$  are hermitian  $N \times N$  matrices, ( $m = 1, \dots, 9$  and  $\alpha = 1, \dots, 16$ ). Moreover we employ a real symmetric representation for the Dirac matrices  $\gamma_m$  in which the charge conjugation matrix  $C$  equals unity. The background field effective action is computed as an expansion of the bosonic matrices  $X_{ij}^m$  around diagonal backgrounds

$$X_{ij}^m = \delta_{ij} (b_i^m + v_i^m t) + Y_{ij}^m, \quad i, j = 1, \dots, N, \quad (15)$$

with constant velocities  $\mathbf{v}_i = v_i^m$ , impact parameters  $\mathbf{b}_i = b_i^m$  and fluctuations  $Y_{ij}^m$ . As we will focus on the leading spin-independent terms in scattering amplitudes, we do not consider fermionic background fields. Manifestly, this background solves the classical equations of motion. Thanks to the decoupling of the free  $U(1)$  centre of mass sector of the model, all one and higher loop results may be expressed in terms of the relative quantities  $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$  and  $\mathbf{r}_{ij}(t) = \mathbf{b}_{ij} + \mathbf{v}_{ij}t$ .

One proceeds by fixing a background field gauge and adding appropriate ghost couplings and kinetic terms. The propagators for all fluctuations may be expressed in terms of the inverse of a kinetic operator  $-\partial_t^2 + r_{ij}(t)^2$  which, in proper time representation, reads

$$[-\partial_t^2 + r_{ij}(t)^2]^{-1} \circ \delta(t_1 - t_2) = \int_0^\infty d\sigma \Delta(\sigma, t_-, \mathbf{r}_{ij}(t_+)), \quad (16)$$

where  $t_- = (t_1 - t_2)/2$  and  $t_+ = (t_1 + t_2)/2$  along with



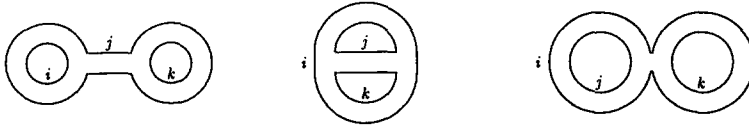


Fig. 3. The planar two-loop Matrix theory graphs: dumbbell, setting sun and figure eight.

$$\Delta(\sigma, t_-, \mathbf{r}_{ij}(t_+)) = \sqrt{\frac{v_{ij}}{2\pi \sinh(2\sigma v_{ij})}} \exp \left[ -v_{ij}^2 t_-^2 \coth(\sigma v_{ij}) - \sigma \mathbf{r}_{ij}(t_+)^2 - \left( \frac{\mathbf{v}_{ij} \cdot \mathbf{r}_{ij}(t_+)}{v_{ij}} \right)^2 \frac{1}{v_{ij}} (\tanh(\sigma v_{ij}) - \sigma v_{ij}) \right], \quad (17)$$

using  $v_{ij} = |\mathbf{v}_{ij}|$ . The two-loop calculation is then rather standard, yet tedious. One computes the three and four-point vertices from the expansion of the action (14) about the background. There are three possible topologies, the dumbbell, setting sun and figure eight denoted  $\Gamma_{0-0}$ ,  $\Gamma_Y$  and  $\Gamma_V$ , respectively<sup>9</sup> as depicted in Fig. 3 in 't Hooft double line notation. We remark that, as can clearly be seen from the Matrix theory LSZ formalism formulated in [8,9,25], one must compute *all* Matrix theory graphs, one-particle irreducible, connected-reducible and disconnected<sup>10</sup>. The latter we disregard since it is easy to see that they can only correspond to disconnected graphs on the supergravity side. However, as we shall see, graphs of the connected-reducible type (such as the dumbbell graph) reproduce re-scattering processes in supergravity [12].

The Okawa–Yoneya result may be stated (somewhat implicitly) as the effective action

$$\Gamma_{2\text{loop}} = \Gamma_{0-0} + \Gamma_V + \Gamma_Y \quad (18)$$

where

$$\Gamma_{0-0} = -\frac{1}{2} \sum_i \int dt_1 dt_2 \langle \partial_{t_1}^2 Y_{ii}^m(t_1) \rangle \Delta(t_1 - t_2) \langle \partial_{t_2}^2 Y_{ii}^m(t_2) \rangle, \quad (19)$$

with

$$\begin{aligned} \langle \partial_t^2 Y_{ii}^m(t) \rangle = & -32 \sum_j \int_0^\infty d\sigma \left[ r_{ij}^m(t) \sinh^4 \left( \frac{\sigma v_{ij}}{2} \right) \right. \\ & \left. + \frac{v_{ij}^m}{v_{ij}} \cosh \left( \frac{\sigma v_{ij}}{2} \right) \sinh^3 \left( \frac{\sigma v_{ij}}{2} \right) \frac{\partial}{\partial t} \right] \Delta(\sigma, 0, \mathbf{r}_{ij}(t)), \end{aligned} \quad (20)$$

and  $\Delta(t_1 - t_2) = \int_0^\infty d\sigma \Delta(\sigma, t_-, 0)$  is the propagator for a free massless scalar field in one dimension. Further

<sup>9</sup>To be precise, note that any terms from the setting sun diagram that may be written as a total derivative  $d/d\sigma_i$  of a polynomial times three propagators are included in  $\Gamma_V$  rather than  $\Gamma_Y$ , see [7] for details.

<sup>10</sup>This is easily seen as follows. In quantum mechanics the  $S$ -matrix reads  $S_{fi} = \int dx' dx \Phi_f^*(x') \langle x' | \exp(-iHT) | x \rangle \Phi_i(x)$  for incoming and outgoing wavefunctions  $\Phi_i$  and  $\Phi_f^*$ . The transition element from  $|x\rangle$  to  $\langle x'|$  may be represented as a path integral for which clearly one must compute all diagrams.

$$\begin{aligned} \Gamma_V = & -128 \sum_{ijk} \int dt \int_0^\infty d\sigma_1 d\sigma_2 \sinh^3 \left( \frac{\sigma_1 v_{ij}}{2} \right) \sinh^3 \left( \frac{\sigma_2 v_{jk}}{2} \right) \\ & \times \left[ \frac{2 \mathbf{v}_{ij} \cdot \mathbf{v}_{jk}}{v_{ij} v_{jk}} \cosh \left( \frac{\sigma_1 v_{ij}}{2} \right) \cosh \left( \frac{\sigma_2 v_{jk}}{2} \right) - \sinh \left( \frac{\sigma_1 v_{ij}}{2} \right) \sinh \left( \frac{\sigma_2 v_{jk}}{2} \right) \right] \\ & \times \Delta(\sigma_1, 0, \mathbf{r}_{ij}(t)) \Delta(\sigma_2, 0, \mathbf{r}_{jk}(t)), \end{aligned} \tag{21}$$

along with

$$\begin{aligned} \Gamma_Y = & - \sum_{ijk} \int dt_+ dt_- \int_0^\infty d\sigma_1 d\sigma_2 d\sigma_3 \\ & \times P_Y(\sigma_1, \sigma_2, \sigma_3, \mathbf{r}_{ij}(t_+), \mathbf{r}_{jk}(t_+), \mathbf{r}_{ki}(t_+), \mathbf{v}_{ij}, \mathbf{v}_{jk}, \mathbf{v}_{ki}, t_-) \\ & \times \Delta(\sigma_1, t_-, \mathbf{r}_{ij}(t_+)) \Delta(\sigma_2, t_-, \mathbf{r}_{jk}(t_+)) \Delta(\sigma_3, t_-, \mathbf{r}_{ki}(t_+)). \end{aligned} \tag{22}$$

The Okawa–Yoneya computation of the function  $P_Y$  is an impressive technical achievement and the result is a quadratic polynomial in the variables  $\mathbf{r}_{ij}(t_+)$  and  $t_-$  (the result itself is given by Eq. (3.47) of [7] along with three pages of the appendices of that work). Its correctness (at least to leading order in  $\mathbf{v}_{ij}$ ) is well tested by comparison with supergravity.

A remark on the  $N$  dependence of the two-loop effective action  $\Gamma_{2\text{loop}}$  is in order. The planar two-loop graphs of Fig. 3 carry three independent  $U(N)$  indices  $(i, j, k)$  thus giving rise to three body interactions. For backgrounds consisting of three blocks proportional to unit matrices of size  $N_i$  ( $i = 1, 2, 3$ , with  $\sum_i N_i = N$ ) the sums  $\sum_{ijk}$  reduce to  $N_1 N_2 N_3 \sum_{ijk=1}^3$  and  $\Gamma_{2\text{loop}}$  scales homogeneously like  $N^3$  to all orders in  $\mathbf{v}_{ij}$ , precisely like the corresponding supergravity term (10). This procedure, however, has from our viewpoint no real justification and we will therefore take  $N_i = 1$  in the following.

Up to now we have simply restated the results of [7]. In what follows we compare these results with the tree level supergravity  $S$ -matrix and in doing so show how to relax the restriction  $\{\mathbf{b} \cdot \mathbf{v}\} = 0$ . Thereafter, the same techniques will be employed to compare the next to leading order in  $\mathbf{v}_{ij}$  Matrix theory prediction with one-loop supergravity.

### 3.2. $\Gamma_Y$ contribution to the Matrix theory $S$ -matrix

Let us begin with the most difficult contribution  $\Gamma_Y$  of (22). One might suspect that since the result depends on three proper time parameters  $\sigma_1, \sigma_2$  and  $\sigma_3$  the result ought correspond to the triple pole structure of the Y-type diagrams in supergravity and indeed this naive suspicion will be borne out in the following. According to the Matrix theory LSZ formalism [8,9,25] the leading spin independent  $1 + 2 + 3 \rightarrow 1' + 2' + 3'$  Matrix theory  $S$ -matrix is given by

$$S^{3 \rightarrow 3} = \int d^9 b_1 d^9 b_2 d^9 b_3 \exp(iq_1 \cdot b_1 + iq_2 \cdot b_2 + iq_3 \cdot b_3) \Gamma_{2\text{loop}}. \tag{23}$$

Note that we have dropped contributions corresponding to disconnected processes (so that  $\Gamma_{2\text{loop}}$  no longer appears in the exponent). The transverse kinematics described by (23) are initial and final momenta

$$\begin{aligned} \mathbf{p}_i &= \mathbf{v}_i - \mathbf{q}_i/2, \\ \mathbf{p}'_i &= \mathbf{v}_i + \mathbf{q}_i/2, \quad i = 1, 2, 3, \end{aligned} \quad (24)$$

in accord with the supergravity kinematics (4). Note that at this stage  $\mathbf{v}_i$  is *not* a velocity anymore, but rather the average momentum of the  $i$ th particle  $\mathbf{v}_i = (\mathbf{p}_i + \mathbf{p}'_i)/2$ , for details see [8,9,25]. Since  $\Gamma_{2\text{loop}}$  only depends on relative quantities, the integral over the average impact parameter  $(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3)/3$  yields the usual momentum conserving  $\delta^{(9)}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)$  which we drop from now on. Concentrating on the  $\Gamma_Y$  contribution we then have

$$\begin{aligned} S_Y^{3 \rightarrow 3} &= - \int d^9 b_{13} d^9 b_{23} \exp(iq_1 \cdot b_{13} + iq_2 \cdot b_{23}) \int dt_+ dt_- \int_0^\infty d^3 \sigma \\ &\quad \times P_Y(\sigma_i, \mathbf{r}_{ij}(t_+), \mathbf{v}_{ij}, t_-) \Delta(\sigma_1, t_-, \mathbf{r}_{12}(t_+)) \\ &\quad \times \Delta(\sigma_2, t_-, \mathbf{r}_{23}(t_+)) \Delta(\sigma_3, t_-, \mathbf{r}_{31}(t_+)). \end{aligned} \quad (25)$$

The leading contribution to three body scattering should depend on the sixth power of velocities  $\mathbf{v}_{12}$ ,  $\mathbf{v}_{23}$  and  $\mathbf{v}_{31}$  as can be seen from the supergravity amplitude (10). However, if one examines the polynomial  $P_Y$ , its leading behaviour is quadratic in velocities and the “propagators”  $\Delta$  are to leading order velocity independent. In order to see explicitly how the cancellations of the terms quadratic and quartic in velocities occur, two observations are needed. Firstly, examining the  $t_-$  dependence of the exponent in (25) arising from the three propagators  $\Delta$  defined in (17)

$$-t_-^2 (v_{12}^2 \coth(\sigma_1 v_{12}) + v_{23}^2 \coth(\sigma_2 v_{23}) + v_{31}^2 \coth(\sigma_3 v_{31})) \equiv -t_-^2 P, \quad (26)$$

one sees that under the Gaussian  $t_-$  integral, all terms linear in  $t_-$  can be discarded by symmetric integration and terms proportional to  $t_-^2$  may be replaced by  $1/(2P)$ . Secondly, observe that the operator  $d/dt_+$  acting on the three propagators  $\Delta$  in (25) yields the factor

$$\begin{aligned} -2 \left[ \frac{\mathbf{v}_{12} \cdot \mathbf{r}_{12}(t_+)}{v_{12}} \tanh(\sigma_1 v_{12}) + \frac{\mathbf{v}_{23} \cdot \mathbf{r}_{23}(t_+)}{v_{23}} \tanh(\sigma_2 v_{23}) \right. \\ \left. + \frac{\mathbf{v}_{31} \cdot \mathbf{r}_{31}(t_+)}{v_{31}} \tanh(\sigma_3 v_{31}) \right] \equiv Q. \end{aligned} \quad (27)$$

Now recall that  $P_Y$  is a polynomial quadratic in  $\mathbf{r}_{12}(t_+)$ ,  $\mathbf{r}_{23}(t_+)$  and  $\mathbf{r}_{31}(t_+)$ . However, intuitively one may expect that terms of order two and four in velocity should not depend on the impact parameters  $\mathbf{b}_i$  since, in the case of one- and two-particle kinematics, shifts of the zero of  $t_+$  can always be made in such a fashion as to arrange that  $\mathbf{r}_{ij}(t_+) \rightarrow \mathbf{v}_{ij} t_+$ . This in fact is the case since at orders two and four in velocity, the  $\mathbf{r}_{ij}$  dependence of  $P_Y$  can be expressed as  $Q \times (\text{terms order one in velocity})$ . Writing  $Q$  as  $d/dt_+$  acting

on the  $\Delta$ 's and subsequently integrating by parts removes all dependence on  $\mathbf{r}_{12}$ ,  $\mathbf{r}_{23}$  and  $\mathbf{r}_{31}$ . Coupled with the first observation, one in fact finds miraculously that all terms proportional to squares and the fourth power of velocity cancel [7]. We stress that no restriction involving inner products of velocities and impact parameters must be imposed for this cancellation to take place.

It is now advantageous to interchange the  $dt_+$  and  $d^9\mathbf{b}$  integrals and thereafter shift the integration variable  $\mathbf{b}_{13} \rightarrow \mathbf{r}_{13}(t_+)$  along with  $\mathbf{b}_{23} \rightarrow \mathbf{r}_{23}(t_+)$  so that the  $t_+$  integral may be performed yielding an energy conserving delta function

$$S_Y^{3 \rightarrow 3} = -(2\pi)\delta(\mathbf{q}_1 \cdot \mathbf{v}_{13} + \mathbf{q}_2 \cdot \mathbf{v}_{23}) \times \int d^9\mathbf{r}_{13}d^9\mathbf{r}_{23} \exp(i\mathbf{q}_1 \cdot \mathbf{r}_{13} + i\mathbf{q}_2 \cdot \mathbf{r}_{23}) \int dt_- \int_0^\infty d^3\sigma \times \tilde{P}_Y(\sigma_i, \mathbf{r}_{ij}, \mathbf{v}_{ij}) \Delta(\sigma_1, t_-, \mathbf{r}_{12}) \Delta(\sigma_2, t_-, \mathbf{r}_{23}) \Delta(\sigma_3, t_-, \mathbf{r}_{31}), \quad (28)$$

where the tilde over  $P_Y$  indicates that we have performed the manipulations indicated in the two observations above.

So far we have managed to rewrite the  $\Gamma_Y$  contribution to the Matrix theory  $S$ -matrix as (suppressing from now on the energy conserving delta function  $(2\pi)\delta(\mathbf{q}_1 \cdot \mathbf{v}_{13} + \mathbf{q}_2 \cdot \mathbf{v}_{23})$ )

$$S_Y^{3 \rightarrow 3} = - \int d^9\mathbf{r}_{13}d^9\mathbf{r}_{23} \exp(i\mathbf{q}_1 \cdot \mathbf{r}_{13} + i\mathbf{q}_2 \cdot \mathbf{r}_{23}) \int_0^\infty d^3\sigma \frac{1}{\sqrt{4\pi P}} \times (\tilde{P}_Y + \tilde{P}_Y^m r^m + r^m \tilde{P}_Y^{mn} r^n) \Delta(\sigma_1, 0, \mathbf{r}_{12}) \Delta(\sigma_2, 0, \mathbf{r}_{23}) \Delta(\sigma_3, 0, \mathbf{r}_{31}). \quad (29)$$

Note that we have performed the integral over  $t_-$  as explained above. Furthermore,  $\tilde{P}_Y$ ,  $\tilde{P}_Y^m$  and  $\tilde{P}_Y^{mn}$  are functions of the  $\sigma_i$ ,  $\mathbf{r}_{ij}$  and  $\mathbf{v}_{ij}$  only ( $ij = (12, 23, 31)$ ) and their leading behaviour goes with the sixth power of velocity. Also their coupling to  $\mathbf{r}_{ij}$  has been schematized.

We proceed by interchanging the Fourier integrals over  $\mathbf{r}_{13}$  and  $\mathbf{r}_{23}$  with those over proper time  $\sigma_i$  parameters. If we content ourselves with leading order in velocities, the  $\mathbf{r}$  dependence in the exponent of (29) reads

$$\exp(i\mathbf{q}_1 \cdot \mathbf{r}_{13} + i\mathbf{q}_2 \cdot \mathbf{r}_{23} - r_A^m \mathcal{O}_{AB} r_B^m), \quad \mathcal{O} = \begin{pmatrix} \sigma_1 + \sigma_3 & -\sigma_1 \\ -\sigma_1 & \sigma_1 + \sigma_2 \end{pmatrix}, \quad (30)$$

where the index  $A = (13, 23)$ . The matrix  $\mathcal{O}$  has determinant  $p = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$  and the Gaussian integral over  $\mathbf{r}_{13}$  and  $\mathbf{r}_{23}$  may now be performed. Remarkably, we find that all terms not proportional to inner products of momentum transfers  $\mathbf{q}_i$  and velocities  $\mathbf{v}_{ij}$  cancel amongst themselves and to leading order in velocities we are left with

$$S_Y^{3 \rightarrow 3} = \frac{\pi^8}{4} \int_0^\infty d^3\sigma \frac{1}{p^3} \exp\left(-\frac{1}{4p}(q_1^2\sigma_2 + q_2^2\sigma_3 + q_3^2\sigma_1)\right) \mathcal{T}^2, \quad (31)$$

where  $\mathcal{T}$  is the same as defined in (8). Finally doing the  $d^3\sigma$  integral yields our result

$$S_Y^{3 \rightarrow 3} = 32 \pi^9 \frac{T^2}{q_1^2 q_2^2 q_3^2}. \quad (32)$$

Although we leave the orchestration of the two-loop leading velocity Matrix theory result to the end of this section, we remark that (32) already has precisely the correct form to match with tree level supergravity graphs of the Y-type (7) in the triple pole sector.

### 3.3. $\Gamma_V$ contribution to the Matrix theory $S$ -matrix

Compared with the  $\Gamma_Y$  contribution, the computation of the  $S$ -matrix elements arising from the  $\Gamma_V$  terms are very straightforward. The leading contribution from  $\Gamma_V$  as given in (21) is seen by inspection to be order six in velocity. Hence, interchanging  $dt$  and  $d^9\mathbf{b}$  integrals as above and thereafter performing the Fourier transforms and proper time  $\sigma_i$  integrations we find (suppressing delta functions over energy and momentum)

$$S_V^{3 \rightarrow 3} = -64\pi^9 \frac{v_{12}^2 v_{31}^2 \mathbf{v}_{12} \cdot \mathbf{v}_{31}}{q_2^2 q_3^2} + \text{cyclic}. \quad (33)$$

We emphasize that the result (32) mixes with terms arising from dumbbell graphs  $\Gamma_{o-o}$  which we will consider next. Thus a comparison to supergravity is not possible until we consider the sum of *all* Matrix theory Feynman diagrams, which has been the source of some confusion in the literature [26,27].

### 3.4. $\Gamma_{o-o}$ contribution to the Matrix theory $S$ -matrix

The final Matrix theory contribution to the leading order  $3 \rightarrow 3$   $S$ -matrix is given by the dumbbell diagrams. In [12] it has been shown that these graphs can be given the interpretation of recoil corrections to a source probe approximation. In Feynman diagram language there is, of course, no artificial distinction into recoil and non-recoil terms (physically since one finds that  $\Gamma_V$  and  $\Gamma_{o-o}$  contributions mix, this is certainly the case).

To extract the  $S$ -matrix contribution from  $\Gamma_{o-o}$  as given in (19) and (20) we begin by writing the free massless propagator for a scalar field in one dimension as

$$\Delta(t_1 - t_2) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1 - t_2)}}{\omega^2 + i\epsilon}. \quad (34)$$

The explicit time derivatives appearing in the truncated tadpoles (20) may, integrating by parts, be converted to  $\omega$ 's. Then, in the same fashion explained above, interchanging  $d^9\mathbf{b}$  and time integrals and shifting  $\mathbf{b} \rightarrow \mathbf{r}(t)$ , then performing the resulting Fourier transforms and proper time integrals we find

$$\begin{aligned}
 S_{0\rightarrow 0}^{3\rightarrow 3} &= \pi^7 \sum_{i \neq j \neq k} \int dt_1 dt_2 \int d\omega \exp(-iq_j \cdot v_{ji} t_1 - i\omega(t_1 - t_2) - iq_k \cdot v_{ki} t_2) \\
 &\times \frac{1}{\omega^2 + i\epsilon} \frac{v_{ij}^2 v_{ki}^2}{q_j^2 q_k^2} [\mathbf{q}_j v_{ij}^2 - 4\omega \mathbf{v}_{ij}] \cdot [\mathbf{q}_k v_{ki}^2 - 4\omega \mathbf{v}_{ki}].
 \end{aligned}
 \tag{35}$$

Note that we have kept only the leading velocity dependence and discarded terms in the sum over  $U(N)$  indices  $i, j$  and  $k$  in which the inner loop running around each end of the dumbbell takes the same value since one may convince oneself that these terms can only correspond to disconnected processes.<sup>11</sup> Now, the integral over  $t_- = (t_1 - t_2)/2$  yields  $\delta(2\omega + q_j \cdot v_{ji} - q_k \cdot v_{ki})$  and the  $t_+ = (t_1 + t_2)/2$  integral yields the usual energy conserving delta function which we suppress as usual. The integral over  $\omega$  is then trivial and gives the final result

$$S_{0\rightarrow 0}^{3\rightarrow 3} = 4\pi^9 \left[ 16 \frac{v_{12}^2 v_{31}^2 \mathbf{v}_{12} \cdot \mathbf{v}_{31}}{q_2^2 q_3^2} + 8\mathcal{T} \frac{v_{12}^2 v_{31}^2}{q_2^2 q_3^2 \mathbf{q}_2 \cdot \mathbf{v}_{12}} + \frac{v_{12}^4 v_{31}^4 q_1^2}{q_2^2 q_3^2 (\mathbf{q}_2 \cdot \mathbf{v}_{12})^2} + \text{cyclic} \right].
 \tag{36}$$

Observe in particular that here the first term and its permutations exactly cancels the contribution from  $S_V^{3\rightarrow 3}$  in (33). Clearly then, one sees that from a physical viewpoint the split into recoil and non-recoil terms is an artifact of one’s approximation scheme. In a Feynman graph approach, where one simply computes all terms contributing at a given order in velocity there is no need to make such a distinction so long as one also computes all Feynman diagrams on the Matrix theory side.

Finally, we see that the sum  $S_V^{3\rightarrow 3} + S_V^{3\rightarrow 3} + S_{0\rightarrow 0}^{3\rightarrow 3}$  as given in Eqs. (32), (33) and (36) reproduces the tree level supergravity result (10). No restriction upon impact parameters or velocities has been made in this comparison and this result represents the completion of the leading order spin-independent three-graviton scattering problem whose tortuous history may be followed in the sequence of articles [26,29,30,7].

#### 4. Next to leading order: can Matrix theory see $\mathcal{R}^4$ corrections?

Armed with the above clear-cut scheme for the computation of Matrix theory S-matrix elements and given the precise agreement of the tree level supergravity amplitude with the leading Matrix theory result, we now turn to the question of whether Matrix theory is sensitive to the one-loop corrections to the M-theory effective action discussed in Section 2.2. A simple-dimensional analysis indicates that the next to leading order contributions to the two-loop Matrix theory effective action, i.e. the terms of order  $v^8/(r^{18} R^7 M^{24})$ , have the correct dependence on  $v, r$ , the eleven-dimensional Planck mass  $M$  and compactification radius  $R$  to match the  $\mathcal{R}^4$  correction of (11) [13].

As mentioned in the introduction, this question has been already studied for two-graviton scattering in [15], where a mismatch of factors  $N$  between supergravity and

<sup>11</sup> Such terms have been analyzed in a recent preprint [28].

Matrix theory was found. However, our philosophy here is quite different, since we perform an analysis of tensorial structures in both theories which will allow us to give more definite and stronger conclusions.

The setup of the computation is now clear. We simply expand all terms in the two-loop effective action  $\Gamma_{2\text{loop}}$  of (18) to order  $v^8$  and apply the same manipulations discussed in the last section to obtain the Matrix theory amplitudes.

#### 4.1. Next to leading order results and disagreement

The order  $v^8$  result of the spin independent  $1 + 2 + 3 \rightarrow 1' + 2' + 3'$  amplitude is again comprised of the three terms

$$S^{3 \rightarrow 3}|_{v^8} = S_{0 \rightarrow 0}^{3 \rightarrow 3}|_{v^8} + S_V^{3 \rightarrow 3}|_{v^8} + S_Y^{3 \rightarrow 3}|_{v^8}. \quad (37)$$

Dropping the overall energy and momentum conserving delta function we find

$$\begin{aligned} S_{0 \rightarrow 0}^{3 \rightarrow 3}|_{v^8} = & -\frac{\pi^9}{6} \left[ \frac{v_{12}^4 v_{31}^4}{\mathbf{q}_2 \cdot \mathbf{v}_{12}} \left( \left\langle \frac{1}{\sigma_1^2} \right\rangle \mathbf{q}_3 \cdot \mathbf{v}_{12} + \left\langle \frac{1}{\sigma_2^2} \right\rangle \mathbf{q}_2 \cdot \mathbf{v}_{31} \right) \right. \\ & + \mathbf{q}_2 \cdot \mathbf{q}_3 v_{12}^4 v_{31}^4 \left( \left\langle \frac{1}{\sigma_1} \right\rangle + \left\langle \frac{1}{\sigma_2} \right\rangle \right) \\ & - 4\mathbf{v}_{12} \cdot \mathbf{v}_{31} v_{12}^2 v_{31}^2 \left( \left\langle \frac{1}{\sigma_1^2} \right\rangle v_{12}^2 + \left\langle \frac{1}{\sigma_2^2} \right\rangle v_{31}^2 \right) \\ & - 4\mathbf{q}_2 \cdot \mathbf{v}_{12} v_{12}^2 v_{31}^2 (v_{12}^2 \mathbf{q}_2 \cdot \mathbf{v}_{31} + v_{31}^2 \mathbf{q}_3 \cdot \mathbf{v}_{12}) \left( \left\langle \frac{1}{\sigma_1} \right\rangle + \left\langle \frac{1}{\sigma_2} \right\rangle \right) \\ & \left. + 16\mathbf{v}_{12} \cdot \mathbf{v}_{31} (\mathbf{q}_2 \cdot \mathbf{v}_{12})^2 v_{12}^2 v_{31}^2 \left( \left\langle \frac{1}{\sigma_1} \right\rangle + \left\langle \frac{1}{\sigma_2} \right\rangle \right) + \text{cyclic} \right] \quad (38) \end{aligned}$$

along with

$$\begin{aligned} S_V^{3 \rightarrow 3}|_{v^8} = & \frac{\pi^9}{2} v_{31}^4 v_{12}^4 \left\langle \frac{1}{\sigma_1 \sigma_2} \right\rangle - \frac{\pi^9}{3} \mathbf{v}_{12} \cdot \mathbf{v}_{31} v_{12}^2 v_{31}^2 \left[ v_{12}^2 \left\langle \frac{1}{\sigma_1^2} \right\rangle + v_{31}^2 \left\langle \frac{1}{\sigma_2^2} \right\rangle \right. \\ & \left. - 4(\mathbf{q}_2 \cdot \mathbf{v}_{12})^2 \left( \left\langle \frac{1}{\sigma_1} \right\rangle + \left\langle \frac{1}{\sigma_2} \right\rangle \right) \right] + \text{cyclic}, \quad (39) \end{aligned}$$

where we have defined

$$\langle f(\sigma_1, \sigma_2) \rangle = \int_0^\infty d^2 \sigma f(\sigma_1, \sigma_2) e^{-\sigma_1 q_2^2 - \sigma_2 q_3^2} \quad (40)$$

that is the proper time integrals remain to be performed<sup>12</sup>. We first note that none of the terms in (38) and (39) displays a genuine two pole structure  $\langle 1 \rangle = 1/(q_2^2 q_3^2)$  as found

<sup>12</sup> As a matter of fact all integrals in (38) and (39) are divergent, but exist in a distributional sense. See for example [31]; one must interpret the logarithm in (41) as  $\log(q^2/\Lambda^2)$  for some momentum scale  $\Lambda$  which can only be determined by some physical principle.

in the supergravity amplitude (13), such terms, however, will arise from the  $S_Y^{3 \rightarrow 3}|_{v^8}$  contribution to be studied.

An immediate disagreement arises from the first term of (38) with a “re-scattering pole”  $1/(\mathbf{q}_2 \cdot \mathbf{v}_{12})$ , whereas on the supergravity side re-scattering diagrams of the type (c) of Fig. 1 are absent since there are no  $\mathcal{R}^2$  and  $\mathcal{R}^3$  curvature corrections to the effective M-theory action, as argued in Section 2.2. Note also that  $S_Y^{3 \rightarrow 3}|_{v^8}$  does not give rise to re-scattering poles, as we shall see shortly. Performing the corresponding  $\sigma$  integrals for this term in a distributional sense

$$\int_0^\infty d\sigma \frac{1}{\sigma^2} e^{-\sigma q^2} = \frac{1}{16} q^2 (\log q^2 + \gamma - 1), \tag{41}$$

where  $\gamma$  is the Euler constant, the re-scattering contributions of  $S^{3 \rightarrow 3}|_{v^8}$  take the form

$$\frac{v_{12}^4 v_{31}^4}{\mathbf{q}_2 \cdot \mathbf{v}_{12}} \left( \frac{q_2^2}{q_3^2} \mathbf{q}_3 \cdot \mathbf{v}_{12} + \frac{q_3^2}{q_2^2} \mathbf{q}_2 \cdot \mathbf{v}_{31} \right) + \log \text{ terms.} \tag{42}$$

Hence it is clear that Matrix theory produces terms with no counterpart on the supergravity side. However, taking a conservative viewpoint one could argue that only the “truly eikonal” terms with a double pole  $1/(q_2^2 q_3^2)$  structure should be compared on both sides. A similar phenomenon occurred in the computation of polarization dependent two-graviton scattering amplitudes [8], where the spin dependent contributions to the Matrix theory amplitude gave rises to terms cancelling the  $1/q^2$  pole and had to be dropped.

Taking this viewpoint we would have to conclude that all terms in (38) and (39) are spurious and we need to go on to the rather involved computation of  $\Gamma_Y$  at order  $v^8$ .

The outcome of this computation is the amplitude (recall that  $p = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$ )

$$S_Y^{3 \rightarrow 3}|_{v^8} = \int_0^\infty d^3 \sigma \frac{1}{p^5} \exp(-q_1^2 \sigma_2 - q_2^2 \sigma_3 - q_3^2 \sigma_1) (\mathcal{T}^2 p^2 \Pi_2 + \mathcal{T} p \Pi_1 + \Pi_0), \tag{43}$$

where  $\mathcal{T}$  was introduced in (8) and  $\Pi_n$  ( $n = 0, 1, 2$ ) are polynomials order  $7 - n$  in the  $\sigma$ 's and order  $n$  in  $\mathbf{q} \cdot \mathbf{v}$ 's. In particular

$$\Pi_2 = -\frac{8\pi^9}{3} ((\mathbf{v}_{12} \cdot \mathbf{q}_1)^2 (\sigma_1 + \sigma_2) (\sigma_1 \sigma_2)^2 - 2\mathbf{v}_{12} \cdot \mathbf{q}_1 \mathbf{v}_{23} \cdot \mathbf{q}_2 \sigma_1 \sigma_2^3 \sigma_3) + \text{cyclic} \tag{44}$$

and

$$\begin{aligned} \Pi_1 = & \frac{16\pi^9}{3} \mathbf{v}_{12} \cdot \mathbf{q}_1 \\ & \times \left( (\sigma_1^2 v_{12}^4 + \sigma_2^2 v_{23}^4) \sigma_1 \sigma_2 [\sigma_1 \sigma_2 - 2(\sigma_1 + \sigma_2) \sigma_3] + 3v_{31}^4 (\sigma_1 \sigma_2 \sigma_3)^2 \right. \\ & \left. + 2v_{12}^2 v_{23}^2 \sigma_2 \sigma_1 (\sigma_1^3 \sigma_2 + \sigma_2^3 \sigma_3 + 3\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_2 \sigma_3 + \sigma_1^3 \sigma_3 + \sigma_1 \sigma_2^2 \sigma_3 + \sigma_1 \sigma_2^3) \right) \end{aligned}$$



$$\begin{aligned}
 & -v_{23}^2 v_{31}^2 \sigma_2^2 (3\sigma_1^2 \sigma_3^2 - \sigma_2^2 \sigma_3^2 - 2\sigma_1^2 \sigma_2 \sigma_3 + \sigma_1^2 \sigma_2^2 + 2\sigma_2 \sigma_1 \sigma_3^2) \\
 & -v_{12}^2 v_{31}^2 \sigma_1^2 (2\sigma_2 \sigma_1 \sigma_3^2 + \sigma_1^2 \sigma_2^2 - \sigma_1^2 \sigma_3^2 - 2\sigma_1 \sigma_2^2 \sigma_3 + 3\sigma_2^2 \sigma_3^2) \Big) + \text{cyclic} \quad (45)
 \end{aligned}$$

along with

$$\begin{aligned}
 \Pi_0 = & -\frac{8\pi^9}{3} \left[ v_{12}^8 \sigma_1^3 (-2\sigma_2^2 \sigma_2^2 - \sigma_1 \sigma_3 \sigma_2^2 - \sigma_1 \sigma_3^2 \sigma_2 - 4\sigma_1^2 \sigma_3 \sigma_2 + \sigma_3^2 \sigma_1^2 + \sigma_1^2 \sigma_2^2) \right. \\
 & -4v_{12}^6 v_{23}^2 \sigma_1^2 (3\sigma_3^2 \sigma_2^3 + \sigma_1 \sigma_3^2 \sigma_2^2 + 5\sigma_1^2 \sigma_3 \sigma_2^2 - \sigma_1^2 \sigma_3^2 \sigma_2 - 3\sigma_1^2 \sigma_2^3 - \sigma_1^3 \sigma_3 \sigma_2 \\
 & + \sigma_3^2 \sigma_1^3 - 2\sigma_3^3 \sigma_2^2) - 2v_{12}^4 v_{23}^2 v_{31}^2 \sigma_1 (\sigma_1 \sigma_3 - 2\sigma_3 \sigma_2 + \sigma_1 \sigma_2) (3\sigma_3^2 \sigma_1^2 - 3\sigma_3^2 \sigma_2^2 \\
 & + \sigma_1^3 \sigma_3 - 5\sigma_1^2 \sigma_3 \sigma_2 + \sigma_1^3 \sigma_2 + 3\sigma_1^2 \sigma_2^2) + v_{12}^4 v_{23}^4 (2\sigma_1 \sigma_3 \sigma_2^5 - \sigma_1 \sigma_3^2 \sigma_2^4 \\
 & + 11\sigma_1^2 \sigma_3 \sigma_2^4 + 10\sigma_1^2 \sigma_3^2 \sigma_2^3 + \sigma_1^2 \sigma_2^5 + 10\sigma_1^3 \sigma_3^2 \sigma_2^2 + 12\sigma_1^3 \sigma_2^4 \\
 & \left. + 11\sigma_1^4 \sigma_3 \sigma_2^2 - \sigma_1^4 \sigma_3^2 \sigma_2 + 12\sigma_1^4 \sigma_2^3 + 2\sigma_1^5 \sigma_3 \sigma_2 + \sigma_1^5 \sigma_3^2 + \sigma_1^5 \sigma_2^2 + \sigma_3^2 \sigma_2^5) \right] \\
 & + \text{permutations.} \quad (46)
 \end{aligned}$$

Note that the permutations in the above formula act on the “objects”  $(\mathbf{v}_1, \mathbf{q}_1, \sigma_2)$ ,  $(\mathbf{v}_2, \mathbf{q}_2, \sigma_3)$  and  $(\mathbf{v}_3, \mathbf{q}_3, \sigma_1)$  (because of the coupling of the proper times  $\sigma_i$  and momenta  $\mathbf{q}_i$  in the exponent of (43)).

Amongst these terms it is now instructive to focus on a specific class of terms in the supergravity amplitude (13). We choose to study terms with the structure

$$(\mathbf{q} \cdot \mathbf{v})^4 \frac{v^4}{q^4}. \quad (47)$$

On the Matrix theory side these terms are easily isolated from  $S_Y^{3 \rightarrow 3}|_{v^8}$  of (43), in particular

$$S_Y^{3 \rightarrow 3}|_{v^4(\mathbf{q} \cdot \mathbf{v})^4} = \mathcal{V}^2 \int_0^\infty d^3 \sigma \frac{\Pi_2}{p^3} \exp(-q_1^2 \sigma_2 - q_2^2 \sigma_3 - q_3^2 \sigma_1). \quad (48)$$

Of course it is rather difficult to perform this integral exactly. Being interested only in the poles  $1/(q_1^2 q_2^2)$  and permutations thereof we proceed as follows. First perform the integral over (say)  $\sigma_3$  exactly and thereafter expand the integrand in powers of  $1/\sigma_1$  and  $1/\sigma_2$ . Using

$$\int_0^\infty d\sigma \frac{1}{\sigma} e^{-\sigma q^2} = -\log q^2 - \gamma \quad (49)$$

we obtain the final result contributing to the structure (47) (up to overall factors, dropping the logarithms)

$$S_2^{3 \rightarrow 3}|_{\text{loop}}|_{v^4(\mathbf{q} \cdot \mathbf{v})^4} = \mathcal{V}^2 (\mathbf{q}_1 \cdot \mathbf{v}_{12})^2 \left( \frac{1}{q_2^2 q_3^2} + \frac{1}{q_1^2 q_2^2} \right) + \text{cyclic} \quad (50)$$

which is astonishingly close, but nevertheless *not* equal to the corresponding terms in the supergravity amplitude of (13)

$$\mathcal{A}_{\mathcal{R}^4}|_{\mathcal{O}(\mathbf{q}, \mathbf{v})^4} = T^2(\mathbf{q}_1 \cdot \mathbf{v}_{12})^2 \frac{1}{q_3^2 q_1^2} + \text{cyclic}. \quad (51)$$

This constitutes the above-mentioned definite disagreement of the two results and concludes our study of the  $\mathcal{R}^4$  contributions to the three-graviton amplitudes.

## 5. Conclusions

In this work we have presented detailed comparisons between three-graviton scattering amplitudes in Matrix theory and  $d = 11$  supergravity along with its leading M-theoretic curvature corrections. On the one hand we have been able to complete and unify the results of [7,12] showing that the leading order  $v^6$  eikonal spin independent  $S$ -matrices of tree level supergravity and two-loop Matrix theory exactly agree. On the other hand, the moment one studies the next to leading order  $v^8$  Matrix theory amplitude, the result fails to match the corresponding (conjectured) term in  $\mathcal{R}^4$ -corrected supergravity. Why does such a mismatch occur?

In trying to answer this most pressing of questions, let us begin by noting that our results pertain most strongly to the Susskind finite  $N$  formulation of the Matrix theory conjecture [4]. Susskind's conjecture has been proven to be literally true in [32], i.e. M-theory on a light-like circle with  $N$  units of compactified momentum is described by  $U(N)$  Matrix theory. The real issue is what it implies for comparison with  $d = 11$  supergravity. M-theory on a lightlike circle is Lorentz equivalent to M-theory on a vanishing spacelike circle [32]. On the contrary, supergravity is a good effective description of M-theory at low energy and at the same time when the radius of compactification is large (so that all possible wrapped membranes are decoupled). In terms of the string coupling constant  $g_s$ , for instance, this shows that perturbative Matrix theory and supergravity computations are really trust-able in two different regions (respectively at small and large values of  $g_s$ ).<sup>13</sup> It is then evident that no agreement should be expected *a priori*, except for those amplitudes which are somehow protected from receiving any correction as one moves from one regime to the other. In view of the agreement found for tree level two- and three-particle scattering amplitudes, this appears to be the case for the terms of order  $v^4$  and  $v^6$  in the Matrix theory effective action as has been shown in [34] for the  $U(2)$  and  $U(3)$  models. From this viewpoint the finite  $N$  Matrix theory conjecture, extended to the supergravity regime, would require the existence of an infinite number of non-renormalization theorems. However, our two-loop order  $v^8$  result indicates that there exists *no* non-renormalization theorem for these terms in the super Yang–Mills quantum mechanics.

<sup>13</sup> Roughly speaking this is due to the fact that Matrix theory is a good description of physics at substringy distances, whereas supergravity is a good description at long wavelengths.

The underlying type IIA string theory itself can be employed to understand the relationship between perturbative Matrix theory and low energy M-theory. In particular the extensive agreement of one-loop spin dependent terms for  $2 \rightarrow 2$  scattering can be easily understood by considering the scale independence of the string theory cylinder/annulus amplitude between two D0 particles [33,8,9]. Indeed arguments supported also by the string theory picture suggest that, if visible perturbatively, the effects due to the  $\mathcal{R}^4$  term may correspond to a five-loop non-planar contribution in Matrix theory<sup>14</sup> [35]. On the other hand there is no perturbative “string derivation” of a correspondence between the next-to-leading  $v^8$  two-loop term and the  $\mathcal{R}^4$  amplitude given in (13).

Aside from the possibility of discovering new non-renormalization theorems and although there were no real expectations for an agreement between two-loop Matrix theory and  $\mathcal{R}^4$  supergravity corrections, neither was there a definitive argument or computation to rule it out. We believe that our work gives a final (negative) answer to this question.

Finally, an obvious question to ask is whether one should find further agreements with tree level supergravity. Interestingly enough, in light of the simple Feynman diagrammatic understanding of semi-classical recoil effects given in this work, further comparison between Matrix theory and tree-level supergravity amplitudes can be contemplated for four-graviton scattering (i.e. three-loop level in the quantum mechanical model). In [27] it has been argued that at this order disagreement is possible, but there is no definite answer yet. As we have seen within our formalism of comparing directly S-matrices, it is quite easy to single out particular tensorial sub-structures. In this way the analysis could be greatly simplified and yet remain conclusive.

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<sup>14</sup> Although again the agreement would require a very non-trivial matching of amplitudes computed in different regimes.

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