On a class of consistent linear higher spin equations on curved manifolds

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We analyze a class of linear wave equations for odd half spin that have a well posed initial value problem. We demonstrate consistency of the equations in curved space-times. They generalize the Weyl neutrino equation. We show that there exists an associated invariant exact set of spinor fields indicating that the characteristic initial value problem on a null cone is formally solvable, even for the system coupled to general relativity. We derive the general analytic solution in flat space by means of Fourier transforms. Finally, we present a twistor contour integral description for the general analytic solution and assemble a representation of the group $O(4,4)$ on the solution space.

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I. INTRODUCTION

It is a well known fact that many spinor equations that are perfectly well behaved in (flat) Minkowski space can not be translated to a general four-dimensional curved background manifold. This happens e.g., for the zero rest-mass (zrm) fields with spin $s > 1$ and for the twistor equation. In these cases the appearance of the Buchdahl conditions ([2], [12]) imposes algebraic conditions relating any solution of the field equations to the (conformal) curvature of the manifold. In the zrm case, these conditions are very restrictive in that they limit the solution space of the equations for a general given background manifold. In the case of the twistor equation one finds that solutions can exist only on algebraically special manifolds of type N or O. Recently [11], there has been some interest in the case $s = 3/2$, the Rarita-Schwinger system, for which the consistency condition is just Ricci-flatness of the manifold such that one can regard the vacuum Einstein equations as integrability conditions for this system of spinor equations.

Despite this very interesting approach to the vacuum Einstein equations and its relation to twistor theory we want to present here a class of spinor equations which does not have the drawback of being well defined only on flat space-time (cf. also [14]). In particular, these are linear equations for a spinor field of half integer spin s, which include the Weyl neutrino equation as the case $s = (1/2)$. The general equation we shall consider may be written as follows:

$$\partial^{A(A'}\phi_{AB\ldots CD}^{B'\ldots C'D')} = 0. \tag{1.1}$$

Here the operator $\partial_{AA'}$ is the standard Levi-Cività spin connection and the spinor field $\phi_{AB\ldots CD}^{B'\ldots C'D'}$ is totally symmetric and has $m+1$ unprimed indices and n primed indices, with m and n non-negative integers. A simple count shows that the field $\phi_{AB\ldots CD}^{B'\ldots C'D'}$ has $(m+2)(n+1)$ components at every point, whereas the number of field equations is $(m+1)(n+2)$. So the excess number of field equations vis à vis components is $m-n$. In the case $m < n$, we have fewer field equations than components. Typically this leads to gauge freedom (at least in flat space): for example in the case $(m,n) = (0,1)$, equation (1.1) gives the self-dual Maxwell equations for a potential ϕ_a which has the gauge freedom $\phi_a \mapsto \phi_a + \partial_a\phi$ with ϕ an arbitrary scalar field. Next in the case $m > n$, we have an over determined system. This leads to integrability conditions and possible inconsistencies in a general curved spacetime. The classic example is the case $n = 0$, in which case equation (1.1) becomes just the standard zrm for spin $(m+1)/2$, which is inconsistent in general, at least for $m > 4$. The subject of this work is the case $m = n$, where there are exactly as many equations as there are field components, so one might expect that there are no non-progagating degrees of freedom and no constraints. Indeed we will show that these equations possess the following properties:

– They can be derived from a variational principle;

– They are conformally invariant;

– The Cauchy problem is well posed in flat and in an arbitrary curved space;

– There exists an equivalent exact set of spinor fields [10] for the fields when propagating on a curved background and also when they are coupled to gravity, which means that the characteristic initial value problem is formally well posed;

– The fields propagate along null hypersurfaces in flat space. However, they do not satisfy the wave equation $\Box\phi = 0$ but an equation $\Box^{m+1}\phi = 0$ instead, where $\Box$ is the d'Alembert operator corresponding to the spin connection;

– The general solution in Minkowski space can be given using a variation of Fourier transforms;

– A twistor description for analytic flat space solutions can be given.

The plan of the paper is as follows: in section 2. we present the variational principle and derive the energy momentum tensor. To do this we need to explain and extend a formalism given elsewhere [4]. In section 3. we discuss the Cauchy problem and show the existence of solutions. In section 4. we explain the relation with exact sets and discuss the characteristic initial value problem. Section 5. is devoted to the general solution in Minkowski space and in section 6. we show how to obtain solutions by performing contour integrals in twistor space thereby establishing an isomorphism between analytic flat space solutions and certain cohomology groups for suitable domains in projective twistor space.

II. THE VARIATIONAL PRINCIPLE AND THE ENERGY MOMENTUM TENSOR

In this section we will derive the spinor equations from a variational point of view. We will use a formalism described in [4] to derive our results. In order to introduce the notation and also to transcribe the various formulae to apply to spinors we will briefly review its basic features.

The starting point is an $SO(1,3)$ principal bundle $O(\mathcal{M})$ over space-time $\mathcal{M}$ carrying a tensorial one-form θ^a with values in $\mathbb{R}^4$ and a connection form $\theta^a{}_b$ with values in the Lie algebra $so(1,3)$ of the structure group. There also exists a constant matrix η_{ab} of signature $(1,3)$ that is used to construct the Lorentz metric on $\mathcal{M}$. We will require that the connection be torsion free, i.e., $D\theta^a = 0$; D is the exterior covariant derivative. Making use of the standard 2−to−1 epimorphism of $SL(2,\mathbb{C})$ onto the orthochronous Lorentz group to enlarge the structure group of the bundle and employing abstract index notation we may write $\theta^a = \theta^{AA'}$, $\theta^a{}_b = -\epsilon^A{}_B\theta^{A'}{}_{B'} - \epsilon^{A'}{}_{B'}\theta^A{}_B$, thus defining the unprimed and primed spin connections, symmetric in their respective indices (the sign is chosen in order to conform with other references). Then the torsion free condition is (note, that we suppress the wedge because it will be the only product we use between forms)

$$0 = d\theta^{AA'} + \theta^A{}_B\theta^{BA'} + \theta^{A'}{}_{B'}\theta^{AB'}. \tag{2.1}$$

We introduce a set of vector fields $\partial_{AA'}$, $\partial_A{}^B$ dual to the forms $\theta^{AA'}$ and $\theta^A{}_B$. Their action will be extended from functions to indexed quantities by requiring that they be derivations of the algebra of indexed forms annihilating $\theta^{AA'}$ and $\theta^A{}_B$, see [5] for further details.

We define a variation at θ^a and $\theta^a{}_b$ as the derivative at $\lambda = 0$ of one-parameter families $\theta^a(\lambda)$ and $\theta^a{}_b(\lambda)$ with $\theta^a(0) = \theta^a$ and $\theta^a{}_b(0) = \theta^a{}_b$. Denoting the variations by χ^a and $\chi^a{}_b$ we find that χ^a is a tensorial one-form with values in $\mathbb{R}^4$, $\chi^a{}_b$ is a tensorial one-form with values in $so(1,3)$. Using spinor indices and decomposing into irreducible parts we have

$$\chi^{AA'} = \sigma^A{}_B{}^{A'}{}_{B'}\theta^{BB'} + \tfrac{1}{4}\sigma\theta^{AA'} + \tau^A{}_B\theta^{AB'} + \bar{\tau}^{A'}{}_{B'}\theta^{BA'}. \tag{2.2}$$

The variations are not independent; in fact, the torsion free condition entirely fixes the variation $\chi^A{}_B = \chi^A{}_{BCC'}\theta^{CC'}$ of the connection for given χ^a:

$$\chi_{ABCC'} = -\partial^{E'}_{(A}\sigma_{B)CE'C'} + \tfrac{1}{4}\epsilon_{C(B}\partial_{A)C'}\sigma + \partial_{CC'}\tau_{AB}. \tag{2.3}$$

Spinor fields can now be considered as tensorial functions with values in the appropriate representation of $SL(2,\mathbb{C})$. In particular we consider totally symmetric fields $\psi^{AB\ldots C}_{B'\ldots C'}$ with m primed and $m+1$ unprimed indices. Note however, that we will switch back and forth between viewing the fields as tensorial functions on the bundle and as spinor fields on space-time $\mathcal{M}$ (section in an appropriate associated bundle). With these preparations we can now write down the following horizontal four-form

$$\mathcal{L} \equiv \mathrm{Im}\left(\bar{\psi}^{A'B'\ldots C'}_{B\ldots C} D\psi^{AB\ldots C}_{B'\ldots C'}\right)\Sigma_{AA'}, \tag{2.4}$$

with $\Sigma_a = (1/6)\epsilon_{abcd}\theta^b\theta^c\theta^d$. Note that we consider our spinors to be commuting quantities. If instead we would need to have anticommuting spinors we would use the real part instead of the imaginary part in equation (2.4). Since $\mathcal{L}$ can be considered as the pullback of a unique globally defined four-form on $\mathcal{M}$ we may define the action $\mathcal{A}$ as the integral of $\mathcal{L}$ over $\mathcal{M}$: $\mathcal{A} \equiv \int_{\mathcal{M}} \mathcal{L}$. It is then easily verified that the variations of $\mathcal{A}$ with respect to $\bar{\psi}$ (ψ) give the equation (and its complex conjugate):

$$D\psi^{AB\dots C}_{(B'\dots C'}\Sigma_{A')A} = 0. \tag{2.5}$$

Now $D = \theta^a\partial_a$ and $\theta^a\Sigma_b = \delta^a_b\Sigma$ with $\Sigma = (1/24)\epsilon_{abcd}\theta^a\theta^b\theta^c\theta^d$. Using this and stripping off the form Σ, we obtain the desired field equation

$$\partial_{A(A'}\psi^{AB\dots C}_{B'\dots C')} = 0. \tag{2.6}$$

Let us now discuss some of the basic properties of this field equation. First, we note that for $m = 0$ this is just the Weyl neutrino equation $\partial_{AA'}\nu^A = 0$, which is the zrm field equation for spin (1/2). Just as the neutrino equation the general equations are conformally invariant if we assume a transformation of the fields with conformal weight -2. We define the other irreducible parts of the covariant derivative of ψ,

$$\lambda^{AB\dots CD}_{B'\dots C'D'} = \partial^{(D}_{(D'}\psi^{AB\dots C)}_{B'\dots C')}, \tag{2.7}$$
$$\mu^{AB\dots C}_{B'\dots C'} = \partial^{E'(A}\psi^{B\dots C)}_{E'B'\dots C'}, \tag{2.8}$$
$$\nu^{AB\dots C}_{B'\dots C'} = \partial^{E'}_{E}\,\psi^{EAB\dots C}_{E'A'\dots C'}. \tag{2.9}$$
$$\tag{2.10}$$

Then we have the expansion

$$\partial^E_{E'}\psi^{AB\dots C}_{B'\dots C'} = \lambda^{EAB\dots C}_{E'B'\dots C'} - \frac{m}{m+1}\epsilon_{E'(B'}\mu^{EAB\dots C}_{\dots C')} - \frac{m}{m+2}\epsilon_{E'(B'}\epsilon^{E(A}\nu^{B\dots C)}_{\dots C')}. \tag{2.11}$$

By virtue of the field equation these fields satisfy the following relations (among others):

$$\frac{m}{m+1}\partial_{A(A'}\nu^{AC\dots D}_{C'\dots D')} = \Box_{AB}\psi^{ABC\dots D}_{A'C'\dots D'}, \tag{2.12}$$
$$\frac{m}{m+1}\partial^{(A}_{(A'}\nu^{BC\dots D)}_{C'\dots D')} = \Box^{(A}{}_{E}\psi^{BC\dots D)E}_{A'C'\dots D'} - \tfrac{1}{2}\Box\psi^{ABC\dots D}_{A'C'\dots D'}, \tag{2.13}$$
$$\partial_{A(A'}\lambda^{AB\dots CD}_{B'\dots C'D')} = \Box_{(A'B'}\psi^{BC\dots D}_{C'\dots D')}, \tag{2.14}$$
$$\partial^{A'}_{A}\lambda^{AB\dots CD}_{A'C'\dots D'} = -\frac{1}{2(m+1)}\Box\psi^{BC\dots D}_{C'\dots D'} - \Box_A{}^{(B}\psi^{C\dots D)A}_{C'\dots D'} + \tag{2.15}$$
$$\frac{m}{m+1}\partial^{(B}_{(C'}\psi^{C\dots D)}_{\dots D')} + \frac{m}{m+1}\Box^{E'}_{(C'}\psi^{BC\dots D}_{\dots D')E'}, \tag{2.16}$$
$$\partial_{A(A'}\mu^{AB\dots CD}_{B'\dots C')} = \frac{1}{2(m+2)}\Box\psi^{B\dots CD}_{A'B'\dots C'} + \Box^{D'}_{(A'}\psi^{BC\dots D}_{B'\dots C')D'} + \tag{2.17}$$
$$\frac{m+1}{m+2}\Box_{A(B}\psi^{C\dots D)A}_{A'B'\dots C'} + \frac{1}{m+2}\partial^{(B}_{(A'}\nu^{C\dots D)}_{B'\dots C')}, \tag{2.18}$$
$$\partial^{B'}_{A}\mu^{ABC\dots D}_{B'C'\dots D'} = -\frac{m+1}{m+2}\left\{\partial^{B'(B}\nu^{C\dots D)}_{B'C'\dots D'} + \Box^{C'B'}\psi^{BC\dots D}_{B'C'\dots D'}\right\}. \tag{2.19}$$

In these formulae we have used the spinor curvature derivations $\Box_{AB}$ and $\Box_{A'B'}$ as defined in [12]. To further analyze the situation it is very convenient to introduce four differential operators L, M, M' and N acting on irreducible spinor fields by taking the covariant derivative and then projecting onto one of the four possible irreducible parts (see [5] for a rigorous definition and further details). Thus, for a field ϕ with p unprimed and p' primed indices and all its indices down, we identify $L\phi \doteq \partial_{(E'(E}\psi_{A\dots B)C'\dots D')}$, $M'\phi \doteq -p'\partial^{E'}_{(E}\psi_{A\dots B)C'\dots E'}$, $M\phi \doteq -p\partial^{E}_{(E'}\psi_{C'\dots D')A\dots E}$, $N\phi \doteq pp'\partial^{EE'}\psi_{A\dots EC'\dots E'}$. These operators obey certain commutation rules, most of which are trivial in flat space. The nontrivial ones are $[L,N]\,\phi = -\frac{1}{2}(p+p'+2)\Box\phi$ and $[M,M']\,\phi = -\frac{1}{2}(p-p')\Box\phi$. There is a further relation: $LN\phi - MM'\phi - \frac{1}{2}p(p'+1)\Box\phi = 0$. The wave operator commutes with all the derivative operators. In flat space, the equations (2.14), (2.12), (2.18) and (2.13) above can be written as follows:

$$M'\lambda = 0, \quad M'\nu = 0, \quad M'\mu = \tfrac{1}{2}\Box\psi, \quad N\lambda = -m(m+1)L\nu + \tfrac{1}{2}(2m+3)\Box\psi, \quad L\nu = -\frac{m+1}{2m}\Box\psi. \tag{2.20}$$

We observe that ν and λ are spinor fields of the same class as ψ obeying the same equation. In contrast to the zrm case, the field ψ does not obey the wave equation $\Box\psi = 0$ (unless $m = 0$, because then $\mu = 0$). However, with these preparations it is now easy to prove the

Proposition II.1 *Given a smooth spinor field ψ with m primed and $m+1$ unprimed indices subject to the field equation (1.1), then $\Box^{m+1}\psi = 0$ in flat space.*

Proof: We prove this by induction on m. The case $m = 0$ is the Weyl neutrino equation for which the assertion is true. Now assume it is true for $(m-1)$, then $\Box^m \nu = 0$. But then $0 = L\Box^m\nu = \Box^m LN\psi = \Box^m(\frac{1}{2}(m+1)^2\Box)\psi = \frac{1}{2}(m+1)^2\Box^{m+1}\psi$.

Finally, we want to derive the energy momentum tensor of these spinor fields. This is usually done by considering the action as depending on the metric of the background manifold and then varying with respect to that metric. The result is the natural object that would appear on the right hand side of the Einstein equations if the system were coupled to gravity. In our case we can not regard the action as depending on the metric, we have to take it depending on the canonical one-form. Then the variation of the action with respect to θ^a contains terms proportional to $\sigma_{ABA'B'}$, τ_{AB} and σ. The functional derivative of the action with respect to $\sigma_{ABA'B'}$ is the trace free part of the energy momentum tensor while taking the functional derivative with respect to σ gives the trace part. However, in the present case, we expect this term to vanish due to the conformal invariance of the equation and the functional derivative with respect to τ_{AB} will be seen to vanish as well. This is related to the fact that the connection is required to be torsion free. The variation of the action with respect to $\theta^{AA'}$ is $\delta\mathcal{A} = \mathrm{Im}\int\delta\mathcal{L}$ with

$$\delta\mathcal{L} = \left\{(m+1)\bar\psi^{A'B'\ldots C'}_{B\ldots C}\chi^{(A}{}_E\psi^{B\ldots C)E}_{B'\ldots C'} - m\bar\psi^{A'B'\ldots C'}_{B\ldots C}\bar\chi^{E'}{}_{(B'}\psi^{AB\ldots C}_{C'\ldots)E'}\right\}\Sigma_{AA'} + \left\{\bar\psi^{A'B'\ldots C'}_{B\ldots C}D\psi^{AB\ldots C}_{B'\ldots C'}\right\}\delta\Sigma_{AA'} \qquad (2.21)$$

Using a formula from [4] (which however contains a misprint) we find $\delta\Sigma_a = 2\chi^b{}_{[b}\Sigma_{a]}$. If we now put all the σ-terms in $\chi^A{}_B$ equal to zero retaining only the τ terms we get

$$\delta\mathcal{L} = \left\{(m+1)\bar\psi^{A'B'\ldots C'}_{B\ldots C}D\tau^{(A}{}_E\psi^{B\ldots C)E}_{B'\ldots C'} - m\bar\psi^{A'B'\ldots C'}_{B\ldots C}D\bar\tau^{E'}{}_{(B'}\psi^{AB\ldots C}_{C'\ldots)E'} + \bar\psi^{E'B'\ldots C'}_{B\ldots C}D\psi^{EB\ldots C}_{B'\ldots C'}\tau^{AA'}{}_{EE'}\right\}\Sigma_{AA'}. \qquad (2.22)$$

Integrating by parts and using the field equation several times gives

$$\delta\mathcal{L} = -\bar\psi^{A'B'\ldots C'}_{B\ldots C}\tau^A{}_E D\psi^{BC\ldots E}_{B'\ldots C'}\Sigma_{AA'} + m\bar\psi^{A'B'\ldots C'}_{B\ldots C}\bar\tau^{E'}{}_{B'}D\psi^{AB\ldots C}_{\ldots C'E'}\Sigma_{AA'} \qquad (2.23)$$

$$+\bar\psi^{A'B'\ldots C'}_{B\ldots C}\tau^E{}_A D\psi^{AB\ldots C}_{B'\ldots C'}\Sigma_{EA'} + \bar\psi^{A'B'\ldots C'}_{B\ldots C}\bar\tau^{E'}{}_{A'}D\psi^{AB\ldots C}_{B'\ldots C'}\Sigma_{AE'} \qquad (2.24)$$

Now the first and third term cancel while the second and fourth term combine to a multiple of the field equation. Hence the functional derivative of the action with respect to the τ terms vanishes. By a similar argument one can show that the terms proportional to σ also vanish so that one has to consider only the trace free parts proportional to $\sigma_{ABA'B'}$. In this case the calculation is similar but more complicated, so we only state the result. The energy momentum tensor of the spinor fields subject to equation (1.1) is

$$T^{ABA'B'} = \mathrm{Im}\left\{(2m-1)\bar\psi^{C'\ldots D'(A'}_{C\ldots D}\lambda^{B')BAC\ldots D}_{C'\ldots D'} - \frac{m(2m+1)}{m+1}\bar\psi^{A'B'C'\ldots D'}_{C\ldots D}\mu^{ABC\ldots D}_{C'\ldots D'}\right.$$
$$\left. - \frac{m^2(2m+7)}{(m+1)(m+2)}\bar\psi^{A'B'C'\ldots D'(A}_{C\ldots D}\nu^{B)C\ldots D}_{C'\ldots D'}\right\}. \qquad (2.25)$$

By construction, it is divergence free and due to the conformal invariance it is also trace free. Note, that it is made up of the fields and all the non vanishing irreducible parts of its first derivative. The case $m = 0$ agrees with the conventional energy momentum tensor for the Weyl equation.

III. THE CAUCHY PROBLEM

In this section we will prove that equation (1.1) has a well-posed Cauchy problem, i.e., we will show that given appropriate Cauchy data on a spatial hypersurface S there will exist a unique solution of equation (1.1) on a small enough neighbourhood of S. So existence and uniqueness will hold (only) locally in time.

We will first examine the hyperbolicity properties of equation (1.1). Let us write the field equation in the form

$$\left(\delta^{B'}_{(D'}\ldots\delta^{C'}_{E'}\delta^{P'}_{A')}\delta^P_{(A}\ldots\delta^D_B\delta^E_{C)}\right)\partial_{PP'}\psi^{AB\ldots C}_{B'\ldots C'} = 0. \qquad (3.1)$$

We abbreviate the product of δ's by $A^a_{\bar\mu\nu}$ thus introducing the clumped indices $a \sim AA'$ and ν, $\bar\mu$, indicating elements of the spin space $S^{AB\ldots C}_{B'\ldots C'}$ and its complex conjugate dual space. Then equation (3.1) has the form

$$A^a_{\bar\mu\nu}\partial_a\psi^\nu = 0. \qquad (3.2)$$

For each covector p_a, $p_a A^a_{\bar\mu\nu}$ defines a map P from S^ν into $S_{\bar\mu}$ which is easily seen to define a sesquilinear form on S^ν. Before proceeding further, we will prove two useful lemmas concerning the map P.

Lemma III.1 *The map P is an anti-isomorphism if and only if p_a is not a null vector.*

Proof: For the sufficiency we use induction on the number m of primed indices. For $m = 0$ the map is $\nu^A \mapsto p_{A'A}\nu^A$. If $p_{A'A}\nu^A = 0$ then $p^{A'}_D p_{A'A}\nu^A = -\frac{1}{2}p^2\nu_D = 0$, where $p^2 = p_a p^a$. Hence, if $p^2 \neq 0$ then P is injective and therefore bijective. Now suppose the statement is true for an integer $m-1$; we will show that this implies that it is also true for m. In this case the map is $\lambda^{AB\ldots C}_{B'\ldots C'} \mapsto p_{A(A'}\lambda^{AB\ldots C}_{B'\ldots C')}$. If $p_{A(A'}\lambda^{AB\ldots C}_{B'\ldots C')} = 0$ then also $0 = p^{A'}{}_B p_{A(A'}\lambda^{AB\ldots C}_{B'\ldots C')} = \frac{m}{m+1}p_{A(B'}\lambda^{AB\ldots C}_{\ldots C')A'}p^{A'}{}_B$. The induction hypothesis implies that $p_A{}^{B'}\lambda^{AB\ldots C}_{B'\ldots C'} = 0$ and therefore $p_{AA'}\lambda^{AB\ldots C}_{B'\ldots C'} = 0$ but this implies $p^{A'D}p_{AA'}\lambda^{AB\ldots C}_{B'\ldots C'} = \frac{1}{2}p^2\lambda^{AB\ldots C}_{B'\ldots C'} = 0$. So, if $p^2 \neq 0$ the map is injective and therefore bijective. If $p^2 = 0$ then we may write $p_{AA'} = p_A p_{A'}$ for some spinor p_A and its complex conjugate and then $\lambda^{AB\ldots C}_{B'\ldots C'} = p^A p^B \ldots p^C p_{B'} \ldots p_{C'}$ is a nonvanishing spinor that is mapped to zero.

Lemma III.2 *The determinant of the sesquilinear form defined by P is $\det(p_a A^a_{\bar{\mu}\nu}) = c\,(p_a p^a)^{\frac{N}{2}}$ where $N = (m+1)(m+2)$ is the dimension of S^ν and c is a non-zero real number depending on the choice of basis.*

Proof: Consider the "characteristic polynomial" $Q(p) = \det(p_a A^a_{\bar{\mu}\nu})$. As a determinant it is a Lorentz scalar and since the only available scalar for a given p_a is $p_a p^a$ it follows that $Q(p)$ is a function of $p_a p^a$. Since $Q(p)$ is a homogeneous polynomial in p_a of degree N the result follows. Another way (which is useful later) to see this is the following: The determinant is proportional to $\bar{\epsilon}^{\bar{\mu}_1\ldots\bar{\mu}_N} p_a A^a_{\bar{\mu}_1\nu_1} \ldots p_a A^a_{\bar{\mu}_N\nu_N} \epsilon^{\nu_1\ldots\nu_N}$ where $\epsilon^{\nu_1\ldots\nu_N}$ is a (dual) volume form on S^ν. As such it is built up from the volume form ϵ_{AB} of spin space. After contracting away all the ϵ's in the expression we are left with N p_a's which are all contracted with each other. Due to the identity $p_{AA'}p^{AB'} = \frac{1}{2}p_a p^a \epsilon_{A'}{}^{B'}$ each pair of p_a's contributes one factor $p_a p^a$ to the determinant. Since there are $N/2$ pairs, the result follows.

From these two lemmas, we see that the system (3.2) is not symmetric hyperbolic unless $m = 0$. If this were the case, then there would exist a timelike future pointing covector p_a such that $p_a A^a_{\bar{\mu}\nu}$ was hermitian and positive definite. Although the system is symmetric, it is not definite. For let p_a be any timelike future pointing covector with $p_a p^a = 2$ and choose a spin frame (o^A, ι^A) such that $p_a = o_A o_{A'} + \iota_A \iota_{A'}$. Then we choose $\lambda^{AB\ldots C}_{B'\ldots C'} = o^A o^B \ldots o^C o_{B'} \ldots o_{C'}$ and find that $\bar{\lambda}^{\bar{\mu}} p_a A^a_{\bar{\mu}\nu}\lambda^\nu = 0$ if $m > 0$. This, of course, implies that not all (real and nonvanishing) eigenvalues of $p_a A^a_{\bar{\mu}\nu}$ can be of strictly one sign. In the Weyl case it is well known that the equation can be written in a symmetric hyperbolic way.

To proceed further we determine the characteristics of the system (3.2). These are surfaces locally described by the vanishing of a function ϕ such that it is not possible to determine the outward derivatives of a function from given Cauchy data on the surface. Hence, on these surfaces the "characteristic equation" $\det(A^a_{\bar{\mu}\nu}\partial_a\phi) = 0$ holds. Because of lemma (III.2) each characteristic surface is a null surface. At each point of M the normal characteristic cone defined by $Q(p) = 0$ coincides with the null cone at that point. However, unless $m = 0$ the characteristic cone has multiple sheets which implies that the system is not strictly hyperbolic. The theory for non-strictly hyperbolic differential operators is not as well developped as for strictly hyperbolic or symmetric hyperbolic operators. However, in our case we can apply a theorem of Leray and Ohya [7] on non-strictly hyperbolic systems of partial differential equations. Their main assumption is that the characteristic determinant $Q(p)$ factorizes such that each factor is a strictly hyperbolic polynomial[1] which certainly is the case here because $p_a p^a$ is a strictly hyperbolic polynomial of degree 2. They show that for Cauchy data on an initial nowhere characteristic surface S which belong to a Gevrey class of functions with index α the system has a unique solution in that class. This solution admits a domain of dependence, i.e., the value of the solution at a point depends only on the Cauchy data in the past of that point.

Let S be a spacelike hypersurface in M and t^a a timelike vector field on M. Define a time function t on M by the requirement that $t = 0$ on S and that $t^a\partial_a t = 1$. We define spacelike surfaces S_t as the surfaces of constant t. Given coordinates (x^1, x^2, x^3) on S we can continue them off S along t^a by Lie transport, i.e., by requiring that the lines $x^i =$ const. are the integral curves of t^a. Thus, we obtain a coordinate system $(t = x^0, x^1, x^2, x^3)$ on an open submanifold Σ of M that is topologically $S \times \mathbb{R}$. We choose a spin frame (o^A, ι^A) such that $o^A o^{A'} + \iota^A \iota^{A'} = \sqrt{2}n^a$, the unit normal to S_t. Then $\psi_{AB\ldots CB'\ldots C'}$ has components $\psi^\alpha = \psi_{kk'}$ with $\alpha = (m+1)k + k'$, $0 \le k < m+1$, $0 \le k' < m$ (such that $0 \le \alpha < (m+1)(m+2)$) and

$$\psi_{kk'} = \psi_{A\ldots BC\ldots DA'\ldots B'C'\ldots D'} \underbrace{o^A \ldots o^B}_{k} \iota^C \ldots \iota^D \underbrace{o^{A'} \ldots o^{B'}}_{k'} \iota^{C'} \ldots \iota^{D'}. \tag{3.3}$$

[1] A polynomial $Q(\xi)$ of degree n is strictly hyperbolic iff the cone $C = \{\xi : Q(\xi) = 0\}$ has a nonempty interior such that each line through an interior point not including $\xi = 0$ intersects C in exactly n distinct real points.

By taking components of equation (1.1) we obtain a system of equations of the form (D_a denoting the partial derivative with respect to x^a)

$$G^{a\alpha}_{\beta}(t,x)D_a\psi^{\beta} + \Gamma^{\alpha}_{\beta}(t,x)\psi^{\beta} = 0. \tag{3.4}$$

Here the functions $G^{a\alpha}_{\beta}(t,x)$ are functions on M which are algebraic expressions in the metric components g_{ab} with respect to the coordinates $(t,x) = (x^0, x^1, x^2, x^3)$. The functions $\Gamma^{\alpha}_{\beta}(t,x)$ are algebraic expressions in the coefficients of the spin connection. We define $a^{\alpha}_{\beta}(x,D)$ as the linear differential operator in (3.4). Then we consider the following Cauchy problem:

$$\begin{aligned} &a^{\alpha}_{\beta}(x,D)\psi^{\beta}(x) = 0, \\ &\psi^{\beta}|_S \quad \text{is given.} \end{aligned} \tag{3.5}$$

It follows from lemma (III.2) above that up to operators of lower order than $N = (m+1)(m+2)$ we have $\det(a^{\alpha}_{\beta}(x,D)) = \Box^{\frac{N}{2}}$ where we define the determinant of noncommuting quantities by the usual formula $\det(a^{\alpha}_{\beta}(x,D)) = \sum_{\pi \in S_N} \mathrm{sign}(\pi)\ a^1_{\pi(1)} \dots a^N_{\pi(N)}$. $\Box$ is the wave operator with respect to the metric g_{ab} expressed in the coordinates (x^a). This operator is strictly hyperbolic with respect to the hypersurfaces S_t. We are now in a position to prove the

Theorem III.3 *Let α be a real number with $1 \le \alpha \le \frac{N}{N-2}$. If the metric coefficients are in the Gevrey class $\gamma_{\infty}^{\frac{3}{2}N,(\alpha)}(\Sigma)$ and if the initial data ψ^{β} are in the Gevrey class $\gamma_2^{(\alpha)}(S)$ then in a sufficiently narrow strip $\Sigma' = \{(t,x) : 0 \le t \le T\}$ around S the Cauchy problem (3.5) has a unique solution $\psi^{\beta} \in \gamma_2^{1+\frac{N}{2},(\alpha)}(\Sigma)$, whose support is contained in the domain of influence of the support of the initial data. $\Sigma' = \Sigma$ if $1 \le \alpha < \frac{N}{N-2}$.*

Proof: This is a straightforward application of theorems of existence and uniqueness in §6 of [7]. We only need to determine the various integers needed in the theorem. We associate the integers $m^{\beta} = 1$ with each unknown function ψ^{β} and the integers $n^{\alpha} = 0$ with each of the equations such that $\mathrm{order}(a^{\alpha}_{\beta}(x,D)) \le m^{\beta} - n^{\alpha} = 1$. Then $m = \sum_{\beta}(m^{\beta} - n^{\beta}) = N$ is the order of $\det(a^{\alpha}_{\beta}(x,D))$. Each of the factors in the principal part in $\det(a^{\alpha}_{\beta}(x,D))$ is equal to the wave operator such that $a_j = \Box$, $m_j = 2$ for $(j = 1, \dots, N)$ and the number of factors is $p = N/2$. Since in our case $r = 1$ we need to add to each of the integers m^{β} and n^{α} the same integer $N/2$ in order to satisfy the chain of inequalities

$$0 \le r \le p \le n \le n^{\alpha} \le \bar{n}, n \le m^{\beta}, p \le m, \tag{3.6}$$

as required in [7]. With our choice of the integers we have $m^{\beta} = 1 + (N/2)$, $n^{\alpha} = 0$, $p = (N/2) = n = \bar{n}$ and the inequalities are satisfied. According to the theorem the index α of the appropriate Gevrey classes lies in the interval $1 \le \alpha \le \frac{N}{N-2}$. The coefficients of a^{α}_{β} are assumed to be in $\gamma_{\infty}^{3N/2-1,(\alpha)}(\Sigma)$, those of a_j are in $\gamma_{\infty}^{j+N/2,(\alpha)}(\Sigma)$ for $0 \le j \le N/2 - 1$. Since all the factors are the same, $a_j = \Box$, this implies that the coefficients are in fact in the smallest possible class which is $\gamma_{\infty}^{N-1,(\alpha)}(\Sigma)$. The coefficients of the operator a have to be in the class $\gamma_{\infty}^{N/2,(\alpha)}(\Sigma)$. Taking all this together and remembering how the coefficients in the operators are constructed from the metric this implies that the metric coefficients have to be in $\gamma_{\infty}^{3N/2,(\alpha)}(\Sigma)$. Then the conclusion of the theorem implies that the solution of the Cauchy problem is in the Gevrey class $\gamma_{\infty}^{1+N/2,(\alpha)}(\Sigma)$.

According to this theorem there exists a strong correlation between the spin of the field and the degree of smoothness of the space-time that admits a solution of the equation. The higher the spin, the "smoother" the space-time has to be. The smoothness is controlled by the number of components of the field, N, which depends quadratically on the spin $m + \frac{1}{2}$ of the field. We can improve on this relationship somewhat by using a simplification due to Bruhat [1] which is based on the observation that if all the minors in $\det(a^{\alpha}_{\beta})$ have a common factor then this factor can be ignored which results in a reduction of the number p of factors of $\det(a^{\alpha}_{\beta}(x,D))$ and therefore in the Gevrey index α. To be more precise, we need to prove the

Lemma III.4 *In the adjoint matrix of $p_a A^a_{\bar{\mu}\nu}$ all the entries have the factor $(p_a p^a)^{m(m+1)/2}$ in common.*

Proof: As in the proof of lemma (III.2) this result can be obtained by "index counting". The adjoint matrix is given by the expression $\bar{\epsilon}^{\bar{\mu}\bar{\mu}_2\dots\bar{\mu}_N} p_{a_2} A^{a_2}_{\bar{\mu}_2\nu_2} \dots p_{a_N} A^{a_N}_{\bar{\mu}_N\nu_N} \epsilon^{\nu\nu_2\dots\nu_N}$, homogeneous of degree $N-1$ in p_a. If we contract over all the indices contained in all the ϵ's in the volume forms we are left with an expression that contains $N-1 = (m+1)(m+2)-1$ p_a's, each with one unprimed and one primed spinor index and has $(2m+1)$ free indices of either kind. Therefore, $(N-1)-(2m+1)$ of the p_a's are contracted together, resulting in a factor $(p_a p^a)^{m(m+1)/2}$ in each component.

This lemma allows us to prove the

Corollary III.5 *In the statement of the theorem we can extend the range of the Gevrey index α to $1 \leq \alpha \leq 1 + \frac{1}{m}$. More precisely, if the metric coefficients are in the Gevrey class $\gamma_\infty^{3m+3,(\alpha)}(\Sigma)$ and if the Cauchy data are in $\gamma_2^{(\alpha)}(S)$ then the Cauchy problem* (3.5) *has a unique solution $\psi^\beta \in \gamma_2^{m+2,(\alpha)}(\Sigma')$ in a sufficiently narrow strip Σ' around S.*

Proof: We observe that in proving the existence theorem Leray and Ohya use a theorem for systems with diagonal principal part (see §5 of [7]). To apply this theorem one multiplies the system (1.1) with the differential operator corresponding to the adjoint matrix of $p_a A^a_{\mu\nu}$. This renders the principal part of the resulting system diagonal. Due to lemma (III.4) it is enough to multiply with the operator (of lower order) obtained from the adjoint matrix by dropping the common factor. Then one obtains the result in a straightforward manner by applying the theorem for diagonal systems.

To end this section we want to make several remarks.

— The case $m = 0$ (the Weyl neutrino equation) is the strictly hyperbolic case where we can choose $\alpha = \infty$. This implies that it is possible to prescribe initial data with only a finite number of continuous derivatives which is a known result for strictly hyperbolic systems.

— It is also worth to mention that in all the cases there exists a domain of influence, a fact which is taken to indicate the hyperbolic character of partial differential equations by many authors.

— The fact that the smoothness of space-time is strongly linked with the spin of the field is an interesting feature of this class of equations that is not present in other spinor equations. One has to say, though, that it is not known whether this is a necessary consequence since the theorems only provide sufficient conditions for existence and uniqueness.

— The diagonal system used in the proof of the corollary corresponds to the equation $\Box^{m+1}\psi^{AB\ldots C}_{B'\ldots C'} = 0$ that we derived in the flat case in section 2. This equation is distinguished by the fact that it is a linear equation for $\psi^{AB\ldots C}_{B'\ldots C'}$ such that the coefficients are functions not of the connection but of the curvature and its derivatives only.

— The inhomogeneous equation $\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} = \chi^{B\ldots C}_{A'B'\ldots C'}$ can be treated in a straightforward way and one obtains existence and uniqueness of solutions in the same Gevrey class as for the homogeneous case provided that the right hand side is in an appropriate Gevrey class, see [7].

IV. THE FORMAL CHARACTERISTIC INITIAL VALUE PROBLEM

In this section we want to discuss the formal aspects of the characteristic initial value problem on a null cone for this class of spinor equations. Due to the inherent singularity at the vertex of a null cone this problem is very difficult to analyze and, in fact, there are no existence results for many partial differential equations appearing in physics, most notably the vacuum Einstein equations. So one has to resort to formal methods to obtain at least results about the feasability of existence theorems. A very useful method to achieve this which is adapted to four dimensions is the method of exact sets of spinor fields developed by Penrose [9]. It is based on the observation that in Taylor expansions of spinor fields around a point it is exactly the totally symmetric derivatives of the field that determine the restriction of the field to the null cone of that point (the null datum). Roughly speaking, if a system of field equations for a collection of spinor fields has the properties that the totally symmetric derivatives are algebraically independent and if they determine algebraically all possible derivatives of the fields then the collection of fields is said to be exact (see [10], [12] for the rigorous definition).

It has been useful to employ an algebraic formalism based on the four derivative operators L, M, M', N (already mentioned in section 2) which correspond to taking the four possible irreducible components of the covariant derivative of an irreducible spinor field. We will not describe the full formalism here because it would take up too much space. Instead we will only give a brief summary and refer for further details to [5]. The totally symmetric derivatives of a spinor field correspond to applying powers of L to the field. We will call an irreducible spinor to be of type (k, k') if and only if it has k unprimed and k' primed indices (irrespective of their position). Acting on a spinor field of type (k, k') the operators L, M, M', N produce fields of respective type $(k+1, k'+1)$, $(k+1, k'-1)$, $(k-1, k'+1)$, $(k-1, k'-1)$. We define the operators H and H' by $H\phi = k\phi$ and $H'\phi = k'\phi$ for a type (k, k') field ϕ.

As we have already mentioned in section 2, the commutator of two covariant derivatives induces commutation relations between the derivative operators which in general involve the curvature of the manifold and in addition the wave operator. The curvature is characterized by three spinor fields Ψ, Φ and Λ of respective types $(4, 0)$, $(2, 2)$ and $(0, 0)$. Before we present these relations we need to define an algebraic operation between two spinor fields ϕ and χ

of respective types (p,p') and (q,q'). The only possible way to combine two spinor fields within the class of totally symmetric fields in a bilinear way is by contracting over some of the indices and then symmetrizing over the remaining lot. This operation is entirely characterized by the numbers of contracted primed and unprimed indices. So we define the bilinear pairings $C_{kk'}$ by the correspondence

$$C_{kk'}(\phi,\chi) \cong \phi^{A_1\ldots A_k B'_{1'}\ldots B'_{k'}}_{(C\ldots D(C'\ldots D'}\chi_{A_1\ldots A_k B'_{1'}\ldots B'_{k'}E'\ldots F')E\ldots F)}. \tag{4.1}$$

Then the commutation relations between the derivative operators can be given explicitly as

$$\begin{aligned} [L,N] &= -(H+1)T' - (H'+1)T - \tfrac{1}{2}(H+H'+2)\Box, \\ [M,M'] &= -(H+1)T' + (H'+1)T - \tfrac{1}{2}(H-H')\Box, \\ [L,M] &= -(H'+1)S, \qquad [L,M'] = -(H+1)S', \\ [N,M] &= (H+1)U', \qquad [N,M'] = (H'+1)U. \end{aligned} \tag{4.2}$$

The operators S, T and U and their primed versions are curvature derivations and act on a field ϕ of type (p,p') according to

$$S\phi = pC_{10'}(\Psi,\phi) + p'C_{01'}(\Phi,\phi), \tag{4.3}$$

$$T\phi = p(p-1)C_{20'}(\Psi,\phi) + pp'C_{11'}(\Phi,\phi) - p(p+2)C_{00'}(\Lambda,\phi), \tag{4.4}$$

$$U\phi = p(p-1)(p-2)C_{30'}(\Psi,\phi) + p(p-1)p'C_{21'}(\Phi,\phi). \tag{4.5}$$

The action of the primed operators can be inferred from these by formal complex conjugation. There exists an additional relation between the derivative operators, the wave operators and the curvature:

$$LN - MM' = -(H'+1)T + \tfrac{1}{2}H(H'+1)\Box. \tag{4.6}$$

The formulae describing the action of a derivative operator on a bilinear pairing are quite lengthy and it is not necessary for what follows to present them in detail (see [5]). Symbolically they are given by

$$OC(\phi,\chi) = \sum_O \alpha_1 C(Of,g) + \alpha_2 C(f,Og), \tag{4.7}$$

where O is any of the derivative operators L, M', M or N and α_1 and α_2 are rational numbers determined by the bilinear product and the type of ϕ and χ.

In [5] we showed that a collection of fields $\{\phi_j\}$ is exact if and only if two conditions are satisfied:

(i) all the "powers" $L^l\phi_j$ are algebraically independent,

(ii) all the "derivatives" of the fields, i.e., all the expressions $s\phi_j$ where s is an arbitrary string of derivative operators are algebraically determined by the powers.

By algebraic independence we mean that there are no relations between the powers involving only the bilinear pairings (and possibly the curvature). In the same spirit we mean that the derivatives are determined by exactly such relations in terms of the powers. Several examples have been treated in [3], [5], [12]. In a certain sense, the powers form a complete and independent set of functions that generate the solution space of the equations considered.

Before we consider the general case we want to study the flat case to find the exact set structure underlying equation (1.1). So let ψ be a field of type $(m+1,m)$ satisfying the equation $M'\psi = 0$. Referring again to [5], we see that in that case the expressions $L^lM^kM'^jN^n\Box^i\psi$ for positive integers l, k, j, n and i generate the solution space. By use of the field equation and the commutation relations we have $j = 0$ since all terms with $j > 0$ vanish (note that in flat space $\Box$ commutes with every derivative operator). Similarly, $i \le m$ because of proposition (II.1) and $n + k \le m$ because M and N each contract over one primed index. Due to the additional relation (4.6) we have the following: $L^lM^kN^n\Box^i\psi \sim L^lM^k\Box N^n\Box^{i-1}\psi \sim L^{l+1}M^kN^{n+1}\Box^{i-1}\psi \sim \ldots \sim L^{l+i}M^kN^{n+i}\psi$ where $a \sim b$ means "a is expressible in terms of b". So we find that in the flat case the solution space is generated by the powers of the functions $\psi_{ki} = M^{k-i}N^i\psi$, with $0 \le k \le m$ and $0 \le i \le k$. Note that there are $\frac{1}{2}(m+1)(m+2)$ of those functions. This number is in agreement with the general observation that the number of null data per point for a partial differential equation is half the number of Cauchy data per point.

We now claim that the same set of fields is also a generating set in the nonflat case. Before we prove this statement we need some more preparation. Let s_n denote any string of derivative operators of length n. We say that s_n is in normal order if and only if it has the form $s_n = L^lM^kN^iM'^j$ with $l+k+i+j = n$. With each string s_n we can associate a unique normally ordered string $\widetilde{s_n}$ of the same length in the following way: if s_n does not contain M' then $\widetilde{s_n}$ is the unique normally ordered string that contains the same number of operators as s_n does. If there are M' operators in s_n we first replace each pair (M, M') which need not be adjacent with (L, N) until there is no M or M' left. Then we bring the result into normal order to obtain $\widetilde{s_n}$. Thus we get normally ordered strings containing either M or M' but not both. Upon applying these normally ordered strings to ψ we get zero for all strings containing M' and for those strings with $k + i > m$. The others result in $L^l M^k N^i \psi = L^l \psi_{k+i,i}$; these functions and their complex conjugates will be called a normally ordered derivative or a "power". For each power $L^l \psi_{ki}$ we call $l + k$ its order. Furthermore, we need to formalize the structure of the relevant terms that will be encountered.

Definition IV.1 *A t-term ("t" for "tree") is recursively defined either*

(i) as a power $L^l \psi_{ki}$ or $L^l \bar{\psi}_{ki}$ for nonnegative integers l, k, i or

(ii) to be of the form $C_{kk'}(R, t)$ where R is any derivative of any of the curvature spinors and their complex conjugates and t is a t-term.

A *t-expression* is the formal finite sum $\sum_j \alpha_j t_j$ with coefficients $\alpha_j \in \mathbb{Q}$ and t-terms t_j. A t-term that is not a power will be called a *pure t-term* and a *pure t-expression* is a $\mathbb{Q}$-linear combination of pure t-terms.

The pure t-terms are binary trees with $C_{kk'}$ as nodes and with powers or (derivatives of) curvature spinors as leaves. In fact, exactly one leaf is a power all other ones are curvature derivatives. This reflects the linearity of the system. We are now ready to prove the

Lemma IV.2 *Let ψ be of type $(m + 1, m)$ and satisfying the equation $M'\psi = 0$. Let s_n be an arbitrary string of length n of derivative operators. Then $s_n\psi$ can be written in a unique way as $s_n\psi = \alpha\widetilde{s_n}\psi + t$ where $\alpha \in \mathbb{Q}$ and t is a pure t-expression which contains only powers of order strictly less than $n - 1$.*

Proof: We use induction on the length of the string. With $n = 1$ there are four possibilities: $L\psi = L\psi_{00}$, $M\psi = \psi_{10}$, $M'\psi = 0$ and $N\psi = \psi_{11}$; so the statement is true.

Let O denote any of the derivative operators and assume the statement to be true for all strings s of length less than or equal to n. Then consider $(Os_n)\psi = O(s_n\psi)$. By the induction hypothesis and linearity of O we need to consider only two cases, namely $s_n\psi$ is (i) a pure t-term or (ii) a power. In case (i) we need to employ (4.7) to apply O to a bilinear pairing, thus bringing O inside the $C_{kk'}$ to act on each of its arguments. Note, that then O is not necessarily the same operator we started with. When it hits the left argument, O converts a curvature derivative into a higher one thus producing a t-term of the required type. The other argument is again either a power or a pure t-term. In the latter case we continue descending down the tree structure until we finally hit the power. Then we need to consider $OL^l\psi_{ki}$. By the induction hypothesis this is a derivative of ψ of order $l + k + 1 \le n - 1$ and therefore equal to the sum of the corresponding normally ordered derivative of ψ and a pure t-expression with powers of order less than $n - 3$. The normally ordered derivative is (when non vanishing) equal to a power of order $n - 1$, hence the application of O to a pure t-expression yields a pure t-expression of the required type.

In case (ii) we have $s_n\psi = L^l\psi_{ki}$ with $l + k = n$. Let us first suppose that $l \ge 1$. Then $OL^l\psi_{ki} = [O, L]\, L^{l-1}\psi_{ki} + LOL^{l-1}\psi_{ki}$. In the second term we can replace $OL^{l-1}\psi_{ki}$ with the sum of the normally ordered derivative and a t-expression by the induction hypothesis. Then applying L yields a normally ordered derivative of order $n + 1$ and, as was just shown, a t-expression of the required type. So we are left with the commutator term. If $O = L$ we are done. If $O = M$ or $O = M'$ the commutator term is equal to a linear combination of curvature terms by (4.2) and (4.3)–(4.5) which are t-terms of the required type. When $O = N$ we obtain apart from curvature terms as before a term involving the wave operator. This term can be rewritten using (4.6) as a linear combination of curvature terms and the terms $LNL^{l-1}\psi_{ki}$ and $MM'L^{l-1}\psi_{ki}$. The first term has been shown above to be of the correct type and with a similar argument one shows that the second term is also.

Now suppose $l = 0$ and $k > i$. Then we need to look at a term of the form $OMM^{k-i-1}N^i\psi$. The only nontrivial cases are $O = M'$ and $O = N$. In the latter case we find that the commutator term is a curvature term and therefore of the correct type. The other term is shown to be correct by similar arguments as above using the induction hypothesis. In the case $O = M'$ only the wave operator term appearing in the commutator term needs a different treatment. But this has been shown above also to lead to correct terms.

The last case is $l = 0$ and $k = i$. Then we are looking at $ONN^{i-1}\psi$. Here all cases are trivial except for $O = N$ and this case is treated as above. So, in summary, we have shown that all the appearing terms are of the stated type and hence the lemma is proved.

From this result we can obtain a set of equations satisfied by the functions ψ_{ki}. Consider $M'\psi_{ki}$, a derivative of ψ of order $k + 1$; therefore, there exist equations

$$M'\psi_{ki} = \alpha_{ki} L\psi_{ki+1} + t'_{ki} \tag{4.8}$$

and similarly

$$M\psi_{ki} = \psi_{k+1,i}, \tag{4.9}$$
$$N\psi_{ki} = \psi_{k+1,i+1} + t_{ki}, \tag{4.10}$$

where $\alpha_{ki} = 0$ if $k = i$ and where t_{ki} and t'_{ki} are pure t-expressions which contain ψ_{lj} with $l \leq k-1$ and possibly $L\psi_{lj}$ with $l \leq k-2$. If we regard these equations as the field equations for the fields ψ_{ki} then we can state the

Theorem IV.3 *A formal solution of (1.1) gives rise to a formal solution of (4.8)–(4.10) and vice versa. The set of spinor fields $\{\psi_{ki} : 0 \leq k \leq m, 0 \leq i \leq k\}$ is exact.*

Proof: The equivalence of the two systems is obvious. We need to show the exactness. Here, condition (ii) concerning the completeness of the powers is an immediate consequence of the lemma. The condition (i) concerning the independence of the powers can be verified as follows. Any relation between the powers has to be generated by the application of the commutation relations and (4.6) to the field equation (1.1) and all its derivatives. From looking at the structure of these relations one finds that they can not link any derivatives that contain more than two adjacent L's. So all the relations that can be generated must already be conditions on the derivatives of the field equation. But this is a condition only on derivatives of the form $s_n M'\psi$ and not on any power. The other possible source for conditions on powers come from the (derivatives of the) defining equations of the ψ_{ki}: $\psi_{ki} = M^{k-i}N^i\psi$. However, these are not algebraic relations but differential relations between the functions and — upon taking derivatives — between the powers. So there can not be any relations between the powers, which therefore are independent.

This theorem shows that the characteristic initial value problem for the equation (1.1) is formally well posed. This is, of course, a rather weak statement, implying only that one can prescribe certain components of derivatives of ψ on the null cone of a point in an arbitrary way and that this is just enough information for a unique solution to exist on the level of formal power series.

The exact set $\{\psi_{ki}\}$ is not invariant (cf. [12]) because in the expressions for the derivatives in terms of the powers there appear the curvature spinors together with their derivatives which are taken to be known background quantitites. Thus, these expressions depend on the actual point in space-time that is the vertex of the null cone. Since the field equation comes from a variational principle and since, therefore, there exists an energy momentum tensor we can couple the system via Einstein's equation to the curvature. Thus, we write $G_{ab} = 8\pi T_{ab}$ with T_{ab} from (2.25). Then we know how to express the curvature spinors Φ and Λ in terms of ψ and its first derivatives. In fact, $\Lambda = 0$ due to the conformal invariance of the equation. We can interpret ψ as describing some kind of matter field whose energy content creates the curvature of the manifold. We have one more unknown function to consider, the Weyl spinor Ψ which is subject to the equation (a part of the Bianchi identity) $M'\Psi = 2M\Phi$. Referring to a theorem in [5] we see that the enlarged set $\{\psi_{ki}, \Psi\}$ will be an invariant exact set on $\mathcal{M}$ provided that we can show that $M\Phi$ is a t-expression and that $N\Phi = 0$. We need to interpret the term "t-expression" a little different now because whenever Λ or Φ appear in the expressions we need to substitute their resp. representations in terms of the fields ψ_{ki}. Thus, we obtain an actual tree structure built from the bilinear pairings whose leaves consist only of powers $L^l\psi_{ki}$ and $L^l\Psi$ and their complex conjugates. This reflects the nonlinear nature of the coupling to gravity. The conditions above are easily verified, in fact, $N\Phi = 0$ is just the condition that the energy momentum tensor be divergence free and since Φ itself is a t-expression its derivative is also a t-expression as was shown above. So we have effectively proven the

Theorem IV.4 *The set $\{\psi_{ki}, \Psi\}$ subject to the equations (4.8)–(4.10), Einstein's equation and the Bianchi identity is an invariant exact set.*

Thus we can make a similar statement as before concerning the system coupled to gravity. The formal characteristic initial value problem is well posed. In this case, we do not have a similar result for the Cauchy problem.

V. THE GENERAL SOLUTION IN MINKOWSKI SPACE-TIME

Our aim in this section is to present the general solution of the field equation (1.1) in flat space subject to suitable initial and boundary conditions. Since each such solution is also a solution of $\Box^m\psi = 0$ for some positive integer m we will first derive the general solution of that equation. Since this does not depend on the existence of spinors and on the dimension of space-time we present the result in a slightly generalized form for arbitrary space-time dimension. So we are working in $\mathbb{M} = \mathbb{R}^{1,n-1}$. Then we will specialize to four dimensions and restrict the kernel of $\Box^m$ to those spinor fields that do satisfy (1.1). Since in flat space we are dealing with a partial differential equation with constant coefficients the general solution could be found using methods from the theory of distributions. We will, however, not pursue this here but present a different approach which fits better with the applications we have in mind.

We begin by introducing certain rings of functions associated to the null cone of momentum space. A function $f(k)$ defined on the null cone in "k"-space will be said to be admissible if and only if it obeys the following conditions:

(i) $f(k)$ is defined for all null vectors k^a,

(ii) $f(0) = 0$,

(iii) the function f is smooth on the complement of the origin,

(iv) $\lim_{t\to 0}(t^r f(tk)) = 0$, for any real number r and for any non-zero null vector k^a; this limit must be uniform on compact subsets of momentum space.

Condition (iv) controls both the "infrared" and "ultraviolet" behaviour of the function $f(k)$. Condition (iv) is needed to guarantee the differentiability and integrability of Fourier transforms involving the function $f(k)$.

Denote by K, K^+ and K^- the rings of all admissible functions, all admissible functions that vanish identically on the past null cone, all admissible functions that vanish on the future null cone, respectively. Note that we have the vector space direct sum decomposition: $K = K^+ \oplus K^-$.

For any subspace R of a ring and any vector variable X denote by $R[X]$ the space of all polynomials in the vector X with coefficients in the subspace R. In particular if R is itself a (sub)-ring, then $R[X]$ is a ring. For any non-negative integer m, denote by $R_m[X]$ the subspace of the space $R[X]$ consisting of polynomials of degree less than $m+1$ and by $R^{(m)}[X]$ the subspace of $R_m[X]$ consisting of all polynomials homogeneous of degree m in the variable X. In particular, for x a space-time vector-valued variable, we have that every element $\phi(x,k)$ of the ring $K[x]$ has an explicit expression as a polynomial in the variable x of the following form:

$$\phi(x,k) = \sum_{r=0}^{\infty} x^{a_1}x^{a_2}\dots x^{a_r}\phi_{a_1a_2\dots a_r}(k). \tag{5.1}$$

Here each coefficient tensor, $\phi_{a_1a_2\dots a_r}(k)$, is completely symmetric and is an indexed element of the ring K. Also only a finite number of these coefficient tensors is non-zero. Henceforth, each infinite sum we encounter will have only a finite number of non-zero terms.

Denote by ∂_a the derivative with respect to the variable x and by $\Box \equiv g_{ab}\partial^a\partial^b$ the wave operator, regarded as an endomorphism of the space $K[x]$. Denote by $L[x]$ the kernel of this endomorphism and define $L^+[x]$, $L_m[x]$ and $L^+_m[x]$ as the intersections of the space $L[x]$ with the spaces $K^+[x]$, $K_m[x]$ and $K^+_m[x]$, respectively. Consider the operator $k_a\partial^a$ as an endomorphism of $K[x]$. Since the operators $\Box$ and $k_a\partial^a$ commute, $k_a\partial^a$ restricts to an endomorphism, denoted D, of $L[x]$. Note that D is the derivative operator along the generators of the null cone restricted to solutions of the wave equation in $K[x]$.

A. The operator D

Proposition V.1 *The operator $D : L[x] \to L[x]$ is surjective provided the space-time is at least three-dimensional.*

The proof of this first technical result is rather lengthy and proceeds in several steps. We first perform a decomposition into space and time to obtain an expression for a general element of $L[x]$. First pick a unit timelike future pointing vector t^a and denote by S the orthogonal complement in space-time of the vector t^a. We shall use lower case Latin indices from the middle of the alphabet to label the (spatial) tensors of S and shall write γ_{ik} for the negative of the (flat) metric induced on S from the ambient space-time metric. Then the position vector x^a decomposes as $x^a = (t, \xi^i)$, where $t \equiv x^a t_a$ and one has the relation $x^a x_a = t^2 - \xi^i\xi_i$, where the tensor γ_{ik} and its inverse are used for index lowering and raising for spatial tensors. Correspondingly, the operators $\Box$ and D decompose as $\Box = \partial_t^2 - \Delta$, where $\Delta \equiv \gamma_{ik}\partial^i_\xi\partial^k_\xi$, and $D = \kappa\partial_t - (\kappa\cdot\partial_\xi)$, where $k^a = (\kappa, \kappa^i)$, $\kappa \equiv k^a t_a$ and $\kappa\cdot\partial_\xi \equiv \kappa_i\partial^i_\xi$. Note that since the vector k^a is null, we have the relation $\kappa^2 = \kappa^i\kappa_i$.

Given $\phi(x,k) \in L[x]$, define $\phi_0(\xi,k) \in K[\xi]$ and $\phi_1(\xi,k) \in K[\xi]$ to be the restrictions to the subspace S of the functions $\phi(x,k)$ and $t_a\partial^a\phi(x,k)$, respectively.

Lemma V.2 *The mapping $\rho_S : L[x] \to K[\xi]^2$, $\phi \mapsto (\phi_0, \phi_1)$ is an isomorphism, mapping each solution of the wave equation to its initial data on S.*

Proof: Given the pair $(\phi_0(\xi,k), \phi_1(\xi,k)) \in K[\xi]^2$, the function $\phi(x,k) \equiv \rho_S^{-1}((\phi_0(\xi,k),\phi_1(\xi,k))$ may be given by the following explicit formula:

$$\phi((t,\xi),k) = \cosh(t\Delta^{1/2})\,\phi_0(\xi,k) + \Delta^{-1/2}\sinh(t\Delta^{1/2})\,\phi_1(\xi,k). \tag{5.2}$$

Here the functions $\cosh(u)$ and $u^{-1}\sinh(u)$, with u an operator, are to be interpreted as formal power series. Note that in equation (5.2) there are no problems with the square root of the Laplacian, since the functions $\cosh(u)$ and $u^{-1}\sinh(u)$ are both even. Also there are no convergence problems, since the functions ϕ_0 and ϕ_1 are polynomials in the variable ξ^i.

Define the operator $\Lambda : K[\xi]^2 \to K[\xi]^2$ by $\Lambda \equiv \rho_S D \rho_S^{-1}$. Then in view of lemma (V.2) we have to show that Λ is surjective. We derive from equation (5.2) the explicit formula for the operator Λ, valid for any pair $(\phi_0, \phi_1) \in K[\xi]^2$:

$$\Lambda(\phi_0, \phi_1) = (\kappa\phi_1 - (\kappa \cdot \partial_\xi)\phi_0, \kappa\Delta\phi_0 - (\kappa \cdot \partial_\xi)\phi_1). \tag{5.3}$$

So we must now solve the following pair of equations:

$$\kappa\beta - (\kappa \cdot \partial_\xi)\alpha = \gamma, \tag{5.4}$$

$$\kappa\Delta\alpha - (\kappa \cdot \partial_\xi)\beta = \delta. \tag{5.5}$$

In equations (5.4) and (5.5) the pair (γ, δ) is a given element of the space $K[\xi]^2$ and the desired solution is the pair (α, β) which must be shown to lie in $K[\xi]^2$. Now it is clear from its definition that $K[\xi]^2$ is closed under multiplication or division by κ, so we may use equation (5.4) to eliminate the function β from equation (5.5). This gives the following equation:

$$\left(\Delta - (n \cdot \partial_\xi)^2\right)\alpha = \sigma. \tag{5.6}$$

Here we have put $n \cdot \partial_\xi \equiv n_i \partial_\xi^i$, with $n_i \equiv \kappa^{-1}\kappa_i$, a unit vector and $\sigma \equiv \kappa^{-2}(\kappa\delta + (\kappa \cdot \partial_\xi)\gamma) \in K[\xi]$. Note that the desired result is false in two space-time dimensions since the left hand side of equation (5.6) then vanishes identically, but the right hand side need not vanish. So we have reduced the problem to solving equation (5.6), given $\sigma \in K[\xi]$ such that the solution α must also lie in the space $K[\xi]$.

Proof of Proposition (5.1): We first prove the proposition for the special case with σ of the form

$$\sigma = (\xi \cdot n)^p \left(\xi^2 - (\xi \cdot n)^2\right)^q \upsilon_r. \tag{5.7}$$

Here the numbers p, q and r are non-negative integers and the function $\upsilon_r \in K[\xi]$ is homogeneous of degree r in the vector variable ξ^i and obeys both of the differential equations $\Delta\upsilon_r = 0$ and $(n \cdot \partial_\xi)\upsilon_r = 0$. It is easy to solve equation (5.6) in this case explicitly: a solution is just $\alpha = ((q+1)(n+2r+2q-2))^{-1}(\xi^2 - (\xi \cdot n)^2)\sigma$, as is easily checked, by differentiation. This solution clearly lies in the space $K[\xi]$ and is of the form $(\xi \cdot n)^p\tau$, where τ statisfies the equation $(n \cdot \partial_\xi)\tau = 0$.

The rest of the proof consists in a demonstration that the general case can be reduced to this special case by decomposing σ into a sum of appropriate terms and then using linearity of the operator Λ. We first decompose σ as a sum of terms as follows:

$$\sigma = \sum_{r=0}^{\infty} \frac{1}{r!}(\xi \cdot n)^r \sigma_r. \tag{5.8}$$

Each coefficient σ_r is required to obey the differential equation $(n \cdot \partial_\xi)\sigma_r = 0$. Explicitly one has the following formula for the quantity σ_r, valid for any non-negative integer r:

$$\sigma_r = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}(\xi \cdot n)^s (n \cdot \partial_\xi)^{r+s}\sigma. \tag{5.9}$$

In particular it is clear from equation (5.9) that each function σ_r belongs to the space $K[\xi]$. Note that the operator $\Delta - (n \cdot \partial_\xi)^2$ commutes with the multiplication operator $(\xi \cdot n)$. Also if $\alpha \in K[\xi]$, then we also have $(\xi \cdot n)^r\alpha \in K[\xi]$ for any non-negative integer r. So using the linearity of equation (5.6) and the decomposition of equation (5.8) and (5.9) it suffices to prove the solvability of equation (5.6) with both the functions σ and α lying in the kernel of the operator $n \cdot \partial_\xi$. Denote this kernel (a subspace of the space $K[\xi]$) by $N[\xi]$.

Next we use a standard fact from tensor theory that any symmetric tensor may be decomposed into tracefree parts. In the present language one has, for any $\tau \in N[\xi]$, a decomposition:

$$\tau = \sum_{r=0}^{\infty} \frac{1}{2^r r!}\left(\xi^2 - (\xi \cdot n)^2\right)^r \tau_r. \tag{5.10}$$

Here we have put $\xi^2 \equiv \xi_i \xi^i$. In equation (5.10), the coefficients τ_r must obey the differential equation: $\left(\Delta - (n \cdot \partial_\xi)^2\right) \tau_r = 0$ and must lie in the space $N[\xi]$. Indeed for the case that τ is a homogeneous function of non-negative integral degree m in the vector variable ξ, each function τ_r may be given explicitly by the following formula:

$$\tau_r = \sum_{s=0}^{\infty} (-1)^s \frac{(\lambda - 2r)\Gamma(\lambda - 2r - s)}{2^{r+2s}\Gamma(\lambda - r + 1)\Gamma(s+1)} \left(\xi^2 - (\xi \cdot n)^2\right)^s (\Delta - (n \cdot \partial_\xi)^2)^{r+s} \tau, \tag{5.11}$$

with $\lambda \equiv (n - 4 + 2m)/2$. It is easily checked by differentiation that the function τ_r of equation (5.11) lies in the kernel of both the operators $\Delta - (n \cdot \partial_\xi)^2$ and $n \cdot \partial_\xi$ as required. The proof of compatibility of equations (5.10) and (5.11) follows immediately from the lemma (V.3) given below.

Since every $\tau \in K[\xi]$ is uniquely a sum of its homogeneous parts and each of its homogeneous parts lies also in the space $K[\xi]$, equations (5.10) and (5.11) hold also for inhomogeneous functions $\tau \in N[\xi]$, provided that equation (5.11) is rewritten as follows:

$$\tau_r = \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{r+2s}} \frac{\mu \Gamma(\mu - s)}{\Gamma(\mu + r + 1)\Gamma(s+1)} \left(\xi^2 - (\xi \cdot n)^2\right)^s (\Delta - (n \cdot \partial_\xi)^2)^{r+s} \tau, \tag{5.12}$$

where $\mu \equiv (n - 4 + 2\xi_A \partial_\xi^A)/2$.

Combining these results and again using the linearity of equation (5.6), we see that it is sufficient to prove the solvability of equation (5.6) with the function σ being of the form given in (5.7) above. This completes the proof of proposition (5.73).

Note that since the above proof is compatible with homogeneity, it also shows that the restriction of the map D to the subspace $L_m[x]$ has range the subspace $L_{m-1}[x]$, for every positive integer m and that the restriction of D to the subspace of $L^{(m)}[x]$ is surjective onto the subspace $L^{(m-1)}[x]$.

Lemma V.3 *For j a non-negative integer, define a function $g_j(z)$ of the complex variable z as follows:*

$$g_j(z) = \sum_{r=0}^{j} (-1)^r \frac{(z - 2r)\Gamma(z - r - j)}{\Gamma(z - r + 1)\Gamma(r + 1)\Gamma(j - r + 1)}. \tag{5.13}$$

Then the function g_j vanishes identically unless $j = 0$ and the function g_0 is the constant function with value one.

Proof: First, the case $j = 0$ is easily checked by inspection. So henceforth assume, for convenience, that j is a fixed positive integer. From its definition it is clear that the function $g_j(z)$ is a rational function of the variable z and that $lim_{z \to \infty} g_j(z) = 0$. Therefore, the result will follow if it is proved that the function $g_j(z)$ is periodic. But one has the following relations, using equation (5.13):

$$\begin{aligned} g_j(z) &= \sum_{r=0}^{j} (-1)^r \frac{(z - r)\Gamma(z - r - j)}{\Gamma(z - r + 1)\Gamma(r + 1)\Gamma(j - r + 1)} - \sum_{r=1}^{j} (-1)^r \frac{\Gamma(z - r - j)}{\Gamma(z - r + 1)\Gamma(r)\Gamma(j - r + 1)} \\ &= \sum_{r=0}^{j} (-1)^r \frac{\Gamma(z - r - j)}{\Gamma(z - r)\Gamma(r + 1)\Gamma(j - r + 1)} + \sum_{r=0}^{j-1} (-1)^r \frac{\Gamma(z - r - j - 1)}{\Gamma(z - r)\Gamma(r + 1)\Gamma(j - r)} \\ &= \sum_{r=0}^{j} (-1)^r \frac{(z - r - j - 1)\Gamma(z - r - j - 1)}{\Gamma(z - r)\Gamma(r + 1)\Gamma(j - r + 1)} + \sum_{r=0}^{j} (-1)^r \frac{(j - r)\Gamma(z - r - j - 1)}{\Gamma(z - r)\Gamma(r + 1)\Gamma(j - r + 1)} \\ &= \sum_{r=0}^{j} (-1)^r \frac{(z - 2r - 1)\Gamma(z - r - j - 1)}{\Gamma(z - r)\Gamma(r + 1)\Gamma(j - r + 1)} = g_j(z - 1). \end{aligned} \tag{5.14}$$

So the function $g_j(z)$, for $j > 0$ is periodic and therefore vanishes identically as required.

B. The Fourier transform operator $\mathcal{F}$.

Consider the $(n - 1)$-form Ξ defined on the null cone in momentum space defined by the formula $\Xi \equiv dk^{a_1} dk^{a_2} \dots dk^{a_{n-1}} = k_{a_0} \epsilon^{a_0 a_1 a_2 \dots a_{n-1}} \Xi$. Restricted to the null cone one has $k_a k^a = k_a dk^a = 0$, so the left hand side of this equation is orthogonal to the null vector k^a. The $(n-1)$-form Ξ factorizes according to the formula $k^a \Xi = dk^a \omega$, where the $n-2$-form ω is defined by the formula: $k^{[a_1} dk^{a_2} \dots dk^{a_{n-1}]} = k_{a_0} \epsilon^{a_0 a_1 a_2 \dots a_{n-1}} \omega$. Both the forms Ξ and ω are closed: $d\Xi = d\omega = 0$.

Given any $\lambda(x,k) \in K[x]$, we consider its generalized Fourier transform $\mathcal{F}(\lambda)$, which is a space-time field given by the following formula, valid at any point x of $\mathbb{M}$:

$$\mathcal{F}(\lambda)(x) \equiv \int e^{ik_b x^b} \lambda(x,k)\, \Xi. \tag{5.15}$$

The integration in equation (5.15) is to be carried out over the complete (past and future) null cone, equipped with the induced orientation from its embedding in k-space, which in turn is oriented by the volume form, $\epsilon_{a_1 a_2 \dots a_n} dk^{a_1} dk^{a_2} \dots dk^{a_n}$. Note that by definition of the space $K[x]$ the convergence of the integral of equation (5.15) is automatic and the resulting field $\mathcal{F}(\lambda)$ is everywhere smooth on space-time.

The linear operator $\mathcal{F} : \lambda \mapsto \mathcal{F}(\lambda)$ is defined on $K[x]$ and we denote its range by $\Gamma[x]$. Also denote by Γ, Γ^+ and $\Gamma^+[x]$ the images under the operator $\mathcal{F}$ of the spaces K, K^+ and $K^+[x]$, respectively.

It is clear that the space $\Gamma[x]$ consists of certain polynomials in the variable x^a with coefficients in the space Γ, so to understand the range of the operator $\mathcal{F}$ it is sufficient to identify the space Γ.

To this end, we first introduce for any α and β, solutions of the wave equation in spacetime the $n-1$-form $\omega(\alpha,\beta) \equiv \alpha(*d\beta) - \beta(*d\alpha)$, where $*$ is the Hodge star operator on forms for the given Lorentzian metric. Since the wave equation for a field ϕ may be written $d*(d\phi) = 0$, it is clear that the form $\omega(\alpha,\beta)$ is closed. Define $\Omega(\alpha,\beta)$ to be the integral of the form $\omega(\alpha,\beta)$, over a spacelike hypersurface, oriented towards the future, asymptotic to spacelike infinity, for given fields α and β, which are required to be such that the integral converges and is independent of the choice of that hypersurface. Denote by W the space of all solutions of the scalar wave equation with initial data, on any spacelike hypersurface, asymptotic to spacelike infinity, in the Schwarz class (the initial data for a solution ϕ on a hypersurface is by definition the restriction of ϕ and $*d\phi$ to that hypersurface). Denote by $M[x]$ the space of all polynomial solutions of the wave equation and by $M'[x]$ its dual space. Then for each $\phi(x) \in W$, we obtain an element $\mu(\phi)$ of the space $M'[x]$, defined by the formula $\mu(\phi)(f) = \Omega(\phi, f)$, for each $f \in M[x]$. This gives a moment map $\mu : W \to M'[x]$, $\phi \mapsto \mu(\phi)$. Then we have the following result:

Proposition V.4 $\Gamma = Ker(\mu)$.

The proof of this result follows immediately from the Fourier inversion formula.

Note that the information in the moment map μ is completely contained in the formal power series defined by the formula: $\rho(\phi)(p_a) = \Omega(e^{ip_a x^a}, \phi)$, where the exponential $e^{ip_a x^a}$ is understood as a formal power series in the null covector p_a. The quantity $\rho(\phi)(p_a)$ is then a formal power series whose coefficients are tracefree symmetric tensors, representing the various moments of the field ϕ. In terms of initial data, the quantity $\rho(\phi)$ represents all moments of the data for the field ϕ. In this language the space Γ is the subspace of the space W consisting of all fields ϕ, for which $\rho(\phi) = 0$.

Our final aim in this subsection is to determine the kernel of $\mathcal{F}$ and to prove the

Proposition V.5 *For $\lambda \in K[x]$ we have*

$$\mathcal{F}(\lambda)(x) = \int e^{ik_b x^b} \lambda(x,k)\, \Xi = 0 \quad \textit{for all } x \tag{5.16}$$

$$\iff \lambda(i\partial_k + tk) = 0. \tag{5.17}$$

Here the scalar t is an indeterminate. Also in writing equation (5.17) it is to be understood that in each term of the expression of the function $\lambda(x,k)$ as a polynomial in the variable x, the expression is ordered by placing all the factors of the variable x to the left, before replacing the variable x by the operator $i\partial_k + tk$.

Before we proceed to the proof of the proposition we want to clarify the structure of equation (5.17) with an example. In the case $\lambda \in K_2[x]$, we have the expression $\lambda(x,k) = x^a x^b \beta_{ab} + x^a \beta_a + \beta$, for some symmetric tensor β_{ab}, vector β_a and scalar β, each of which depends only on the variable k. For this case equation (5.17) reads as follows:

$$0 = (i\partial_k^a + tk^a)(i\partial_k^b + tk^b)\beta_{ab} + (i\partial_k^a + tk^a)\beta_a + \beta. \tag{5.18}$$

Note that the commutator $(i\partial_k^a + tk^a)(i\partial_k^b + tk^b) - (i\partial_k^b + tk^b)(i\partial_k^a + tk^a)$ vanishes identically, so there is no factor ordering problem. Expanding equation (5.18) in powers of the indeterminate t, equation (5.18) is equivalent to the following three equations:

$$0 = k^a k^b \beta_{ab}, \tag{5.19}$$
$$0 = (\partial_k^a k^b + k^a \partial_k^b)\beta_{ab} - ik^a\beta_a, \tag{5.20}$$
$$0 = \partial_k^a \partial_k^b \beta_{ab} - i\partial_k^a \beta_a - \beta. \tag{5.21}$$

Note that equations (5.17) – (5.21) implicitly require that one extend the function $\lambda(x, k)$ off the null cone of momentum space before writing these equations since the formulation of the equations uses the full derivative operator ∂_k. However it must be possible to rewrite the equations so that they are purely intrinsic to the null cone. In the case of equations (5.19) – (5.21), one can see easily that these equations are equivalent to the following three equations:

$$0 = k^a k^b \beta_{ab}, \tag{5.22}$$
$$0 = (L^a k^b + k^a L^b)\beta_{ab} - 3p^a k^b \beta_{ab} - ip^c k_c k^a \beta_a, \tag{5.23}$$
$$\begin{aligned} 0 = {} & L^a L^b \beta_{ab} - 3p^a L^b \beta_{ab} - ip^a k_a L^b \beta_b + 2p^a p^b \beta_{ab} \\ & + ip^a k_a p^b \beta_b + (1/2)p^a p_a (ik_b \beta^b + \beta_b^b) - (p^a k_a)^2 \beta. \end{aligned} \tag{5.24}$$

Here the operator L^a is defined for any fixed vector p^a by the relation:

$$L^a \equiv 2p_b k^{[b} \partial_k^{a]}. \tag{5.25}$$

It is clear from its definition that the operator L^a is intrinsic to the null cone. Then equations (5.22)–(5.24) hold for arbitrary vectors p^a. Alternatively equations (5.23) and (5.24) can be rewritten without using the vector p^a as follows:

$$0 = (L_c{}^a k^b + k^a L_c{}^b)\beta_{ab} - 3k^b \beta_{cb} - ik_c k^a \beta_a, \tag{5.26}$$
$$0 = L_{(c}{}^a L_{d)}{}^b \beta_{ab} - 3L_{(c}{}^b \beta_{d)b} - ik_{(c} L_{d)}{}^b \beta_b + 2\beta_{cd} + ik_{(c}\beta_{d)} + (1/2)g_{cd}(ik_b \beta^b + \beta_b{}^b) - k_c k_d \beta. \tag{5.27}$$

Here we have introduced the intrinsic operator L^{ab}, given by the formula: $L^{ab} \equiv 2k^{[a}\partial_k^{b]}$, in terms of which one has the relation $L^a = p_b L^{ba}$. Note that provided that equation (5.22) also holds, each of the equations (5.26) and (5.27) amounts to just one scalar equation, since one may verify that the right hand side of equations (5.26) and (5.27) are proportional to the quantities k_c and $k_c k_d$, respectively.

Proof of the proposition (5.77): We shall prove the statement for $\lambda \in K_m[x]$, for every non-negative integer m by induction on the natural number m. First the required result holds for $m = 0$, since in this case the function $\lambda(x, k) = \beta(k)$ for some function $\beta(k) \in K$. When equation (5.16) holds, the integral $\int e^{ik_a x^a}\beta(k)\,\Xi$ gives the zero solution of the wave equation $\Box\Phi = 0$ and it is well known in this case that this entails that the function β must vanish identically.

Next suppose the required result is true for all $\lambda \in K_m[x]$ for all $m < s$, for some positive integer s. We prove the result for $m = s$. So consider equation (5.16) with the function $\lambda(x, k) \in K_s[x]$ now a polynomial in the variable x of degree not more than s. Then we can decompose the function $\lambda(x, k)$ as $\lambda(x, k) = \alpha(x, k) + \beta(x, k)$, where $\alpha(x, k) \in K^{(s)}[x]$ and $\beta(x, k) \in K_{s-1}[x]$.

Applying the wave operator to equation (5.16), we get the following equation:

$$0 = \int e^{ik_b x^b}(2ik_a\partial^a + \Box)\lambda(x, k)\,\Xi. \tag{5.28}$$

Since the function $(2ik_a\partial^a + \Box)\lambda(x, k) \in K_{s-1}[x]$, we get by the inductive hypothesis, the equation:

$$0 = \left[(2ik_a\partial^a + \Box)\,\lambda(x, k)\right]_{x\to(i\partial_k + tk)}. \tag{5.29}$$

Next take the partial derivative of equation (5.16) with respect to the variable x. We get the following equation:

$$0 = \int e^{ik_b x^b}(ik^a + \partial^a)(\alpha + \beta)\,\Xi. \tag{5.30}$$

Now we have $\alpha(x, k) = x^e\alpha_e(x, k)$, where the function $\alpha^e(x, k) \equiv s^{-1}\partial^e\alpha(x, k) \in K_{s-1}[x]$. Then equation (5.30) may be rewritten, using an integration by parts as follows:

$$\begin{aligned} 0 &= \int e^{ik_b x^b}(ik^a x^e\alpha_e + ik^a\beta + \partial^a\lambda)\,\Xi \\ &= \int e^{ik_b x^b}(2ik^{[a}x^{e]}\alpha_e + ik^a\beta + \partial^a\lambda)\,\Xi + \int e^{ik_b x^b} ix^a k^e\alpha_e\,\Xi \\ &= \int e^{ik_b x^b}(-2k^{[a}\partial_k^{e]}\alpha_e + ik^a\beta + \partial^a\lambda)\,\Xi + x^a\int e^{ik_b x^b} ik^e\alpha_e\,\Xi. \end{aligned} \tag{5.31}$$

Now using equation (5.28), we have the following:

$$0 = \int e^{ik_b x^b} (2ik_a \partial^a + \Box)\lambda \, \Xi = \int e^{ik_b x^b} (2isk^a \alpha_a + 2ik_a \partial^a \beta + \Box\lambda) \, \Xi. \tag{5.32}$$

Hence one has the following equation:

$$\int e^{ik_b x^b} ik^a \alpha_a \, \Xi = -\frac{1}{2s} \int e^{ik_b x^b} (2ik_a \partial^a \beta + \Box\lambda) \, \Xi. \tag{5.33}$$

This equation is used to replace the last integral of equation (5.31). We then obtain

$$0 = \int e^{ik_b x^b} (-2k^{[a} \partial_k^{e]} \alpha_e + ik^a \beta + \partial^a \lambda) - \frac{x^a}{2s} (2ik_a \partial^a \beta + \Box\lambda) \, \Xi. \tag{5.34}$$

Now, by inspection, each term multiplying the quantity $e^{ik_b x^b}$ of equation (5.34) is of degree at most $s-1$ in the variable x. Therefore we may invoke the inductive hypothesis again to deduce

$$\begin{aligned} 0 &= \left[-2k^{[a} \partial_k^{b]} \alpha_b + ik^a \beta + \partial^a \lambda - \frac{x^a}{2s} (2ik_a \partial^a \beta + \Box\lambda) \right]_{x \mapsto (i\partial_k + tk)} \\ &= \left[2ik^{[a} x^{b]} \alpha_b + ik^a \beta + \partial^a \lambda - \frac{x^a}{2s} (2ik_a \partial^a \beta + \Box\lambda) \right]_{x \mapsto (i\partial_k + tk)}. \end{aligned} \tag{5.35}$$

Now equation (5.28), when written out gives the equation:

$$[2ik_a \partial^a \beta + \Box\lambda]_{x \mapsto (i\partial_k + tk)} = [-2ik_a \partial^a \alpha]_{x \mapsto (i\partial_k + tk)} = [-2isk_a \alpha^a]_{x \mapsto (i\partial_k + tk)}. \tag{5.36}$$

Substituting equation (5.36) into the last part of equation (5.35) gives the following equation:

$$\begin{aligned} 0 &= \left[2ik^{[a} x^{b]} \alpha_b + ik^a \beta + \partial^a \lambda + ix^a k_b \alpha^b \right]_{x \mapsto (i\partial_k + tk)} \\ &= \left[ik^a x^b \alpha_b + ik^a \beta + \partial^a \lambda \right]_{x \mapsto (i\partial_k + tk)} = [ik^a \alpha + ik^a \beta + \partial^a \lambda]_{x \mapsto (i\partial_k + tk)} \\ &= [i(k^a - i\partial^a)\lambda]_{x \mapsto (i\partial_k + tk)} = ik^a \, [\lambda]_{x \mapsto (i\partial_k + tk)}. \end{aligned} \tag{5.37}$$

In the transition from the penultimate to the last line of equation (5.37), we have used the fact that the terms arising from the commutator of the operator of multiplication by k^a and the operator $i\partial_k + tk$ exactly cancel the derivative term, the quantity $i\partial^a \lambda$. That this is correct may be seen as follows. Consider the quantity $\left[(k^a - i\partial^a) \left((x^b p_b)^n f(k) \right) \right]_{x \mapsto (i\partial_k + tk)}$, for n a non-negative integer, p a constant covector and for $f \in K$. We then have the following equation:

$$\begin{aligned} &\left[(k^a - i\partial^a) \left((x^b p_b)^n f(k) \right) \right]_{x \mapsto (i\partial_k + tk)} = \left[(x^b p_b)^n k^a f(k) - inp^a (x^b p_b)^{n-1} f(k) \right]_{x \mapsto (i\partial_k + tk)} \\ &= ((i\partial_k^b + tk^b) p_b)^n k^a f(k) - \left[inp^a (x^b p_b)^{n-1} f(k) \right]_{x \mapsto (i\partial_k + tk)} \\ &= k^a \left((i\partial_k^b + tk^b) p_b \right)^n f(k) + inp^a \left((i\partial_k^b + tk^b) p_b \right)^{n-1} f(k) - inp^a \left[(x^b p_b)^{n-1} f(k) \right]_{x \mapsto (i\partial_k + tk)} \\ &= k^a \left[(x^b p_b)^n f(k) \right]_{x \mapsto (i\partial_k + tk)}. \end{aligned} \tag{5.38}$$

Since any polynomial $\lambda \in K[x]$ may be written as a finite linear combination of terms of the form $(x^b p_b)^n f(k)$, we have the relation $[(k^a - i\partial^a)\lambda]_{x \mapsto (i\partial_k + tk)} = k^a [\lambda]_{x \mapsto (i\partial_k + tk)}$, for any $\lambda \in K[x]$, as required.

Finally we remove the factor ik^a from equation (5.37) giving the required result and the induction is complete.

C. The general solution of $\Box^m \psi = 0$

In this subsection we provide the general solution of the equation $\Box^m \psi = 0$ in the space $\Gamma[x]$. As the construction will show this is equivalent to finding the general solution subject only to the condition that the zrm field $\Box^{m-1}\psi$ lies in the space Γ (i.e. has zero moment map). We begin by proving that there is no loss of generality in restricting the domain of the Fourier transform operator $\mathcal{F}$ from the space $K[x]$ to its subspace $L[x]$. More specifically, we have

Proposition V.6 *For each $\lambda(x,k) \in K[x]$ there exists a $\mu(x,k) \in L[x]$ such that $\mathcal{F}(\lambda) = \mathcal{F}(\mu)$.*

Proof: For any $\lambda \in K[x]$, we have the decomposition, directly analogous to that of equation (5.12) above:

$$\lambda = \sum_{r=0}^{\infty} \frac{(x^2)^r}{2^r r!} \lambda_r. \tag{5.39}$$

$$\lambda_r = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2^s}}{2^{2s+r}} \frac{\Gamma(\nu - 2r - s)(\nu - 2r)}{\Gamma(\nu - r + 1)\Gamma(s+1)} \Box^{r+s} \lambda. \tag{5.40}$$

Here we have put $x^2 \equiv x_a x^a$ and $\nu \equiv (n - 2 + 2x^a \partial_a)/2$. From equation (5.40), by differentiation, it follows that the coefficients λ_r belong to the space $L[x]$. In particular it follows that the space $K[x]$ is the sum of the space $L[x]$ with the module generated over the ring $K[x]$ by the function x^2. In view of this decomposition, to prove the required result it suffices to show that for any function $\alpha \in K_m[x]$ there exists a function $\beta \in K_{m+1}[x]$ such that $\mathcal{F}(x^2\alpha) = \mathcal{F}(\beta)$. Now we have the integration by parts identity, valid for any $y_a \in K[x]$, such that $y_a k^a = 0$:

$$0 = \int \partial^a (e^{ik_b x^b} y_a)\, \Xi. \tag{5.41}$$

Rephrasing equation (5.41) in terms of the operator $\mathcal{F}$, we have $\mathcal{F}(\partial^a y_a + i x^a y_a) = 0$, valid for any $y_a \in K[x]$, such that $y_a k^a = 0$. In particular consider the case that $y_a \equiv -ik_0^{-1}(k_0 x_a - t_a x^b k_b + k_a x^b t_b)\alpha$, where t^a is a fixed unit vector and $k_0 \equiv k^a t_a$. Then it is clear that $y_a \in K_{m+1}[x]$ and that y_a satisfies the identity $y_a k^a = 0$, so one has $\mathcal{F}(-\partial^a y_a) = \mathcal{F}(i x^a y_a)$. But by contracting the vectors x^a and iy_a we also have the relation $i x^a y_a = x^2 \alpha$, which yields the relation $\mathcal{F}(x^2\alpha) = \mathcal{F}(\beta)$, where $\beta \equiv -\partial^a y_a$. Since from its definition it is clear that the function β lies in the space $K_{m+1}[x]$, the proof is complete.

In view of this result we henceforth assume without loss of generality that $\mathcal{F}$ is defined on the space $L[x]$.

The wave operator of space-time, $\Box$ acts naturally on the space $\Gamma[x]$ and one has the relation, immediate from the definition of $\mathcal{F}$ in equation (5.15), valid for any $\lambda \in L[x]$:

$$\Box \mathcal{F}(\lambda) = 2i\mathcal{F}(D\lambda). \tag{5.42}$$

We next wish to determine the kernel of the operator $\Box$ acting on the space $\Gamma[x]$. By equation (5.16) above and using equation (5.42), we have $\Box\mathcal{F}(\lambda) = 0$, for $\lambda \in L[x]$ if and only if the function $\lambda(x,k)$ obeys the equation:

$$0 = (D\lambda)(i\partial + tk, k). \tag{5.43}$$

Let the power series expansion of the function $\lambda \in L[x]$ be given as follows:

$$\lambda(x,k) = \sum_{r=0}^{\infty} \frac{1}{r!} x^{a_1} x^{a_2} \dots x^{a_r} \lambda_{a_1 a_2 \dots a_r}(k). \tag{5.44}$$

In equation (5.44), each coefficient tensor $\lambda_{a_1 a_2 \dots a_r}$ belongs to the space K and is completely symmetric and tracefree. Writing out equation (5.43) in terms of this expansion gives the following system of equations, one for each positive integer q:

$$0 = \sum_{p=0}^{\infty} \frac{1}{p!} \lambda_{p,q}(k). \tag{5.45}$$

Here the quantity $\lambda_{p,q}$ is by definition $\lambda_{p,q} \equiv i^p \partial^{a_1} \dots \partial^{a_p} k^{b_1} \dots k^{b_q} \lambda_{a_1 \dots a_p b_1 \dots b_q}$. Note that there are no factor ordering problems for the quantity $\lambda_{p,q}$, since the tensor coefficients are all tracefree. Now expanding in powers of the indeterminate t we have the identity, derived from equation (5.44):

$$\lambda(i\partial + tk, k) = \sum_{p,q=0}^{\infty} \frac{t^q}{p!q!} \lambda_{p,q}(k). \tag{5.46}$$

Comparing equations (5.46) and (5.45), we see that if we define $\mu(x,k) \equiv \lambda(x,k) - \lambda(i\partial, k)$, then we have the relation:

$$0 = \mu(i\partial + tk, k). \tag{5.47}$$

Note that $\mu \in L[x]$, so from equation (5.47), one has $\mu \in \ker(\mathcal{F})$. This gives the relation $\mathcal{F}(\lambda) = \mathcal{F}(\nu)$, where $\nu \equiv \lambda(i\partial, k) \in L_0[x]$. So we have shown that if $\Box\mathcal{F}(\lambda) = 0$, then $\lambda = 0 \bmod (\ker(\mathcal{F}) + L_0[x])$. Conversely, if $\lambda \in \ker(\mathcal{F}) + L_0[x]$, then it is clear from equation (5.42) that $\Box\mathcal{F}(\lambda) = 0$. So we have proved the relation $\ker(\Box\mathcal{F}) = \ker(\mathcal{F}) + L_0[x]$. Rephrasing we have proved the relation $\ker(\Box) = \mathcal{F}(L_0[x])$.

This generalizes immediately to our first main result

Theorem V.7 *For any integer $m > 0$ and any space-time dimension $n > 2$, the kernel of the operator $\Box^m$ when acting on $\Gamma[x]$ is given by the relation*

$$\ker(\Box^m\mathcal{F}) = \ker(\mathcal{F}) + L_{m-1}[x]. \tag{5.48}$$

Equivalently this may be stated as:

$$\ker(\Box^m) = \mathcal{F}(L_{m-1}[x]). \tag{5.49}$$

Proof: The required result has just been proved in the case $m = 1$, so henceforth assume m is a fixed integer greater than one. Suppose that $\Phi \in \ker(\Box^m)$ and $\Phi = \mathcal{F}(\phi)$, for some $\phi \in L[x]$. Then $\Box^{m-1}\Phi \in \ker(\Box)$, so we have $\Box^{m-1}\Phi = \mathcal{F}(\alpha)$, for some $\alpha \in L_0[x]$. By proposition (5.73) the operator D is surjective as a linear map from the space $L_s[x]$ to the space $L_{s-1}[x]$, for any positive integer s. It immediately follows that the operator D^r is also surjective as a linear map from the space $L_s[x]$ to the space $L_{s-r}[x]$, for any positive integers r and s, with s not less than r. Therefore we may put $\alpha(x,k) = (2iD)^{m-1}\beta(x,k)$ for some $\beta \in L_{m-1}[x]$. Then we have $\Box^{m-1}\Phi = \mathcal{F}(\alpha) = \mathcal{F}\left((2iD)^{m-1}\beta\right) = \Box^{m-1}(\mathcal{F}(\beta))$. So we have $\Phi - \mathcal{F}(\beta) \in \ker(\Box^{m-1})$ and the required result now follows immediately by induction.

We have shown that the general solution of the equation $\Box^{m+1}\phi = 0$ in the space $\Gamma[x]$ is given by an integral formula

$$\phi(x) = \int e^{ik_b x^b} \left\{ \lambda(k) + \sum_{i=1}^{m} \lambda_{a_1 \dots a_i}(k) x^{a_1} \dots x^{a_i} \right\} \Xi \tag{5.50}$$

where the polynomial inside the integral satisfies the wave equation. The solutions of this type are automatically $\mathcal{C}^\infty$. So with this formula we can not reach functions which are merely $\mathcal{C}^k$ differentiable. However, this class of solutions is sufficient for our purposes. More general solutions can be obtained using functional analytic methods by starting with this integral formula on an appropriate function space and then taking limits. We will not pursue this here.

D. Spinor momentum space

We now specialize to the case of four dimensions and introduce spinor variables. A future pointing null momentum covector k_a may be factorized as $k_a = k_A k_{A'}$, for k_A a two component spinor, with complex conjugate spinor $k_{A'}$. More precisely, we have a surjective map from the momentum spin space to the future null cone of momentum space, which maps the spinor k_A to the null vector $k_A k_{A'}$. The inverse image of the vector $k_a \equiv k_A k_{A'}$ is the circle of spinors αk_A, (with $\alpha \in \mathbb{C}$ and $|\alpha| = 1$) for k_a non-zero, and is the zero spinor only, when $k_a = 0$. We pull back our previous constructions along this surjection. The pullback of the ring K^+ is then the ring of functions $f(k_A, k_{A'})$ which are everywhere smooth and vanish to all orders at the origin, decay faster than any power at infinity and which obey the differential equation $(k_A\partial_k^A - k_{A'}\partial_k^{A'})f = 0$. We shall need to multiply by components of the spinors k_A and $k_{A'}$, so it is natural to enlarge the ring K^+ in the spinor case to the ring $\widehat{K}$ which is by definition the ring of all functions $f(k_A, k_{A'})$, such that f has a decomposition as an infinite sum: $f = \sum_{r=-\infty}^{\infty} f_r$, with only a finite number of the functions f_r non-zero and such that for each integer r we have:

(i) f_r is globally defined and smooth on the momentum spin space;

(ii) $\lim_{t\to\infty} f_r(e^{t\alpha}k_A, e^{t\alpha'}k_{A'}) = 0$, for all $\alpha \in \mathbb{C}$, such that $\Re(\alpha)$ is non-zero; here the limit is taken with t real and the limit must be uniform on compact subsets of the momentum spin space;

(iii) $(k_A\partial_k^A - k_{A'}\partial_k^{A'})f_r(k_A, k_{A'}) = r f_r(k_A, k_{A'})$.

For each integer s, denote by K^s the subspace of $\widehat{K}$ consisting of all $f \in \widehat{K}$ with f_r vanishing for all r different from s. Then the pullback of the ring K^+ is the ring K^0 and one has $K^p K^q$ included in K^{p+q}, for all integers p and q. In particular K^0 is a subring of $\widehat{K}$ and every space K^j is a K^0-module. Denote by $K^j[x]$ the subspace of $\widehat{K}[x]$

consisting of all polynomials in x with coefficients in the space K^j. Denote by $\widehat{L}[x]$ the subspace of $\widehat{K}[x]$ annihilated by the wave operator $\Box$ and by $L^j[x]$ the intersection of the spaces $K^j[x]$ and $\widehat{L}[x]$. Denote by $\widehat{K}[\pi]$ the space of all polynomials in the spinor variables π_A and $\pi_{A'}$ with coefficients in the ring $\widehat{K}$. For p and q any non-negative integers and for j any integer, denote by $\widehat{K}_{p,q}[\pi]$ and $K^j_{p,q}[\pi]$ the spaces of all polynomials in the spinor variables π_A and $\pi_{A'}$, homogeneous of degrees (p,q) in the pair $(\pi_A, \pi_{A'})$, with coefficients taken from the spaces $\widehat{K}$ and K^j, respectively. For every element $f(x)$ of the space $\widehat{L}[x]$ and $g(\pi)$ of the space $\widehat{K}[\pi]$, we have unique expansions of the following form:

$$f(x) = \sum_{r=0}^{\infty} f_{A_1A_2\ldots A_rA'_1A'_2\ldots A'_r} x^{A_1A'_1}x^{A_2A'_2}\ldots x^{A_rA'_r}, \tag{5.51}$$

$$g(\pi) = \sum_{p,q=0}^{\infty} g_{A_1A_2\ldots A_pA'_1A'_2\ldots A'_q} \pi^{A_1}\pi^{A_2}\ldots\pi^{A_p}\pi^{A'_1}\pi^{A'_2}\ldots\pi^{A'_q}. \tag{5.52}$$

In equations (5.51) and (5.52), the coefficient spinors $f_{A_1A_2\ldots A_rA'_1A'_2\ldots A'_r}$ and $g_{A_1A_2\ldots A_pA'_1A'_2\ldots A'_q}$ lie in the space $\widehat{K}$ and are completely symmetric in all indices. Denote by $E^\pi_x : \widehat{L}[x] \to \widehat{K}[\pi]$ the evaluation operator which substitutes the spinor $\pi^A\pi^{A'}$ for x^a in any element of $\widehat{L}[x]$. Then it is clear that the map E^π_x is an isomorphism of the space $\widehat{L}[x]$, with range the subspace of $\widehat{K}[\pi]$ consisting of all polynomials $g(\pi) \in \widehat{K}[\pi]$, which obey the differential equation $(\pi_A\partial^A_\pi - \pi_{A'}\partial^{A'}_\pi)g(\pi) = 0$.

To proceed we need the spinor analogues of our previous technical results. First the analogue of the surjectivity of the operator D of proposition (V.1).

Proposition V.8 *The operators $k_A\partial^a, k_{A'}\partial^a : \widehat{L} \to \widehat{L}$ and $k_A\partial^A_\pi, k_{A'}\partial^{A'}_\pi : \widehat{K} \to \widehat{K}$ are surjective.*

Proof: Using the isomorphism E^π_x it is easily seen that it is sufficent to prove surjectivity for the operators $k_A\partial^A_\pi$ and $k_{A'}\partial^{A'}_\pi$. Further by formal conjugation the proof of surjectivity for the operator $k_A\partial^A_\pi$ will yield a proof of surjectivity for the operator $k_{A'}\partial^{A'}_\pi$. So we just need to prove that the operator $k_A\partial^A_\pi$ is surjective, when acting on the space $\widehat{K}[\pi]$. Using the expansion of equation (5.52), we reduce to proving that given a totally symmetric spinor $g_{B\ldots CB'\ldots C'} \in \widehat{K}$, there exists a totally symmetric spinor $f_{AB\ldots CB'\ldots C'} \in \widehat{K}$, such that $k^A f_{AB\ldots CB'\ldots C'} = g_{B\ldots CB'\ldots C'}$. By taking components with a fixed primed spinor basis, we reduce further to the case that the spinor $g_{B\ldots CB'\ldots C'}$ has only unprimed indices. By contracting throughout with a spinor variable π^A, we reduce to solving the differential equation $k_A\partial^A_\pi f = g$, given $g \in \widehat{K}[\pi]$, such that the solution f lies in the space $\widehat{K}[\pi]$ and both f and g are independent of the variable $\pi_{A'}$.

Let t^a denote a fixed unit timelike vector and put $n_A \equiv t_a k^{A'}$. Note that $n_A k^A = t_a k^A k^{A'}$ is always a positive real number unless $k_A = 0$. Then one has the following decomposition of the function g:

$$g = \sum_{p,q=0}^{\infty} \frac{(-1)^p}{p!q!}(k_A\pi^A)^p(n_B\pi^B)^q g_{p,q}. \tag{5.53}$$

This decomposition follows from the expression of the spinor π_A in terms of the spinor basis n_A and k_A: $\pi_A = (t^c k_C k_{C'})^{-1}(-k_B\pi^B n_A + n_B\pi^B k_A)$. Then by the binomial theorem, we have the following explicit formula for the quantities $g_{p,q}$:

$$g_{p,q} = \frac{1}{(t^c k_C k_{C'})^{p+q}} \left[(n_A\partial^A_\pi)^p(k_B\partial^B_\pi)^q g\right]_{\pi=0}. \tag{5.54}$$

It is clear from equation (5.54) that each coefficient $g_{p,q}$ lies in the ring $\widehat{K}$, so by linearity it suffices to prove the required result for the case $g = (k_A\pi^A)^p(n_B\pi^B)^q$, with p and q non-negative integers. But then we have the following relation:

$$k_A\partial^A_\pi \left[\frac{(k_A\pi^A)^p(n_B\pi^B)^{q+1}}{(q+1)t^c k_C k_{C'}}\right] = (k_A\pi^A)^p(n_B\pi^B)^q. \tag{5.55}$$

So $f \equiv ((q+1)t^c k_C k_{C'})^{-1}(k_A\pi^A)^p(n_B\pi^B)^{q+1}$ provides a solution in this case. Since it is clear that this function f belongs to the space $\widehat{K}[\pi]$ and is independent of the variable $\pi_{A'}$, the proof is complete. Note that by tracking homogeneities through the proof we find that if g belongs to the space $K^j_{p,q}$, then we may take the solution f to lie in the space $K^{j-1}_{p+1,q}$.

Second we need to analyze the kernel of the pullback of the Fourier transform operator $\mathcal{F}$. This Fourier transform, still called $\mathcal{F}$, is defined now as follows, when acting on any $\phi \in \widehat{K}[x]$:

$$\mathcal{F}(\phi) \equiv \int e^{ik_a x^a} \phi(x^a, k_A, k_{A'})\, \Omega. \tag{5.56}$$

Here one has $\Omega \equiv \epsilon^{AB}\epsilon^{A'B'} dk_A dk_B dk_{A'} dk_{B'}$ and the integral is carried out over all of spin space. It is easily shown that the operator $\mathcal{F}$ maps K^0 isomorphically onto Γ^+ (the range of $\mathcal{F}$ of section 5.2 acting on the space K^+) and annihilates all the spaces K^j, for j non-zero. Furthermore, acting on the space $K^0[x]$ the operator $\mathcal{F}$ agrees with the pullback of our original Fourier transform operator (restricted to the domain $K^+[x]$), up to a fixed non-zero multiplicative constant.

Proposition V.9 *: For each $\phi \in L^0[x]$:*

$$\phi \in \ker \mathcal{F} \iff \phi(x^a, k_A, k_{A'}) = (\partial_k^A + ix^a k_{A'})\phi_A + (\partial_k^{A'} + ix^a k_A)\phi_{A'}, \tag{5.57}$$

with $\phi_A \in L^1[x]$ and $\phi_{A'} \in L^{-1}[x]$ obeying the spinor zrm-field equations: $\partial^a \phi_A = 0$ and $\partial^a \phi_{A'} = 0$. If ϕ has degree at most m in x, then ϕ_A and $\phi_{A'}$ may be taken to have degree at most $m-1$ in x.

Proof: The "if"-part of this result is a trivial integration by parts, so we assume that $\mathcal{F}(\phi)$ vanishes and we establish the formula of equation (5.57) for the function ϕ. First if ϕ is independent of the variable x, then $\mathcal{F}(\phi) = 0$ entails that $\phi = 0$, so the result holds if we take $\phi_A = \phi_{A'} = 0$. So now we assume that the required result is true for ϕ any polynomial of degree at most $m-1$ and take ϕ to have degree at most m. Then applying the wave operator to the equation $\mathcal{F}(\phi) = 0$, we obtain the equation $\mathcal{F}(k_a \partial^a \phi) = 0$, so by the inductive assumption, we have the relation:

$$k_a \partial^a \phi = (\partial_k^A + ix^a k_{A'})\psi_A + (\partial_k^{A'} + ix^a k_A)\psi_{A'}. \tag{5.58}$$

Here the quantities $\psi_A \in L^1[x]$ and $\psi_{A'} \in L^{-1}[x]$ are polynomials in x of degree at most $m-2$, belong to the spaces $L^1[x]$ and $L^{-1}[x]$ and obey the field equations $\partial^a \psi_A = 0$ and $\partial^a \psi_{A'} = 0$, respectively. Write $\phi = \alpha + \beta$, where α is homogeneous of degree exactly m and β is of degree at most $m-1$ in the variable x. Similarly decompose the fields ψ_A and $\psi_{A'}$ as $\psi_A = -i\rho_A + \sigma_A$ and $\psi_{A'} = -i\rho_{A'} + \sigma_{A'}$, where ρ_A and $\rho_{A'}$ are homogeneous of degree $m-2$, whereas σ_A and $\sigma_{A'}$ have degree at most $m-3$. Then the terms of highest degree in the variable x of equation (5.58) give the following equation:

$$k_a \partial^a \alpha = x^a k_{A'} \rho_A + x^a k_A \rho_{A'}. \tag{5.59}$$

Note that by the inductive hypothesis, the functions $\rho_A \in L^1[x]$ and $\rho_{A'} \in L^{-1}[x]$ obey the zrm field equations: $\partial^a \rho_A = 0$ and $\partial^a \rho_{A'} = 0$. Note that the quantities σ, ρ and ρ' are respectively of homogeneity (m, m), $(m-1, m-2)$ and $(m-2, m-1)$ in the variables π_A and $\pi_{A'}$, respectively. Now put $x^a = \pi^A \pi^{A'}$ in equation (5.59). We obtain the equation:

$$k_a \partial_\pi^A \partial_\pi^{A'} \sigma = m(k_{A'} \pi^{A'} \rho + k_A \pi^A \rho'). \tag{5.60}$$

Here we have put $\sigma \equiv E_x^\pi(\alpha) \in K^0_{m,m}[\pi]$, $\rho \equiv E_x^\pi(\pi^A \rho_A) \in K^1_{m-1,m-2}[\pi]$ and $\rho' \equiv E_x^\pi(\pi^{A'} \rho_{A'}) \in K^{-1}_{m-2,m-1}[\pi]$. Next write $\rho = m^{-1} k_A \partial_\pi^A \tau$ and $\rho' = m^{-1} k_{A'} \partial_\pi^{A'} \tau'$, for some $\tau \in K^0_{m,m-2}[\pi]$ and $\tau' \in K^0_{m-2,m}[\pi]$. This we can do by the surjectivity of the operator $k_A \partial_\pi^A$ proved above. Then equation (5.60) may be rewritten as follows:

$$0 = k_a(\partial_\pi^A \partial_\pi^{A'} \sigma - \pi^{A'} \partial_\pi^A \tau - \pi^{A'} \partial_\pi^{A'} \tau'). \tag{5.61}$$

Now suppose that the quantity $v^a \in \widehat{K}[\pi]$ obeys the equation $k_a v^a = 0$. We may expand the vector v^a in terms of the spinors k^A, n^A and their conjugates $k^{A'}$ and $n^{A'}$ as follows:

$$v^a = k^A k^{A'} U + k^A n^{A'} V \;+\; n^A k^{A'} W + n^A n^{A'} X, \tag{5.62}$$

$$U \equiv \frac{(v^a n_A n_{A'})}{(t_c k^C k^{C'})^2}, \qquad V \equiv -\frac{(v^a n_A k_{A'})}{(t_c k^C k^{C'})^2}, \tag{5.63}$$

$$W \equiv -\frac{(v^a k_A n_{A'})}{(t_c k^C k^{C'})^2}, \qquad X \equiv \frac{(v^a k_A k_{A'})}{(t_c k^C k^{C'})^2}. \tag{5.64}$$

It is clear that each of the quantities U, V, W and X lies in the space $\widehat{K}[\pi]$. When we have the relation $v^a k_a = 0$, this implies that the quantity X vanishes. This in turn entails that the quantity v^a may be expressed as $v^a = k^A v^{A'} + k^{A'} v^A$, for some $v^A \in \widehat{K}[\pi]$ and $v^{A'} \in \widehat{K}[\pi]$: indeed one may take $v^A = Uk^A + Wn^A$ and $v^{A'} = Vn^{A'}$. Note that if $v^a \in K^j_{p,q}[\pi]$, then by equation (5.64) the quantities v^A and $v^{A'}$ may be taken to lie in the spaces $K^{j+1}_{p,q}[\pi]$ and $K^{j-1}_{p,q}[\pi]$, respectively.

Applying this result to equation (5.61), we obtain

$$\partial^A_\pi \partial^{A'}_\pi \sigma = \pi^{A'} \partial^A_\pi \tau + \pi^A \partial^{A'}_\pi \tau' - m^2 k^A v^{A'} - m^2 k^{A'} v^A, \tag{5.65}$$

for some $v^A \in K^1_{m-1,m-1}[\pi]$ and $v^{A'} \in K^{-1}_{m-1,m-1}[\pi]$. Contracting equation (5.65) through with the spinors π_A and $\pi_{A'}$ gives the following equation:

$$\sigma = k^A \pi_A v' + k^{A'} \pi_{A'} v. \tag{5.66}$$

Here we have put $v \equiv \pi_A v^A \in K^1_{m,m-1}[\pi]$ and $v' \equiv \pi_{A'} v^{A'} \in K^{-1}_{m-1,m}[\pi]$. Rewriting equation (5.66) in terms of the variable x, we find:

$$\alpha = x^a (k_A \alpha_{A'} + k_{A'} \alpha_A). \tag{5.67}$$

Here the fields $\alpha_A(x)$ and $\alpha_{A'}(x)$ are determined by the formulas $E^\pi_x(\alpha^A) = m^{-1}\partial^A_\pi v$ and $E^\pi_x(\alpha^{A'}) = m^{-1}\partial^{A'}_\pi v'$. Also we have $\alpha_A(x) \in L^1[x]$ and $\alpha_{A'}(x) \in L^{-1}[x]$ and both the spinor fields $\alpha_A[x]$ and $\alpha_{A'}[x]$ obey the zrm field equations and are homogeneous of degree $m-1$ in the variable x. By equation (5.67), we have the following relation, using an integration by parts:

$$0 = \mathcal{F}(\phi) = \mathcal{F}(\alpha + \beta) = \mathcal{F}(x^a(k_A \alpha_{A'} + k_{A'} \alpha_A) + \beta) \tag{5.68}$$

$$= \mathcal{F}(i\partial^{A'}_k \alpha_{A'} + i\partial^A_k \alpha_A + \beta) \tag{5.69}$$

By the inductive hypothesis, we obtain from equation (5.69) the relation:

$$i\partial^{A'}_k \alpha_{A'} + i\partial^A_k \alpha_A + \beta = (\partial^A_k + ix^a k_{A'})\omega_A + (\partial^{A'}_k + ix^a k_A)\omega_{A'}. \tag{5.70}$$

Here the fields $\omega_A \in L^1[x]$ and $\omega_{A'} \in L^{-1}[x]$ obey the spinor zrm field equations and are polynomials of degree at most $m-2$ in the variable x. Combining equations (5.67) and (5.70), we get:

$$\phi = \alpha + \beta = x^a(k_A \alpha_{A'} + k_{A'} \alpha_A) + (\partial^A_k + ix^a k_{A'})\omega_A + (\partial^{A'}_k + ix^a k_A)\omega_{A'} - i(\partial^{A'}_k \alpha_{A'} + \partial^A_k \alpha_A) \tag{5.71}$$

$$= (\partial^A_k + ix^a k_{A'})\phi_A + (\partial^{A'}_k + ix^a k_A)\phi_{A'}. \tag{5.72}$$

In equation (5.72) we have put $\phi_A \equiv \omega_A - i\alpha_A$ and $\phi_{A'} \equiv \omega_{A'} - i\alpha_A$. Since it is clear that the fields ϕ_A and $\phi_{A'}$ have all the requisite properties, we have proved the validity of equation (5.57) for any field $\phi(x) \in L^0[x]$ of degree at most m in the variable x. Therefore by induction we have the validity of equation (5.57) in general and the proof is complete.

E. The general solution of $M'\Phi = 0$

If we wish to construct a space-time field from elements of K^j with j non-zero, we first need to multiply by spinors k_A or $k_{A'}$ as appropriate to map the element to an (indexed) element of K^0, before applying the Fourier transform operator $\mathcal{F}$. The result is a spinor indexed field on space-time. For example consider the standard zrm equation $\partial^{AA'}\Phi_{AB\ldots CD}(x) = 0$ for a totally symmetric spinor field $\Phi_{AB\ldots CD}(x)$ of r indices. Taking another derivative and contracting, we immediately find that the field $\Phi_{AB\ldots CD}(x)$ obeys the wave equation $\Box\Phi_{AB\ldots CD}(x) = 0$. Therefore by the theorem (V.7), we may write its general solution, (after pulling back to the momentum spin space) in the space Γ^+ , as follows:

$$\Phi_{AB\ldots CD}(x) = \int e^{ik_a x^a} \alpha_{AB\ldots CD}(k_E, k_{E'})\,\Omega. \tag{5.73}$$

Here the Fourier coefficients $\alpha_{AB\ldots CD}$ lie in the space K^0. Applying the field equation we get the equation $k^A \alpha_{AB\ldots CD} = 0$, whence it follows that $\alpha_{AB\ldots CD}(k_E, k_{E'}) = k_A k_B \ldots k_C k_D \phi(k_E, k_{E'})$, for some function $\phi(k_E, k_{E'}) \in K^{-j}$. So now equation (5.73) reads as follows:

$$\Phi_{AB\ldots CD}(x) = \int e^{ik_ax^a} k_A k_B \ldots k_C k_D \phi(k_E, k_{E'})\, \Omega. \tag{5.74}$$

Next we shall derive explicitly the solution by Fourier transform of the equation $\partial^{A(A'}\Phi^{B'\ldots C')}_{AB\ldots C} = 0$ in the special case of one primed index ($m = 1$) and then later generalize to arbitrary positive m. We know that the field $\Phi_{ABB'}$ lies in the kernel of the operator $\Box^2$ by proposition (II.1), so by theorem (V.7) it admits a Fourier representation of the following form:

$$\Phi(X) = \int e^{ik_ax^a}(a_bx^b + a)\, \Omega. \tag{5.75}$$

Here the variable X is an abbreviation: $X \equiv (x^a, \pi_A, \pi_{A'})$ and we have put $\Phi(X) \equiv \Phi_{ABB'}(x)\pi^A\pi^B\pi^{B'}$. The Fourier coefficients a_b and a depend on the spinors k_A and π_A and their conjugates, but not on the variable x^a. Defining the operator $M' \equiv \pi^{A'}\partial^A_\pi\partial_a$ the field equation may be written as $M'\Phi = 0$. This operator agrees with the operator M' defined in section 2 in its action on the spinor indexed coefficients of Φ. Applying the field equation, we get the following equation:

$$0 = \mathcal{F}\left(\pi^{B'}\partial^B_\pi(ik_b + \partial_b)(a_cx^c + a)\right) = \mathcal{F}\left((\pi^{B'}\partial^B_\pi)(ik_ba_cx^c + ik_ba + a_b)\right). \tag{5.76}$$

Using equation (5.57) above, we deduce the following equation from equation (5.76):

$$(\pi^{B'}\partial^B_\pi)(ik_ba_cx^c + ik_ba + a_b) = (\partial^A_k + ix^ak_{A'})\alpha_A + (\partial^{A'}_k + ix^ak_A)\alpha_{A'}. \tag{5.77}$$

Here the quantities α_A and $\alpha_{A'}$ are independent of the variable x. Equating the coefficients of x in equation (5.77), we get

$$k_{B'}\pi^{B'}k_B\partial^B_\pi a_c = k_{C'}\alpha_C + k_C\alpha_{C'}. \tag{5.78}$$

Equation (5.78) gives immediately the equation $k_B\partial^B_\pi a^ck_c = 0$, which is solved by $a^ck_c = (\pi^Bk_B)^2\alpha$, for some α, independent of the variable π_A. This gives the relation: $a_c = k_B\pi^B\pi_C\beta_{C'} + k_C\gamma_{C'} + k_{C'}\gamma_C$, for some $\beta_{C'}$, $\gamma_{C'}$ and γ_C, with $\alpha = k^{A'}\beta_{A'}$. After an integration by parts applied to equation (5.75), the terms involving $\gamma_{C'}$ and γ_C may be eliminated, so one may take just $a_c = k_B\pi^B\pi_C\beta_{C'}$, without loss of generality.

Equation (5.77) now becomes

$$\begin{aligned}
&(\pi^{B'}\partial^B_\pi)(ik_bk_D\pi^D\pi_C\beta_{C'}x^c + ik_ba + k_D\pi^D\pi_B\beta_{B'}) \\
&= ik_{B'}\pi^{B'}k_D\pi^Dk_C\beta_{C'}x^c + ik_{B'}\pi^{B'}k_B\partial^B_\pi a + 3k_D\pi^D\pi^{B'}\beta_{B'} \\
&= (\partial^A_k + ix^ak_{A'})\alpha_A + (\partial^{A'}_k + ix^ak_A)\alpha_{A'}.
\end{aligned} \tag{5.79}$$

This gives the relation $k_{A'}\alpha_A + k_A\alpha_{A'} = k_{B'}\pi^{B'}k_D\pi^Dk_A\beta_{A'}$. So we have $\alpha_{A'} = k_{B'}\pi^{B'}k_D\pi^D\beta_{A'} + \delta k_{A'}$ and $\alpha_A = -k_A\delta$ for some δ. But then the contribution of the terms involving the quantity δ to the right hand side of equation (5.79) is just $(\partial^A_kk_A - \partial^{A'}_kk_{A'})\delta = (k_A\partial^A_k - k_{A'}\partial^{A'}_k)\delta$, which vanishes, since by tracking homogeneities we find that δ lies in the space K^0. Therefore without loss of generality, we may take $\delta = 0$. Then if we put $x^a = 0$ in equation (5.79), we obtain the relation:

$$ik_{B'}\pi^{B'}k_B\partial^B_\pi a + 2k_D\pi^D\pi^{B'}\beta_{B'} = k_{B'}\pi^{B'}k_D\pi^D\partial^{A'}_k\beta_{A'}. \tag{5.80}$$

Next write $\beta_{A'} = \beta_{A'B'}\pi^{B'} + \pi_{A'}\beta$, where $\beta_{A'B'}$ is symmetric and both $\beta_{A'B'}$ and β are independent of the variables π_A and $\pi_{A'}$. Then putting $\pi_{A'} = k_{A'}$ in equation (5.80) gives the relation: $\beta_{A'B'}k^{A'}k^{B'} = 0$, which entails that $\beta_{A'B'} = k_{A'}\delta_{B'}$, for some spinor $\delta_{B'}$. Then $\beta_{A'} = k_{A'}\delta_{B'}\pi^{B'} + \pi_{A'}\epsilon$, for some scalar ϵ. The term in $\beta_{A'}$ proportional to $k_{A'}$ may be eliminated by an integration by parts, applied to equation (5.75), so we may take without loss of generality: $\beta_{A'} = \pi_{A'}\epsilon$. Then equation (5.80) reduces to the equation:

$$ik_B\partial^B_\pi a = k_D\pi^D\pi_{A'}\partial^{A'}_k\epsilon. \tag{5.81}$$

Next we put $a = a_{AB}\pi^A\pi^B$, for some symmetric spinor a_{AB}, independent of the spinor π_A. Putting $\pi_A = k_A$ in equation (5.81) gives the relation: $a_{AB}k^Ak^B = 0$, so $a_{AB} = k_A\epsilon_B$, for some spinor ϵ_B. Also put $\epsilon_B = \epsilon_{BB'}\pi^{B'}$, where ϵ_b is independent of the spinors π_A and $\pi_{A'}$. Then equation (5.81) reduces to the following equation:

$$\epsilon_{BB'}k^B = i\partial_k^{A'}\epsilon. \tag{5.82}$$

Then we have the relation $a_b x^b + a = k_D\pi^D(x^b\pi_B\pi_{B'}\epsilon + \epsilon_{BB'}\pi^B\pi^{B'})$. Summarizing we have found that the general solution by Fourier transform of the equation $\partial^{A(A'}\Phi_{AB}^{B')} = 0$ is given by the formula:

$$\Phi(X) = \int e^{ik_a x^a} k_B\pi^B(x^c\pi_C\pi_{C'}\epsilon + \epsilon_{CC'}\pi^C\pi^{C'})\,\Omega. \tag{5.83}$$

Here the quantities ϵ and $\epsilon_{BB'}$ depend only on the momentum spinors k_A and $k_{A'}$ and are subject to equation (5.82). Next equation (5.82) may be solved by writing $\epsilon = -i\phi_A k^A$, for some ϕ_A. Then one has $\epsilon^{AA'} = \partial_k^{A'}\phi^A - k^A\phi^{A'}$, for some $\phi^{A'}$. The quantities ϕ_B and $\phi^{A'}$ are freely specifiable. Writing out equation (5.83) in terms of the spinors ϕ_A and $\phi_{A'}$, we get the following formula:

$$\Phi(X) = \int e^{ik_a x^a} k_B\pi^B(-ix^c\pi_C\pi_{C'}\phi_B k^B + (\partial_k^{C'}\phi^C - k^C\phi^{C'})\pi_C\pi_{C'})\,\Omega. \tag{5.84}$$

Finally we may integrate by parts in equation (5.84) to eliminate the derivative term. This gives the equation:

$$\begin{aligned}\Phi(X) &= \int e^{ik_a x^a} k_B\pi^B(-ix^{CC'}\pi_C\pi_{C'}\phi_B k^B + ix^{C'C}k_C\phi_B\pi^B\pi_{C'} - k^C\pi_C\phi^{C'}\pi_{C'})\,\Omega \\ &= \int e^{ik_a x^a}(k_B\pi^B)^2(ix^{C'C}\phi_C\pi_{C'} + \phi^{C'}\pi_{C'})\,\Omega. \end{aligned} \tag{5.85}$$

Note that there is gauge freedom in the pair of Fourier coefficients $\phi_\alpha \equiv (\phi_A, \phi^{A'})$: the quantity ϵ of equation (5.82) is unchanged under the transformation $\phi_A \mapsto \phi_A + k_A\gamma$. Then the quantity $\epsilon^{AA'}$ is also unchanged provided we make the transformation $\phi^{A'} \mapsto \phi^{A'} + \partial_k^{A'}\gamma$. So the complete gauge transformation is $\phi_\alpha \mapsto \phi_\alpha + K_\alpha\gamma$, where K_α is the operator pair: $K_\alpha \equiv (k_A, \partial_k^{A'})$.

Note that equation (5.85) may be rewritten in the following compact form:

$$\Phi(X) = \int e^{ik_a x^a}(k_B\pi^B)^2 Z^\alpha\phi_\alpha\,\Omega. \tag{5.86}$$

Here Z^α is the twistor $Z^\alpha \equiv (ix^a\pi_{A'}, \pi_{A'})$. This result may be generalized immediately:

Theorem V.10 *The integral*

$$\Phi(X) = \int e^{ik_a x^a}\phi(Z^\alpha, k_B\pi^B)\,\Omega, \tag{5.87}$$

is a solution of the equation $M'\Phi = 0$ for every $\phi(Z^\alpha, k_B\pi^B) \in \widehat{K}[Z, k_A\pi^A]$. Conversely, let $\Phi \in \Gamma^+[x,\pi,\bar{\pi}]$ be a polynomial homogeneous of degree (p,q) in the spinors $(\pi,\bar{\pi})$ with $p > q$. If Φ satisfies the equation $M'\Phi = 0$ then it has the representation (5.87) for some $\phi(Z, k_A\pi^A) \in \widehat{K}[Z, k_A\pi^A]$ homogeneous of degree (q,p) in the variables $(Z, k_A\pi^A)$.

The restriction $p > q$ is necessary, because the theorem is false when $p = q$, or when $p < q$, because in each of these cases the equation $M'\Phi = 0$ possesses a gauge freedom: if $\Phi = (\pi_{A'}\pi_A\partial^a)^p\rho$, where ρ is a polynomial homogeneous of degree $(0,q)$ in the variables $(\pi_A, \pi_{A'})$, then the equation $\pi^{A'}\partial_\pi^A\partial_a\Phi = 0$ is automatically obeyed, for arbitrary such functions ρ, as is checked easily , since the operators $\pi_{A'}\pi_A\partial^a$ and M' commute and since ρ is annihilated by the operator M'. Because of this gauge freedom, no Fourier transform formula based on the null cone of momentum space is possible, unless one first fixes the gauge freedom in some way.

Proof: By straightforward differentiation we see immediately that the function Φ obeys the equation $\pi^{A'}\partial_\pi^A\partial_a\Phi = 0$.

Conversely we prove next that the general solution of the equation $M'\Phi = 0$ may be put in the form of equation (5.87). The proof is by induction on the integer q. First consider the case $q = 0$. Then the field $\Phi(X)$ is independent of the variable $\pi_{A'}$, so the field equation $M'\Phi = 0$ is equivalent to the equation $\partial_\pi^A\partial_a\Phi = 0$, which is just the standard zrm field equation, for a totally symmetric spinor field with $p > 0$ indices. By equation (5.74) above the solution may be written as follows:

$$\Phi(X) = \mathcal{F}((k_B\pi^B)^p\phi). \tag{5.88}$$

Here the function ϕ is independent of the variables $(x^a, \pi_A, \pi_{A'})$. Therefore the required result holds in this case, with the function $f(Z^\alpha)$ independent of the variable Z^α.

Next consider the case $q > 0$. Let $\Phi(X)$ of homogeneity (p, q) satisfy the field equation $M'\Phi = 0$ and put $\Psi(X) \equiv \partial_\pi^{A'}\partial_\pi^A\partial_a\Phi$. Then $\Psi(X)$ is of homogeneity $(p-1, q-1)$ so, since $q-1$ is non-negative and since $p-1 > q-1$, we may use the inductive hypothesis and write the field $\Psi(X)$ as follows: $\Psi = \mathcal{F}(ik_a x^a(k_B\pi^B)^{p-1}\psi(Z^\alpha))$. Define a field $F(X)$ by the following formula:

$$F(X) = \mathcal{F}\left((k_B\pi^B)^p f(Z^\alpha)\right). \tag{5.89}$$

We wish to choose the function $f(Z^\alpha)$, such that the field $F(X)$ is of homogeneity (p, q) and obeys the equation: $\Psi = \partial_\pi^{A'}\partial_\pi^A\partial_a F$. Applying the differential operator $\partial_\pi^{A'}\partial_\pi^A\partial_a$ to equation (5.89), we get

$$\begin{aligned} &\partial_\pi^{A'}\partial_\pi^A\partial_a F(X) = \mathcal{F}((\partial_\pi^{A'}\partial_\pi^A(\partial_a + ik_a))(k_B\pi^B)^p f(Z^\alpha)) \\ &= \mathcal{F}(\partial_\pi^{A'}\partial_\pi^A\partial_a(k_B\pi^B)^p f(Z^\alpha)) = -p\mathcal{F}\left((k_B\pi^B)^{p-1}\partial_\pi^{A'}k^A\partial_a f(Z^\alpha)\right) \\ &= i\mathcal{F}((k_B\pi^B)^{p-1}\partial_\pi^{A'}\pi_{A'}k_A(\partial_Z)^A f(Z^\alpha)) = ip(q+1)\mathcal{F}((k_B\pi^B)^{p-1}k_A(\partial_Z)^A f(Z^\alpha)). \end{aligned} \tag{5.90}$$

Therefore we have a solution to the equation $(\partial_\pi^{A'}\partial_\pi^A\partial_a)F = \Psi$ provided that the function $f(Z^\alpha)$ obeys the equation $k_A(\partial_Z)^A f(Z^\alpha) = (ip(q+1))^{-1}g(Z^\alpha)$.

Having found the function $F(X)$, we write $\Phi(X) = F(X) + g(X)$, for some function $g(X)$. Then the function $g(X)$ must obey both the equations $(\partial_\pi^{A'}\partial_\pi^A\partial_a)g = 0$ and $\pi^{A'}\partial_\pi^A\partial_a g = 0$. Combining these equations, we get the single equation: $\partial_\pi^A\partial_a g = 0$. By contracting this equation on the left with the operator $\pi_B\partial^{BA'}$, we see that the function $g(X)$ lies in the kernel of the operator $\Box$ whence it admits a representation analogous to that of equation (5.74): $g(X) = \mathcal{F}(\gamma)$, for some function $\gamma(k_A, k_{A'}, \pi_B, \pi_{B'})$. Then the field equation $\partial_\pi^A\partial_a g = 0$ gives the equation $k_A\partial_\pi^A\gamma = 0$, so $\gamma = (k_A\pi^A)^p\eta$, for some function $\eta(k_A, k_{A'}, \pi_{B'})$. Putting $\phi(Z^\alpha) \equiv f(Z^\alpha) + \eta$ now gives the desired representation of the function Φ and the complete proof follows by induction.

If we apply this theorem to our special case we obtain the explicit representation given in the following

Corollary V.11 *For any non-negative integer m the integral*

$$\psi^{AB\ldots C}_{B'\ldots C'}(x) = \int e^{ik_a x^a} k^A k^B \ldots k^C \left\{\phi_{B'\ldots C'} + \phi_{E(B'\ldots}x^E{}_{C')} \cdots + \phi_{E\ldots F}x^E{}_{B'} \cdots x^F{}_{C'}\right\}\Omega, \tag{5.91}$$

is a positive frequency solution of the equation $\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} = 0$. Conversely, every such solution is represented in the above form.

VI. TWISTOR SOLUTION OF THE FIELD EQUATIONS

Let us first introduce the structures which are relevant for the purposes of this work. For more details see [13] and references therein. Twistor space $\mathbb{T}$ is by definition a four dimensional complex affine space. Denote by $\mathbb{V}$ the underlying vector space of $\mathbb{T}$. At any $z \in \mathbb{T}$, denote by $\theta(z)$ the natural isomorphism of the tangent space of $\mathbb{T}$ at z with the vector space $\mathbb{V}$ and denote by θ the $\mathbb{V}$-valued one form on $\mathbb{T}$ whose value at any $z \in \mathbb{T}$ is $\theta(z)$. It is clear that the one form θ is exact: $\theta = d\zeta$. Here the quantity ζ is a $\mathbb{V}$ valued function globally defined on $\mathbb{T}$. The function ζ is unique up to the transformation: $\zeta \mapsto \zeta + \alpha$, with α constant. The function ζ serves as a vector valued global coordinate for the space $\mathbb{T}$.

Naturally associated to the affine space $\mathbb{T}$ is the space $S(\mathbb{T})$ which is the space of all two dimensional affine subspaces of the space $\mathbb{T}$. Naturally associated to the vector space $\mathbb{V}$ is the space $M(\mathbb{T})$ which is the space of all two dimensional subspaces of the space $\mathbb{V}$. There is a natural surjection, $\mu : S(\mathbb{T}) \to M(\mathbb{T})$, which takes each element of $S(\mathbb{T})$ to its tangent space. The map μ renders $S(\mathbb{T})$ a two dimensional fibre bundle over $M(\mathbb{T})$ with fibre a two dimensional affine space. The space $M(\mathbb{T})$ is provided with a natural holomorphic conformal structure, which is such that $x, y \in M(\mathbb{T})$ are null related if and only if the intersection $x \cap y$ is non-trivial. It is isomorphic to (compactified, complexified) Minkowski space and the space $S(\mathbb{T})$ can be regarded in a natural way as the affine (unprimed) spin bundle over the space $M(\mathbb{T})$. The primed cospin bundle $S'(\mathbb{T})$ is by definition the space of all pairs $(X, z) \in S(\mathbb{T}) \times \mathbb{V}$ with z tangent to X. It is a two dimensional vector bundle over the space $S(\mathbb{T})$. Denote by $L'(\mathbb{T})$ the line bundle $\Omega^2(S'(\mathbb{T}))$ over $S(\mathbb{T})$. Note that the restriction of the form θ^2 to any $X \in S(\mathbb{T})$ naturally takes values in the line bundle $L'(\mathbb{T})$ at X, pulled back to the space X.

Let f denote a holomorphic function defined on some domain U in $\mathbb{T}$. Then the form $f\theta^2$ is a holomorphic two form on $\mathbb{T}$ with values in $\Omega^2(\mathbb{V})$. For suitable $X \in S(\mathbb{T})$, consider the following contour integral:

$$\mathcal{S}(f)(X) \equiv \int_{\gamma(X)} f\theta^2. \tag{6.1}$$

Here $\gamma(X)$ is a closed oriented contour of two real dimensions, which is required to lie in the intersection of the space X with the domain of the function f and to vary smoothly with X. It is clear that the quantity $\mathcal{S}(f)$ represents a holomorphic section of the line bundle $L'(\mathbb{T})$ over its domain of definition, $M(U)$ (this domain is an open subset of the space of subspaces X, for which the intersection with the open subset U has non-trivial second homology). By definition, if the integration contours are regarded as given, the section $\mathcal{S}(f)$ is the (unprimed) spinor field associated to the twistor function f.

More generally let $F(z,\zeta)$ denote a holomorphic function on the tangent bundle of the domain U with $(z,\zeta) \in U \times \mathbb{V}$. Then consider the following contour integral for $(X,z) \in S'(\mathbb{T})$, such that $X \in M(U)$:

$$\mathcal{S}(F)(X,z) \equiv \int_{\gamma(X)} F(z,\zeta)\theta^2. \tag{6.2}$$

This defines a function $\mathcal{S}(F)$ on $S'(\mathbb{T})$ taking values in the line bundle $L'(\mathbb{T})$ (pulled back to the space $S'(\mathbb{T})$).

Lemma VI.1 *The contour integrals of equation (6.1) and (6.2) give coordinate independent formulations of solutions of the zrm equation and the equation* $M'\Phi(x^a, \pi^A, \pi_{A'}) = \pi^{A'}\partial^A_\pi \partial_a \Phi(x^a, \pi^A, \pi_{A'}) = 0$*, respectively.*

Proof: We use lower case greek indices for tensors based on the vector space $\mathbb{V}$ and introduce the standard representation $\zeta^\alpha = (\zeta^A, \zeta_{A'})$ of a twistor ζ in terms of an unprimed spinor ζ^A and a primed cospinor $\zeta_{A'}$. A point X, not at infinity, of the space $S(\mathbb{T})$ is labelled by the pair (x^a, π^A). The two dimensional affine subspace of $\mathbb{T}$ corresponding to X is the set of all twistors $\zeta^\alpha(\rho_{A'})$ of the form :

$$\zeta^\alpha(\rho_{A'}) = (ix^{AA'}\rho_{A'} + \pi^A,\, \rho_{A'}) = X^{\alpha A'}\rho_{A'} + \Pi^\alpha, \tag{6.3}$$

for arbitrary cospinors $\rho_{A'}$. Defining $X^{\alpha B'} \equiv (ix^{AB'},\, \delta^{B'}_{A'})$, $\Pi^\alpha \equiv (\pi^A,\, 0)$ and $X^{\alpha\beta} \equiv X^{\alpha C'}X^{\beta D'}\epsilon_{C'D'}$, we note that the restriction of the one form θ^α to the space X is given by the formula: $X^*(\theta^\alpha) = X^{\alpha B'}d\rho_{B'}$ and therefore the restriction of the form $\theta^\alpha\theta^\beta$ to the space X is just $X^*(\theta^\alpha\theta^\beta) = X^{\alpha\beta}d^2\rho$, with $d^2\rho \equiv (1/2)\epsilon^{A'B'}d\rho_{A'}d\rho_{B'}$. Therefore, the field $\mathcal{S}(f)(X)$ of equation (6.1) factorizes as $\mathcal{S}(f)(x^c,\pi^C)^{\alpha\beta} = X^{\alpha\beta}\phi(f)(x^c,\pi^C)$, where we have the following explicit formulas for the function $\phi(f)(x^a,\pi^A)$:

$$\begin{aligned}\phi(f)(x^a,\pi^A) &\equiv \int_{\rho(X)} f(X^{\alpha B'}\rho_{B'} + \Pi^\alpha)d^2\rho \\ &= \int_{\rho(X)} f(ix^{AB'}\rho_{B'} + \pi^A, \rho_{A'})d^2\rho.\end{aligned} \tag{6.4}$$

In equation (6.4) the two dimensional contour $\rho(X)$ lies in the primed spin space of the variable $\rho_{A'}$ and varies smoothly with X, avoiding the singularities of the integrand. Differentiating equation (6.4), we immediately obtain the zrm field equations in the form $\partial^A_\pi\partial_a\phi(f) = 0$.

Similarly, the integral of equation (6.2) gives rise to a field $\Phi(F)(x^a,\pi^A,\pi_{A'})$ given by the following formula:

$$\begin{aligned}\Phi(F)(x^a,\pi^A,\pi_{A'}) &\equiv \int_{\rho(X)} F(X^{\alpha B'}\pi_{B'}; X^{\alpha B'}\rho_{B'} + \Pi^\alpha)d^2\rho \\ &= \int_{\rho(X)} F(ix^{AB'}\pi_{B'}, \pi_{A'}; ix^{AB'}\rho_{B'} + \pi^A, \rho_{A'})d^2\rho.\end{aligned} \tag{6.5}$$

Differentiation of equation (6.5) immediately gives the field equation $M'\Phi(F) = 0$, as required.

Note that, depending on the properties of the twistor functions f or F, the fields $\phi(f)$ and $\Phi(F)$ may contain many different helicities or irreducible spinor parts.

Denote by $\mathcal{O}(p,q)$ the sheaf of germs of holomorphic sections of rank p totally symmetric covariant tensors on projective twistor space $P(\mathbb{V})$, taking values in the sheaf $\mathcal{O}(q)$ (the sheaf of germs of holomorphic functions $h(\zeta)$ homogeneous of degree q in the variable ζ). Such a section is described non-projectively by a tensor with p indices: $f_{\alpha_1\alpha_2\ldots\alpha_p}(\zeta^\alpha)\theta^{\alpha_1}\otimes\theta^{\alpha_2}\otimes\ldots\otimes\theta^{\alpha_p}$, such that $f_{\alpha_1\alpha_2\ldots\alpha_p}$ is completely symmetric, holomorphic, homogeneous of integral degree $q-p$ in the variable ζ^α and such that $0=\zeta^{\alpha_1}f_{\alpha_1\alpha_2\ldots\alpha_p}(\zeta^\alpha)$. Our main result is that the sheaf cohomology group $H^1(U,\mathcal{O}(p,2p-1))$, for p a positive integer describes the general analytic solution to our higher spin equations (the spin is $p+1/2$), for suitable domains U in twistor space. This restriction to analytic solutions is not mandatory: by replacing ordinary cohomology by C.R. cohomology one can obtain non-analytic solutions from the twistor theory. We do not discuss this further here.

A. Twistor description for the spin (3/2) case

As before we begin with the spin (3/2) case and treat the general case later. In this case the object of study is the group $H^1(U,\mathcal{O}(1,1))$. We shall use the contour integral description to get at the results. Each calculation that we do then corresponds, according to well established procedures, to an appropriate calculation using sheaf cohomology as described for example in the books of Penrose and Rindler. For $H^1(U,\mathcal{O}(1,1))$ we use functions $f_\alpha(\zeta)$, which are homogeneous of degree zero in the variable ζ. For a function $g(\zeta)$, homogeneous of degree zero, the corresponding spacetime field g_{BC} is spin one and is given by a contour integral according to the standard formula of Hughston:

$$g_{BC}(x^a)=\int_{\gamma(x)}\partial_B\partial_C g(ix^{AB'}\rho_{B'},\rho_{A'})\rho^{C'}d\rho_{C'}. \tag{6.6}$$

Here the operator ∂_B denotes the partial derivative with respect to the unprimed spinor part ζ^B of the twistor variable ζ^β. Also the one dimensional contour $\gamma(x)$ is closed, avoids the singularities of the integrand and varies smoothly with the point x.

Next we need the explicit action of the twistor operator ζ^α on the field g_{BC}. Multiplication of g by ζ gives a function homogeneous of degree one for which the corresponding field ζg is spin (3/2) and is given by the following formulas:

$$\begin{aligned}(\zeta g)^\alpha_{BCD}(x^a)&=\int_{\gamma(x)}\partial_B\partial_C\partial_D(\zeta^\alpha g)(ix^{EF'}\rho_{F'},\rho_{E'})\rho^{G'}d\rho_{G'}\\&=\int_{\gamma(x)}\Big(3\delta^\alpha_B\partial_C\partial_D g(ix^{EF'}\rho_{F'},\rho_{E'})+X^{\alpha B'}\rho_{B'}\partial_B\partial_C\partial_D g(ix^{EF'}\rho_{F'},\rho_{E'})\Big)\rho^{G'}d\rho_{G'}\\&=3\delta^\alpha_B g_{CD}-iX^{\alpha B'}\partial_{B'B}g_{CD}.\end{aligned} \tag{6.7}$$

In equation (6.7), the twistor $\delta^\alpha_B\equiv(\delta_B{}^A,0)$. Also we have used the fact that inside the twistor integral the operator ∂_b is represented by the operator $i\rho_{B'}\partial_B$. Applying these results to the indexed function $f_\beta(\zeta)$, allowing for the extra index, equations (6.6) and (6.7) become

$$\phi_{\beta CD}(x^a)=\int_{\gamma(x)}\partial_C\partial_D f_\beta(ix^{AB'}\rho_{B'},\rho_{A'})\rho_{C'}d\rho^{C'}. \tag{6.8}$$

$$(\zeta\phi)^\alpha_{\beta CDE}=3\delta^\alpha_C\phi_{\beta DE}-iX^{\alpha C'}\partial_{C'C}\phi_{\beta DE}. \tag{6.9}$$

If we now impose the condition $\zeta^\alpha f_\alpha=0$, we see that the trace over the twistor indices of equation (6.9) must vanish. From the right hand side of equation (6.9) this gives the following condition:

$$\begin{aligned}0&=3\phi_{CDE}-iX^{\alpha C'}\partial_{CC'}\phi_{\alpha DE}=3\phi_{CDE}+x^{AC'}\partial_{CC'}\phi_{ADE}-i\partial_{A'C}\phi^{A'}_{DE}\\&=3\phi_{CDE}-2\phi_{CDE}+\partial_{CC'}(x^{AC'}\phi_{ADE}-i\phi^{C'}_{DE})=\psi_{CDE}-2\epsilon_{C(D}\psi_{E)}-i\partial_{A'C}\psi^{A'}_{DE},\end{aligned} \tag{6.10}$$

with $\phi_{\alpha BC}=(\phi_{ABC},\,\phi^{A'}_{DE})$, $\phi_{ABC}=\psi_{ABC}+\epsilon_{A(B}\psi_{C)}$ and $\psi^{A'}_{DE}\equiv\phi^{A'}_{DE}+ix^{A'C}\phi_{CDE}$, where the spinors $\phi^{A'}_{DE}$, ψ_{ABC} and $\psi^{A'}_{DE}$ are completely symmetric.

Equation (6.10) gives in particular the equations $\psi_{CDE}=i\partial_{A'(C}\psi^{A'}_{DE)}$ and $\psi_A=(-i/3)\partial^{BB'}\psi_{ABB'}$. Hence the field ϕ_{ABC} is completely determined given the field $\psi^{A'}_{AB}$. Once the field ϕ_{ABC} is known, the field $\phi^{A'}_{AB}$ can be recovered from the formula: $\phi^{A'}_{AB}(x^e)=\psi^{A'}_{AB}(x^e)-ix^{A'C}\phi_{CAB}(x^e)$. So knowledge of the single field $\psi^{A'}_{AB}$ (and its derivatives) is completely equivalent to knowledge of the original field $\phi_{\alpha BC}$. Finally the field equation for the field $\phi_{\alpha BC}$ is just the standard zrm field equation $\partial^{B'B}\phi_{\alpha BC}=0$. This equation immediately implies (by straightforward differentiation) the equation $\partial^{B(A'}\psi^{B')}_{BC}=0$ and conversely it is seen easily that the equation $\partial^{B(A'}\psi^{B')}_{BC}=0$ implies the field equation $\partial^{B'B}\phi_{\alpha BC}=0$. So we have established that the cohomology group $H^1(U,\mathcal{O}(1,1))$, for appropriate domains U in projective twistor space is isomorphic to the space of solutions of the equation $\partial^{B(A'}\psi_{BC}^{B')}=0$, on the corresponding domain in space-time, with the field $\psi_{BC}^{B'}$ being totally symmetric. Finally from the definition of the field $\psi_{AB}^{A'}$ and from equation (6.8), we have the following contour integral expression for the field $\psi_{AB}^{A'}$:

$$\psi_{AB}^{A'}(x^a)=\int_{\gamma(x)}\partial_A\partial_B f^{A'}(ix^{AB'}\rho_{B'},\rho_{A'})+ix^{A'C}\partial_A\partial_B f_C(ix^{AB'}\rho_{B'},\rho_{A'})\rho_{C'}d\rho^{C'}. \tag{6.11}$$

Contracting equation (6.11) through with $\pi^A\pi^B\pi_{A'}$, and using Cauchy's integral formula to reduce the integral to a one dimensional integral, we find complete agreement with equation (6.5), where the function $F(z,\zeta)\equiv z^\alpha f_\alpha$ and the field $\Phi(F)(x^a,\pi^A,\pi_{A'})$ is then a fixed constant multiple of the field $\psi_{AB}^{A'}(x^e)\pi^A\pi^B\pi_{A'}$. So we have the shown the

Proposition VI.2 *There exists an isomorphism between the sheaf cohomology group $H^1(U,\mathcal{O}(1,1))$ and the space of holomorphic solutions of the equation $\partial_{AA'}\phi_{B'}^{AB}=0$.*

B. The general spin case

Consider the interpretation of the twistor cohomology group $H^1(U,\mathcal{O}(p,q))$. We assume that the integers p and $s\equiv q-p+2$ are positive (later we shall restrict further by requiring that $s>p$). As discussed above, a representative element is an indexed function $f_{\alpha_1\alpha_2\ldots\alpha_p}(\zeta)$, which is completely symmetric, holomorphic, and homogeneous of degree $q-p$ in the twistor variable ζ and such that $0=\zeta^{\alpha_1}f_{\alpha_1\alpha_2\ldots\alpha_p}(\zeta)$. The unprimed spinor field corresponding to the function $f_{\alpha_1\alpha_2\ldots\alpha_p}$ may be given as follows:

$$\phi(Z^\alpha,x^a,\pi^A)\equiv\int_{\gamma(X)}F(Z^\alpha,X^{\alpha B'}\rho_{B'}+\Pi^\alpha)d^2\rho. \tag{6.12}$$

Here we have put $F(Z,\zeta)\equiv Z^{\alpha_1}Z^{\alpha_2}\ldots Z^{\alpha_p}f_{\alpha_1\alpha_2\ldots\alpha_p}(\zeta)$. The field $\phi(Z^\alpha,x^a,\pi^A)$ is a homogenous polynomial of degree (p,s) in the pair of variables (Z^α,π^A), with coefficients obeying the zrm field equation by lemma (VI.1). Our first objective is to obtain a formula for the action of the twistor variable ζ^β on such a field. Denote the result of this action by $(\zeta\phi)^\beta$. Then for this field we have the expression: $(\zeta\phi)^\beta(Z,x,\pi)=X^{\beta C'}\phi_{C'}(Z,x,\pi)+\Pi^\beta\phi$, where the field $\phi_{B'}$ is given by the following formula:

$$\phi_{C'}(Z,x,\pi)=\int_{\gamma(X)}\rho_{C'}F(Z^\alpha,X^{\alpha B'}\rho_{B'}+\Pi^\alpha)d^2\rho. \tag{6.13}$$

Multiplying both sides of equation (6.13) by $s+1$, we manipulate equation (6.13) as follows:

$$\begin{aligned}(s+1)\phi_{C'}(Z,x,\pi)&=\pi_A\partial_\pi^A\int_{\gamma(X)}\rho_{C'}F(Z^\alpha,X^{\alpha B'}\rho_{B'}+\Pi^\alpha)d^2\rho\\&=\pi^A\int_{\gamma(X)}\rho_{C'}((\partial_\zeta)_AF)(Z^\alpha,X^{\alpha B'}\rho_{B'}+\Pi^\alpha)d^2\rho=-i\pi^A\partial_{AC'}\phi(Z,x,\pi).\end{aligned} \tag{6.14}$$

Summarizing we have established the formula

$$(\zeta\phi)^\beta=\pi^A\left(-i(s+1)^{-1}X^{\beta A'}\partial_a+\delta_A^\beta\right)\phi. \tag{6.15}$$

Next, we consider the condition $0=\zeta^{\alpha_1}f_{\alpha_1\alpha_2\ldots\alpha_p}(\zeta)$, which is equivalent to the condition $0=\zeta^\alpha(\partial_Z)_\alpha F$. When applied to equation (6.15), with the field ϕ replaced by the field $(\partial_Z)_\alpha\phi$ this implies

$$0=\pi^A\left(iX^{\beta A'}\partial_a-(s+1)\delta_A^\beta\right)(\partial_Z)_\beta\phi=-\pi^Ax^{BA'}\partial_a(\partial_Z)_B\phi-(s+1)\pi^B(\partial_Z)_B\phi+i\pi^A\partial_a\partial_Z^{A'}\phi. \tag{6.16}$$

Next we perform a change of variables and write the twistor $Z^\alpha=(Z^A,Z_{A'})$ as follows: $Z^\alpha=X^{\alpha B'}z_{B'}+z^B\delta_B^\alpha$, so we have $Z^A=ix^az_{A'}+z^A$ and $Z_{A'}=z_{A'}$. Then the field ϕ becomes a function of the variables (x^a,π^A,z^α), where $z^\alpha\equiv(z^A,z_{A'})$. Under this change of variables, we make the derivative replacements: $\partial_a\mapsto\partial_a-iz_{A'}(\partial_z)_A$, $\partial_\pi^A\mapsto\partial_\pi^A$, $(\partial_Z)_A\mapsto(\partial_z)_A$ and $\partial_Z^{A'}\mapsto\partial_z^{A'}-ix^a(\partial_z)_A$. Equation (6.16) then becomes

$$0 = -\pi^A x^{BA'} \left(\partial_a - iz_{A'}\partial_A\right) \partial_B\phi - (s+1)\pi^B\partial_B\phi + i\pi^A \left(\partial_a - iz_{A'}\partial_A\right) \left(\partial^{A'} - ix^{BA'}\partial_B\right)\phi$$
$$= \pi^A\partial_A(-s+1+\gamma')\phi + i\pi^A\partial_a\partial^{A'}\phi. \tag{6.17}$$

Here we have put $\partial_A \equiv (\partial_z)_A$, $\partial^{A'} \equiv (\partial_z)^{A'}$ and $\gamma' \equiv z_{A'}\partial^{A'}$. Also define $\gamma \equiv z^A\partial_A$. Note that the field ϕ obeys the equation $(\gamma + \gamma' - p)\phi = 0$.

Equation (6.17) may be regarded as giving a partial propagation of the field ϕ in the z^A directions. But we also have the field equation obeyed by ϕ, which in terms of the original variables is the zrm equation $\partial_\pi^A\partial_a\phi = 0$. In terms of the new variables this equation becomes the equation:

$$0 = \partial_\pi^A\partial_a\phi - iz_{A'}\partial_\pi^A\partial_A\phi. \tag{6.18}$$

Removing the factor π^A from equation (6.17), we get the following equation, valid for some field χ:

$$\partial_A(-s+1+\gamma')\phi + i\partial_a\partial^{A'}\phi = \pi_A\chi. \tag{6.19}$$

Applying the operator ∂_π^A to (6.19), we get

$$(s+1)\chi = \partial_\pi^A\partial_A(-s+1+\gamma')\phi + i\partial_\pi^A\partial_a\partial^{A'}\phi. \tag{6.20}$$

Then, applying the operator $i\partial^{A'}$ to (6.18), gives

$$0 = (\gamma'+2)\partial_\pi^A\partial_A\phi + i\partial^{A'}\partial_\pi^A\partial_a\phi, \tag{6.21}$$

and, finally, subtracting equation (6.20) from equation (6.21) yields

$$\chi = -\partial_\pi^A\partial_A\phi. \tag{6.22}$$

From equations (6.19), (6.21) and (6.22), we get the following relation:

$$(2+\gamma')(s-1-\gamma')\partial_A\phi = i(2+\gamma')\partial_a\partial^{A'}\phi - i\pi_A\partial^{B'}\partial_\pi^B\partial_b\phi, \tag{6.23}$$

and we also have a field equation which is obtained from equation (6.18) by contraction with the spinor $z^{A'}$:

$$0 = z^{B'}\partial_\pi^B\partial_b\phi. \tag{6.24}$$

For the present purposes, equations (6.23) and (6.24) are the key equations. Note that equation (6.17) is a consequence of equation (6.23), since the operator $(2+\gamma')$ is invertible. To analyze these equations most easily, we henceforth restrict to the case that the quantity $s-p$ is positive. The quantity $\partial_A\phi$ and the right hand side of equation (6.23) are each sums of terms homogeneous of degrees 0 to $p-1$ in the variable $z_{A'}$. So with $s-p$ positive, the operator $(s-\gamma'-1)$ has a well defined inverse acting on such quantities. So we may rewrite equation (6.23) as follows:

$$\partial_A\phi = i(s-1-\gamma')^{-1}(\partial_a\partial^{A'}\phi - (2+\gamma')^{-1}\pi_A\partial^{B'}\partial_\pi^B\partial_b\phi). \tag{6.25}$$

We need to check the compatibility of equations (6.24) and (6.25). First we show that the ∂_A derivative of the righthand side of equation (6.24) vanishes modulo the equations (6.24) and (6.25):

$$\partial_A z^{B'}\partial_\pi^B\partial_b\phi = z^{B'}\partial_\pi^B\partial_b\partial_A\phi = iz^{B'}\partial_\pi^B\partial_b(s-1-\gamma')^{-1}(\partial_a\partial^{A'}\phi - (2+\gamma')^{-1}\pi_A\partial^{C'}\partial_\pi^C\partial_c\phi)$$
$$= i(s-\gamma')^{-1}(1+\gamma')^{-1}\left((1+\gamma')z^{B'}\partial_\pi^B\partial_b\partial_a\partial^{A'}\phi - z^{B'}\partial_\pi^B\partial_b\pi_A\partial^{C'}\partial_\pi^C\partial_c\phi\right)$$
$$= i(s-\gamma')^{-1}(1+\gamma')^{-1}(-(1+\gamma')\partial_\pi^B\partial_b\partial_a\epsilon^{B'A'}\phi - z^{A'}\partial_a\partial^{B'}\partial_\pi^B\partial_b\phi + \pi_A\epsilon^{B'C'}\partial_\pi^B\partial_b\partial_\pi^C\partial_c\phi)$$
$$= (i/2)(s-\gamma')^{-1}(1+\gamma')^{-1}(-(1+\gamma')(\partial_\pi)_A\Box\phi - 2z^{B'}\partial_a\ \partial^{A'}\partial_\pi^B\partial_b\phi - 2\gamma'\epsilon^{A'B'}\partial_a\partial_\pi^B\partial_b\phi)$$
$$= (i/2)(\gamma'-s)^{-1}((\partial_\pi)_A\Box\phi + 2\partial_\pi^B\epsilon^{A'B'}\partial_a\partial_b\phi)$$
$$= (i/2)(\gamma'-s)^{-1}((\partial_\pi)_A\Box\phi + \partial_\pi^B\epsilon_{AB}\Box\phi) = 0.$$

Finally we need to show that applying the operator ∂^A to the righthand side of equation (6.25) gives zero. So it is sufficient to show that the quantity $(2+\gamma')\partial^A\partial_a\partial^{A'}\phi + \pi^A\partial_A\partial^{B'}\partial_\pi^B\partial_b\phi$ vanishes, modulo equations (6.24) and (6.25). We see this as follows:

$$
\begin{aligned}
(2+\gamma')\partial^A\partial_a\partial^{A'}\phi + \pi^A\partial_A\partial^{B'}\partial_\pi^B\partial_b\phi &= (3+\gamma')\partial^A\partial_a\partial^{A'}\phi + \partial^{B'}\partial_\pi^B\partial_b\pi^A\partial_A\phi \\
&= \partial_a\partial^{A'}(2+\gamma')\partial^A\phi + \partial^{B'}\partial_\pi^B\partial_b\pi^A\partial_A\phi \\
&= i\partial^a\partial_{A'}(s-1-\gamma')^{-1}((2+\gamma')\partial_{AC'}\partial^{C'}\phi - \pi_A\partial^{B'}\partial_\pi^B\partial_b\phi) + i\partial^{B'}\partial_\pi^B\partial_b(s-1-\gamma')^{-1}\pi^A\partial_a\partial^{A'}\phi \\
&= -i(s-2-\gamma')^{-1}((\pi^A\partial^{A'}\partial_a)(\partial^{B'}\partial_\pi^B\partial_b)\phi - (\partial^{B'}\partial_\pi^B\partial_b)(\pi^A\partial^{A'}\partial_a)\phi) = 0.
\end{aligned}
$$

Thus equations (6.24) and (6.25) are integrable. If we now write $\phi = \sum_{k=0}^p \phi_k$, with $\gamma\phi_k = k\phi_k$, for $0 \le k \le p$, we get from equations (6.24) and (6.25) give the following equations:

$$0 = z^{B'}\partial_\pi^B\partial_b\phi_k, \tag{6.26}$$

$$\partial_A\phi_{k+1} = i(s-p+k)^{-1} \quad (\partial_a\partial^{A'}\phi_k - (1+p-k)^{-1}\pi_A\partial^{B'}\partial_\pi^B\partial_b\phi_k). \tag{6.27}$$

Equation (6.26) is valid for $0 \le k \le p$. Equation (6.27) is valid for $0 \le k < p$. Note that equation (6.27) entails a recursive formula for the quantities ϕ_k:

$$\phi_{k+1} = i(k+1)^{-1}(s-p+k)^{-1}(z^A\partial_a\partial^{A'}\phi_k - (1+p-k)^{-1}z^A\pi_A\partial^{B'}\partial_\pi^B\partial_b\phi_k), \qquad \text{for } 0 \le k < p. \tag{6.28}$$

Equation (6.28) shows explicitly that the entire field ϕ is uniquely determined by the field ϕ_0. The integrability of equations (6.24) and (6.25) shows that the system of field equations (6.18) for the field $\phi(x^a, \pi^A, z^\alpha)$ is completely equivalent to the single equation $z^{B'}\partial_\pi^B\partial_b\phi_0 = 0$, for the field $\phi_0(x^a, \pi^A, z_{A'})$, in the case $s > p$.

Summarizing, we have outlined a proof of the

Theorem VI.3 *For any pair of positive integers (p, t), there is an isomorphism between the twistor sheaf cohomology group $H^1(U, \mathcal{O}(p, 2p+t-2))$ and the space of holomorphic solutions of the field equation $z^{B'}\partial_\pi^B\partial_b\phi = 0$, where the field ϕ is homogeneous of degrees $(p+t, p)$ in the variables $(\pi^A, z_{A'})$.*

In particular, we have the

Corollary VI.4 *The space of holomorphic solutions of the equations $\partial_{A(A'}\psi^{AB\ldots C}_{B'\ldots C')} = 0$ for spinor fields with $m+1$ unprimed and m primed indices is isomorphic to the twistor sheaf cohomology group $H^1(U, \mathcal{O}(m+1, m))$.*

We would like to make several remarks at this point. Firstly, although we have shown the existence of the twistor correspondence for several different kinds of fields (i.e., with different index structures when considered as spinor fields on space-time) it is only the fields with homegeneity $(p+1, p)$ that can consistently propagate on a curved manifold. All other fields suffer from the existence of consistency conditions. The twistor treatment in this work is to some extent new in that we use an affine twistor space. This allows to incorporate all homogeneities into one formula (see e.g., equation (6.4) in comparison to (6.6)).

C. The group representation

We observe that there are natural operators acting on the twistor cohomology groups $H^1(U, O(p, q))$. Indeed consider the operators $P^{\alpha\beta}$ and $Q_{\alpha\beta}$ and E^α_β, which act on a representative function $F(z, \zeta)$, homogeneous of degrees $(p, q-p)$ in the twistor variables (z, ζ) and obeying the differential equation $\zeta \cdot \partial_z F = 0$, as follows:

$$P(F) = (z \wedge \zeta)F, \tag{6.29}$$

$$Q(F) = (\partial_z \wedge \partial_\zeta)F, \tag{6.30}$$

$$E(F) = (z \otimes \partial_z + \zeta \otimes \partial_\zeta + \delta)F. \tag{6.31}$$

In equation (6.31), the operator δ is the Kronecker delta tensor acting on the representative F by multiplication. Note that each of these operators commutes with the operator $\zeta \cdot \partial_z$. The operator P gives a map from $H^1(U, O(p, q))$ to $H^1(U, O(p+1, q+2))$, the operator Q maps $H^1(U, O(p, q))$ to $H^1(U, O(p-1, q-2))$ and the operator E maps $H^1(U, O(p, q))$ to itself. These operators generate a Lie algebra under commutation. Indeed, by direct calculation, we have the following commutation relations:

$$
\begin{aligned}
[P^{\alpha\beta}, P^{\gamma\delta}] &= 0, & [Q_{\alpha\beta}, Q_{\gamma\delta}] &= 0, \\
[P^{\alpha\beta}, Q_{\gamma\delta}] &= -\delta^{[\alpha}_{[\gamma}E^{\beta]}_{\delta]}, & [E^\gamma_\delta, P^{\alpha\beta}] &= -2\delta^{[\alpha}_\delta P^{\beta]\gamma}, \\
[E^\gamma_\delta, Q_{\alpha\beta}] &= 2\delta^\gamma_{[\alpha}Q_{\beta]\delta}, & [E^\gamma_\delta, E^\alpha_\beta] &= \delta^\alpha_\delta E^\gamma_\beta - \delta^\gamma_\beta E^\alpha_\delta.
\end{aligned}
\tag{6.32}
$$

A dimension count gives dimension twenty-eight for this algebra, six for each of the operators P and Q and sixteen for the operator E. The operator E generates the complex general linear algebra $GL(4,\mathbb{C})$. The algebra $GL(4,\mathbb{C})$ in turn is isomorphic to the conformal orthogonal algebra $CO(6,\mathbb{C})$ (the orthogonal algebra together with a dilation). Adding in the operators P and Q to this algebra gives the complete algebra of $O(8,\mathbb{C})$, regarded as the conformal algebra associated to $O(6,\mathbb{C})$, with P forming the translations, Q the generator of special conformal transformations, the tracefree part of E generating rotations and the trace of E giving the dilation. If we introduce the standard pseudo-hermitian form on twistor space of signature $(2,2)$, then this algebra has the natural real form $O(4,4)$, with the operator iE self-conjugate and Q the pseudo-hermitian conjugate of P. So we have shown that the direct sum over p of all the cohomology groups $H^1(U, O(p, 2p+t-2)$, gives, for each fixed t, a complex representation of the Lie algebra of the group $O(4,4)$. It remains an open question whether or not this representation is unitarizable. Indeed the "natural" inner product, derived from the action of section one above is not positive definite in the case of spin greater than one half. So it would seem that the representation is "naturally" defined on a space with a "natural" inner product, but not a Hilbert space. If one took such representations seriously, it would apparently require enlarging the framework of quantum mechanics to accomodate "negative probabilities".

Finally we note that although this algebra is most easily derived in the twistor picture, one can easily translate into the spacetime picture, using the techniques of this section. In the spacetime picture the operator E acts on the fields as the (complex) conformal algebra of spacetime. The operators P and Q and E act as follows on a field $\phi(x^a, \pi_{A'}, \pi^A)$ obeying the equation $M'\phi = 0$ and homogeneous of degree s in the spinor π^A:

$$P^{\alpha\beta}\phi = \frac{1}{(s+1)}\pi_{C'}\pi^C X^{C'[\alpha} P_C^{\beta]}\phi, \tag{6.33}$$

$$Q_{\alpha\beta}\phi = -\frac{1}{(s+1)}(\partial_\pi)_C \partial_\pi^{C'} X^C_{[\alpha} Q_{\beta]C'}\phi, \tag{6.34}$$

$$E^\alpha_\beta\phi = -iX^{D'\alpha}X^D_\beta \partial_d\phi + X^{D'\alpha}\delta_{C'\beta}\pi_{D'}\partial_\pi^{C'}\phi - X^D_\beta\delta^\alpha_C(\partial_\pi)_D(\pi^C\phi), \tag{6.35}$$

where we have used the following definitions:

$$P^\alpha_B \equiv (s+1)\delta^\alpha_B - iX^{B'\alpha}\partial_b, \quad Q_{\alpha B'} \equiv (s+1)\delta_{\alpha B'} - iX^B_\alpha\partial_b, \tag{6.36}$$

$$X^{B'\alpha} \equiv (ix^{B'A}, \delta^{B'}_{A'}), \qquad X^B_\alpha \equiv (\delta^B_A, -ix^{BA'}), \qquad \delta^\alpha_B \equiv (\delta^A_B, 0), \qquad \delta_{\alpha B'} = (0, \delta^{A'}_{B'}). \tag{6.37}$$

VII. CONCLUSION

We have presented the properties of a class of linear equations for fields with half integer spin $m+\frac{1}{2}$ which generalize the Weyl equation for a neutrino. We have shown how the equations arise as Euler-Lagrange equations for a variational principle. The equations are of hyperbolic type in the sense that the Cauchy problem is well posed and that there exists the notion of a domain of influence. The characteristics of the system are multiply sheeted. The fields propagate freely on any curved background, i.e., there are no constraints on spatial hypersurfaces to be satisfied by the Cauchy data. The solutions lie in certain Gevrey classes provided that the Cauchy data and the metric of the underlying manifold do so. We find a strong relationship between the spin of the fields and the smoothness of the metric, ranging from only $\mathcal{C}^k$ in the neutrino case up to analyticity in the limit $m \to \infty$. We analyzed the characteristic initial value problem using the formal method of exact set and showed that it is well posed in the curved background case as well as when the system is coupled to gravity via the Einstein equation. It is interesting to note, that this is a system of partial differential equations that is not symmetrically hyperbolic (unless $m = 0$) but still allows the description via an exact set. All other examples of exact sets so far have been systems of equations which were also symmetrically hyperbolic. This implies that those two characterizations are not mutually included one in the other. We have given the general solution of the equations in Minkowski space by first solving the equation $\Box^{m+1}\phi = 0$ using Fourier methods and then deriving the Fourier representation for positive frequency fields. Finally, we presented a twistor correspondence between the the space of holomorphic solutions and sheaf cohomology groups on projective twistor space.

The solution space of the equations in flat space is a representation space of the Poincaré group. In contrast to the case of the massless free fields, however, this representation is reducible unless $m = 0$. This can be easily seen from the fact that the entire solution space for spin $2m-1$ is mapped injectively into the solution space for spin $2m+1$ by the operator L. The solution space is also a representation space for the conformal group. It is not yet known whether this representation is irreducible.

It would be interesting to find similar classes of consistent higher spin equations for integer spin generalizing the Maxwell equations. So far attempts have been unsuccesful.

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APPENDIX A: GEVREY CLASSES OF FUNCTIONS

Essential in the proof of existence and uniqueness of solutions of non-strictly hyperbolic systems of partial differential equations is the notion of Gevrey classes of functions. These are sets of $\mathcal{C}^\infty$-functions, labelled by a real number $\alpha \geq 1$ which in some way interpolate between analytic functions ($\alpha = 1$) and functions which are only $\mathcal{C}^k$ (conventionally made to correspond to the case $\alpha = \infty$, see below).

Definition A.1 *Let S be an open set in $\mathbb{R}^l$, $p \geq 1$ and $\alpha \geq 1$. Then $\gamma_p^{(\alpha)}(S)$ is the set of functions $f : S \to \mathbb{C}$ such that*

$$\sup_\sigma \frac{1}{(1+|\sigma|)^\alpha} ||D^\sigma f, S||_p^{\frac{1}{|\sigma|}} < \infty, \tag{A1}$$

where σ is a multi-index $\sigma = (\sigma_1, \ldots, \sigma_l)$, $|\sigma| = \sigma_1 + \ldots + \sigma_l$ and $||f, S||_p$ is the usual L^p-norm of f.

Similar classes are defined to characterise the behaviour of the functions with time.

Definition A.2 *Let $\Sigma := [0, T] \times S$ be a strip in $\mathbb{R}^{l+1}$, $p \geq 1$, $n \geq 1$ and $\alpha \geq 1$. Then $\gamma_p^{n,(\alpha)}(\Sigma)$ is the set of all functions $f : \Sigma \to \mathbb{C}$ such that*

$$\sup_{\sigma,\beta,x^0} \frac{1}{(1+|\sigma|)^\alpha} ||D^{\sigma+\beta} f, S_t||_p^{\frac{1}{|\sigma|}} < \infty, \tag{A2}$$

where, again, σ and β are multi indices with $\sigma_0 = 0$ (i.e., σ refers only to "spatial" derivatives) and $0 \leq x^0 \leq T$. S_t is the slice $x^0 = t$, as usual.

We extend the definition with $\alpha = \infty$ by the rule that $(1 + |\sigma|)^\alpha = 1$ for $|\sigma| = 0$ and $(1 + |\sigma|)^\alpha = 0$ otherwise. Then $\gamma_p^{(\infty)}(S)$ is equal to the function space $L_p(S)$ and $\gamma_p^{n,(\infty)}(\Sigma)$ is equal to the set of all functions for which $||D^\beta f, S_t||$ is a bounded function of t for all β with $|\beta| \leq n$.

For $p = \infty$, $l = 1$ and for real valued functions this is the classical case of Gevrey [6]. In that case $\gamma_\infty^{(\alpha)}(S)$ is an algebra which is closed under the composition of its elements. A similar property holds in the general case (see [8]).

Here are some basic properties of the Gevrey classes: they grow with α; if $\alpha_1 \leq \alpha_2$ then $\gamma_p^{n,(\alpha_1)}(\Sigma) \subset \gamma_p^{n,(\alpha_2)}(\Sigma)$. Also $\gamma_p^{n,(\alpha)}(\Sigma) \subset \gamma_p^{m,(\alpha)}(\Sigma)$ if $m \geq n$. If $\alpha = 1$ the classes consist of functions which are analytic in $x^1, \ldots, x^l$, but for $\alpha \neq 1$ one can show that there exists a partition of unity into elements of the classes with arbitrarily small support; functions with compact support are not necessarily zero.

The essential qualitative distinction within the Gevrey classes seems to be between the case $\alpha = 1$ and $\alpha > 1$, the latter case permitting domains of influence and thus allowing the study of wave propagation. This indicates the hyperbolic character of the equations under consideration. The second distinction is between the cases of finite α and $\alpha = \infty$. Infinite α permits the existence of only a finite number of derivatives and thus the appearance of shocks is possible indicating the strictly hyperbolic case.

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