Holonomy spin foam models: Definition and coarse graining

Benjamin Bahr,[1] Bianca Dittrich,[2,3] Frank Hellmann,[3] and Wojciech Kaminski[3]

[1]*DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[2]*Perimeter Institute, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
[3]*MPI f Gravitational Physics, Am Mühlenberg 1, D-14476 Potsdam, Germany*

We propose a new holonomy formulation for spin foams, which naturally extends the theory space of lattice gauge theories. This allows current spin foam models to be defined on arbitrary 2-complexes as well as to generalize current spin foam models to arbitrary, in particular, finite groups. The similarity with standard lattice gauge theories allows us to apply standard coarse graining methods, which for finite groups can now be easily considered numerically. We will summarize other holonomy and spin network formulations of spin foams and group field theories and explain how the different representations arise through variable transformations in the partition function. A companion paper will provide a description of boundary Hilbert spaces as well as a canonical dynamic encoded in transfer operators.

I. INTRODUCTION

Spin foam models are nonperturbative candidate theories for quantum gravity [1,2], which succeed in combining discreteness of spectra of geometric operators [3,4] with the correct semiclassical behavior of its elementary building blocks [5–7]. One of the key challenges is to obtain a better grasp on the continuum limit of spin foams obtained by considering the limit of (infinitely) many building blocks. This will not only answer questions on the large scale behavior (which here we will understand as the one involving many building blocks) but also address the status of diffeomorphism symmetry in these models [8–12].

To investigate this kind of limit involving many building blocks renormalization and coarse graining methods seem to be most appropriate [13–16]. To this end an understanding of the space of models, on which the renormalization flow takes place, is crucial. This allows us to also distinguish between relevant and irrelevant choices in the construction of spin foam models and helps us reveal the basic dynamical mechanisms of these models.

In this work we therefore propose a unified description of spin foam models based on group or holonomy variables. In this form spin foam models appear to be similar to lattice gauge theories; however, they have a richer structure of dynamical ingredients. In the description with the fewest parameters our models are defined by an edge function E and a face weight ω, whereas lattice gauge theories just feature the face weights ω. All the current spin foam models, namely those of Barrett and Crane (BC), Engle, Pereira, Rovelli, and Livine (EPRL), and that of Freidel and Krasnov (FK) models [17–19] can be described by this choice.

For this description to be useful for coarse graining one needs to show that these structures are stable under coarse graining (within some approximation scheme). Indeed we will investigate different coarse graining schemes for two- and three-dimensional models analytically and numerically

and show that this space of models presented here supports a rich structure for the renormalization flow. To be able to apply numerical techniques we consider spin foam models with finite groups, a concept introduced in Refs. [15,20]. A further advantage of the holonomy description here is that models analogous to the Barrett-Crane or EPRL model can be easily introduced also for finite groups. This will ease the investigation of (for instance symmetry) properties of these models as well as the numerical investigation of the large scale limit.

There are different holonomy representations proposed already in the literature [21–24], the relation to these will be explained in Sec. IV. The advantage of the formulation presented in this work is that compared to the other formulations it features a minimal set of model parameters and makes maximal use of the common properties of the spin foam models proposed so far in the literature. Most prominently that the imposition of the simplicity constraints factorizes in a specific sense, see also Ref. [2].

We will also explain the relation to the so-called operator spin foam models (OSFM) [25], which share with the models proposed here, that they are defined on arbitrary 2-complexes. Indeed the structure of the holonomy representation worked out here makes the definition on a 2-complex completely natural.

Other advantages of the holonomy representations introduced here are the natural definition of boundary Hilbert spaces and a straightforward relation to projected spin networks [26,27] as well as allowing the investigation of the semiclassical limit for arbitrary complexes by exploring the distributional character of the integration kernels of the partition functions. These features will be explained in companion papers [28,29].

In Sec. III we give the first main result of this paper, a definition of spin foam models as generalized lattice gauge theories written entirely in terms of holonomies. Section IV discusses the relation to operator spin foams and other

holonomy formulations for spin foams, including group field theories. The next section gives a description of the Barrett-Crane and EPRL spin foam models in the holonomy formulation. These insights are used to generalize the Barrett-Crane and EPRL models to finite and other Lie groups.

In Sec. VI we discuss coarse graining of these models, in particular in two and three dimensions. We close with an outlook and summary in Secs. VI D and VII.

II. A SHORT MOTIVATION OF THE MODELS

Here we shortly motivate some ideas behind the holonomy representation for spin foams, which will be explained in more detail later on.

Let us start with reviewing standard lattice gauge theory. Such a theory requires the choice of a gauge group G and face weights ω, which are class functions on the group. These face weights encode all the dynamics of the given model, as the partition function is given by

$$Z = \int \left(\prod_e \mathrm{d}g_e\right)\left(\prod_f \omega(g_e g_{e'} \ldots)\right). \tag{1}$$

On the other hand the face weights for the lattice Yang-Mills theory are defined using the background metric of the lattice. This should however be avoided if one wants to quantize gravity, where the metric itself is a variable. Indeed, the starting point for spin foam model construction is the so called background field (BF) theory where the face weights are given by $\omega = \delta_G$ the delta function on the group. This makes the partition function in general divergent (even for compact groups but not for finite ones). This divergence indicates another gauge invariance known as translation symmetry. BF theory is a topological field theory, which coincides with the zero coupling or zero temperature limit of lattice Yang-Mills theory.

To construct a nontopological theory and nevertheless keep the trivial face weights, we have to introduce more structure. First let us just double the variables from one group element g_e per edge to one group element g_{ve} per edge-vertex pair (or half edge). We define $g_{ev} = (g_{ve})^{-1}$. Then the partition function (1) reads

$$Z = \int \left(\prod_{(ve)} \mathrm{d}g_{ve}\right)\left(\prod_f \omega(g_{ve} g_{ev'} g_{v'e'} \ldots)\right), \tag{2}$$

and we can regain the form (1) by variable redefinitions $g_e = g_{ve} g_{ev'}$ and performing $|E|$ trivial integrations. Here and in the following we assume a normalized and left and right invariant group integration measure.

To actually change the model we introduce another set of variables, namely a group element g_{ef} per face-edge pair. These are inserted into the face holonomies so that ω_f is now evaluated on $g_f = g_{ve} g_{ef} g_{ev'} g_{v'e'} \ldots$. In this way the alternative face holonomy $\tilde{G}_f = g_{ve} g_{ev'} g_{v'e'} \ldots$ can nevertheless take nontrivial values even if we choose

$\omega_f = \delta_G$. Allowing all possible g_{ef} would however again trivialize the model. (This time it corresponds to the high temperature fixed point of lattice Yang-Mills, where $\omega_f \equiv 1$ is the constant function.) A nontrivial model can be obtained by weighing the inserted group elements g_{ef} via a function E. The final model will be given by

$$Z' = \int \left(\prod_{(ve)} \mathrm{d}g_{ve}\right)\left(\prod_{(fe)} \mathrm{d}g_{fe}\right)\left(\prod_{(fe)} E(g_{ef})\right)$$
$$\times \left(\prod_f \omega_f(g_{ve} g_{ef} g_{ev'} g_{v'e'} g_{e'f} \ldots)\right). \tag{3}$$

As will be explained in the following the partition functions of all current spin foam models can be expressed in this way. The choice of the function E determines the model, the face weights are usually fixed to be δ_G.

The models can be rewritten in several ways, we will present different formulations in Sec. IV. Here we will just note that one could integrate out the g_{ef} variables in the formulation (3) to obtain effective face weights

$$\omega'_f(g_{ev} g_{ve'}, g_{e'v'} g_{v'e''}, \ldots)$$
$$= \int \left(\prod_{(fe)} \mathrm{d}g_{fe}\right)\left(\prod_{(fe)} E(g_{ef})\right)$$
$$\times \left(\prod_f \omega_f(g_{ev} g_{ve'} g_{e'f} g_{e'v'} g_{v'e''} g_{e''f} \ldots)\right). \tag{4}$$

Such a formulation (for the Barrett-Crane model) appeared for instance in Refs. [21,22] and is nearest to standard lattice gauge theory. See also Refs. F for a geometric interpretation of the holonomy variables and simplicity constraints involved there.

Note that the face weights are not just class functions of a face holonomy, as is the case in standard lattice gauge theory, but will be functions of a certain number of group elements.

III. HOLONOMY FORMULATION OF SPIN FOAM MODELS

Let us describe the models in more detail. These will be defined on an oriented 2-complex $\mathcal{C}$ with vertices v, edges e and faces f. The gauge group will be denoted G, which we will assume to be finite or compact and equipped with a normalized left and right invariant group (Haar) integration measure.

The variables in which the partition function will be expressed are group elements g_{ef} assigned to pairs $e \subset f$ and g_{ve} for $v \subset e$ with the understanding that $g_{ev} = g_{ve}^{-1}$. One can identify them as holonomies of a G connection along certain paths within faces f (see Fig. 1).

From these data we can define the holonomy G_f:

$$G_f = g_{ve} g_{ef} g_{ev'} g_{v'e'} \ldots g_{e^{(n)}v}. \tag{5}$$

With the help of Fig. 1, one can readily see that G_f describes the holonomy associated to a *geometrically*
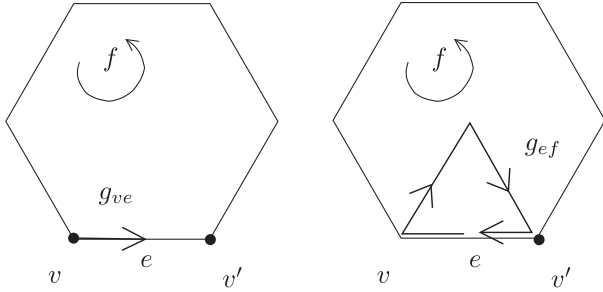
FIG. 1. Holonomy variables associated to half edges (g_{ve}) and wedges (g_{ef}) of a face.

trivial path, i.e., the choice $\omega = \delta_G$ as a weight to this holonomy appears most natural and is also used in the EPRL-FK or BC models. For regularization purposes, this δ_G is often replaced by a heat kernel, see e.g., Ref. [32]. We will therefore leave the face weight ω as a parameter of the theory. As in standard lattice gauge theory ω is required to be a class function of the group, furthermore we ask that $\omega(g) = \omega(g^{-1})$ is satisfied for all $g \in G$.

The class of models we are considering are specified further by a subgroup $G' \subset G$. We will choose the edge function E to be invariant under conjugation with elements in G'. From (4) one can see that E functions which are invariant under conjugation of the full group, i.e., class functions, will lead to effective face weights which can be considered to just depend on the (reduced) face holonomy $g_{ev}g_{ve'}g_{e'v'}\ldots$. This case will therefore lead to a model equivalent to a lattice gauge theory (with face weights ω_f that might explicitly depend on the face, more precisely on the number of edges in the face). In this sense the choice of G' defines an invariance property of the effective face weights (4). As for the face weights we will furthermore require from E the property $E(g) = E(g^{-1})$ for $g \in G$.

Finally the state sum is defined as

$$Z = \int dg_{ev} dg_{ef} \prod_f \omega(G_f) \prod_{ef} E(g_{ef}). \qquad (6)$$

For the choice $\omega = \delta_G$ the partition function may be divergent, even for compact groups. The definition is extendable to noncompact groups, but in this case there is an ambiguity in the normalization of the Haar measure and additional divergences due to gauge orbits. However, for finite groups the delta distribution is just a regular function, thus the above is well defined.

A. Parametrization of models

The space of such holonomy spin foam models is described by a choice of the edge functions E and the face weights ω_f. In the following we will give the spin representation of these entities as such a representation also offers a convenient parametrization of the space of models.

The face weight ω_f is a class function and as usual can be expanded as

$$\omega_f(g) = \sum_\rho \dim(\rho)\omega^\rho \chi_\rho(g), \qquad (7)$$

where $\chi_\rho = \text{tr}_\rho(D_\rho(g))$ is the character of the unitary irreducible representation (irrep) ρ with $D_\rho(g)$ the representation matrix of g.

The edge functions can be expanded in terms of the irreducible unitary representations of G and G', which we denote ρ and k respectively. To this end we utilize injection maps $I(\rho, k)_d$:

$$I(\rho, k)_d \in Hom_{G'}(\bar{\rho}, k), \qquad (8)$$

where the d index labels a basis of $Hom_{G'}(\bar{\rho}, k)$ the space of G' covariant function from the representation space of $\bar{\rho}$, the dual of ρ, to the G' irrep k.

The map $I(\rho, k)_d$ identifies the representation k, which in case there is more than one representation is furthermore labeled by d, in the reduction of the irrep ρ over the group G'. The index d is hence only nontrivial in cases where there are more than one k representations included in ρ.

We assume that the basis is normalized (up to a phase, not relevant in the following)

$$I(\rho, k)_d^\dagger I(\rho, k)_{d'} = \delta_{dd'}\mathbf{1}_k. \qquad (9)$$

This allows us to expand E as

$$E(h) = \sum_{\rho,k} \dim(\rho)e_{k,dd'}^\rho \text{tr}_\rho(D_\rho(h)I(\rho, k)_d I^\dagger(\rho, k)_{d'}). \quad (10)$$

Hence for given groups G, G' the choice of E function can be encoded into the parameters $e_{k,dd'}^\rho$. The index d is a degeneracy index for the case where the G' reducible representation ρ contains more than one copy of the irrep k. For $G = SU(2) \times SU(2)$ and $G' = SU(2)_{\text{diag}}$ this case does not occur, and we have only to deal with the set e_k^ρ.

E encodes the analogue of the simplicity constraints arising in the Plebanski formulation of gravity for the spin foam model at hand, see for instance Ref. [2]. As in the classical Plebanski formulation, where the simplicity constraints are enforced with Lagrange multipliers, we had to introduce here additional variables, the wedge holonomies g_{ef}.

If the face weight is given by the delta function, the product of group elements around the face G_f is flat. This is however not the usual holonomy around the face $\tilde{G}_f = g_{ve}g_{ev'}g_{v'e'}\ldots$, but $\tilde{G}_f$ interwoven with the g_{ef}. If we force $g_{ef} = 1$ by choosing $E = \delta_G$ we have $G_f = \tilde{G}_f$, and obtain a theory of flat connections that is BF theory. Choosing a different E function relaxes the constraints on flatness, just as we expect from the simplicity constraints.

B. Boundaries and Hilbert spaces

We can also include boundaries into our models, which are represented as graphs Γ in the 2-complex $\mathcal{C}$. The set of edges of Γ will be denoted by $\Gamma_e \subset \mathcal{C}_e$ and the set of vertices $\Gamma_v \subset \mathcal{C}_v$. We write Γ_{ev} for the set of pairs $v \in e$ in $\Gamma_v \times \Gamma_e$.

By dropping integration over the group elements associated to the pairs of Γ_{ev} we arrive at the partition function

$$Z^\Gamma(\mathcal{C})[g_{ev}] = \int \Big(\prod_{e \subset f} \mathrm{d}g_{ef}\Big)\Big(\prod_{\substack{v \subset e \\ ef \notin \Gamma_{ev}}} \mathrm{d}g_{ev}\Big)\Big(\prod_{e \subset f} E(g_{ef})\Big)$$
$$\times \Big(\prod_f \omega_f(g_f)\Big), \tag{11}$$

which can now be interpreted as an element of the Hilbert space $L^2(G^{|\Gamma_{ev}|})$. (Integrability in this general form is only guaranteed for finite groups).

As the integrand of the partition function is invariant under $g_{ev} \to h_e'^{-1} g_{ev} g_v$, $h_{ef} \to h_e'^{-1} h_{ef} h_e'$ for $h_e' \in G' \subset G$ and $g_v \in G$, the partition function can be considered to live in a smaller subspace of $L^2(G^{|\Gamma_{ev}|})$, that is

$$Z \in \mathcal{H}^\Gamma = L^2(G^{|\Gamma_{ev}|}/(G^{|\Gamma_v|} \times G'^{|\Gamma_e|})). \tag{12}$$

This Hilbert space also describes the effective face weights ω' defined in (4).

We call this space the universal boundary space for the holonomy spin foams, which is fixed by a choice of graph and the two groups G, G'. This Hilbert space and its relation to projected spin networks [26,27] will be further explored in the companion paper [28].

IV. RELATION TO OTHER FORMALISMS

The form of the state sum (6) was chosen such that it mimics the main features of the EPRL (and other known) spin foam models, but has a minimal set of free parameters. This appears to be a very convenient choice, but it is by far not the only one. There have been other forms used in the literature, see for instance Refs. [21–24], which include weights for different geometric objects, which might appear under coarse graining. The aim of this section is to give a precise relation between the E-function formalism used in this paper, and the other formulations.

A. Operator spin foam models

The operator spin foam models introduced in Ref. [25] were originally formulated with the aim of allowing for arbitrary operators P_e associated to edges e of $\mathcal{C}$. The state sum Z is evaluated by contracting the P_e at the vertices v of $\mathcal{C}$ according to the combinatorics of the 2-complex.

Starting with a 2-complex with irreps ρ_f of G associated to faces f, an OSFM is specified by operators mapping the intertwiner space onto itself

$$P_e: \mathrm{Inv}_G(\rho_{f_1} \otimes \ldots \rho_{f_n}) \to \mathrm{Inv}_G(\rho_{f_1} \otimes \ldots \rho_{f_n}), \tag{13}$$
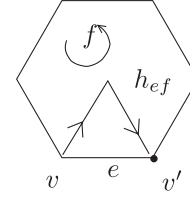

FIG. 2. Holonomy variables in the dual description of operator spin foam models.

as well as face weights ω^{ρ_f}. There is a dual formulation of this model [24] using holonomies $h_{ef} \in G$ for all pairs $(e \subset f)$. This holonomy can be thought of as starting at one vertex v going to the middle of the face f, and going to another vertex v' so that e goes from v to v' in the orientation induced by f, see Fig. 2. The Fourier transform of the P_e operators is given by functions C_e,

$$C_e(h_1, \ldots, h_n) = \sum_{\rho_k} \Big(\prod_k \dim \rho_k D_{\bar{\rho}_k}(h_k)_{m_k}{}^{n_k}\Big)$$
$$\times (P_e)^{m_1 \ldots m_n}{}_{n_1 \ldots n_n}, \tag{14}$$

where we assume for simplicity that the edge orientation agrees with all orientations from all faces meeting at it. Then, the state sum in terms of holonomies is given by

$$Z_{\mathrm{OSFM}} = \int dh_{ef}\Big(\prod_f \omega(G_f)\Big)\prod_e C_e(h_{ef_1}, \ldots, h_{ef_n}). \tag{15}$$

The G_f is the ordered product of holonomies h_{ef} around a face f. This forms the same path as (5).

The E-function formalism is a special case of the OSFM, in which the edge operator P_e factorizes over the edges, and the invariance property $C_e(h_1, \ldots, h_n) = C_e(g_L h_1 g_R, \ldots, g_L h_n g_R)$, which follows from (13) and (14), is explicitly enforced by projectors onto the intertwiner space. One can write the E-function state sum (6) in the form (15) by performing the variable transformation

$$h_{ef} = g_{ve} g_{ef} g_{ev'}. \tag{16}$$

This removes the g_{ev} dependence of the face functions ω, and one can carry out the integration over the g_{ev}, defining

$$C_e(h_{ef_1}, \ldots, h_{ef_n}) := \int_{G^2} dg_{ev} dg_{ev'} \prod_{f_k \supset e} E(g_{ev} h_{ef_k} g_{v'e}) \tag{17}$$

for each edge. This results in (15), as one can readily check.

Hence one can see that the E-function state sum (6) is a special case of the OSFM state sum (15), in which the operator P_e, which is a map from the invariant subspace of $\rho_{f_1} \otimes \cdots \rho_{f_n}$ to itself, can be written as a part which factorizes over every face, and where the right and left invariance are enforced by sandwiching the product between two Haar intertwiners. This Haar projector is just
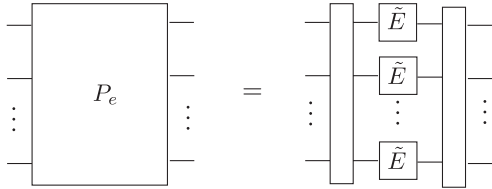
FIG. 3. Graphical notation of the relation between the OSFM version of the edge propagator P_e and the one used in (6), which is also used in the BC and EPRL models.

FIG. 4. Variables in the group field theory formulation.

the group averaging in (17) translated into the spin representation. The factorizing property means that P_e can be written in the following way

$$P_e = P_{\text{Haar}} \cdot P'_e \cdot P_{\text{Haar}}$$
$$= P_{\text{Haar}} \cdot (\tilde{E}_{\rho_1} \otimes \tilde{E}_{\rho_2} \otimes \cdots \tilde{E}_{\rho_n}) \cdot P_{\text{Haar}}, \quad (18)$$

which again is (17) group-Fourier transformed (see Fig. 3). Here $\tilde{E}_{\rho_i}$ are (usually not covariant under the group action) maps acting on ρ_i and with matrix components given by the group Fourier transform of the functions E, i.e.,

$$(\tilde{E}_\rho)^m{}_n = \sum_{d,d',i} e^\rho_{kdd'} I(\rho, k)^m{}_i (I^\dagger(\rho, k))^i{}_n \quad (19)$$

using the notation from (10).

B. Group field theory formulation

The group field theory (GFT) formulation [33–36] of the spin foam state sum is based on a quantum theory of fields defined on the group G, such that the Feynman graphs are spin foam amplitudes associated to 2-complexes.[1] For one of these 2-complexes the group holonomies are pairs of holonomies g^e_{vf}, $g^{e'}_{vf}$, where $v \subset e$, $e' \subset f$. Both are interpreted as going from the vertex v to the center of the face f, and each is associated with one of the two edges e, e' that border f and meet at v (see Fig. 4).

As usual, we denote the inverse by $(g^e_{vf})^{-1} =: g^e_{fv}$. A natural class of models arises if one indeed imposes $g^e_{vf} =^! g^{e'}_{vf}$, as Fig. 4 suggests, but the GFT models allow for relaxation of this condition.

The ingredients for the GFT amplitude associated to $\mathcal{C}$ are edge propagator functions Π_e, each depending on all g^e_{fv} for a given e, and vertex functions $\mathcal{V}_v$, each depending on all g^e_{vf} for a given v. The functions have to satisfy the following symmetries:

$$\mathcal{V}_v(\ldots, g^e_{vf}, g^{e'}_{vf}, \ldots) = \mathcal{V}_v(\ldots, gg^e_{vf}, gg^{e'}_{vf}, \ldots),$$
$$\Pi_e(\ldots, g^e_{vf}, g^e_{v'f'}, \ldots) = \Pi_e(\ldots, g^e_{vf}g, g^e_{v'f}g, \ldots). \quad (20)$$

The symmetries (20) make sure that $\mathcal{V}$ effectively only depends on the combinations $g^e_{fv}g^{e'}_{vf}$, and the propagator

---

[1] Where the specific form of the action is designed to control the class of appearing 2-complexes, see e.g., Ref. [37].
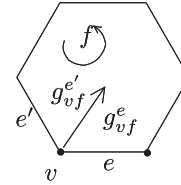
functions Π depend only on $g^e_{vf}g^e_{fv'}$. Also, both functions $\mathcal{V}_v$ and Π_e have to be gauge invariant with respect to gauging at an edge, i.e., $g^e_{vf} \to gg^e_{vf}$ for fixed v, e, and all f which appear as variables in the respective function. As a result, the Π_e will be invariant under left or right shift of its arguments. The $\mathcal{V}_v$ can be decomposed into spin network functions of the boundary graph of the vertex v, which gives the usual spin representation of the vertex amplitudes.

The state sum for the GFT models is given by

$$Z_{\text{GFT}} = \int dg^e_{vf} \prod_e \Pi_e(g^e_{vf_1}, \ldots, g^e_{v'f_n}) \prod_v \mathcal{V}_v(g^{e_1}_{vf_1}, \ldots, g^{e_m}_{vf_n}). \quad (21)$$

The E-function form (6) for the state sum Z can be written as a GFT state sum (21) for $\omega = \delta$. If the product of the h_{ef} around the faces is constrained to be the unit element, one can write (uniquely up to global right shift of all g^e_{vf} for some fixed f by the same $g \in G$)

$$h_{ef} = g^e_{vf}g^e_{fv'}, \quad (22)$$

imposing $g^e_{vf} =^! g^{e'}_{vf}$ in the state sum. This leads to

$$Z = \int dg^e_{vf} \prod_e C(g^e_{vf}g^e_{fv'}, \ldots) \prod_{vf} \delta(g^e_{vf}g^{e'}_{fv})$$
$$= \int dg^e_{vf} \prod_e \Pi_e(g^e_{vf_1}, \ldots, g^e_{v'f_n}) \prod_v \mathcal{V}_v(g^{e_1}_{vf_1}, \ldots, g^{e_m}_{vf_n}) \quad (23)$$

with vertex function

$$\mathcal{V}_v(g^{e_1}_{vf_1}, \ldots, g^{e_m}_{vf_n}) := \prod_{f \supset v} \delta(g^e_{vf}g^{e'}_{fv}) \quad (24)$$

and propagator

$$\Pi_e(g^e_{vf_1}, \ldots, g^e_{v'f_n}) := C(g^e_{vf}g^e_{fv'}, \ldots). \quad (25)$$

C. Vertex amplitude formulation

A formulation closely related to the GFT formalism can be given in terms of holonomies labeled by wedges, i.e., by a face f, and pairs e, e' that meet at a common vertex (see Fig. 5). The holonomy can be thought of as going from the center of e' to the vertex v, and from there to the center of e.
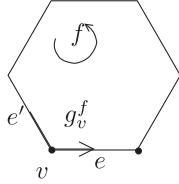
FIG. 5. Holonomy variables in the vertex amplitude formulation.

One can arrive at a formalism using the g_v^f by imposing $g_{e'v}g_{ve} \overset{!}{=} g_v^f$ (as suggested by Figs. 1 and 5) in the state sum (6) and integrating out the gauge holonomies g_{ev}, arriving at

$$Z_{\text{Vertex}} = \int dg_{ef}dg_v^f\prod_f\omega(G_f)\prod_{ef}E(g_{ef})\prod_v W_v(g_v^{f_1},\ldots,g_v^{f_n}),$$
(26)

where $G_f = g_v^f g_{ef}g_{v'}^f\ldots g_{e'f}$ is the ordered product of group elements around a face f according to its orientation. The vertex function W_v for a vertex v is given by

$$W_v(g_v^{f_1},\ldots,g_v^{f_n}) = \int dg_{ev}\prod_{f\supset v}\delta(g_{e'v}g_{ve}(g_v^f)^{-1}). \quad (27)$$

The vertex functions W_v have a symmetry that corresponds to gauge transformation associated to edges, i.e., for an edge e beginning at vertex v, V_v is invariant under the transformation of its arguments

$$g_v^f \rightarrow \begin{cases} g_v^f h_e & \text{if } f \supset e \\ g_v^f & \text{if } f \not\supset e. \end{cases} \quad (28)$$

This invariance allows us to interpret W_v as a G-spin network function for the boundary graph of the neighboring 2-complex of the vertex v. A very similar formulation that uses G'-spin networks instead was given in Ref. [23].

Writing the state sum model in the form (26) allows for generalizations of the model by relaxing the condition (27), allowing for more complicated interactions at the vertex. These nontrivial vertex functions actually do appear in a renormalization group flow via the $n-1$ Pachner moves of state sum models, which is why using the space of all models of the form (26) allows for computing an exact renormalization group flow in this case. The cost of this is that this space is much larger than the space of E-function models using state sums of the form (6). Still, the example in Sec. VI C 1 shows that truncating the flow to this subspace can still capture essential features of the phase space, as well as its fixed point structure.

The form (26) can be simplified by absorbing the E functions into the vertex functions W_v. To this end, we write the E functions as convolution of two functions, i.e.,
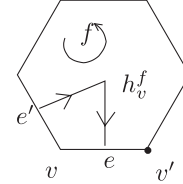
$$E(g) = \int dkF(k)F(gk). \quad (29)$$

FIG. 6. Wedge holonomies in the alternative vertex amplitude description.

The function F has the same symmetries as E, i.e., $F(g) = F(g'g(g')^{-1}) = F(gh)$ for $g, g' \in g, h \in G'$.[2] Thus we get one more integration over G for each pair ef, resulting in

$$Z_{\text{Vertex}} = \int dg_v^f dg_{ef}dk_{ef}\prod_f\omega(G_f)$$
$$\times \prod_{ef}F(g_{ef}k_{ef})F(k_{ef})\prod_v W_v(g_v^f,\ldots). \quad (30)$$

Performing the variable transformation $g_{ef} \rightarrow g_{ef}k_{ef}$ and $g_v^f \rightarrow h_v^f \equiv k_{ef}g_v^f g_{e'f}^{-1}$ and carrying out the integration over the g_{ef} and k_{ef}, one arrives at the form

$$Z_{\text{Vertex}} = \int dh_v^f\prod_f\omega(G_f)\prod_v V_v(h_v^f,\ldots), \quad (31)$$

where G_f is the ordered product of the k_v^f for all $v \subset f$, according to the orientation of f. These h_v^f can be thought of geometrically as the wedge holonomies illustrated in Fig. 6. The new vertex functions V_v are given by

$$V_v(h_v^f,\ldots) = \int dg_{ef}\left(\prod_{v\subset e\subset f}F(g_{ef})\right)W_v(g_{ef}h_v^f g_{e'f}^{-1},\ldots).$$
(32)

The vertex functions V_v inherit properties from both W_v and F. Firstly, it can be readily seen that V_v can—just as W_v—be interpreted as a G-spin network function on the boundary graph around the vertex v. Furthermore, the right invariance of F under the subgroup G' implies that V_v is invariant under the symmetry

$$h_v^f \rightarrow h_1 h_v^f h_2, \qquad h_1, h_2 \in G'. \quad (33)$$

Hence V_v is not only a function on elements of G, but rather on $\tilde{G} = G'\backslash G/G'$. If G' happens to be a normal subgroup of G, $\tilde{G}$ in itself is again a group that carries a right and left G action. In this case V_v can be interpreted as function on a graph, depending on $\tilde{G}$ elements on the edges, invariant under gauge transformations at the vertices with elements from G.

It should be noted that one can arrive at the same version of the state sum (31) by starting from a version of C_e

---

[2]It should be noted that F is not unique but depends on choosing a collection of phase functions. If, however, E is positive, then F can be chosen positive as well.

TABLE I. Different formulations of the spin foam state sum using different variables and parametrizing functions.

State sum	Symbol	Variables	Parametrizing functions
E function	Z	g_{ev}, g_{ef}	ω, E
OSFM	Z_{OSFM}	h_{ef}	ω, C_e
GFT	Z_{GFT}	g_{vf}^e	$\Pi_e, \mathcal{V}_v$
Vertex function	Z_{Vertex}	g_v^f, g_{ef}	ω, E, W_v
Vertex function (alternative)	Z_{Vertex}	h_v^f	ω, V_v

functions and vertex functions W_v. Since the C_e functions are more general than the E functions, in this case the resulting vertex functions V_v are just G-spin network functions, without any specific properties regarding the subgroup G'. It is this form, i.e., Eq. (31) with general G-spin network functions V_v and class functions ω, which is the most general of all the presented formalisms. The various different formulations are summarized briefly in Table I.

V. THE BF, BC AND EPRL MODELS

Here we will shortly describe the structure of the current spin foam models [5,18,19] in the holonomy language. To do so we specify now to $G = \text{Spin}(4)$ and $G' = SU(2) = \text{Spin}(4)_{\text{diag}}$ the diagonal $SU(2)$ subgroup, with irreps labeled by ρ and k respectively. Also, the face weights are usually set to be the delta function on the group.

The operators $\tilde{E}$ described in Sec. IV A for the various models are given by the following, see for instance Ref. [2]:

(i) For BF theory we have

$$\tilde{E}^{\text{BF}} = \mathbf{1}, \tag{34}$$

which reduces the edge projector P_e to be the Haar projector.

(ii) In the Barrett-Crane model the maps $\tilde{E}_\rho$ which act on representation space ρ restrict to representations $\rho = (k, k)$. Furthermore $\tilde{E}_\rho$ just involves the injection maps (described in Sec. III A) $I(\rho, 0)$ from $\bar{\rho}$ to the representation $k = 0$:

$$\tilde{E}_\rho^{\text{BC}} = d_e(\rho) \sum_k \delta(\rho, (k, k)) I(\rho, 0) I(\rho, 0)^\dagger, \tag{35}$$

where $d_e(\rho)$ is an unspecified edge measure factor.

(iii) For the EPRL model the maps $\tilde{E}_\rho$ also restrict to a k-parameter family of representations, which is now of the form $\rho_\gamma(k) = (\frac{1+\gamma}{2} k, \frac{|1-\gamma|}{2} k)$ with γ being the Barbero-Immirzi parameter:

$$\tilde{E}_\rho^{\text{EPRL}} = d_e(\rho) \sum_k \delta(\rho, \rho_\gamma(k)) I(\rho, k) I(\rho, k)^\dagger. \tag{36}$$

The injection map picks out a representation k_ρ which is determined by ρ via the relation $\rho = \rho_\gamma(k_\rho)$.

(iv) The FK model is constructed using coherent states [38], a description of the corresponding edge operators and functions can be found in Refs. [2,28].

The edge functions $E(h)$ can be obtained from the edge operators $\tilde{E}$ by Fourier transform and using the parametrization in Sec. III A:

(i) For the BF model we just have $E^{\text{BF}}(h) = \delta_G(h)$, i.e., the E function is given by the delta function on the group.

(ii) For the Barrett-Crane model we obtain

$$E^{\text{BC}}(h) = \sum_{\rho,k} d_e(\rho) \dim(\rho) \delta(\rho, (k, k)) \text{tr}_\rho$$
$$\times (D_\rho(h) I(\rho, 0) I(\rho, 0)^\dagger). \tag{37}$$

For the choice $d_e(\rho) = 1$ this results in

$$E^{\text{BC}}(h) = \delta_{SU(2)}(h^+ h^{-^{-1}}) = \Theta_{\text{Spin}(4)_{\text{diag}}}(h), \tag{38}$$

where, for a general subgroup $G' \subset G$ we write

$$\Theta_{G'}(h) = \int_{G'} dh' \delta(hh'^{-1}) \tag{39}$$

for the delta function that forces a group element to lie in the subgroup. Note that the E function has an enhanced symmetry with respect to the subgroup $G' = \text{Spin}(4)_{\text{diag}}$, it is not only invariant under conjugation with elements in G' but also invariant under left and right multiplications. This explains some of the special properties of the BC model as compared to more general models.

(iii) For the EPRL model the function $E^{\text{EPRL}}(h)$ is given by

$$E^{\text{EPRL}}(h) = \sum_{\rho,k} d_e(\rho) \dim(\rho) \delta(\rho, \rho_\gamma(k)) \text{tr}_\rho$$
$$\times (D_\rho(h) I(\rho, k) I(\rho, k)^\dagger). \tag{40}$$

As for the BC model this will be a distribution with singular support described below.

The E functions are distributional for all the cases described here and have support on the following critical manifolds:

(i) BF: $\mathbf{1}$,

(ii) BC: $SU(2)_{\text{diag}}$,

(iii) EPRL: $M_\gamma = \{g \in \text{Spin}(4) | \exists L \in \mathfrak{su}(2): g = (\exp(L) \exp(\gamma L))\}$.

For the EPRL model this is shown in Ref. [29]. For our purposes it will be convenient to characterize the EPRL support in terms of the group. This can be done for example by choosing a $U(1)$ subgroup of $U(1)_\gamma \in \text{Spin}(4)$ generated by $(L, \gamma L)$. The critical manifold is then simply $SU(2)_{\text{diag}} \rhd_{\text{ad}} U(1)_\gamma$, where $\rhd_{\text{ad}}$ is the adjoint action.

A. General (finite) groups

We can now mimic the three types of spin foam models defined above for finite groups. To do so we assume that the

most important part of the amplitude is the divergent, distributional part. To define models for finite groups we choose functions with support on the appropriate subsets of $M \subset G$, that is by replacing the delta functions by normalized $\Theta_M(g)$ functions that are defined to be $\frac{1}{|M|} \forall \ g \in M$ and 0 otherwise. The E functions are then:

(i) BF: $E(h) = \Theta_e(h)$ with e being the trivial subgroup consisting of the identity element,

(ii) BC: $E(h) = \Theta_{G'}(h)$,

(iii) EPRL: $E_C(h) = \Theta_{M_C}(h)$, where C is a cyclic subgroup of G and $M_C = G' \rhd_{\mathrm{ad}} C$.

This generalizes the BC and the EPRL model to arbitrary groups $G' \subset G$.

1. Example: S_3

Let us consider as an example the finite group $G = S_3$, the group of permutations of three elements, as gauge group. It can be written as

$$S_3 = \langle a, b | a^2 = b^3 = (ab)^2 = 1 \rangle. \tag{41}$$

The group has six elements $\{e, a, bab^{-1}, b^2ab^{-2}, b, b^2\}$, where e is the unit element, $\{a, bab^{-1}, b^2ab^{-2}\}$ are odd permutations (two cycles) and $\{b, b^2\}$ are three cycles.

The group has subgroups isomorphic to $\mathbb{Z}_2$ generated by either a, ab or ab^2. Furthermore there is a $\mathbb{Z}_3$ subgroup e, b, b^2 which is normal in S_3. To specify the models further we have to choose one of the subgroups and consider E functions which are invariant under the adjoint action of the subgroup G' on G. The orbits under the different subgroups, including the group $G = S_3$ are

$$G' \equiv S_3: \{e\}, \{a, bab^{-1}, b^2ab^{-2}\}, \{b, b^2\};$$

$$G' = \{e, a\} \equiv \mathbb{Z}_2: \{e\}, \{a\}, \{bab^{-1}, b^2ab^{-2}\}, \{b, b^2\};$$

$$G' = \{e, b, b^2\} \equiv \mathbb{Z}_3: \{e\}, \{a, bab^{-1}, b^2ab^{-2}\}, \{b\}, \{b^2\}. \tag{42}$$

Choosing $G' = \mathbb{Z}_2$ as subgroup, we have *a priori* a four parameter space of E functions

$$E(g) = \beta_1 \delta(g, e) + \beta_2 \delta(g, a) + \beta_3 (\delta(g, bab^{-1})$$
$$+ \delta(g, b^2ab^{-2})) + \beta_4 (\delta(g, b) + \delta(g, b^2)). \tag{43}$$

Taking some normalization condition into account, we obtain three parameters. As will be explained in Sec. VI A, models in which the E function is invariant under conjugation of the full group can be rewritten as standard lattice gauge theory models, i.e., the E functions can be absorbed into the face weights ω. This will be the case for $\beta_2 = \beta_3$.

The BC model for the choice of this subgroup $\mathbb{Z}_2$ corresponds to $\beta_1 = \beta_2 = 1$ and $\beta_3 = \beta_4 = 0$. Hence it will in general define a nontrivial model, which cannot be rewritten into a lattice gauge theory.

We can define an EPRL-like model with $\mathbb{Z}_3$ as a cyclic subgroup, which leads to $\beta_1 = \beta_4 = 1$ and $\beta_2 = \beta_3 = 0$. This model is also a $\mathbb{Z}_3$ subgroup model (the BC model with subgroup $\mathbb{Z}_3$) and actually rewritable as a standard lattice gauge theory. For this choice of parameters we have

$$E(g) = \Theta_{\mathbb{Z}_3}(g), \tag{44}$$

and as $\mathbb{Z}_3$ is a normal subgroup the function E will be invariant under conjugation. Indeed the BC model with a normal subgroup will define a topological theory which is of BF type on the quotient group (here $\mathbb{Z}_2$) and of high temperature type (i.e., E is a constant function) on the subgroup itself. We can define two different one-parameter families which connect this topological theory to either the high temperature/strong coupling fixed point of the full group ($\beta_1 = \beta_2 = \beta_3 = \beta_4$),

$$E(g) = \Theta_{\mathbb{Z}_3}(g) + \gamma \Theta_{\mathrm{odd}}(g), \tag{45}$$

(here $\Theta_{\mathrm{odd}} = 1$ on odd permutations and vanishes on even permutations) or to the BF theory on the full group,

$$E(g) = \delta(g, e) + \beta_4 (\delta(g, b) + \delta(g, b^2)). \tag{46}$$

Another example to define a BC model more closely connected to the $SU(2) \times SU(2)$ model is to take $G = S_3 \times S_3$ as gauge group and the diagonal group as a subgroup. We will discuss this model shortly in Sec. VI C.

VI. COARSE GRAINING

In this section we will consider coarse graining of the holonomy spin foam models in dimensions two and three. To this end we will utilize in the first part techniques from standard lattice gauge theory, that is use the similarity of holonomy spin foams to standard gauge theories. Indeed, as for gauge theories, coarse graining in two dimensions can be done exactly. The holonomy spin foam models feature in addition to the face weights, which are the central dynamical entities for standard lattice gauge theory, the edge weights or edge functions E, which will lead to additional factors in the coarse graining formulas.

In three dimensions certain models including BC-type models can be rewritten into models of standard lattice gauge theory type. We will identify this class of models and coarse grain these models by applying the Migdal-Kadanoff scheme.

These results can be compared to another type of coarse graining scheme, which involves a hierarchical lattice on which coarse graining can be performed by applying repeated $4 - 1$ Pachner moves. This allows us to do an exact coarse graining. In particular we will test whether the parametrization of the holonomy models with E function captures the relevant dynamical data. That is, although the general coarse graining flow leaves the model space described by the E functions, we can project back to this

space and test whether we still find the fixed point structure of the exact flow.

Attempting concrete (numerical) calculations we face the difficulty that the models are not necessarily finite for (compact) Lie groups. To avoid these difficulties numerical investigations will be (mostly) performed with finite groups, in particular S_3, for which we defined a class of models in the last section.

A. Coarse graining of edges

Let us consider the formulation of the models (14), where to every edge we associate a C function and to every face, face weights ω. One important difference of these models compared to standard lattice gauge theories is that in general, these are not invariant under edge subdivision, or the inverse operation, coarse graining of edges by removing two-valent vertices.

Therefore we will briefly discuss the coarse graining of edges. This is most conveniently done in the spin basis where the C function is expressed as follows:

$$
\begin{aligned}
&C(h_1, h_2, \ldots h_n)\\
&= \sum_{\rho_1,\ldots,\rho_n,dd'} \prod_{i=1,\ldots,n} \dim \rho_i \tilde{C}_{\rho_1,\ldots,\rho_n,dd'} \langle \iota_{d'} | D_{\rho_1}(h_1)\\
&\quad \times \otimes \cdots \otimes D_{\rho_n}(h_n) \iota_d \rangle.
\end{aligned}
\tag{47}
$$

Here $\{\iota_d\}$ is an orthonormal basis of the intertwiners and $\langle \cdot | \cdot \rangle$ is the inner product on $L^2(G^n)$.

The coefficients $\tilde{C}$ can be computed from the coefficients $e^k_{\rho dd'}$ defined in (10) if the C function is derived from an E function as in

$$
\begin{aligned}
&C(h_{ef_1}, \ldots, h_{ef_n})\\
&= \int dg_L dg_R \prod_i d\tilde{h}_{ef_i} \prod_i E(\tilde{h}_{ef_i}) \prod_i \delta(g_L h_{ef_i} g_R \tilde{h}_{ef_i}^{-1}).
\end{aligned}
\tag{48}
$$

Expanding the δ function $\delta(g) = \sum_\rho \dim \rho \chi_\rho(g)$ and using the identity

$$
\begin{aligned}
&\int dg_L dg_R \prod_i \chi_{\rho_i}(g_L h_{ef_i} g_R \tilde{h}_{ef_i}^{-1})\\
&= \sum_{dd'} \text{tr} |\iota_{d'} \rangle \langle \iota_{d'} | \otimes D_{\rho_i}(h_{ef_i}) |\iota_d \rangle \langle \iota_d | \otimes D_{\rho_i}(\tilde{h}_{ef_i}^{-1}),
\end{aligned}
\tag{49}
$$

we obtain

$$
\tilde{C}_{\rho_1\ldots,\rho_n dd'} = \prod_i d\tilde{h}_{ef_i} \prod_i E(\tilde{h}_{ef_i}) \langle \iota_d | \otimes D_{\rho_i}(\tilde{h}_{ef_i}^{-1}) \iota_{d'} \rangle.
\tag{50}
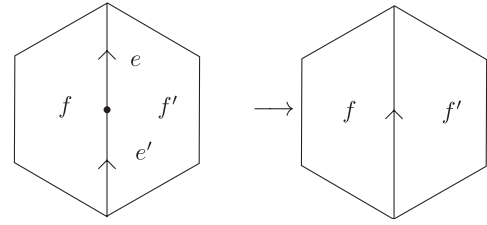$$

Using the explicit form for the E function (10) yields

FIG. 7. Removing a trivial subdivision of an edge.

$$
\begin{aligned}
\tilde{C}_{\rho_1\ldots,\rho_n dd'} &= \sum_{k_i,d_i d_i'} \prod_i e^{\rho_i}_{k_i d_i d_i'} \langle \iota_d | \otimes_i I(\rho_i, k_i)_{d_i} I^\dagger\\
&\quad \times (\rho_i, k_i)_{d_i'} | \iota_{d'} \rangle.
\end{aligned}
\tag{51}
$$

With the form (47) of our edge function C at hand we can now consider the coarse graining, where a subdivided edge, that is two edges e, e' joined by a two-valent vertex, is coarse grained into one edge n. All edges bound the same faces $f_1, \ldots, f_m$, see Fig. 7. As C functions are composed by delta functions, the C function for the new edge n is given by

$$
\begin{aligned}
&C^{(n)}(h_{nf_1}, \ldots, h_{nf_m})\\
&= \int \prod_i dh_{ef_i} \prod_i dh_{e'f_i} C^{(e)}(h_{ef_1}, \ldots, h_{ef_m}) C^{(e')}\\
&\quad \times (h_{e'f_1}, \ldots, h_{e'f_m}) \prod_i \delta(h_{nf_i}^{-1} h_{ef_i} h_{e'f_i}).
\end{aligned}
\tag{52}
$$

Expanding all functions involved as in (47) and using the orthogonality of the basis ι_d we arrive at

$$
\tilde{C}^{(n)}_{\rho_1,\ldots,\rho_n,dd'} = \sum_{d''} \tilde{C}^{(e)}_{\rho_1,\ldots,\rho_n,dd''} \tilde{C}^{(e')}_{\rho_1,\ldots,\rho_n,d''d'}.
\tag{53}
$$

This gives the renormalization flow equation for the coarse graining of edges by removing two-valent vertices. Note that the flow equations for this operation are local.

In 2D and many 3D models the C function simplifies drastically, so that the models can be reduced to standard gauge theory. To explain how this happens, we define an n-valent C function to be factorizable, if it can be written as

$$
\begin{aligned}
&C(h_{ef_1}, h_{ef_2}, \ldots, h_{ef_n})\\
&= \int dg_L \epsilon(g_L h_{ef_1}) \epsilon(g_L h_{ef_2}) \ldots \epsilon(g_L h_{ef_n}),
\end{aligned}
\tag{54}
$$

where ϵ is a class function. Let us notice that every function of the form (54) can be rewritten into a left and right group averaging of E functions since due to the symmetries of ϵ we have

$$
\begin{aligned}
&C(h_{ef_1}, h_{ef_2}, \ldots, h_{ef_n})\\
&= \int dg_L dg_R \epsilon(g_L h_{ef_1} g_R) \epsilon(g_L h_{ef_2} g_R) \ldots \epsilon(g_L h_{ef_n} g_R),
\end{aligned}
\tag{55}
$$

hence $E = \epsilon$. The form introduced above allows us to integrate out h_{ef} and to absorb the ϵ factors into the face weights ω. If we define the effective face weights

$$\omega'_f = \int \prod_{e \subset f} \mathrm{d}h_{ef}\epsilon(h_{ef})\omega_F(\dots g_{ve}h_{ef}g_{ev'}\dots), \quad (56)$$

these are again class functions of the group element $\dots g_{ve}g_{ev'}\dots$. (Note that the face weights ω'_f might depend on the face f, in particular if faces differ in the number of adjacent edges.) Explicitly, using the Fourier transform

$$\omega'(g) = \sum_\rho \dim\rho\, \omega'^\rho \chi_\rho(g), \quad (57)$$

$$\epsilon(g) = \sum_\rho \dim\rho\, \epsilon^\rho \chi_\rho(g), \quad (58)$$

we obtain

$$\tilde{\omega}'^\rho = (\epsilon^\rho)^{\sharp(e \subset f)} \tilde{\omega}^\rho. \quad (59)$$

In this way every spin foam with factorizable C functions can be rewritten as a standard lattice gauge theory. This factorization property holds for every two-valent edge and in the case of BC/EPRL/FK also for every three-valent edge.

If we do not integrate out the h_{ef} and keep the C functions, the Fourier transform of these will be given by

$$\tilde{C}_{\rho_1,\dots,\rho_n,dd'} = \left(\prod_i \epsilon^{\rho_i}\right)\delta_{dd'}. \quad (60)$$

This will simplify the renormalization flow (53).

B. Coarse graining in two dimensions

Here we will consider a 2-complex for which all edges are two valent, that is, each edge bounds exactly two faces. In this case the coarse graining flow can be computed exactly. The procedure and result are quite similar to real space renormalization of 2D lattice gauge theories [39]. As pointed out in Sec. VI A, compared to standard lattice gauge theory one obtains however an additional renormalization flow from the edge weights. Starting with the form of the partition function

$$Z = \int\left(\prod_{(fe)} \mathrm{d}h_{fe}\right)\left(\prod_{(e)} C(h_{ef}, h_{ef'})\right)\left(\prod_f \omega_f(h_{ef}h_{e'f}\dots)\right), \quad (61)$$

we have to discuss the renormalization flow of the C functions and the face weights ω_f.

The C function is by construction invariant under left and right multiplication (in both entries). Hence it can be expanded into a basis of gauge-invariant functions, that is

$$C(h_1, h_2) = \sum_\rho \tilde{C}_\rho \dim(\rho)\chi_\rho(h_1 h_2^{-1}). \quad (62)$$

Such a C function is factorable in the sense of (54): the Fourier expansion of the factorization property gives

$$C(h_1, h_2) = \int \mathrm{d}g\,\epsilon(gh_1)\epsilon(gh_2)$$
$$= \sum_\rho \epsilon^\rho \epsilon^{\bar\rho} \dim(\rho)\chi_\rho(h_1 h_2^{-1}). \quad (63)$$

Hence we just need to choose the Fourier coefficients ϵ^ρ such that $\epsilon^\rho\epsilon^{\bar\rho} = \tilde{C}_\rho$ is satisfied.

The coefficients $\tilde{C}_\rho$ can be expressed using the parametrizations through $e^\rho_{k,dd'}$ as

$$\tilde{C}_\rho = \sum_k \frac{\dim k}{\dim\rho}\sum_{dd'} e^\rho_{kdd'} e^\rho_{kd'd}, \quad (64)$$

where we used that $e^{\bar\rho}_{kdd'} = e^\rho_{kd'd}$, which can be shown to hold in general [28]. That is from the choices $e^\rho_{kdd'}$ for the basis coefficients of E only the contraction (64) matters in a two-dimensional theory.

The renormalization flow equations for the coarse graining of edges (53) are in this case given by

$$\tilde{C}^{(n)}_\rho = \tilde{C}^{(e)}_\rho \tilde{C}^{(e')}_\rho. \quad (65)$$

The other coarse graining step that will appear in two dimensions is integrating out the variables over one edge e that divides two faces f, f', see Fig. 8. The face holonomies starting at the source of e will be denoted by $g_f = h_{ef}H_f$ and $g_{f'} = h_{ef'}H_{f'}$. The new face weight is given by

$$\omega^{(n)}_{f\cup f}(H_f H_{f'}^{-1}) = \int \mathrm{d}h_{ef}\mathrm{d}h_{ef'}\,\omega_f(h_{ef}H_f)\omega_{f'}$$
$$\times (h_{ef'}H_{f'})C(h_{ef}, h_{ef'}), \quad (66)$$

which for the Fourier coefficients gives

$$\tilde{\omega}^{(n)}_{f\cup f} = \tilde{\omega}_f(\rho)\tilde{\omega}_{f'}(\rho)\tilde{C}^{(e)}_\rho. \quad (67)$$

In two dimensions all coarse graining steps can be decomposed into the two operations we discussed, that is the inverse of subdividing a face with one edge and the inverse of subdividing an edge with one vertex. This allows us to write the complete partition function of a two-dimensional complex as

$$Z = \sum_\rho (\tilde{\omega}(\rho))^F(\tilde{C}_\rho)^E \quad (68)$$

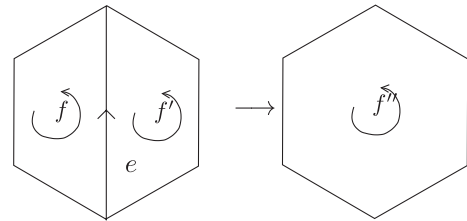(assuming that the face weights and edge functions are initially the same for all faces and edges).

FIG. 8. Removing a subdivision of a face.

The recursion relations (65) and (67) enable us to discuss completely the renormalization flow in two dimensions. Starting with a "heat kernel" ($\mathcal{C}_\rho$ is the Casimir of ρ)

$$\tilde{C}_\rho = \exp\left(-\beta^{-1}\mathcal{C}_\rho\right) \tag{69}$$

for the edge weights, we flow from zero temperature $\beta = \infty$ to infinite temperature $\beta = 0$. Note that zero temperature $\beta = \infty$, $\tilde{C}_\rho = 1$ corresponds to a C function $C(h_1, h_2) = \delta(h_1 h_2^{-1})$. This converts the partition function (61) into one for standard lattice gauge theory. Choosing nonzero temperature corresponds to a regularization (introducing a nontrivial propagator for the edges) for instance in group field theory [35]. One should however be aware that this choice leads to a flow into the high temperature fixed point, both for the edge factors and the face weights. In this fixed point only the values for the trivial representation contribute (assuming $\mathcal{C}_{\rho=0} = 0$).

The renormalization flow for the face weights behaves similarly, with the addition that it can be set off from the zero temperature fixed point $\omega(g) = \delta(g)$ by a nontrivial choice for the edge factors alone. Note that the zero and high temperature fixed points are not the only fixed points, as will be obvious from the discussion below.

In higher dimensions the subdivision of an edge by adding a two-valent vertex, as well as the subdivision of a face by inserting a two-valent edge can be argued to be operations under which the partition function should remain invariant [40]. Indeed this notion has been connected to the anomaly freeness of the path integral measure in spin foams with respect to diffeomorphisms [10,41,42] and has the advantage of fixing uniquely the edge and face measure factors [40]. Such an invariance would require $\tilde{C}_\rho$ and $\tilde{\omega}(\rho)$ to take values either equal to 1 or to 0. Any choice would lead to a fixed point in the two-dimensional theory. Also note that for the two-dimensional theory it is equivalent to either choose $\omega(\rho) = 0$ or $\tilde{C}_\rho = 0$ for a given ρ. That is all $\tilde{C}_\rho$ can be chosen to be equal to 1. Hence we just need to consider the lattice gauge theory parameter space reduced to the choice $\tilde{\omega}(\rho) = \theta(\rho)$ where θ takes only values 0 and 1.

C. Dual simplicial lattices in three dimensions

We have discussed that the renormalization flow for the two-dimensional theory agrees with the one in lattice gauge theory in case that one adopts the condition of invariance under edge subdivisions. This condition is also a reasonable requirement to hold for higher dimensions and higher-valent edges. Indeed the Barrett-Crane model can be easily made to satisfy this condition. For the EPRL model, [25,43] give a construction which satisfies this requirement. However, this is more complicated than the one considered here, and leads to a model that no longer falls into the description of E functions (it would however still be in the more general class of models parametrized by C functions).
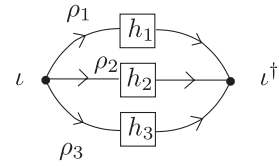

FIG. 9. The structure of the C function for an edge shared by three faces.

Let us assume that we can construct a spin foam model where such an invariance holds. How much freedom is left for the parameters in this spin foam model? In a dual simplicial lattice in three dimensions the edges (dual to triangles) would be three valent, i.e., adjacent to always three faces (as a triangle has three edges). The C function associated to such three-valent edges is a function of three arguments, invariant under left and right group multiplication. Hence it can be expanded into a basis of gauge-invariant spin networks, which are based on three edges with common source and target vertex, see Fig. 9.

These spin networks are labeled with representations at the edges and with intertwiners between the three representations, that is we have

$$C(h_1, h_2, h_3) = \sum_{\rho_1,\rho_2,\rho_3,dd'} \prod_i \dim \rho_i \tilde{C}_{\rho_1,\rho_2,\rho_3,dd'} \langle \iota_{d'}, D_{\rho_1}(h_1)$$
$$\otimes D_{\rho_1}(h_2) \otimes D_{\rho_n}(h_3) \iota_d \rangle, \tag{70}$$

where ι_d labels orthonormal basis of invariants. We will consider groups, such as $SU(2)$, $SO(4)$ and the finite group S_3 for which the intertwiner between three representations is canonical, or in other words for which the tensor product of two representations is multiplicity free. In this case the intertwiner labels d, d' in (70) can be omitted. Also the sum in (70) is only over triples of representations admitting an intertwiner.

Gluing two edges with a two-valent vertex leads to new $\tilde{C}$ coefficients

$$\tilde{C}^{(n)}_{\rho_1,\rho_2,\rho_3} = \tilde{C}^{(e)}_{\rho_1,\rho_2,\rho_3} \tilde{C}^{(e')}_{\rho_1,\rho_2,\rho_3}. \tag{71}$$

Demanding invariance under edge subdivision fixes the coefficients to be equal to 1 or 0 depending on the three representation labels. In the BC and EPRL models (with G isomorphic to $G' \times G'$ group) this condition even factorizes as in Eq. (60), that is we have

$$\tilde{C}_{\rho_1,\rho_2,\rho_3} = \epsilon^{\rho_1} \epsilon^{\rho_2} \epsilon^{\rho_3}, \tag{72}$$

where ϵ_ρ takes only the values 0 or 1. One might expect such a factorization property to hold from the construction of the edge propagator C as group average of a product of E functions. Note however, that even in this class of theories the factorization property does not hold in general. For a counter example see the Appendix. Also the BC model for S_3 with $\mathbb{Z}_2$ as a subgroup provides a counterexample. As one can check explicitly, the corresponding three-valent

edge function $C(h_1, h_2, h_3)$ does not lie in the subspace generated by the conjugation invariant E functions, which define factorable models.

The following considerations restrict therefore to the class of theories where the coefficients factorize as in (72). In this case we can absorb the choice function ϵ_ρ into the face weights $\tilde{\omega}_f$, whereas the $\tilde{C}$ can be chosen to be the same as for standard lattice gauge theory (where invariance under edge subdivision trivially holds). In other words we are reduced again to consider the class of theories described by standard lattice gauge theories. This motivates us to test approximation methods developed for lattice gauge theory, notably the Migdal-Kadanoff approximation [39,44]. There is one drawback which is that the Migdal-Kadanoff approximation was developed for regular (cubical) 2-complexes and for hierarchical lattices [45] (where it can be made to be exact). That is we have to assume that the difference of a simplicial versus a cubical lattice does not influence the large scale dynamics.

The Migdal-Kadanoff approximation is a truncation to local couplings for the real space renormalization flow of lattice gauge theories. As for the flow in two dimensions it results in a recursion relation for the face weights ω_f. In the version of Migdal [39] we have for the coarse graining of a three-dimensional cubical lattice under the doubling of the lattice constant

$$\tilde{\omega}_\rho^{(n)} = \sum_{\rho_1, \rho_2} (\tilde{\omega}_{\rho_1})^4 (\tilde{\omega}_{\rho_2})^4 \frac{\dim(\rho_1)\dim(\rho_2)}{\dim(\rho)} n_{\rho_1, \rho_2}^\rho. \quad (73)$$

Here n_{ρ_1, ρ_2}^ρ is the number of copies of representations ρ in the tensor product of ρ_1 with ρ_2. For the case we are considering here, this number is either one or zero.

Note the essential difference of (73) to the two-dimensional recursion relation (66) which is that the coefficients of different representation labels are coupled with each other. In the BC and EPRL spin foam models for a group $G = G' \times G'$ only a (one-parameter) subset of all possible representations is allowed initially, however the recursion relation (73) will spread the range of allowed values.

The implementation of (73) for a finite group is straightforward. Here we consider the group $S_3 \times S_3$ with a BC-like amplitude. There are three representations for S_3 which we denote by $j = 0, 1, 2$. The representation $j = 2$ is two dimensional whereas the others are one dimensional given by the trivial and the (permutation) sign representation. For $S_3 \times S_3$ we have representations (j^+, j^-) with $j^+, j^- = 0, 1, 2$. The initial face weights are given by

$$\tilde{\omega}_{j^+, j^-} = \delta_{j^+, j^-}, \quad (74)$$

which we put into the recursion relation (73). The coupling coefficients are given by

$$n_{(j_1^+, j_1^-),(j_2^+, j_2^-)}^{(j^+, j^-)} = n_{j_1^+, j_2^+}^{j^+} n_{j_1^-, j_2^-}^{j^-}, \quad (75)$$

where $n_{j_1, j_2}^{j_3}$ is invariant under permutations of the three labels and equal to 1 for the following combinations of representation labels

$$(0, 0, 0), (0, 1, 1), (0, 2, 2), (1, 2, 2), (2, 2, 2). \quad (76)$$

For all other combinations the coefficients are vanishing. After each renormalization step we normalize the amplitude so that $\tilde{\omega}_{0,0} = 1$. The result is that after ca. ten iterations the theory is at a fixed point

$$\tilde{\omega}_{j^+, j^-} = (\delta_{j^+, 0} + \delta_{j^+, 1})(\delta_{j^-, 0} + \delta_{j^-, 1}). \quad (77)$$

These weights correspond (modulo normalization) to a face weight

$$\omega(g_1, g_2) = \Theta_{\text{even}}(g_1)\Theta_{\text{even}}(g_2), \quad (78)$$

which defines a BF($\mathbb{Z}_2$) theory, where this Z_2 can be understood to be the the quotient group of the diagonal subgroup with respect to the $\mathbb{Z}_3$ subgroup of even diagonal elements.

Let us also discuss the $SU(2) \times SU(2)$ BC model. Representations are given by (j^+, j^-) where the spins can take integer or half integer non-negative values and the dimension of the representation (j^+, j^-) is given by $(2j^+ + 1)(2j^- + 1)$. The coefficients $n_{j_1, j_2}^{j_3}$ are equal to 1 if the triangle inequalities $j_i + j_j \geq j_k$ are satisfied and if the sum $j_1 + j_2 + j_3$ is an entire number, otherwise it is 0. The initial face weights are again given by (74).

However the sum (73) diverges. Therefore we introduce a cutoff and sum values only up to $j \leq \lambda$. Again we normalize after each iteration step, such that $\tilde{\omega}_{0,0} = 1$. An impression of the behavior of the flow can be obtained from Fig. 10.

As one can see the face weights flow very fast into the high temperature fixed point and the result does almost not vary with the cutoff. Note also that whereas initially only a one-parameter subset of face weights is nonzero, this range is spread during the recursion process. Therefore one can also expect a similar flow for the EPRL model, where the initial face weights are also only supported on a one-parameter set of labels.

Of course the introduction of the cutoff can be questioned. On the other hand this result is supported by a general theorem stating that in 3D gauge theories with groups $U(1)$, $SU(N)$ flow under the Migdal-Kadanoff relations to the strong coupling/high temperature fixed point [45,46].

Now, we made several approximations which could be questioned. The first is to replace a simplicial lattice with a regular one, the other is the Migdal-Kadanoff approximation itself. To support the findings here, we considered an exact renormalization flow for a special simplicial lattice, which will be discussed more extensively in the next section. This lattice is obtained by iteratively applying 1–4 Pachner moves to a tetrahedron, see Fig. 11.
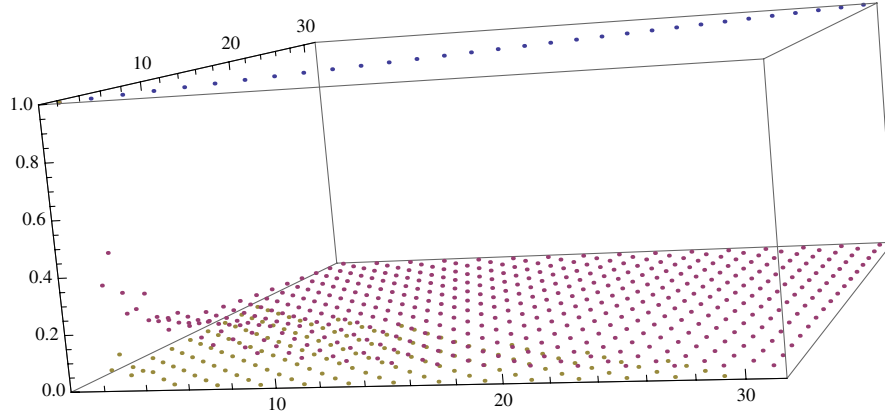
FIG. 10 (color online). This figure shows the face weights $\tilde{\omega}_{j^+,j^-}$ for the $SU(2) \times SU(2)$ Barrett-Crane model as a function of $2j^+$, $2j^-$, which label the x and y axes respectively. The initial configuration $\tilde{\omega}_{j^+,j^-} = \delta_{j^+,j^-}$ of face weights is the diagonal line of dots at the top (blue). The dots starting at around 0.2 (red) and spreading out flatly give the configuration after one iteration, where we show only a dot for values of $\tilde{\omega}_{j^+,j^-} > 10^{-4}$ from the picture the initial diagonal configuration is spread to nondiagonal arguments, but the values for the face weights are decreasing fast. After two iterations we arive at the dots on the floor of the graph (yellow-green). A large portion of the $\tilde{\omega}_{j^+,j^-}$ is already smaller than 10^{-4} and the configuration is almost at the high temperature fixed point.

Such a 1–4 Pachner move subdivides one tetrahedron into four. To these four, one can again apply the Pachner moves obtaining 16, and so on. Going backwards, we can coarse grain blocks of four tetrahedra into one. The advantage of such a hierarchical lattice is that the renormalization flow stays in the set of local couplings. Truncations, which are usually unavoidable in real space renormalization due to the generations of nonlocal couplings, are not necessary. This renormalization flow on the simplicial lattice is most convenient to consider in the spin representation with vertex weights. Here one is actually working with the space of vertex weights for an S_3 model. In the spin representation the BC model with $S_3 \times S_3$ is included in this space. The flow of this model terminates in a fixed point describing a $BF(\mathbb{Z}_2)$ theory, which confirms our findings with the Migdal-Kadanoff truncation.

Another example, which we can consider within $G = S_3$ is with weights $\tilde{\omega}_0 = 1$ and $\tilde{\omega}_2 = 1$.[3] With the Migdal-Kadanoff recursion relation (73) it flows to BF theory for the group S_3. With the hierarchical lattice flow explained in the next section, we obtain a different result: there the model flows to the strong coupling/high temperature fixed point, which is given by having only the coefficient $\tilde{\omega}_0 = 1$ nonvanishing.

This difference can be expected already due to the different structure of the Migdal-Kadanoff flow and the hierarchical flow. A typical feature of the Migdal-Kadanoff flow is that the $\tilde{\omega}$ are convoluted with each other, which leads to a spreading effect of $\tilde{\omega}$ as a function of the representation labels. For the S_3 example it is therefore possible to flow from a configuration with $\tilde{\omega}_1 = 0$, i.e.,

where the sign representation does not appear, to the BF fixed point, in which $\tilde{\omega}_1 = 1$. In the flow defined by $n - 1$ Pachner moves this does not happen. The flow is described by vertex functions which depend on (six) representation labels. If initially the vertex functions are zero for any of the labels giving the sign (or any other) representation, this will be preserved by the flow. This is however a special feature of the $n - 1$ Pachner moves. To get a more complete picture of the flow for more general lattice one has also to consider other Pachner moves, which will require some implementation of a truncation. The Migdal-Kadanoff method is one particular truncation available for lattice gauge theories. Other methods such as tensor network renormalization [15,47] are more general and in principle also applicable to spin foam models [15]. Tensor network renormalization can be motivated by using Pachner moves [47], and indeed the Pachner moves different from the $n - 1$ move are the ones where the truncation
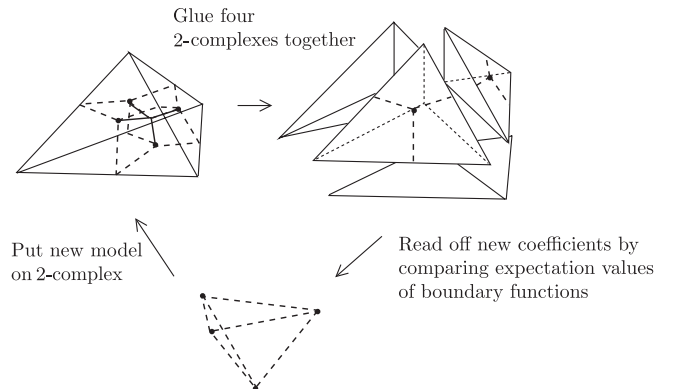

FIG. 11. The 1–4 Pachner move can be iterated to generate an actual flow in the space of models.

---

[3]This model corresponds to the choice of parameters $\beta_1 = 5$, $\beta_2 = \beta_3 = 1$, $\beta_4 = -1$ in (43) and $e_1 = 0$, $e_{20} = e_{21} = 1$ in the next section.

is implemented in tensor network renormalization. This choice of a truncation can be also encoded into embedding maps which are closely related to the concept of cylindrical consistency in loop quantum gravity [48].

1. Renormalization group flow of an S_3 model via a 1–4 move

In the following we will consider the flow of 3D models with group S_3 generated under the 1–4 move. We described this group in Sec. VA 1. We also choose $G' = \{1, a\} \simeq \mathbb{Z}_2$ as a subgroup.

We will define the model on the 2-complex κ_1 dual to a tetrahedron. Its topology allows us to glue four of them together to form the 2-complex κ_2 dual to four tetrahedra which are the result of a 1–4 move. The boundary of that 2-complex will be the same as the one dual to one tetrahedron, apart from trivial subdivisions of edges. The effective model on this arising from integrating over internal degrees of freedom can then be compared to the original model by evaluating Z_{κ_1} and Z_{κ_2} on the same S_3-boundary spin network functions. Iterating this process generates a flow in the space of models.

If we define the model in terms of vertex functions, the flow can be carried out exactly. This is a special feature of all 1-$(n - 1)$ moves, since they do not generate any non-localities. These moves have another interesting property, which is present for all models and dimensions: the vertex function can be interpreted as a function depending on spins and intertwiners. If one starts the flow with a function that is constrained to vanish whenever a specific spin or specific intertwiner is present, then this constraint is conserved under the 1-$(n - 1)$ move. Therefore, if certain degrees of freedom are switched off, they stay switched off during the flow. This observation allows for a very convenient investigation of the RG flow, since it is automatically constrained to lie in certain submanifolds of the model space, all of which intersect in the high temperature fixed point. Note that this is a direct result of working on a hierarchic lattice because the flow does not generate any nonlocalities. It is quite different from the hypercubic lattice flow described in the last section, where nonlocalities are truncated by the Migdal-Kadanoff approximation.

Let us turn to S_3 once again. As previously described there are two one-dimensional representations (denoted $\mathbf{0}_{S_3}$ and $\mathbf{1}_{S_3}$), and one two-dimensional representation (denoted by $\mathbf{2}_{S_3}$). Since $\mathbf{0}_{S_3}$ is the trivial representation, the only nontrivial tensor products are given by

$$\mathbf{1}_{S_3} \otimes \mathbf{1}_{S_3} = \mathbf{0}_{S_3}, \qquad \mathbf{1}_{S_3} \otimes \mathbf{2}_{S_3} = \mathbf{2}_{S_3},$$
$$\mathbf{2}_{S_3} \otimes \mathbf{2}_{S_3} = \mathbf{0}_{S_3} \oplus \mathbf{1}_{S_3} \oplus \mathbf{2}_{S_3}.$$

The group $\mathbb{Z}_2$ has two one-dimensional representations denoted by $\mathbf{0}_{\mathbb{Z}_2}$ and $\mathbf{1}_{\mathbb{Z}_2}$ satisfying $\mathbf{1}_{\mathbb{Z}_2} \otimes \mathbf{1}_{\mathbb{Z}_2} = \mathbf{0}_{\mathbb{Z}_2}$. Decomposing the representation spaces of S_3 with respect to $\mathbb{Z}_2$ yields

$$\mathbf{0}_{S_3} = \mathbf{0}_{\mathbb{Z}_2}, \qquad \mathbf{1}_{S_3} = \mathbf{1}_{\mathbb{Z}_2}, \qquad \mathbf{2}_{S_3} = \mathbf{0}_{\mathbb{Z}_2} \oplus \mathbf{1}_{\mathbb{Z}_2}. \quad (79)$$

After gauge invariance and discrete symmetries of the tetrahedron, there are 11 gauge-invariant combinations of S_3 irreps along the edges of a tetrahedron. In other words, a vertex function is given by specifying the 11 different values of the $\{6j\}$ symbol for S_3.[4]

If we specify a model in terms of E functions, then there are fewer parameters: Due to the decomposition (79) the most general E function is—after normalization—of the form

$$E(h) = 1 + e_1 \text{sgn}(h) + 2e_{20}\rho_{\mathbf{2}_{S_3}}(h)_{00} + 2e_{21}\rho_{\mathbf{2}_{S_3}}(h)_{11}, \quad (80)$$

where $\text{sgn}(h) = \text{tr}\rho_{\mathbf{1}_{S_3}}(h)$ is the signum of the permutation $h \in S_3$, and (00) and (11) specify the upper left and the lower right entry of the 2×2 matrix $\rho_{\mathbf{2}_{S_3}}(h)$. A model is then given by the three parameters e_1, e_{20} and e_{21}.[5]

Note that every model in terms of E functions can be rewritten in terms of vertex functions, but since these are more general, the converse is not necessarily true. Moreover, it turns out that the RG flow does not preserve the class of models of the form (80), but takes place in the 11-dimensional space of all vertex functions instead. This puts us in the favorable position of comparing the exact flow to the truncated flow, which is given by projecting onto the space of models of the form (80) after each step.[6] In the following we present the results.

First of all, there are three obvious fixed points, which exist both under the exact and the truncated flow. All three are models which can be expressed in terms of E functions as in (80). In these parameters, they are given by

(I) S_3 BF theory: $e_1 = e_{20} = e_{21} = 1$,
(II) $\mathbb{Z}_2$-BF theory: $e_1 = 1$, $e_{20} = e_{21} = 0$,
(III) High-temperature fixed point: $e_1 = e_{20} = e_{21} = 0$.

Point I has only unstable directions—a slight derivation will either lead to the model flowing to diverging couplings or into one of the other two fixed points. In point I all couplings associated to terms containing $\mathbf{2}_{S_3}$ are turned off. Because of (79), the two remaining irreps of S_3 can be interpreted as irreps of $\mathbb{Z}_2$, so the point corresponds to $\mathbb{Z}_2$-BF theory. This point has both stable and unstable directions. In point III, which is completely attractive, also $\mathbf{1}_{S_3}$ is switched off. The resulting theory can be either interpreted as BF theory for the trivial group, or in the language of measure theory as the Ashtekar-Lewandowski measure for S_3 functions on the boundary. On it the kinematical and physical Hilbert space coincide, and in

---

[4]Face and edge amplitudes can in principle be absorbed into the vertex functions in this case, so we will fix them to equal those of the S_3 BF theory for convenience.

[5]These parameters are connected to the parameters in (43) by $\beta_1 = 1 + e_1 + 2(e_{20} + e_{21})$, $\beta_2 = 1 - e_1 + 2(e_{20} - e_{21})$, $\beta_3 = 1 - e_1 + 4(e_{20} - e_{21})$, $\beta_4 = 1 + e_1 - (e_{20} - e_{21})$.

[6]There is no unique way of projecting, but different choices do not seem to yield qualitatively different results.
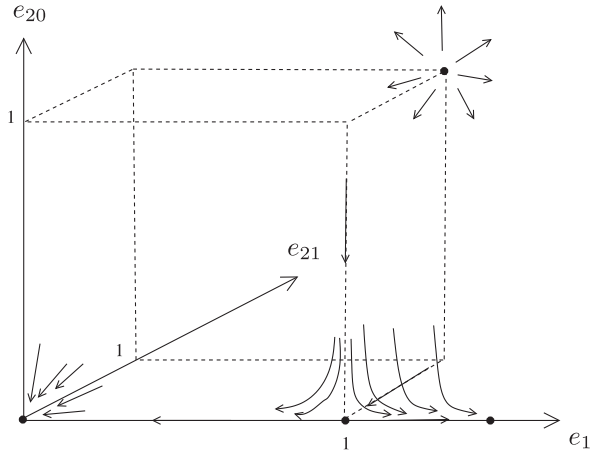
FIG. 12. Truncated RG flow of the S_3 model under a 1–4 move. The diagram includes four fixed points of the flow.

statistical field theory, it corresponds to the high temperature fixed point. The flow diagram for these fixed points is sketched in Fig. 12.

Looking at the exact flow equations for the 11 parameters of the vertex functions, one can find, however, several more fixed points. These do not correspond to any topological theory, but can be interpreted as nontrivial. One of the points, which is rather close to the other three fixed points, is given in terms of the vertex function couplings (i.e., the values of the $\{6j\}$ symbols) all being zero, apart from

$$\alpha_1 = \begin{Bmatrix} \mathbf{0}_{S_3} & \mathbf{0}_{S_3} & \mathbf{0}_{S_3} \\ \mathbf{1}_{S_3} & \mathbf{1}_{S_3} & \mathbf{1}_{S_3} \end{Bmatrix}, \qquad \alpha_2 = \begin{Bmatrix} \mathbf{0}_{S_3} & \mathbf{1}_{S_3} & \mathbf{1}_{S_3} \\ \mathbf{0}_{S_3} & \mathbf{1}_{S_3} & \mathbf{1}_{S_3} \end{Bmatrix}.$$

The exact recursion equations for this system are given by

$$\alpha_1 \rightarrow \frac{\alpha_1^3 + \alpha_1 \alpha_2^3}{1 + \alpha_1^4}, \qquad \alpha_2 \rightarrow \frac{2\alpha_1^2 \alpha_2^2}{1 + \alpha_1^4}. \qquad (81)$$

The obvious fixed point solution is given by $\alpha_1 = \alpha_2 = 1$, which corresponds to point II. Another nontrivial solution is e.g., given by

(IV) $\alpha_1 = 2.54605$, $\alpha_2 = 3.31832$.

The values for α_1 and α_2 can be expressed as roots of polynomials of order five, and we are not aware of a closed expression. The point IV seems to have several attractive directions, and the correct perturbation from point I can let the flow run into point IV. The point is *not* topological, in particular it defines a model which is not invariant under trivial subdivisions of bulk edges by a two-valent vertex, unlike all the BF theories. Also, the resulting model is *not* invariant under the 2–3 move, as one can readily check.

Since the models given by (80) always lead to coefficients of the vertex functions satisfying $\alpha_1 = e_1^3$ and $\alpha_2 = e_1^4$, one can readily check that point IV is not of the form (80). It lies, however, quite close to the three-dimensional submanifold of models given by an E function.

Under the truncated flow, in which one projects onto the space of E-function models as in (80) after each step, one obtains a flow in the three-dimensional space of parameters (e_1, e_{20}, e_{21}). One does not only recover the three obvious fixed points I, II and III, but remarkably there is another fixed point which lies at roughly

(IV*) $e_1 = 1.4021$, $e_{20} = e_{21} = 0$.

It can be readily checked that the vertex function derived from fixed point IV*, which has the only nonzero couplings $\alpha_1 = e_1^3$ and $\alpha_2 = e_1^4$ is rather close to the fixed point under the exact flow IV. This demonstrates that truncating the theory to models of the form (80) can reveal characteristic nontrivial features of the phase space.

D. Outlook

In particular the example in the last section gives us hope that the parametrization with E function presented in this work captures the relevant parameter for the renormalization flow. More generally we have seen that there is a number of features similar to the behavior of lattice gauge theories [39], in particular the appearance of the strong coupling/high temperature fixed point and the BF (low temperature) fixed points. (If there is a normal subgroup, a BF theory on the quotient group will also define a fixed point.) The BF fixed points are unstable in 2D and 3D, whereas the high temperature fixed points have attractive directions. It is conjectured that non-Abelian lattice gauge theories do not show phase transitions in 4D, hence that as in 3D all models flow to the high temperature fixed point.

The spin foam models provide a larger phase space, and a crucial question for future research will be if this larger phase space allows for additional phases/fixed points. This question can be investigated with spin nets models [15,20], which are "dimensionally reduced spin foams." Interestingly spin nets can be formulated with the same algebraic ingredients as spin foams, i.e., any spin foam model formulated with E functions leads to a corresponding spin net model. The advantage is that these models are already nontrivial in two dimensions which simplifies enormously coarse graining considerations. Additionally, it is known that 4D lattice gauge theories show a similar flow behavior as the corresponding 2D (spin net/edge) models [44]. If this similarity extends to spin foam models, these investigations in 2D could already give information on the 4D spin foam models.

In this work we discussed $n - 1$ Pachner moves whose corresponding renormalization flow can be obtained exactly if vertex amplitudes are used. Classical Regge calculus is invariant under $n - 1$ Pachner moves, but 4D Regge calculus is not invariant under the $3 - 3$ move [49]. Finding theories (fixed points) invariant under the $n - 1$ Pachner moves could therefore lead to a quantum theory better mimicking the symmetry properties of the classical one.

On the other hand the flow generated by the $n - 1$ Pachner moves is quite special. It leads not only to a

closed, exact flow for the vertex amplitudes but also features invariant submanifolds resulting from the set of vertex amplitudes that are zero if a given representation appears as argument. Correspondingly one can find a number of fixed points for this flow, which do not appear e.g., in the Migdal-Kadanoff scheme for lattice gauge theories. The question arises whether these fixed points have any significance, i.e., flow to other fixed points, if we consider more general coarse grainings.

Therefore also the other Pachner moves have to be considered, or alternatively the blocking of a regular lattice, which incorporates the other Pachner moves. To this end truncations have to be used, otherwise the number of coupling parameters will grow without bound. A truncation which is suggested by this work is given by vertex amplitudes that can be formulated through E functions. Again the main question will be whether this truncation captures the relevant (local) couplings.

In this work we considered either coarse graining procedures which only lead to local couplings (two-dimensional theories, $n - 1$ Pachner moves) or a truncation of the flow to local couplings (Migdal-Kadanoff procedure). For some questions, like the restoration of diffeomorphism symmetry via coarse graining [12,50], or in order to obtain a more reliable phase diagram, the consideration of nonlocal couplings is necessary. The difficulty in dealing with nonlocal couplings is that these will proliferate and are hard to interpret and to control.

One way to give an interpretation of truncations to parameter spaces including nonlocal couplings is to introduce nonelementary building blocks which carry more boundary data than the building blocks one starts with [48]. The growing number of coupling parameters is now reflected in the growing number of boundary data, between which the couplings can occur. These couplings can be encoded into a functional which acts on the boundary Hilbert space of the (nonelementary) building block in question. The universal boundary Hilbert space defined in Sec. III B provides in this sense a more general parameter space describing also models with nonlocal couplings. (Here nonlocal is to be understood to include more than just the nearest neighbor couplings.) Again to obtain a closed coarse graining flow and not a flow which rather maps to larger and larger Hilbert spaces (leading to the continuum limit, i.e., a path integral over a region with continuum boundary data), one has to implement truncations. Such truncations can be described by maps [48] which embed the boundary Hilbert spaces into each other.

The examples we considered for coarse graining included finite groups, in particular the permutation group S_3. Indeed, one advantage of the holonomy formulation presented in this work is that the models can be extended to finite groups but preserve the key dynamical ingredient of spin foams, the simplicity constraints. For example one can study the coarse graining of these constraints also in the finite group case. As we have also seen, finite groups allow us to apply numerical algorithms which still need to be developed and tested for spin foams. In this work we considered $n - 1$ Pachner moves for the finite group S_3 and found (unexpectedly) a number of fixed points. We conjecture that $SU(2)$ models will show even more fixed points, however the dynamical relevance of these fixed points is not clear yet. As explained above, to uncover the relevance of these fixed points, one needs to incorporate also other Pachner moves or consider coarse graining on a regular lattice.

Testing numerical algorithms is however not the only purpose of finite groups. Experience in lattice gauge theory has shown that models with finite groups allowed one to uncover many dynamical mechanisms of lattice gauge theories and to formulate conjectures on e.g., the phase diagrams of these theories. Thus finite groups were in particular important in the early developments of lattice gauge theories. Given the current knowledge on the statistical behavior of spin foams we envisage a similar role for finite group spin foam models. Additionally finite groups avoid technical difficulties, for instance divergencies, which are otherwise hard to deal with.

In the best case the finite group models allow us through their numerical investigation to extract the relevant parameters for coarse graining. The next step would be to identify the analogue parameters in the full models and to devise a coarse graining algorithm which keeps these relevant parameters and truncates away the dynamically unimportant ones. Such a strategy might overcome the challenge of an infinite dimensional parameter space present for the full models.

VII. SUMMARY

In this work we proposed a new holonomy formulation for spin foam models, which is able to describe all[7] the current models. Compared to other formulations, this class of models features a minimal set of parameters, and is naturally extendable to finite groups and arbitrary 2-complexes. We related this formulation to other holonomy formulations and the operator formulation of spin foams. Further applications of this holonomy formulation will appear in Refs. [28,29]

The choice between different formulation[8] of a class of models becomes most relevant for coarse graining and the

---

[7]With the exception of the Kaminski, Kieselowski, and Lewandowski model [43].

[8]Holonomy and spin representations underly in lattice gauge theory the weak and strong coupling expansion respectively. Hence these different representations help to identify the appropriate degrees of freedom in the different regimes. For pure gravitational spin foam models such a distinction of regimes is much less clear, but could corresponds to a phase where geometry is mostly degenerate (strong coupling) and a (near) topological phase (weak coupling).

construction of renormalization group flows, as it defines the space of models in which the renormalization flow takes place (or to which it is projected/truncated). Here the question is still open which spin foam formulation is the most appropriate, at least for the four-dimensional case.

The holonomy formulation suggested the implementation of coarse graining techniques from lattice gauge theories, and we followed this strategy for the 2D and 3D theories that are in a certain sense equivalent to gauge theories. On the other hand spin foams have a richer structure than standard lattice gauge theories: in the E function formulation introduced here this can be made precise, as in spin foams we permit E functions that are not class functions. The main question for future research will be how relevant the parameters that describe the deviations of the E functions from a class function are under coarse graining. This will reveal whether the algebraic structures specific to spin foams could lead to macroscopic phases different from those of standard lattice gauge theory.

APPENDIX: NONFACTORIZABLE MODEL FOR A 3D DUAL SIMPLICIAL LATTICE

We will prove in this Appendix that not all three-valent edge C functions coming from E have the factorization property (54).

For a three-valent edge in the case of $SU(2) \subset \mathrm{Spin}(4)$ we know that every C function can be written in the form

$$C(g_1, g_2, g_3) = \sum_{j_i^\pm} \tilde{C}_{j_1^+, j_1^-, j_2^+, j_2^-, j_3^+, j_3^-} \, \psi_{j_i^\pm}(g_1, g_2, g_3), \quad (A1)$$

where $g_i \in \mathrm{Spin}(4)$ and

$$\psi_{j_i^\pm} = \iota_{j_i^\pm}^{m_1 m_2 m_3} D_{j_1}(g_1)_{m_1}^{m'_1} D_{j_2}(g_2)_{m_2}^{m'_2} D_{j_3}(g_3)_{m_3}^{m'_3} \iota_{j_i^\pm \, m'_1 m'_2 m'_3} \tag{A2}$$

and Clebsches ι are normalized.

Suppose that we are starting from the E function with $e_{(j^+ j^-)k}$ equal to

$$e_{(j,j)0} = 1 \quad \text{for } j \neq 1, \qquad e_{(1,1)1} = 1, \tag{A3}$$

and otherwise 0.

Then we can compute $\tilde{C}_{j_i^\pm}$ from E functions using (51) as follows:

$$\tilde{C}_{j_i^\pm} = \left(\prod_i e_{(j_i^+ j_i^-) k_i} \right) \left\langle \iota_{j_i^\pm} \left| \prod_i P_{(j_i^+, j_i^-) k_i} \right| \iota_{j_i^\pm} \right\rangle, \tag{A4}$$

where k_i are the only k when e is nonzero and $P_{j^+ j^- k}$ is the projection onto $SU(2)$ subrepresentation with spin k. This in fact is the product of $\prod_i e_{(j_i^+ j_i^-) k_i}$ and a square of suitable contraction of Clebsches since

$$\left\langle \iota_{j_i^\pm} \left| \prod_i P_{(j_i^+, j_i^-) k_i} \right| \iota_{j_i^\pm} \right\rangle = \left| \iota_{j_i^\pm} \left(\prod_i \iota_{j_i^+, j_i^- k_i} \right) \iota_{k_1 k_2 k_3} \right|^2. \tag{A5}$$

One can check the following properties:

(1) If $j_i \neq 1$ and (j_1, j_2, j_3) satisfied triangle inequalities, $\sum j_i$ is an integer then

$$\tilde{C}_{j_1, j_1, j_2, j_2, j_3, j_3} \neq 0. \tag{A6}$$

(2) We have

$$\tilde{C}_{j_1, j_1, j_2, j_2, 1, 1} = 0, \tag{A7}$$

for $j_i \neq 1$ because k_i do not satisfy triangle inequality.

(3) We have

$$\tilde{C}_{2, 2, 1, 1, 1, 1} \neq 0 \tag{A8}$$

proved by use of diagrams.

Suppose that $\tilde{C}_{j_i^\pm}$ has the factorization property, then it can be written in the form

$$\tilde{C}_{j_1, j_1, j_2, j_2, j_3, j_3} = \epsilon_{j_1, j_1} \epsilon_{j_2, j_2} \epsilon_{j_3, j_3}, \tag{A9}$$

but from (A6) we know that

$$\epsilon_{j,j} \neq 0 \quad \text{for } j \neq 1. \tag{A10}$$

By the property (A7) we thus obtain

$$\epsilon_{1,1} = 0, \tag{A11}$$

but then (A8) cannot be satisfied.

We thus proved that for this E the function C cannot be written in the factorized form.

[1] C. Rovelli, arXiv:1102.3660.
[2] A. Perez, arXiv:1205.2019 [Living Rev. Relativity (to be published)].
[3] C. Rovelli and L. Smolin, Nucl. Phys. **B442**, 593 (1995).
[4] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **14**, A55 (1997).

[5] J. W. Barrett and R. M. Williams, Adv. Theor. Math. Phys. **3**, 209 (1999).

[6] J. W. Barrett, R. Dowdall, W. J. Fairbairn, H. Gomes, F. Hellmann, and R. Pereira, Gen. Relativ. Gravit. **43**, 2421 (2011).

[7] F. Conrady and L. Freidel, Phys. Rev. D **78**, 104023 (2008).

[8] B. Dittrich, arXiv:0810.3594.

[9] B. Bahr and B. Dittrich, Classical Quantum Gravity **26**, 225011 (2009).

[10] B. Bahr and B. Dittrich, Phys. Rev. D **80**, 124030 (2009).

[11] C. Rovelli, arXiv:1107.2310.

[12] B. Dittrich, arXiv:1201.3840 [Proc. Sci. (to be published)].

[13] F. Markopoulou, Classical Quantum Gravity **20**, 777 (2003).

[14] R. Oeckl, Nucl. Phys. **B657**, 107 (2003).

[15] B. Dittrich, F. C. Eckert, and M. Martin-Benito, New J. Phys. **14**, 035008 (2012).

[16] B. Dittrich and F. C. Eckert, J. Phys. Conf. Ser. **360**, 012004 (2012).

[17] J. W. Barrett and L. Crane, J. Math. Phys. (N.Y.) **39**, 3296 (1998).

[18] J. Engle, E. Livine, R. Pereira, and C. Rovelli, Nucl. Phys. **B799**, 136 (2008).

[19] L. Freidel and K. Krasnov, Classical Quantum Gravity **25**, 125018 (2008).

[20] B. Bahr, B. Dittrich, and J. P. Ryan, arXiv:1103.6264.

[21] H. Pfeiffer, Classical Quantum Gravity **19**, 1109 (2002).

[22] V. Bonzom and E. R. Livine, Phys. Rev. D **79**, 064034 (2009).

[23] E. Magliaro and C. Perini, Int. J. Mod. Phys. D **21**, 1250090 (2012).

[24] B. Bahr, J. Phys. Conf. Ser. **360**, 012042 (2012).

[25] B. Bahr, F. Hellmann, W. Kaminski, M. Kisielowski, and J. Lewandowski, Classical Quantum Gravity **28**, 105003 (2011).

[26] S. Alexandrov, Phys. Rev. D **66**, 024028 (2002).

[27] E. R. Livine, Classical Quantum Gravity **19**, 5525 (2002).

[28] B. Dittrich, F. Hellmann, and W. Kaminski, arXiv:1209.4539.

[29] F. Hellmann and W. Kaminski, arXiv:1210.5276.

[30] V. Bonzom, Classical Quantum Gravity **26**, 155020 (2009).

[31] B. Dittrich and S. Speziale, New J. Phys. **10**, 083006 (2008).

[32] V. Bonzom and M. Smerlak, Lett. Math. Phys. **93**, 295 (2010).

[33] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli, Nucl. Phys. **B574**, 785 (2000).

[34] M. P. Reisenberger and C. Rovelli, Classical Quantum Gravity **18**, 121 (2001).

[35] J. B. Geloun, R. Gurau, and V. Rivasseau, Europhys. Lett. **92**, 60008 (2010).

[36] D. Oriti, arXiv:gr-qc/0607032.

[37] R. Gurau, Commun. Math. Phys. **304**, 69 (2011).

[38] E. R. Livine and S. Speziale, Phys. Rev. D **76**, 084028 (2007).

[39] A. A. Migdal, Sov. Phys. JETP **42**, 413 (1975).

[40] B. Bahr, Classical Quantum Gravity **28**, 045002 (2011).

[41] M. Bojowald and A. Perez, Gen. Relativ. Gravit. **42**, 877 (2010).

[42] B. Bahr, B. Dittrich, and S. Steinhaus, Phys. Rev. D **83**, 105026 (2011).

[43] W. Kaminski, M. Kisielowski, and J. Lewandowski, Classical Quantum Gravity **27**, 095006 (2010).

[44] L. Kadanoff, Ann. Phys. (N.Y.) **100**, 359 (1976).

[45] K. Ito, Commun. Math. Phys. **95**, 247 (1984).

[46] V. Muller and J. Schiemann, Commun. Math. Phys. **97**, 605 (1985).

[47] M. Levin and C. Nave, Phys. Rev. Lett. **99**, 120601 (2007).

[48] B. Dittrich, New J. Phys. **14**, 123004 (2012).

[49] B. Dittrich and S. Steinhaus, Phys. Rev. D **85**, 044032 (2012).

[50] B. Bahr, B. Dittrich, and S. He, New J. Phys. **13**, 045009 (2011).