Geometric evolution equations for hypersurfaces

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1 Introduction

Let $F_0 : \mathcal{M}^n \to (N^{n+1}, \bar{g})$ be a smooth immersion of a hypersurface $\mathcal{M}_0^n = F_0(\mathcal{M}^n)$ in a smooth Riemannian manifold $(N^{n+1}, \bar{g})$. We study one–parameter families $F : \mathcal{M}^n \times [0,T] \to (N^{n+1}, \bar{g})$ of hypersurfaces $\mathcal{M}_t^n = F(\cdot, t)(\mathcal{M}^n)$ satisfying an initial value problem

$$\frac{\partial F}{\partial t}(p,t) = -f\nu(p,t), \qquad p \in \mathcal{M}^n,\ t \in [0,T], \tag{1.1}$$

$$F(p,0) = F_0, \qquad p \in \mathcal{M}^n, \tag{1.2}$$

where $\nu(p,t)$ is a choice of unit normal at $F(p,t)$ and $f(p,t)$ is some smooth homogeneous symmetric function of the principal curvatures of the hypersurface at $F(p,t)$.
We will consider examples where $f = f(\lambda_1, \cdots, \lambda_n)$ is monotone with respect to the principal curvatures $\lambda_1, \cdots, \lambda_n$ such that (1.1) is a nonlinear parabolic system of second order. Although there are some similarities to the harmonic map heatflow, this deformation law is more nonlinear in nature since the leading second order operator depends on the geometry of the solution at each time rather than the initial geometry. There is a very direct interplay between geometric properties of the underlying manifold $(N^{n+1}, \bar{g})$ and the geometry of the evolving hypersurface which leads to applications both in differential geometry and mathematical physics.
Here we investigate some of the general properties of (1.1) and then concentrate on the mean curvature flow $f = -H = -(\lambda_1 + \cdots + \lambda_n)$, the inverse mean curvature flow $f = H^{-1}$ and fully nonlinear flows such as the the Gauss curvature flow $f = -K = -(\lambda_1 \cdots \lambda_n)$ or the harmonic mean curvature flow, $f = -(\lambda_1^{-1} + \cdots + \lambda_n^{-1})^{-1}$. We discuss some new developments in the mathematical understanding of these evolution equations and include applications such as the use of the inverse mean curvature flow for the study of asymptotically flat manifolds in General Relativity.

In section 2 we introduce notation for the geometry of hypersurfaces in Riemannian manifolds and derive the crucial commutator relations for the second derivatives of the second fundamental form.
In section 3 we study the general evolution equation (1.1) and obtain evolution equations for metric, normal, second fundamental form and related geometric quantities. We discuss the parabolic nature of the evolution equations, a shorttime existence result and introduce the main examples.
We study the mean curvature flow in section 4. In this case the evolution law is quasilinear and the knowledge of the flow is more advanced than for all other cases. We give some examples of known results concerning regularity, longtime existence and asymptotic behaviour. In particular we discuss the formation of singularities and give an update of recent new results (joint with C.Sinestrari) concerning the classification of singularities in the mean convex case. The section concludes with an isoperimetric estimate for the one-dimensional case, ie the curve shortening flow.
Section 5 deals with fully nonlinear flows such as the Gauss curvature flow and the harmonic mean curvature flow. Without proof we review in particular results of Ben Andrews concerning an elegant proof of the 1/4-pinching theorem, the affine mean curvature flow, and a conjecture of Firey on the asymptotics of the Gauss curvature flow.
The inverse mean curvature flow is discussed in section 6. We explain the basic properties of this flow in its classical form relating it to the Willmore energy and Hawking mass of a twodimensional surface. In view of these properties the inverse mean curvature flow is particularly interesting in asymptotically flat 3-manifolds which appear as models for isolated gravitating systems in General Relativity. It is briefly explained how in recent joint work with T.Ilmanen an extended notion of the inverse mean curvature flow was used to prove a Riemannian version of the so called Penrose inequality for the total energy of an isolated gravitating system represented by an asymptotically flat 3-manifold.
While the first part of this article just described stems from lectures given by the first author at the CIME meeting at Cetraro 1996, the last section of the article is a previously unpublished part of the doctoral dissertation of Alexander Polden. It provides a selfcontained proof of shorttime existence for a variety of geometric evolution equations including hypersurface evolutions as above, conformal deformations of metrics and higher order flows such as the L^2-gradient flow for the Willmore functional.
The author wishes to thank the organisers of the Cetraro meeting for the opportunity to participate in this stimulating conference triggering joint work with Tom Ilmanen on inverse mean curvature flow, as well as for their patience in waiting for this manuscript.

2 Hypersurfaces in Riemannian manifolds

Let $(N^{n+1}, \bar{g})$ be a smooth complete Riemannian manifold without boundary. We denote by a bar all quantities on N, for example by $\bar{g} = \{\bar{g}_{\alpha\beta}\}$, $0 \leq \alpha, \beta \leq n$, the metric, by $\bar{y} = \{\bar{y}^\alpha\}$ coordinates, by $\bar{\Gamma} = \{\bar{\Gamma}^\gamma_{\alpha\beta}\}$ the Levi-Civita connection, by $\bar{\nabla}$ the covariant derivative and by $\overline{\mathrm{Riem}} = \{\overline{\mathrm{Riem}}_{\alpha\beta\gamma\delta}\}$ the Riemann curvature tensor. Components are sometimes taken with respect to the tangent vectorfields $(\partial/\partial y^\alpha)$, $0 \le \alpha \le n$ associated with a local coordinate chart $y = \{y^\alpha\}$ and sometimes with respect to a moving orthonormal frame $\{e_\alpha\}$, $0 \le \alpha \le n$, where $\bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. We write $\bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\}$ for the inverse of the metric and use the Einstein summation convention for the sum of repeated indices. The Ricci curvature $\bar{\mathrm{Ric}} = \{\bar{R}_{\alpha\beta}\}$ and scalar curvature $\bar{\mathrm{R}}$ of $(N^{n+1}, \bar{g})$ are then given by

$$\bar{R}_{\alpha\beta} = \bar{g}^{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta},$$

and the sectional curvatures (in an orthonormal frame) are given by $\bar{\sigma}_{\alpha\beta} = \bar{R}_{\alpha\beta\alpha\beta}$.

Now let $F : \mathcal{M}^n \to N^{n+1}$ be a smooth hypersurface immersion. For simplicity we restrict attention to closed surfaces, ie compact without boundary. The induced metric on $\mathcal{M}^n$ will be denoted by g, in local coordinates we have

$$\begin{aligned} g_{ij}(p) &= \langle \frac{\partial F}{\partial x^i}(p), \frac{\partial F}{\partial x^j}(p)\rangle_N \\ &= \bar{g}_{\alpha\beta}(F(p))\frac{\partial F^\alpha}{\partial x^i}(p)\frac{\partial F^\beta}{\partial x^j}(p), \qquad p \in \mathcal{M}^n. \end{aligned}$$

Furthermore, $\{\Gamma^i_{jk}\}$, ∇ and $\mathrm{Riem} = \{\mathrm{R}_{ijkl}\}$ with latin indices i, j, k, l ranging from 1 to n describe the intrinsic geometry of the induced metric g on the hypersurface.
If ν is a local choice of unit normal for $F(\mathcal{M}^n)$, we often work in an adapted othonormal frame $\nu, e_1, \cdots, e_n$ in a neighbourhood of $F(\mathcal{M}^n)$ such that $e_1(p), \cdots, e_n(p) \in T_p\mathcal{M}^n \subset T_pN^{n+1}$ and $g(p)(e_i(p), e_j(p)) = \delta_{ij}$ for $p \in \mathcal{M}^n$, $1 \le i, j \le n$.
The second fundamental form $A = \{h_{ij}\}$ as a bilinear form

$$A(p) : T_p\mathcal{M}^n \times T_p\mathcal{M}^n \to \mathrm{I\!R}$$

and the Weingarten map $W = \{h^i_j\} = \{g^{ik}h_{kj}\}$ as an operator

$$W : T_p\mathcal{M}^n \to T_p\mathcal{M}^n$$

are then given by

$$h_{ij} = < \bar{\nabla}_{e_i}\nu, e_j > = - < \nu, \bar{\nabla}_{e_i}e_j > .$$

In local coordinates $\{x^i\}$, $1 \le i \le n$, near $p \in \mathcal{M}^n$ and $\{y^\alpha\}$, $0 \le \alpha \le n$, near $F(p) \in N$ these relations are equivalent to the Weingarten equations

$$\begin{aligned} \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} + \bar{\Gamma}^\alpha_{\beta\delta}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\delta}{\partial x^j} &= -h_{ij}\nu^\alpha, \\ \frac{\partial \nu^\alpha}{\partial x^i} + \bar{\Gamma}^\alpha_{\beta\delta}\frac{\partial F^\beta}{\partial x^i}\nu^\delta &= h_{ij}g^{jl}\frac{\partial F^\alpha}{\partial x^l}. \end{aligned}$$

Recall that $A(p)$ is symmetric, ie W is selfadjoint, and the eigenvalues $\lambda_1(p), \cdots, \lambda_n(p)$ are called the principal curvatures of $F(\mathcal{M}^n)$ at $F(p)$. Also note that at a given point $p \in \mathcal{M}^n$ by choosing normal coordinates and then possibly rotating them we can always arrange that at this point

$$g_{ij} = \delta_{ij}, \qquad \bar{\nabla}^T_{e_i}e_j = 0, \qquad h_{ij} = h^i_j = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

The classical scalar invariants of the second fundamental form are then symmetric homogeneous polynomials in the principal curvatures:
The mean curvature is given by

$$H := tr(W) = h_i^i = g^{ij} h_{ij} = \lambda_1 + \cdots + \lambda_n,$$

the Gauss–Kronecker curvature by

$$K := det(W) = det\{h_j^i\} = \frac{det\{h_{ij}\}}{det\{g_{ij}\}} = \lambda_1 \cdot \cdots \cdot \lambda_n,$$

the total curvature by

$$|A|^2 := tr(W^t W) = h_j^i h_i^j = h^{ij} h_{ij} = g^{ik} g^{jl} h_{ij} h_{kl} = \lambda_1^2 + \cdots + \lambda_n^2,$$

and the scalar curvature (in Euclidean space $\mathbb{R}^{n+1}$) by

$$R = H^2 - |A|^2 = 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n).$$

More general, the mixed mean curvatures S_m, $1 \leq m \leq n$, are given by the elementary symmetric functions of the λ_i,

$$S_m := \sum_{i_1 < \cdots < i_m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m},$$

such that $S_1 = H$, $S_2 = (1/2)R$, $S_n = G$. Other interesting invariants include the harmonic mean curvature

$$\tilde{H} := (\lambda_1^{-1} + \cdots + \lambda_n^{-1})^{-1} = S_n / S_{n-1}$$

as well as other symmetric functions of the principal radii λ_i^{-1}. All the invariants mentioned or powers thereof are candidates for the speed f in our evolution problem (1.1).

For the purposes of analysis it is crucial to know the rules of computation involving the covariant derivatives, the second fundamental form of the hypersurface and the curvature of the ambient space. We assume the reader to have some background in differential geometry, but restate the formulas used in this article for convenience (in an adapted orthonormal frame).
The commutator of second derivatives of a vectorfield X on $\mathcal{M}^n$ is given by

$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = R_{ijlm} g^{kl} X^m,$$

and for a one-form ω on $\mathcal{M}^n$ by

$$\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = R_{ijkl} g^{lm} \omega_m.$$

More generally, the commutator of second derivatives for an arbitrary tensor involves one curvature term as above for each of the indices of the tensor. The corresponding laws of course also hold for the metric $\bar{g}$.

The curvature of the hypersurface and ambient manifold are related by the equations of Gauss

$$
\begin{aligned}
R_{ijkl} &= \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}hjk, \qquad 1 \le i,j,k,l \le n, \\
R_{ik} &= \bar{R}_{ik} - \bar{R}_{oiok} + Hh_{ik} - h_{il}h^l_k, \qquad 1 \le i,k \le n, \\
R &= \bar{R} - 2\bar{R}_{oo} + H^2 - |A|^2,
\end{aligned}
$$

and the equations of Codazzi-Mainardi

$$
\begin{aligned}
\nabla_i h_{jk} - \nabla_k h_{ij} &= \bar{R}_{ojki}, \\
\nabla_i h_{ik} - \nabla_k H &= \bar{R}_{ok}.
\end{aligned}
$$

The following commutator identities for the second derivatives of the second fundamental form were first found by Simons [48] and provide the crucial link between analytical methods and geometric properties of $\mathcal{M}^n$ and N^{n+1}. See also [47] for a derivation of the following facts from the structure equations.

Theorem 2.1 *The second derivatives of A satisfy the identities*

$$
\begin{aligned}
\nabla_k\nabla_l h_{ij} &= \nabla_i\nabla_j h_{kl} + h_{kl}h_{im}h_{mj} - h_{km}h_{il}h_{mj} + h_{kj}h_{im}h_{ml} \\
&\quad - h_{km}h_{ij}h_{ml} + \bar{R}_{kilm}h_{mj} + \bar{R}_{kijm}h_{ml} \\
&\quad + \bar{R}_{mjil}h_{km} + \bar{R}_{oioj}h_{kl} - \bar{R}_{okol}h_{ij} + \bar{R}_{mljk}h_{im} \\
&\quad + \bar{\nabla}_k\bar{R}_{ojil} + \bar{\nabla}_i\bar{R}_{oljk}.
\end{aligned}
$$

The trace of these identities plays an important role in mimimal surface theory and is of particular importance for mean curvature flow and inverse mean curvature flow:

Corollary 2.2 *The Laplacian $\Delta = \sum_i \nabla_i\nabla_i$ of the second fundamental form satisfies*

$$
\begin{aligned}
\Delta h_{ij} &= \nabla_i\nabla_j H + Hh_{im}h_{mj} - h_{ij}|A|^2 + H\bar{R}_{oioj} \\
&\quad - \bar{R}_{oo}h_{ij} + \bar{R}_{kikm}h_{mj} + \bar{R}_{kjkm}h_{im} \\
&\quad + \bar{R}_{kijm}h_{km} + \bar{R}_{mjik}h_{km} + \bar{\nabla}_k\bar{R}_{ojik} + \bar{\nabla}_i\bar{R}_{okjk}, \\
\frac{1}{2}\Delta|A|^2 &= h_{ij}\nabla_i\nabla_j H + |\nabla A|^2 + Htr(A^3) - |A|^4 \\
&\quad + Hh_{ij}\bar{R}_{oioj} - \bar{R}_{oo}|A|^2 + 2\bar{R}_{kikm}h_{mj}h_{ij} - 2\bar{R}_{kimj}h_{km}h_{ij} \\
&\quad + h_{ij}(\bar{\nabla}_k\bar{R}_{ojik} + \bar{\nabla}\bar{R}_{okjk}).
\end{aligned}
$$

Proof. By the Codazzi equations we first get

$$
\nabla_k\nabla_l h_{ij} = \nabla_k(\nabla_i h_{lj} + \bar{R}_{ojil}).
$$

Then compute from the definition of h_{ij}

$$
\begin{aligned}
\nabla_k(\bar{R}_{ojil}) &= \bar{\nabla}_k\bar{R}_{ojil} + h_{km}\bar{R}_{mjil} \\
&\quad - h_{ik}\bar{R}_{ojol} - h_{lk}\bar{R}_{ojio}
\end{aligned}
$$

and commute ∇_i and ∇_k to derive

$$\begin{aligned}\nabla_k\nabla_l h_{ij} &= \nabla_i\nabla_k h_{lj} + R_{kilm}h_{mj} + R_{kijm}h_{ml}\\ &\quad +\bar{\nabla}_k\bar{R}_{ojil} + h_{km}\bar{R}_{mjil} - h_{ik}\bar{R}_{ojol} - h_{lk}\bar{R}_{ojio}.\end{aligned}$$

Then use the Codazzi equations again to get

$$\begin{aligned}\nabla_i\nabla_k h_{lj} &= \nabla_i(\nabla_j h_{kl} + \bar{R}_{oljk})\\ &= \nabla_i\nabla_j h_{kl} + \bar{\nabla}_i\bar{R}_{oljk}\\ &\quad +h_{im}\bar{R}_{mljk} - h_{ij}\bar{R}_{olok} - h_{ik}\bar{R}_{oljo}.\end{aligned}$$

Employing the Gauss equations we finally conclude

$$\begin{aligned}\nabla_k\nabla_l h_{ij} &= \nabla_i\nabla_j h_{kl} + \bar{R}_{kilm}h_{mj} + \bar{R}_{kijm}h_{ml}\\ &\quad +\bar{R}_{mjil}h_{km} - \bar{R}_{ojol}h_{ik} - \bar{R}_{ojio}h_{kl}\\ &\quad +\bar{R}_{mljk}h_{im} - \bar{R}_{olok}h_{ij} - \bar{R}_{oljo}h_{ik}\\ &\quad +\bar{\nabla}_k\bar{R}_{ojil} + \bar{\nabla}_i\bar{R}_{oljk}\\ &\quad +h_{kl}h_{im}h_{mj} - h_{km}h_{il}h_{mj} + h_{kj}h_{im}h_{ml} - h_{km}h_{ij}h_{ml}\end{aligned}$$

and the conclusion follows from the symmetries of $\bar{R}_{\alpha\beta\gamma\delta}$.

3 The evolution equations

Let $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$ be a smooth closed hypersurfaceas as in the introduction in a smooth Riemannian manifold $(N^{n+1}, \bar{g})$, $n \geq 2$. Assume for simplicity that N, $\mathcal{M}$ are orientable and choose a unit normal field ν on $\mathcal{M}$. If $\mathcal{M}^n \subset \mathbb{R}^{n+1}$, we choose the exterior unit normal such that the mean curvature of a sphere is positive. We then consider the initial value problem (1.1), where f is a smooth, homogeneous function of the principal curvatures λ_i.

Shorttime existence for (1.1) can in general only be expected when the system is parabolic. to investigate the linearisation of (1.1), notice that due to the symmetry of f in an equivalent setting we may consider f as a function $\tilde{f}$ of the Weingarten map W or a s a function $\hat{f}$ of the second fundamental form A:

$$\tilde{f}(W) = \tilde{f}(\{h^i_j\}) = \hat{f}(A) = \hat{f}(\{h_{ij}\}) = f(\lambda_1 \cdots \lambda_n).$$

In view of the Weingarten equations the linearisation of (1.1) is then an equation of the form

$$\frac{\partial}{\partial t}G = -\frac{\partial \hat{f}}{\partial h_{ij}} g^{ik} g^{jl} \left\langle \frac{\partial^2 G}{\partial x^k \partial x^l}, \nu \right\rangle \nu + \text{lower order}.$$

Thus the "symbol"

$$\sigma^\alpha_\beta(\xi) = -\frac{\partial \hat{f}}{\partial h_{ij}}\, \xi^i \xi^j \nu^\alpha \nu^\beta$$

of the RHS is always degenerate in tangential directions, reflecting the invariance of the original equation under tangential diffeomorphisms. It is strictly positive definite in normal direction if

$$-\frac{\partial \hat{f}}{\partial h_{ij}}(p)\,\xi^i\xi^j > 0 \qquad \forall\, 0 \neq \xi \in \mathbb{R}^n, \quad p \in \mathcal{M}^n,$$

or equivalently

$$-\frac{\partial f}{\partial \lambda_i}(p) > 0 \qquad \forall\, 1 \leq i \leq n, \quad p \in \mathcal{M}^n. \tag{3.1}$$

The problem with the degeneracy in tangential direction can be overcome in various ways: In [30] Hamilton solves degenerate parabolic equations satisfying an integrability condition. In our case the normal projection $\Pi^N : T_pN^{n+1} \to T_p\mathcal{M}^n$ yields an integrability condition since

$$\frac{\partial}{\partial t}F = f\,\nu \in \operatorname{Kernel}\Pi^N,$$

such that Hamilton's result applies when (3.1) holds. The other approach consists in the choice of some vectorfield transversal to the initial surface. This breaks the gauge invariance of the equation and changes (1.1) to a scalar uniformly parabolic equation provided (3.1) holds. This approach was originally used by DeTurck [14] for the Ricci flow and has been used for the evolution of hypersurfaces in [33], [16]. In the last chapter of the present paper the second author gives a selfcontained proof of shorttime existence for a large class of geometric evolution equations including equations of higher order. For our purposes we note:

Theorem 3.1 *If $F_0 : \mathcal{M}^n \to (N^{n+1}, \bar{g})$ is a smooth, closed hypersurface such that*

$$-\frac{\partial f}{\partial \lambda_i}(p) > 0, \qquad 1 \leq i \leq n, \tag{3.2}$$

holds everywhere on $F_0(\mathcal{M}^n)$, then (1.1) has a smooth solution at least on some short time interval $[0,T)$, $T > 0$.

Examples. i) In the case of mean curvature flow $f = -H$ we have $-(\partial f/\partial\lambda_i) = 1$ and the flow admits a shorttime solution for any smooth initial data.
ii) For Gauss curvature flow $f = -G$ we get $-(\partial f/\partial\lambda_i) = \lambda_i^{-1}G$ and we have shorttime existence if the initial data are convex. More generally, the elementary symmetric functions $-f = S_m$, $1 \leq m \leq n$, satisfy $-(\partial f/\partial\lambda_i) > 0$ on the convex cone $\Gamma_m = \{\lambda \in \mathbb{R}^n | S_l(\lambda) > 0, 1 \leq l \leq m\}$, yielding shorttime existence for corresponding initial data.
iii) The quotients $Q_{k,l} = S_k/S_l$ for $1 \leq l < k \leq n$ again satisfy (3.1) on Γ_k. In particular, this yields a shorttime existence result for the harmonic mean curvature flow on convex initial data, since $-f = \tilde{H} = S_n/S_{n-1}$.
iv) The inverse mean curvature flow with $f = H^{-1}$ satisfies $-(\partial f/\partial\lambda_i) = H^{-2}$, yielding shorttime existence of a classical smooth solution for any initial data of positive mean curvature.

Working in the class of surfaces where shorttime existence is guaranteed, the interesting task is now to understand the longterm change in the shape of solutions, and to characterise their asymptotic behaviour both for large times and near singularities. For this purpose evolution equations have to be established for all relevant geometric quantities, in particular for the second fundamental form.

Theorem 3.2 *On any solution $\mathcal{M}_t^n = F(\cdot,t)(\mathcal{M}^n)$ of (1.1) the following equations hold:*

(i) $\frac{\partial}{\partial t}g_{ij} = 2fh_{ij}$,

(ii) $\frac{\partial}{\partial t}(d\mu) = fH(d\mu)$,

(iii) $\frac{\partial}{\partial t}\nu = -\nabla f$,

(iv) $\frac{\partial}{\partial t}h_{ij} = -\nabla_i\nabla_j f + f(h_{ik}h_j^k - \bar{R}_{oioj})$,

(v) $\frac{\partial}{\partial t}H = -\Delta f - f(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\nu,\nu))$.

Here $d\mu$ is the induced measure on the hypersurface and Δ is the Laplace-Beltrami operator with respect to the time-dependent induced metric on the hypersurface.

Notice that $-\Delta f - f(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\nu,\nu)) = Jf$ is the Jacobi operator acting on f, as is wellknown from the second variation formula for the area.
Proof. The computations are best done in local coordinates $\{x^i\}$ near $p \in \mathcal{M}^n$ and $\{y^\alpha\}$ near $F(p)$ in N. Arranging coordinates at a fixed point p such that $g_{ij}(p) = \delta_{ij}$, $(\partial/\partial x^i)g_{jk}(p) = 0$, $\bar{g}_{\alpha\beta}(F(p)) = \delta_{\alpha\beta}$, $(\partial/\partial y^\alpha)\bar{g}_{\beta\delta}(F(p)) = 0$ all identities are straightforward consequences of the definitions and the Gauss–Weingarten relations. The computations have been carried out in detail for $f = -H$ in [34], [35] and [4]. A short derivation by the second author is also contained in the last section of this article.
We will now use the commutator identities in Theorem 2.1 to convert the evolution equations for the curvature into parabolic systems on the hypersurface. For this pupose we introduce for each speed function f the nonlinear operator L_f by setting

$$L_f u = L_f^{ij}\nabla_i\nabla_j u := -\frac{\partial \hat{f}}{\partial h_{ij}}\nabla_i\nabla_j u,$$

where $\hat{f}$ as before is the symmetric function f considered as a function of the h_{ij}. Note that for mean curvature flow $L_H = \Delta$ is the Laplace–Beltrami operator, for inverse mean curvature flow with $f = H^{-1}$ we have $L_f = (1/H^2)\Delta$ and in general L_f is an elliptic operator exactly when f is elliptic, ie satisfies (3.1). The following form of the evolution equations exhibits their parabolic nature.

Corollary 3.3 *On any solution $\mathcal{M}_t^n = F(\cdot,t)(\mathcal{M}^n)$ of (1.1) the second fundamental form h_{ij} and the speed f satisfy*

$$\frac{\partial}{\partial t}h_{ij} = L_f^{kl}\nabla_k\nabla_l h_{ij} - \frac{\partial^2 f}{\partial h_{kl}\partial h_{pq}}\nabla_i h_{kl}\nabla_j h_{pq}$$

$$
\begin{aligned}
& +\frac{\partial f}{\partial h_{kl}}\{h_{kl}h_{im}h_{mj} - h_{km}h_{il}h_{mj} + h_{kj}h_{im}h_{ml} - h_{km}h_{ij}h_{ml} \\
& +\bar{R}_{kilm}h_{mj} + \bar{R}_{kijm}h_{ml} + \bar{R}_{mijl}h_{km} + \bar{R}_{oioj}h_{kl} - \bar{R}_{okol}h_{ij} + \bar{R}_{mljk}h_{im} \\
& +\bar{\nabla}_k\bar{R}_{ojil} + \bar{\nabla}_i\bar{R}_{oljk}\} + f(h_{ik}h_j^k - \bar{R}_{oioj}), \\
\frac{\partial}{\partial t}f \ = \ & L_f^{ij}\nabla_i\nabla_j f - f\frac{\partial f}{\partial h_{ij}}(h_{ik}h_j^k + \bar{R}_{oioj}).
\end{aligned}
$$

Proof. From $\nabla_i f = (\partial f/\partial h_{kl})\nabla_i h_{kl}$ we see that

$$
\nabla_i\nabla_j f = \frac{\partial f}{\partial h_{kl}}\nabla_i\nabla_j h_{kl} + \frac{\partial^2 f}{\partial h_{kl}\partial h_{pq}}\nabla_j h_{kl}\nabla_i h_{pq}.
$$

This yields the first identity in view of Theorem 2.1 and Theorem 3.2(iv). Similarly we get

$$
\begin{aligned}
\frac{\partial}{\partial t}f \ &= \ \frac{\partial \hat{f}}{\partial h_j^i}\frac{\partial}{\partial t}h_j^i = \frac{\partial \hat{f}}{\partial h_{lj}}g_{li}\frac{\partial}{\partial t}(g^{ik}h_{kj}) \\
&= \ -2fh_i^k h_{kj}\frac{\partial \hat{f}}{\partial h_{ij}} + \frac{\partial \hat{f}}{\partial h_{ij}}\frac{\partial}{\partial t}h_{ij} \\
&= \ L_f^{ij}\nabla_i\nabla_j f + f\frac{\partial f}{\partial h_{ij}}(h_{ik}h_j^k - \bar{R}_{oioj}) - 2f\frac{\partial f}{\partial h_{ij}}h_i^k h_{kj} \\
&= \ L_f^{ij}\nabla_i\nabla_j f - f\frac{\partial f}{\partial h_{ij}}(h_{ik}h_j^k + \bar{R}_{oioj}),
\end{aligned}
$$

as required.
The curvature terms in this nonlinear reaction–diffusion system provide the key for understanding the interaction between geometric properties of the hypersurface and the ambient manifold. They are the tool to study these geometric phenomena with analytical means. For some choices of f we will now describe recent developments.

4 Mean curvature flow

In the case of mean curvature flow $f = -H$ it is well known [34] that for closed initial surfaces the solution of (1.1)–(1.2) exists on a maximal time interval $[0, T[$, $0 < T \leq \infty$. In some cases the behaviour of the flow has been completely understood: For closed convex surfaces in $\mathbb{R}^{n+1}$, $n \geq 2$, it was shown in [34] that the solution contracts smoothly to a point, becoming more and more spherical at the end of the evolution. In [35] this was extended to general Riemannian manifolds under the assumption that the initial hypersurface is sufficiently convex: Each principal curvature λ_i of the initial surface has to be bounded below by a constant depending on the curvature and the derivative of the curvature in the ambient manifold. While the constants are optimal in locally symmetric spaces, the dependence on the derivatives of curvature in the general case is not desirable form a geometric point of view. Some of the fully nonlinear flows discussed in the next section have a better behaviour from this point of view.

Regularity and longtime existence was also obtained for surfaces that can be written as graphs, compare the joint work of the first author with Ecker in [16] and [17].
In the one-dimensional case Grayson proved that any embedded closed curve on a 2-surface of bounded geometry will either smoothly contract to a point in finite time or converge to a geodesic in infinite time, compare [25], [26] and earlier work of Gage and Hamilton in [22].
In higher dimensions it is well known that singularities will in usually occur before the area of the evolving surface tends to zero. If $T < \infty$, as is always the case in Euclidean space, the curvature of the surfaces becomes unbounded for $t \to T$. One would like to understand the singular behaviour for $t \to T$ in detail, having in mind a possible controlled extension of the flow beyond such a singularity. See [37] for a review of earlier results concerning local and global properties of mean curvature flow and its singularities. We will not discuss singularities in weak formulations of the flow, a good reference in this direction is [54] and [43].

Since the shape of possible singularities is a purely local question, we may restrict attention to the case where the target manifold is Euclidean space. Nevertheless, in the light of an abundance even of homothetically shrinking examples with symmetries, the possible limiting behaviour near singularities seems in general beyond classification at this stage. In recent joint work of C. Sinestrari and the author [41], [42] the additional assumption of nonnegative mean curvature is used to restrict the range of possible phenomena, while still retaining an interestingly large class of surfaces. We derive new a priori estimates from below for all elementary symmetric functions of the principal curvatures, exploiting the one-sided bound on the mean curvature. The estimates turn out to be strong enough to conclude that any rescaled limit of a singularity is (weakly) convex.
Recall that

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

are the elementary symmetric functions of the principal curvatures with $S_1 = H$. Then [42] establishes the estimates

Theorem 4.1 *(H.-Sinestrari) Suppose $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ is a smooth closed hypersurface immersion with nonnegative mean curvature. For each k, $2 \leq k \leq n$, and any $\eta > 0$ there is a constant $C_{\eta,k}$ depending only on n, k, η and the initial data, such that everywhere on $\mathcal{M} \times [0, T[$ the estimate*

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{4.1}$$

holds uniformly in space and time.

The proof proceeds by induction on the degree k of S_k and relies heavily on the algebraic properties of the elementary symmetric functions, the structure of the curvature evolution in this particular flow and the Sobolev inequality for hypersurfaces. In each step of the iteration an a priori estimate is proved for a quotient

$$Q_k = \frac{S_k}{S_{k-1}}$$

of consecutive elementary symmetric polynmials, making use of the concavity properties of this function. Using techniques in [35] and [49] the result can be extended to starshaped surfaces in $\mathbb{R}^{n+1}$ and to hypersurfaces in Riemannian manifolds.
Similarly as in the theory of minimal surfaces the structure of singularities is then studied by blowup methods, in this case by parabolic rescaling in space and time, compare [28], [36], [41]. Since η is arbitrary in the above estimate and the mean curvature $S_1 = H$ tends to infinity near a singularity, the scaling invariance is broken in inequality (4.1) and implies that near a singularity each S_k becomes nonnegative after appropriate rescaling:

Corollary 4.2 *(H.-Sinestrari) Let $\mathcal{M}_t$ be a mean convex solution of mean curvature flow on the maximal time interval $[0, T[$ as in Theorem 1.1. Then any smooth rescaling of the singularity for $t \to T$ is convex.*

The structure of the rescaled limit depends on the blowup rate of the singularity: If the quantity $\sup(T-t)|A|^2$ is uniformly bounded (type I singularity), the rescaling will yield a selfsimilar, homothetically shrinking solution of the flow which is completely classified in the case of positive mean curvature, see [36] and [37]. If the quantity $\sup(T-t)|A|^2$ is unbounded (type II singularity), the rescaling of the singularity can be done in such a way that an "eternal solution" (ie defined for all time) of mean curvature flow results where the maximum of the curvature is attained on the surface. In the convex case such solutions were shown by Hamilton to move isometrically by translations, [29]. Hence, combining the classification of type I singularities in [37], the result of Hamilton and the convexity result in Corollary 4.2, one derives a description of all possible singularities (type I and type II) in the mean convex case, compare [42].
Open problems which have to be adressed for the future goal of continuing the flow by surgery concern the classification of convex translating solutions, Harnack estimates for the mean curvature, more precise estimates on the rate of convergence as well as higher order asymptotics near singularities. Some guidance on the possible higher order behaviour near singularities can be taken from the degenerate examples constructed in [7]. A Harnack estimate for the mean curvature has so far only been obtained in the convex case [29], which is too restrictive for many applications. The work of Hamilton on the Ricci flow [28] has a close relation to the mean curvature flow and indicates a strategy for the extension of the flow past singularities once stronger estimates are available [31].

We conclude this section with the one-dimensional case, where an embedded curve is evolving in the plane or on some smooth surface by the curve shortening flow. The remarkable articles of Grayson on this flow [25],[26] show by a number of global arguments that for embedded curves no finite time singularity can occur unless the whole curve contracts to a single point.
The structure of all possible singularities in this case is now well understood: There are no embedded type I singularities except the shrinking circle, which is the desired outcome, and the only possible rescaling of a type II singularity is a so called grim reaper curve given by $y = \log \cos x$. To prove Graysons result it is therefore sufficient to give an argument excluding this last curve as a possible limiting shape. Such an argument is provided both by Hamilton in [32], where an isoperimetric estimate for the area in subdivisions of the

enclosed region is shown, and by the first author in [38], where a lower bound for the ratio between extrinsic and intrinsic distances on the evolving curve is proved.
To describe the last result, let $F: S^1 \times [0,T] \to \mathbb{R}^2$ be a closed embedded curve moving by the curve shortening flow. If $L = L(t)$ is the total length of the curve, the intrinsic distance function l along the curve is smoothly defined only for $0 \leq l < L/2$, with conjugate points where $l = L/2$. We therefore define a smooth function $\psi: S^1 \times S^1 \times [0,T] \to \mathbb{R}$ by setting

$$\psi := \frac{L}{\pi} \sin(\frac{l\pi}{L}).$$

With this choice of ψ, and with d being the extrinsic distance between two points on the curve, the isoperimetric ratio d/ψ approaches 1 on the diagonal of $S^1 \times S^1$ for any smooth embedding of S^1 in $\mathbb{R}^2$ and the ratio d/ψ is identically one on any round circle.

Theorem 4.3 *Let $F: S^1 \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the curve shortening flow (1.1). Then the minimum of d/ψ on S^1 is nondecreasing; it is strictly increasing unless $d/\psi \equiv 1$ and $F(S^1)$ is a round circle.*

Clearly the estimate prevents a grim reaper type singularity. The proof uses the maximum principle on the cross product of the curve with itself. It is an open problem whether similar lower order estimates can be used for the study or exclusion of certain singularities in higherdimensional flows.

5 Fully nonlinear flows

The Gauss curvature flow, where the speed $f = -K = -(\lambda_1 \cdots \lambda_n)$ is the product of the principle curvatures, was first introduced by Firey [20] as a model for the changing shape of a tumbling stone being worn from all directions with uniform intensity. The flow is parabolic only in the class of convex surfaces and much more nonlinear in its analytic behaviour than the mean curvature flow. Tso [52] proved existence, uniqueness and convergence of closed convex hypersurfaces to a point for this flow without however determining the limiting shape of the contracting surface. The conjecture of Firey (1974) that the limiting shape is that of a sphere regardless of the initial data, was only recently confirmed by Andrews [2]:

Theorem 5.1 *(Andrews) Let $\mathcal{M}_0^2$ be a smooth closed strictly convex initial surface in $\mathbb{R}^3$. Then there is a unique smooth solution of (1.1) with $f = -K$ on the time interval $[0,T[$, where $T = V(\mathcal{M}_0^2)/4\pi$ is determined by the enclosed volume of the initial surface, and the surfaces converge to a round sphere after appropriate rescaling.*

The corresponding result for mean curvature flow was obtained earlier by the author in [34] and for a large class of speed functions f including the harmonic mean curvature flow by Andrews in [1]. If the Gauss curvature K is replaced by some power K^α, a whole new range of interesting phenomena appears. If the homogeneity is 1, ie $\alpha = 1/n$, Chow proved contraction to a point and roundness of the limiting shape, [12]. In [5] Andrews

shows that in the interval $1/(n+2) < \alpha \leq 1/n$ there is at least some smooth limiting shape at the end of the contraction, while for small values of α a degeneration of the surface near the end of the contraction is expected.
In the special case $\alpha = 1/(n+2)$, the evolution equation (1.1) becomes affine invariant. In line with the results just mentioned Andrews [3] proves by an extension of Calabi's estimate on the cubic ground form that convex initial data contract smoothly to a point in finite time, with ellipsoids as the natural unique limiting shape. As a consequence he derives an elegant proof of the affine isoperimetric inequality. Compare also the work of Sapiro and Tannenbaum [46] on the affine evolution of curves, which has applications in image processing.
For convex hypersurfaces in general Riemannian manifolds speedfunctions f such as the harmonic mean curvature or other quotients of elementary symmetric functions seem to have the best algebraic behaviour. In mean curvature flow the derivatives of the ambient curvature in the evolution equations of Corollary 2.2 are analytically hard to control, compare the dependance of the main result in [35] on these terms. For harmonic mean curvature flow and flows of similar structure Andrews derives an optimal convergence result for hypersurfaces having sufficiently positive principal curvatures in relation to the ambient curvature, [4]. In particular, he shows that such flows contract convex hypersurfaces in manifolds of positive sectional curvature to a point and gives a new argument for the classical 1/4-pinching theorem.
All speedfunctions considered so far were pointing in the same direction as the mean curvature vector, corresponding to contractions in the case of convex surfaces. In the last section we consider an expanding version of the flow.

6 The inverse mean curvature flow

The inverse mean curvature flow $f = H^{-1}$ is well posed for surfaces of positive mean curvature and characterised by its property that the area element is growing exponentially at each point: From Theorem 1.1(i) we have $\partial/\partial t(d\mu) = d\mu$. In particular, the total area of a smooth closed evolving surface is completely determined by its initial area:

$$|M_t^n| = |M_0^n| \exp(t).$$

The standard example for this behaviour is the exponentially expanding sphere of radius $R(t) = R(0)\exp(t/n)$. Further interesting properties of the flow follow from the evolution equation for the mean curvature H, which we derive from the evolution equation for the speed $f = H^{-1}$.

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - \frac{2|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\bar{R}ic(\nu,\nu)}{H}.$$

Due to the negative sign of the $|A|^2$-term we get from this equation by a simple application of the parabolic maximum principle the remarkable property that the mean curvature H is uniformly bounded in terms of its initial data and the Ricci curvature of the ambient manifold. This is in strong contrast to the mean curvature flow, where the blowup of the mean curvature causes the singularities studied in section 2. For the inverse mean

curvature flow the critical behaviour occurs where $H \to 0$ and the speed becomes infinite. In Euclidean space it is clear that the maximum of the mean curvature is decreasing and the same is true for any L^p-norm.
In case $n = 2$ this property of the flow can be extended to closed surfaces in arbitrary three-manifolds of nonnegative scalar curvature: For any two-surface $\Sigma^2 \subset (N^3, \bar{g})$ the so called Hawking quasi-local mass of Σ^2 is defined as the geometric quantity

$$m_H(\Sigma^2) := \frac{|\Sigma^2|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_{\Sigma^2} H^2 \, d\mu\right),$$

and a computation based on the evolution equation for the mean curvature, the area element of the surface and the Gauss-Bonnet formula shows that for a solution M_t^2 of the inverse mean curvature flow

$$\frac{d}{dt}\int_{M_t^2} H^2 \, d\mu = 4\pi\chi(M_t^2) + \int_{M_t^2} -2\frac{|\nabla H|^2}{H^2} - \frac{1}{2}H^2 - \frac{1}{2}(\lambda_1 - \lambda_2)^2 - \bar{R} \, d\mu.$$

Hence, if the surface M_t^2 is connected and the scalar curvature $\bar{R}$ of the three-manifold is nonnegative, we have

$$\frac{d}{dt}\int_{M_t^2} H^2 \, d\mu \leq \frac{1}{2}\left(16\pi - \int_{M_t^2} H^2 \, d\mu\right)$$

and the Hawking quasi-local mass is nondecreasing along the inverse mean curvature flow:

$$\frac{d}{dt} m_H(M_t^2) \geq 0.$$

A major reason for the interest in the inverse mean curvature flow comes from the interpretation of this purely geometric fact in General Relativity: The spatial part of the exterior of an isolated gravitating system (like a star, black hole or galaxy) is modelled by the end of an asymptotically flat Riemannian 3–manifold with nonnegative scalar curvature as above. Here an end of a Riemannian 3-manifold $(N^3, \bar{g})$ is called *asymptotically flat* if it is realized by an open set that is diffeomorphic to the complement of a compact set K in $\mathbb{R}^3$, and the metric tensor $\bar{g}$ of M satisfies

$$|\bar{g}_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, \qquad |\bar{g}_{ij,k}| \leq \frac{C}{|x|^2}, \qquad \bar{Ric} \geq -\frac{C\bar{g}}{|x|^2},$$

as $|x| \to \infty$. The derivatives are taken with respect to the Euclidean metric $\delta = \{\delta_{ij}\}$ on $\mathbb{R}^3 \setminus K$. On such asymptotically flat ends a concept of total mass or energy is defined by a flux integral through the sphere at infinity,

$$m := \lim_{r\to\infty} \frac{1}{16\pi} \int_{\partial B_r^\delta(0)} (\bar{g}_{ii,j} - \bar{g}_{ij,i}) n^j \, d\mu_\delta,$$

which is a geometric invariant, despite being expressed in coordinates. It is finite precisely when the scalar curvature $\bar{R}$ of $\bar{g}$ satisfies

$$\int_{N^3} |\bar{R}| < \infty,$$

and from a physical point of view it is meant to measure both matter content and gravitational energy of the isolated system. Compare the joint papers [39][40] of the author and T. Ilmanen for references to these facts. The Hawking quasi-local mass defined above is used as a geometric concept for the energy of a three-dimensional region contained inside a two-dimensional surface, motivated by the fact that for large approximately round spheres S^2_R it is true that $m_H(S^2_R) \to m$. Furthermore, since in the physically simplest case the outer boundary of a black hole can be represented by a minimal two-surface inside the given three-manifold, the inverse mean curvature flow can provide a relation between the size of the black hole and the total energy m: If there is a smooth connected solution of the inverse mean curvature flow starting from a minimal surface $M^2_0 \subset N^3$, (the apparent horizon of the black hole) and expanding smoothly to large round spheres where $m_H(M^2_t) \to m$, then by the monotonicity result above we have the inequality

$$\frac{1}{4\sqrt{\pi}}|M^2_0|^{1/2} = m_H(M^2_0) \leq m.$$

This relation between the size of the outermost black hole and the total energy of an isolated gravitating system is the Riemannian Penrose inequality, which sharpens the positive mass theorem. The argument just described was first put forward by Geroch, [24]. Also note the many other contributions to this approach which are refered to in [39]. The crucial question concerns of course the existence of such a solution to the flow by inverse mean curvature. For starshaped surfaces of positive mean curvature in $\mathbb{R}^{n+1}$ Gerhardt [23] and Urbas [53] show that the necessary estimates for complete regularity of the flow can be established and they prove longterm existence as well as asymptotic roundness in this class.

Without an assumption like starshapedness it is quite clear that singularities have to occur in certain situations. For example, the solution evolving from a thin symmetric torus can not exist forever, due to the upper bound on H some blowup in the speed H^{-1} must occur for such initial data. Similar examples can be constructed in the class of two-spheres making it clear that there cannot be a smooth solution for the flow in the general situations that are of natural interest in physics.

To overcome these difficulties, [39] introduces a weak concept of solution for the flow which still retains the crucial monotonicity of the Hawking mass. The weak concept is a level-set formulation of (1.1), where the evolving surfaces are given as level-sets of a scalar function u via

$$M^2_t = \partial\{x|u(x) < t\},$$

and (1.1) is replaced by the degenerate elliptic equation

$$\mathrm{div}_N\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|,$$

where the left hand side describes the mean curvature of the level-sets and the right hand side yields the inverse speed. This formulation in divergence form admits locally Lipschitz continuous solutions and is inspired by the work of Evans-Spruck [19] and Chen-Giga-Goto [11] on the mean curvature flow. Using elliptic regularisation and a minimization principle we show existence of a locally Lipschitz-continuous solution with level-sets of nonnegative mean curvature of class $C^{1,\alpha}$, still satisfying monotonicity of the Hawking quasi-local mass, compare [39]. The solution allows the phenomenon of fattening, which corresponds to jumps of the surfaces and is desirable for our main application. We thus succeed in adapting Geroch's original argument and derive the following sharp lower bound for the mass:

Theorem 6.1 *(H.-Ilmanen) Let N^3 be a complete, connected 3-manifold. Suppose that*

(i) *N^3 has nonnegative scalar curvature,*

(ii) *N^3 is asymptotically flat in the sense above with ADM mass m,*

(iii) *The boundary of N^3 is compact and consists of minimal surfaces, and N^3 contains no other compact minimal surfaces.*

Then $m \geq 0$, and

$$16\pi m^2 \geq |\Sigma^2|,$$

where $|\Sigma^2|$ is the area of any connected component of ∂N^3. Equality holds if and only if N^3 is one-half of the spatial Schwarzschild manifold.

The *spatial Schwarzschild manifold* is the manifold $\mathbb{R}^3 \setminus \{0\}$ equipped with the metric $\bar{g} := (1 + m/2|x|)^4\delta$, representing the spatial exterior region of a single static black hole of mass m.

7 Short-Time Existence Theory

Classically, the existence theory for nonlinear parabolic equations is treated in two stages: first, the use of linearisation techniques to prove that a solution may be found for a short interval of time; and second, derivation of the all-important a priori estimates which enable us to extend the short-time solution to a maximal time interval. In this chapter, we carry out the first half of the process.

7.1 Evolution Equations for Manifolds and Hypersurfaces

This section introduces the primary concern of this work: evolution equations for geometric structures. Typically, we consider motions of manifolds and submanifolds driven by forces which stem from their curvature.
Specifically, we address two problems:

Conformal Deformation of a Manifold: Let (M^n, g) be a smooth Riemannian manifold, and consider the deformation process

$$\frac{\partial}{\partial t} g = \lambda(x,t) \cdot g, \tag{7.1}$$

for some function λ. This defines a continuous, conformal change in the metric — conformal because the metric changes only by a scaling factor; angles are not affected. The best-known examples of this are the Ricci flow on a compact 2-surface (described and solved completely in [27]) and the Yamabe flow on a manifold of dimension at least three [33]; in both these cases, the defining equation is $\frac{\partial}{\partial t}g = -R \cdot g$, where R is the scalar curvature of g.

Normal Deformation of a Hypersurface: Let $F : M^n \hookrightarrow (N^{n+1}, g)$ be a smooth immersion of a hypersurface in a Riemannian manifold. M^n is assumed orientable, so that there is a smoothly varying, globally defined unit normal vector. In this case, we consider deformation of F according to the equation

$$\frac{\partial}{\partial t}F = \lambda(x,t) \cdot n. \tag{7.2}$$

The best-known example of this is the mean-curvature flow of hypersurfaces, where the speed is (up to a sign) the mean curvature of F. This was first introduced by Mullins in [45] (an unjustly little-known work); it was later found independently by Brakke, who expressed the equation in the language of geometric measure theory in [9].

These are the standard examples of such problems, and they share a common structure. In both cases, the deformation process can be shown to be equivalent to a quasilinear scalar partial differential equation on M^n. When the impetus comes from the curvature, as in these examples, the scalar equations are strictly parabolic and of second order. Such are known to possess solutions under very general conditions, at least for some short period of time.

The total curvature problems we wish to study in this work also lead to quasilinear parabolic scalar equations, but of fourth or higher order. It will surprise nobody that such equations still admit short-time solutions. Nevertheless, when the setting is a manifold rather than a euclidean domain, this does not belong to the standard theory and requires proof.

The question of existence will be taken up in later sections; the question of how solutions actually behave will be taken up in later chapters. For the remainder of this section, we assume that we already have a solution to (7.1) or (7.2), and derive a handful of basic properties.

(7.1) and (7.2) imply evolution equations for the curvature and other geometric attributes of g and F. Consider first the conformal deformation.

Lemma 7.1 *Let M^n be a smooth manifold; let g_t be a one parameter family of metrics on M^n varying according to (7.1). Then, g_t can be written as* $\exp 2u(x,t) \cdot g_0$, *where the function u evolves by the equation $\frac{\partial}{\partial t}u = \frac{1}{2}\lambda$.*

Proof. It is obvious that g_t may be so represented. It follows that

$$\frac{\partial}{\partial t}g = \frac{\partial}{\partial t}(\exp 2u \cdot g_0) = 2\frac{\partial u}{\partial t} \cdot \exp 2u \cdot g_0 = 2\frac{\partial u}{\partial t} \cdot g$$

and this provides the equation for u.

Any other metric $g^\natural$ in the same conformal class could take the place of g_0 in this lemma. The equation for the conformal factor is unaffected.

Notation: In what follows, we drop the subscript t from the time-dependent metric. The calculations will relate g to a fixed background metric; for convenience, we take this to be g_0. The covariant derivative and laplacian operators of g and g_0 will be represented as ∇, Δ and ∇^0, Δ^0; the curvatures will be represented in the same way. The zero may appear as a subscript or a superscript, whichever happens to be more convenient for typesetting; the meaning remains clear. This accords with the usage in subsequent chapters.

Lemma 7.2 *The Christoffel symbols and curvature of g may be expressed in terms of those of g_0 and the conformal factor u:*

$$\begin{aligned}
\Gamma_{ij}^k &= (\Gamma_0)_{ij}^k + (\delta_i^k \nabla_j^0 u + \delta_j^k \nabla_i^0 u - g_{ij}^0 g_0^{kl} \nabla_l^0 u) \\
R_{ij} &= R_{ij}^0 - (n-2)\left(\nabla_i^0 \nabla_j^0 u - \nabla_i^0 u \nabla_j^0 u\right) - \left(\Delta^0 u + (n-2)\left|\nabla^0 u\right|^2\right) \cdot g_{ij}^0 \\
R &= e^{-2u}\left(R^0 - 2(n-1)\Delta^0 u - (n-1)(n-2)\left|\nabla^0 u\right|^2\right).
\end{aligned}$$

Similarly, the laplacian operator corresponding to g can be related to that of g_0: for any smooth function $\phi : M^n \to \mathbf{R}$,

$$\Delta\phi = e^{-2u}\Delta^0\phi + (n-2)g^0(\nabla^0 u, \nabla^0\phi)$$

Proof. The first three equations may be found in the discussion of the Yamabe problem in [8]; the fourth follows easily. Let ϕ be a fixed smooth function on M^n; then, in local co-ordinates,

$$\begin{aligned}
\Delta\phi &= g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial\phi}{\partial x^k}\right) \\
&= e^{-2u}\left(g_0^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - (\Gamma_0)_{ij}^k \frac{\partial\phi}{\partial x^k}\right) - g_0^{ij}(\Gamma - \Gamma_0)_{ij}^k \frac{\partial\phi}{\partial x^k}\right) \\
&= e^{-2u}\Delta^0\phi - g_0^{ij}\left(\delta_i^k \frac{\partial u}{\partial x^j} + \delta_j^k \frac{\partial u}{\partial x^i} - g_{ij}^0 g_0^{kl} \frac{\partial u}{\partial x^l}\right)\frac{\partial\phi}{\partial x^k} \\
&= e^{-2u}\Delta^0\phi + (n-2)\, g_0^{kl} \frac{\partial u}{\partial x^l}\frac{\partial\phi}{\partial x^k}
\end{aligned}$$

and this establishes the final equation. Combining the previous two lemmas gives the variation of the curvature under (7.1):

Lemma 7.3 *The change (7.1) produces in the curvature of g is given by:*

$$\frac{\partial}{\partial t}R_{ij} = -\frac{1}{2}\left(\Delta\lambda \cdot g_{ij} + (n-2)\nabla_i\nabla_j\lambda\right) \qquad \text{and} \qquad \frac{\partial}{\partial t}R = -(n-1)\Delta\lambda - R\lambda.$$

Proof. Differentiating the equation above for the Ricci curvature and substituting $\lambda = 2\frac{\partial}{\partial t}u$,

$$\frac{\partial}{\partial t}R_{ij} = \frac{1}{2}\left(-\left(\Delta^0\lambda \cdot g_{ij}^0 + (n-2)\nabla_i^0\nabla_j^0\lambda\right) + 2(n-2)\left(\nabla_i^0\lambda\nabla_j^0 u - g^0\left(\nabla^0\lambda, \nabla^0 u\right) \cdot g_{ij}^0\right)\right).$$

However, for any C^2 function f,

$$\Delta f \cdot g_{ij} + (n-2)\nabla_i\nabla_j f = \Delta^0 f \cdot g^0_{ij} + (n-2)\nabla^0_i\nabla^0_j f + L_f$$

in which the error term is given by

$$L_f = \left((\Gamma^0 - \Gamma)^k_{lm}(g^0)^{lm} g^0_{ij} + (n-2)(\Gamma^0 - \Gamma)^k_{ij}\right) \cdot \nabla^0_k f;$$

and using the transformation rule for the Christoffel symbols of Lemma 7.2, this simplifies to

$$L_f = 2(n-2)\left(g^0\left(\nabla^0\lambda, \nabla^0 u\right) \cdot g^0_{ij} - \nabla^0_i\lambda\nabla^0_j u\right).$$

But this matches precisely the final term in the evolution of R_{ij}, and so, cancelling,

$$\frac{\partial}{\partial t}R_{ij} = -\frac{1}{2}\left(\Delta\lambda \cdot g_{ij} + (n-2)\nabla_i\nabla_j\lambda\right),$$

which proves the first claim of the lemma. The evolution of the scalar curvature is simpler:

$$\begin{aligned}
\frac{\partial}{\partial t}R &= \frac{\partial}{\partial t}\left(e^{-2u}\left(R^0 - 2(n-1)\Delta^0 u - (n-1)(n-2)|\nabla^0 u|^2\right)\right) \\
&= -2\frac{\partial u}{\partial t} \cdot R + e^{-2ut}\left(-2(n-1)\Delta^0\frac{\partial u}{\partial t} - 2(n-1)(n-2)\,g^0\left(\nabla^0 u, \nabla^0\frac{\partial u}{\partial t}\right)\right) \\
&= -(n-1)\Delta\lambda - R\lambda.
\end{aligned}$$

and this proves the second claim. This is as much as we need say for now about the conformal problem.

Now let F_t be a one-parameter family of immersions $M^n \hookrightarrow N^{n+1}$ which vary in accordance with (7.2). Let g_t denote the induced metric $F_t^*\bar{g}$. As above, we shall drop the t subscripts wherever this would not lead to confusion.

From (7.2), we compute evolution equations for the geometric features of F. This is simplified immensely by the use of well-chosen co-ordinates.

These calculations are purely local in nature; so we focus on some point (x^*, t^*) in space-time. Let y^* be the image of x^* under F_{t^*}. We may assume that the co-ordinates on N^{n+1} are normal at y^*, and that those on M^n are normal at x^* relative to the metric induced at this one instant of time. In particular, the Christoffel symbols $\bar{\Gamma}^\alpha_{\beta\gamma}(y^*)$ and $\Gamma^k_{ij}(x^*, t^*)$ all vanish, and the Gauß and Weingarten equations reduce to

$$\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j}(x^*, t^*) = -h_{ij}(x^*, t^*) \cdot n^\alpha(x^*, t^*), \qquad \frac{\partial n^\alpha}{\partial x^i}(x^*, t^*) = h^j_i(x^*, t^*) \cdot \frac{\partial F^\alpha}{\partial x^i}(x^*, t^*).$$

Lemma 7.4 *Under (7.2), the induced metric on M^n evolves according to*

$$\frac{\partial}{\partial t}g_{ij} = 2\lambda h_{ij}.$$

It follows directly that the inverse of the metric and the measure evolve by

$$\frac{\partial}{\partial t}g^{ij} = -2\lambda h^{ij} \qquad \text{and} \qquad \frac{\partial}{\partial t}\,d\mu = \lambda H\,d\mu.$$

Proof. Let (x^*, t^*) be a given point of spacetime, and assume the co-ordinate systems on M^n and N^{n+1} are normal at (x^*, t^*) and $F_{t^*}(x^*)$, as above. We compute the evolution of g_{ij} at the point (x^*, t^*).
The induced metric is by nature given by

$$g_{ij} = \bar{g}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right),$$

and hence, noting that $\bar{g}$ is symmetric and has no covariant derivative of its own,

$$\frac{\partial}{\partial t} g_{ij} = 2\bar{g}\left(\frac{\partial}{\partial t}\left\{\frac{\partial F}{\partial x^i}\right\}, \frac{\partial F}{\partial x^j}\right) = 2\bar{g}\left(\frac{\partial}{\partial x^i}\{\lambda n\}, \frac{\partial F}{\partial x^j}\right),$$

Now expand the product derivative. The derivative term in λ clearly vanishes because of orthogonality, and all that remains is

$$\frac{\partial}{\partial t} g_{ij} = 2\lambda\bar{g}\left(\frac{\partial n}{\partial x^i}, \frac{\partial F}{\partial x^j}\right).$$

and it follows directly from the reduced Weingarten equation at (x^*, t^*) that the final factor is simply h_{ij}; with that, the first claim of the lemma is proved. The rest is easy. To compute the evolution of the inverse of the metric, we differentiate the equation $g^{ik} \cdot g_{kl} = \delta^i_j$:

$$0 = \frac{\partial(g^{ik} \cdot g_{kl})}{\partial t} = \frac{\partial g^{ik}}{\partial t} g_{kl} + g^{ik}\frac{\partial g_{kl}}{\partial t} = \frac{\partial g^{ik}}{\partial t} g_{kl} + 2\lambda h^i_l.$$

Tracing with g^{lj} gives $\frac{\partial}{\partial t} g^{ij} = -2\lambda h^{ij}$, which establishes the the second claim. The final part follows from the rule for differentiating a determinant:

$$\frac{\partial}{\partial t} d\mu = \frac{\partial}{\partial t}(\sqrt{\det g}\, dx) = \frac{1}{2}\sqrt{\det g} \cdot g^{ij}\frac{\partial}{\partial t} g_{ij}\, dx = \lambda H\, d\mu,$$

and this completes the proof.
Next we derive the variation of the normal vector:

Lemma 7.5 *The change in the normal is given by*

$$\frac{\partial n}{\partial t} = -F_*(\text{grad}^{M^n}\lambda).$$

Proof. n is a unit normal vector; thus, $\bar{g}(n, n) = 1$ everywhere. Differentiating this equation, we see that the derivative (*any* derivative) of n must be normal to n itself, and hence tangential to $F(M^n)$. It may therefore be represented in the form

$$\frac{\partial n}{\partial t} = g^{ij}\bar{g}\left(\frac{\partial n}{\partial t}, \frac{\partial F}{\partial x^i}\right) \cdot \frac{\partial F}{\partial x^j}. \tag{7.3}$$

Now differentiating the equation $\bar{g}(n, \frac{\partial}{\partial x^i} F) = 0$, we have

$$0 = \bar{g}\left(\frac{\partial n}{\partial t}, \frac{\partial F}{\partial x^i}\right) + \bar{g}\left(n, \frac{\partial}{\partial t}\frac{\partial F}{\partial x^i}\right) = \bar{g}\left(\frac{\partial n}{\partial t}, \frac{\partial F}{\partial x^i}\right) + \bar{g}\left(n, \frac{\partial(\lambda n)}{\partial x^i}\right) = \bar{g}\left(\frac{\partial n}{\partial t}, \frac{\partial F}{\partial x^i}\right) + \frac{\partial \lambda}{\partial x^i},$$

noting here that another term vanishes because n is orthogonal to all its derivatives. This allows us to substitute for $\bar{g}(\frac{\partial}{\partial t}n, \frac{\partial}{\partial x^i}F)$ in (7.3), giving

$$\frac{\partial n}{\partial t} = -g^{ij}\frac{\partial \lambda}{\partial x^i}\frac{\partial F}{\partial x^j},$$

and this is exactly $-F_*(\mathrm{grad}^{M^n}\lambda)$.

Lemma 7.6 *The variation in the second fundamental form is given by the equations*

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= -\nabla_i\nabla_j\lambda - \lambda(-h_{ik}h_j^k + \overline{\mathrm{Riem}}_{injn}) \\
\frac{\partial}{\partial t}h_i^j &= -\nabla_i\nabla^j\lambda - \lambda(h_{ik}h^{kj} + \overline{\mathrm{Riem}}_{in}{}^j{}_n) \\
\frac{\partial H}{\partial t} &= -\Delta\lambda - (|A|^2 + \overline{\mathrm{Ric}}(n,n)) \\
\frac{\partial}{\partial t}|A|^2 &= -2h^{ij}\nabla_i\nabla_j\lambda - 2\lambda(\mathrm{tr}\ A^3 + h^{ij}\overline{\mathrm{Riem}}_{injn})
\end{aligned}$$

Proof. The second fundamental form is, by definition,

$$h_{ij} = -\bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j}, n\right),$$

and so, differentiating,

$$\frac{\partial}{\partial t}h_{ij} = -\bar{g}\left(\frac{\partial}{\partial t}\left\{\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j}\right\}, n\right) - \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j}, \frac{\partial n}{\partial t}\right).$$

Now we reinstate the assumptions of Lemma 7.4. In the normal co-ordinate system, the rightmost term vanishes altogether because $\frac{\partial}{\partial t}n$ is tangential and the spatial derivative normal. So, expressing the remaining covariant derivative in co-ordinates,

$$\frac{\partial}{\partial t}h_{ij} = -\bar{g}\left(\frac{\partial}{\partial t}\left\{\frac{\partial^2 F}{\partial x^i \partial x^j} + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\frac{\partial}{\partial y^\alpha}\right\}, n\right).$$

Expanding the time derivative, and noting that the terms containing $\bar{\Gamma}^\alpha_{\beta\gamma}$ all vanish, this becomes

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= -\bar{g}\left(\frac{\partial^2(\lambda n)}{\partial x^i\partial x^j} + \frac{\partial}{\partial t}\bar{\Gamma}^\alpha_{\beta\gamma}\cdot\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\frac{\partial}{\partial y^\alpha}, n\right) \\
&= -\bar{g}\left(\frac{\partial^2(\lambda n)}{\partial x^i\partial x^j} + \lambda\bar{\nabla}_n\bar{\Gamma}^\alpha_{\beta\gamma}\cdot\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\frac{\partial}{\partial y^\alpha}, n\right).
\end{aligned}$$

At the point (x^*, t^*), the Weingarten equation for the derivative of the normal gives

$$\frac{\partial n^\alpha}{\partial x^i} = h_i^j\frac{\partial F^\alpha}{\partial x^j} \qquad \text{and} \qquad \frac{\partial^2 n^\alpha}{\partial x^i\partial x^j} = \frac{\partial h_j^k}{\partial x^i}\frac{\partial F^\alpha}{\partial x^k} + h_j^k\frac{\partial^2 F^\alpha}{\partial x^i\partial x^j} - \frac{\partial\bar{\Gamma}^\alpha_{\beta\gamma}}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}n^\gamma.$$

It follows that

$$\bar{g}\left(\frac{\partial^2 n}{\partial x^i \partial x^j}, n\right) = -h_j^k h_{ki} - \bar{g}\left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \bar{\Gamma}^{\alpha}_{\beta\gamma} \frac{\partial F^\beta}{\partial x^i} n^\gamma \frac{\partial}{\partial y^\alpha}, n\right).$$

This enables us to expand the product derivative in (7.4), which gives

$$\frac{\partial}{\partial t} h_{ij} = -\frac{\partial^2 \lambda}{\partial x^i \partial x^j} - \lambda h_{ik} h_j^k + \lambda \bar{g}\left(\left(\bar{\nabla}_n \bar{\Gamma}^{\alpha}_{\beta\gamma} \frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j} - \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \bar{\Gamma}^{\alpha}_{\beta\gamma} \frac{\partial F^\beta}{\partial x^j} n^\gamma\right) \frac{\partial}{\partial x^\alpha}, n\right).$$

However, in our normal co-ordinates, the second partial derivative of λ coincides with the covariant derivative, and the final term is simply the Riemann tensor of N^{n+1}. Permutation of the indices in one or other of the summands allows this term to be rewritten as

$$\bar{g}\left(\left(\bar{\nabla}_{\frac{\partial}{\partial y^\delta}} \bar{\Gamma}^{\alpha}_{\beta\gamma} - \bar{\nabla}_{\frac{\partial}{\partial y^\beta}} \bar{\Gamma}^{\alpha}_{\gamma\delta}\right) \frac{\partial}{\partial y^\alpha}, n\right) \cdot \frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j} n^\delta$$

and now the first factor matches the definition of the Riemann tensor; the product simplifies therefore to

$$\overline{\text{Riem}}_{\alpha\beta\delta\gamma}\, n^\alpha \frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j} n^\delta = \overline{\text{Riem}}_{injn}$$

In view of all this, the evolution equation for h_{ij} may be rewritten as

$$\frac{\partial}{\partial t} h_{ij} = -\nabla_i \nabla_j \lambda - \lambda(-h_{ik} h_j^k + \overline{\text{Riem}}_{injn}) \tag{7.4}$$

which settles the first claim of the lemma. The equation for h_i^j follows quickly:

$$\begin{aligned} \frac{\partial}{\partial t} h_i^j = \frac{\partial}{\partial t}(g^{jk} h_{ik}) &= -2\lambda \cdot h^{jk} h_{ik} - g^{jk}\left(\nabla_i \nabla_k \lambda - \lambda(-h_{il} h_k^l + \overline{\text{Riem}}_{inkn})\right) \\ &= -\nabla_i \nabla^j \lambda - \lambda(h_i^k h_k^j + \overline{\text{Riem}}_{in}{}^j{}_n), \end{aligned}$$

and tracing over i and j gives the equation for H at once. Lastly,

$$\frac{\partial}{\partial t}|A|^2 = \frac{\partial}{\partial t}(h_i^j h_j^i) = -2h^{ij}\nabla_i \nabla_j \lambda - 2\lambda(h_i^j h_j^k h_k^i + h^{ij}\overline{\text{Riem}}_{inij})$$

which is the final claim. The curvature equations hint at the structure concealed in (7.2) and (7.1). In the examples mentioned earlier, where the speed λ was the curvature itself, they are clearly parabolic.

The speeds which interest us in this work feature the laplacian of curvature as their leading term. These too will lead to parabolic equations.

The assumption of orientability demanded in order to make sense of (7.2) is heavy-handed. In real-life examples, it can typically be avoided. The mean curvature flow, for instance, may be rewritten using the Weingarten equation simply as

$$\frac{\partial F}{\partial t} = \Delta F,$$

where the laplacian is computed in the induced metric; and this is now perfectly meaningful even when the manifold is *not* orientable. The hypersurface flows we shall consider later can be redefined in the same way.
The first step towards understanding these problems is to prove that solutions can be found at all. The strategy here is inherited from the second-order theory. The geometric equation is shown to be equivalent to a quasilinear scalar equation, which may be solved for some short interval of time using linearisation techniques; this short-time solution may then be continued as long as it does not become singular.
In the remainder of this chapter, we construct the short-time existence theory. In the problems of higher order we consider, even the linear theory is incomplete, and we have to develop it for ourselves. This is the goal of the following two sections.

2.2 The linear problem

In this section, we prove the existence of solutions to the linear parabolic equation of $2p$-th order on a closed manifold. There are numerous proofs of the corresponding result in a euclidean domain — see, for instance, [18] or [50]. But these typically rely on the construction of a fundamental solution to the equation, a technique which is not easily adapted to the manifold setting. Friedman [21] describes an abstract approach based on a variant of the Lax-Milgram lemma, developing ideas originally due to J. L. Lions [44] and F. Trèves [51].
However, Friedman's account of the result is inaccurate. He proves the existence of a time-$W^{k,2}$ solution to the linear equation $D_t u + A_t u = g, u_0 \equiv 0$ under the unacceptably strong assumption that g vanishes at time zero along with all its derivatives of order up to $k-1$. Considering the equation as a physical process, this is tantamount to assuming there are no external forces at time $t = 0$. This is clearly an undesirable condition, and in no way a natural one.
This is in itself not a mistake, though it limits the usefulness of the theorem. Friedman goes on to claim in a remark, however, that one may prescribe the initial values u_0 freely by considering the equation for $u - u_0$. But this gives an equation whose forcing term no longer satisfies the vanishing condition; the theorem as Friedman states it does not apply.
In this section, we use techniques related to Friedman's to prove a minimal existence result. This will be strengthened later when we prove the natural a priori estimates for the linear problem.

Let (M^n, g) be a smooth, compact Riemannian manifold. Let A be a linear differential operator of order $2p$ on M^n; that is, for a $2p$-times differentiable function $u : M^n \to \mathbb{R}$, we set

$$Au(x) = \sum_{k\leq 2p} A_k^{i_1 i_2 \cdots i_k}(x) \nabla_{i_1 i_2 \cdots i_k} u(x),$$

in local co-ordinates, where the A_k are smooth tensor fields on M^n of type $(0, k)$. Let A be *elliptic* in the very strong sense that the leading term can be factorised as

$$A_{2p}^{i_1 j_1 i_2 j_2 \cdots i_p j_p} = E^{i_1 j_1} E^{i_2 j_2} \ldots E^{i_p j_p},$$

where the 2-form E is strictly positive: $E \geq \lambda g$ for some $\lambda > 0$. In words, the leading part of A should simply be the p-th power of some second-order elliptic operator.

It is possible — easy, even — to define much weaker notions of ellipticity. However, the operators which arise in our geometric problems do turn out to have the structure above; moreover, it is a much simpler matter to prove Gårding's inequality in this class.
Consider such an operator in the usual Sobolev-space way as a bilinear form defined on $W^{p,2}(M^n)$, which we shall also denote by A. Then:

Lemma 7.7 *(Gårding's Inequality for A) For any $\phi \in W^{p,2}(M^n)$,*

$$A(\phi,\phi) \geq \frac{\lambda^p}{2}\,||\phi||^2_{W^{p,2}(M^n)} - Q\,||\phi||^2_{L^2(M^n)},$$

where the constant Q depends on n, λ, and the C^{p-1} norms of the of the tensors A_k and $\mathrm{Riem}^{M^n}_{ijkl}$.

Proof. This result is easy; the only point that requires any explanation is the appearance of the Riemann tensor in the calculation. That arises through perhaps having to permute derivatives in order to ensure that the leading term is given by

$$A(\phi,\phi) = \int_{M^n} E^{i_1j_1}E^{i_2j_2}\ldots E^{i_pj_p}\nabla_{i_1i_2\cdots i_p}\phi\cdot\nabla_{j_1j_2\cdots j_p}\phi + (\text{errorterms})\,d\mu.$$

The very strong ellipticity condition makes it clear at once that the leading term is at least $\lambda^p \int_{M^n} ||\nabla^p\phi||^2_g\,d\mu$, and the usual interpolation argument can then be used to estimate each of the terms of lesser order between a fraction of this and a large multiple of the L^2-norm.
Now we consider the parabolic problem. Let A_t be a smooth family of elliptic operators of order $2p$. 'Smooth' means simply that the component tensor fields should vary smoothly over $M^n \times [0,\infty)$.
To prove the existence of a solution to $D_tu + A_tu = g$, we recast the problem in the natural Hilbert space setting, and solve the resulting operator equation using the following refinement of the Lax-Milgram lemma, which relaxes the continuity assumptions on the bilinear form:

Lemma 7.8 *Let $(H,||\cdot||_H)$ be a Hilbert space and $(\Phi,||\cdot||_\Phi)$ an inner-product space continuously embedded in H. Φ is* not *assumed to be complete. Let $F : H\times\Phi \to \mathbb{R}$ be a bilinear form with the properties that*

- *the mapping $h \to F(h,\phi)$ is continuous for each fixed $\phi\in\Phi$,and*
- *F is coercive on Φ: $F(\phi,\phi)\geq\lambda\,||\phi||^2_\Phi$, for some $\lambda > 0$.*

Then any smooth functional $L\in\Phi^*$ can be realised as a slice through F: there exists $u_L\in H$ such that $L(\phi) = F(u_L,\phi)$ for each $\phi\in\Phi$.

Proof. See [21], Chapter 10, Theorem 16.
For smooth functions $f,g : M^n\times[0,\infty)\to\mathbb{R}$, we introduce the weighted inner products:

$$\begin{aligned}
\langle f,\ g\rangle_{LL_a} &= \int_0^\infty e^{-2at}\,\langle f(\cdot,t),\ g(\cdot,t)\rangle_{L^2(M^n)}\,dt\\
\langle f,\ g\rangle_{LW_a} &= \int_0^\infty e^{-2at}\,\langle f(\cdot,t),\ g(\cdot,t)\rangle_{W^{p,2}(M^n)}\,dt\\
\langle f,\ g\rangle_{WW_a} &= \langle f,\ g\rangle_{LW_a} + \langle D_tf,\ D_tg\rangle_{LL_a};
\end{aligned}$$

we define LL_a, LW_a and WW_a to be the Hilbert spaces formed by completion of $C^\infty(M^n \times [0,\infty))$ in the corresponding norms. Further, let $\Phi = C_C^\infty(M^n \times (0,\infty))$ be the space of smooth functions which vanish for very large and very small times, and let WW_a^0 denote the completion of Φ in WW_a.

Theorem 7.9 *Let A_t be a smooth and uniformly elliptic family of operators of order $2p$. Then, for sufficiently large a (which depends only on A_t), the equation*

$$D_t u + A_t u = g, \qquad u(\bullet, 0) \equiv 0 \tag{7.5}$$

has a unique weak solution in WW_a^0.

Proof. Note first that u is a solution to (7.5) if and only if $\exp(-Mt)\cdot u$ solves the equation $D_t w + (A_t + M\cdot id)w = g\exp(-Mt)$. Choosing $M = Q$, the weight of the error term in the Gårding inequality above, we see that it suffices to solve equations in which the elliptic operator is *strictly* coercive. From here on, we assume this is the case.
Define a bilinear form on $WW_a^0 \times \Phi$ by the formula

$$P(w,\phi) = \langle D_t w,\ D_t\phi\rangle_{LL_a} + \int_0^\infty e^{-2at} A_t(w, D_t\phi)\,dt,$$

and a linear functional on Φ by

$$L(\phi) = \langle g,\ D_t\phi\rangle_{LL_a}.$$

These are simply the results of testing the left and right hand sides of (7.5) with the function $e^{-2at}\cdot D_t\phi$. Fixing ϕ, P is easily seen to be continuous in w. It is just as obvious that L too is continuous with respect to the WW_a-norm. It remains only to show that P is coercive, and Lemma 7.8 will apply.
This is a simple but technical matter. Let $\phi \in \Phi$; then

$$P(\phi,\phi) = ||D_t\phi||^2_{LL_a} + \int_0^\infty e^{-2at} A_t(\phi, D_t\phi)\,dt.$$

Let I denote the second term on the right. Partial integration in time shows that:

$$\begin{aligned} I &= \int_0^\infty e^{-2at}\left(a\cdot A_t(\phi,\phi) - \frac{1}{2}(D_t A_t)(\phi,\phi)\right) dt \\ &\geq \left(a\frac{\lambda^p}{2} - \frac{1}{2}\sup|D_t A_t|\right)\cdot ||\phi||^2_{LW_a}. \end{aligned}$$

If a is chosen large enough, then, this ensures that P is coercive on $\Phi\times\Phi$ with respect to the WW_a-norm; thus, by Lemma 7.7, one can find a $w^* \in WW_a^0$ for which $P(w^*,\phi) = L(\phi)$ for any $\phi \in \Phi$.
It might seem at first that this is insufficient to deliver a weak solution of (7.5), as our test function space is still too small. We are restricted to those functions whose average over time is zero, which would normally mean only that w^* differs from a solution to (7.5) by a time-constant function.

In fact, this problem does not arise because of the weighting given to the measure. Fix some $\psi \in \Phi$, and consider the function

$$\Psi(x,t) = \psi(x,t) - \psi(x,t+B).$$

For B large enough (so large that the support of the second term does not overlap with that of the first), this averages over time to zero, and so it can be represented as $D_t\phi$ for some $\phi \in \Phi$. However, the contributions to $P(w^*,\phi)$ and $L(\phi)$ made by the second term are easily seen to diminish to zero as $B \to \infty$ because of the exponential factor; we therefore have

$$\langle D_t w^*,\ \psi\rangle_{LL_a} + \int_0^\infty e^{-2at} A_t(w^*,\psi)\,dt = \langle g,\ \psi\rangle_{LL_a} \qquad \text{for any } \psi \in \Phi,$$

and with that, w^* is a weak solution to the original equation.

Several points remain open. The solution above has the minimum of regularity needed to make sense of the equation; $W^{1,2}$ in time and $W^{p,2}$ in space. Such a function takes on the zero boundary data continuously only in L^2.

It is possible to accommodate sufficiently smooth nonzero initial data u_0 by considering the equation for $u - u_0$. To apply the result above, this means that $A_t u_0$ needs to be in LL_a, which in turn implies that u_0 has to be a $W^{2p,2}$ function.

This last result is less than optimal. In fact, the natural class for the initial values is $W^{p,2}$; this will follow from the estimates proved in the coming section.

The solution above is unique. This too will follow from the estimates.

7.3 A Priori Estimates for the Linear Equation

This section is concerned with the regularity of the solution obtained above. Crudely, the principal results are first, that the solution is as smooth as the forcing term g allows it to be, and second, that the correspondence between solution u and forcing term is an isomorphism of appropriately defined Banach spaces. The importance of the second of these will become clear in the next section, where we discuss the quasilinear problem.

First, we define the appropriate Hilbert spaces. Let

$$LW_a^s = \left\{ f : M^n \times [0,\infty) \to \mathbb{R} \;\middle|\; \int_0^\infty e^{-2at} \|f\|^2_{W^{s,2}(M^n)}\,dt < \infty \right\}$$

with inner product $\langle f,\ g\rangle_{LW_a^s} = \int_0^\infty e^{-2at}\,\langle f,\ g\rangle_{W^{s,2}(M^n)}\,dt$, and let

$$P_a^m = \{f : M^n \times [0,\infty) \to \mathbb{R} | D_t^i f \text{ exists and is in } LW_a^{2(m-i)p} \text{ for each } i \le m\},$$

where the inner product is the obvious choice:

$$\langle f,\ g\rangle_{P_a^m} = \sum_{i\le m} \langle D_t f,\ D_t g\rangle_{LW_a^{2(m-i)p}}.$$

The description may appear unwieldy, but P_a^m is the natural space forced upon us by the scaling properties of the problem. Accepting the parabolic mantra that one time

derivative corresponds to $2p$ derivatives in space, this somehow describes the space of all functions which are in total $2mp$ times differentiable.
The natural regularity of the boundary values of such a function (and, because M^n has no boundary of its own, we can use the word without ambiguity to refer to the time boundary $t = 0$) is $W^{2p(m-\frac{1}{2}),2}$. The principal result of this section, which we prove below, is this: for any m, the map

$$F(u) = (u_0, g) = (u_0, D_t u - Au) \tag{7.6}$$

is an isomorphism from P_a^m onto $W^{2p(m-\frac{1}{2}),2} \times P_a^{m-1}$.
The technique for proving regularity is standard — we prove Caccioppoli-esque energy estimates first for u and then for its difference quotients. So, suppose u is in some LW_a and solves the equation

$$\langle D_t u,\ \phi\rangle_{LL_a} + \int_0^\infty e^{-2at} A_t(u, \phi)\, dt = \langle g,\ \phi\rangle_{LL_a} \tag{7.7}$$

for any smooth, compactly supported ϕ, and hence by completion for any ϕ in the broader class LW_a. Suppose further that u has initial values u_0, which are taken on only in L^2.

Lemma 7.10 *u satisfies the energy estimate*

$$||u||^2_{LW_a^p} \le C\left(||u_0||^2_{L^2(M^n)} + ||g||^2_{LL_a}\right).$$

Proof. Choose u itself as the test function in (7.7). Then,

$$\langle D_t u,\ u\rangle_{LL_a} + \int_0^\infty e^{-2at} A_t(u, u)\, dt = \langle g,\ u\rangle_{LL_a}. \tag{7.8}$$

For any f which is $W^{1,2}$ in time, partial integration gives

$$\langle D_t f,\ f\rangle_{LL_a} = a\,||f||^2_{LL_a} - \frac{1}{2}\,||f_0||^2_{L^2(M^n)}. \tag{7.9}$$

Thus, returning to (7.6), and using the ellipticity of A_t,

$$\frac{\lambda^p}{2}\,||u||^2_{LW_a^p} + (a - Q)\,||u||^2_{LL_a} \le \frac{1}{2}\,||u_0||^2_{L^2(M^n)} + ||g||_{LL_a} \cdot ||u||_{LL_a}.$$

Now choosing $a > Q$ and handling the forcing term with Young's inequality, the lemma is proved.
To establish that u has derivatives of higher order, we prove estimates for its difference quotients. Defining the difference quotients requires a continuous co-ordinate system, so we have to focus on a single co-ordinate patch. This means multiplying u with a cut-off function in space.

Lemma 7.11 *If $u \in WW_a$ is a solution to (7.7) with initial values $u_0 \in W^{p,2}(M^n)$, then $u \in LW_a^{2p}$, with the estimate*

$$||u||^2_{LW_a^{2p}} \le C\left(||u_0||^2_{W^{p,2}(M^n)} + ||g||^2_{LL_a}\right).$$

Proof. Let $\psi_i : D_1^n \to M^n$, $i = 1, \ldots, N$ be a collection of smooth co-ordinate patches, so chosen that the images $\psi_i(D_{1/2})$ between them cover all of M^n. Let η be a fixed C^∞ cut-off function between $D_{1/2}$ and $D_{3/4}$. Let B_i be the image $\psi_i(D_1)$ in M^n; let B_i^* denote the image of $D_{1/2}$.
Now assume that we already have estimates for u in LW_a^{l+p}, for each $l < k$, and set $\phi = \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u)$ in B_i, where η and the finite differencing operator $\Delta_h f(x) = h^{-1}(f(x + he_j) - f(x))$ are lifted to M^n with the co-ordinate map ψ_i. Outside B_i, we simply extend ϕ to be zero. Notice that, although we don't have uniform estimates in h for the norm of ϕ, the function is at least as regular in space as u itself.
Then, substituting for ϕ in (7.7):

$$\left\langle D_t u,\, \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u)\right\rangle_{LL_a} + \int_0^\infty e^{-2at} A_t(u, \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u))\, dt = \left\langle g,\, \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u)\right\rangle_{LL_a}$$

and so, shifting difference operators with the discrete analogue of partial integration in space,

$$\begin{gathered}\left\langle D_t(\eta^{p+1}\Delta_h^k u),\; \eta^{p+1}\Delta_h^k u\right\rangle_{LL_a} + \textstyle\int_0^\infty e^{-2at} A_t(\Delta_h^k u, \eta^{2p+2}\Delta_h^k u)\, dt = \\ (-1)^k \left\langle g,\, \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u)\right\rangle_{LL_a} - (-1)^k \textstyle\sum_{j=1}^k \binom{k}{j} \int_0^\infty e^{-2at}(\Delta_h^j A_t)(\Delta_h^{k-j} u, \eta^{2p+2}\Delta_h^k u)\, dt,\end{gathered}$$

where the error term on the right arises through applying the product rule for difference quotients to the elliptic term. The discretised product rule is not completely clear-cut; it reads $\Delta_h(fg)(x) = f(x + he_j)\Delta_h g(x) + g(x)\Delta_h f(x)$ — that is, one of the functions is shifted. In our case, this has no bearing; A is completely smooth, and we are concerned only with its pointwise properties. In applying the product rule, then, we always shift the factor in A, never the one in u.
Denote this error term E_1. Then,

$$|E_1| \le C \int_0^\infty \int_{M^n} e^{-2at} \sum_{i,j \le p; 1 \le m \le k} |\nabla^i \Delta_h^{k-m} u| \cdot |\nabla^j(\eta^{2p+2}\Delta_h^k u)|\, d\mu \otimes dt.$$

The first factor in the quadratic part is harmless, because by assumption we already have an estimate for u in LW_a^{k+p-1}. The second may be handled in the same way for $j < p$, while the case $j = p$ is no worse than the helpful term which will be won using the ellipticity of A_t. So, by Young's inequality,

$$|E_1| \le C\,||u||^2_{LW_a^{k+p-1}} + \epsilon \left|\left|\eta^{p+1}\nabla^p(\Delta_h^k u)\right|\right|^2_{LL_a}. \tag{7.10}$$

Notice here that, if we differentiate η^{2p+2} up to p times, there always remain at least $p+1$ powers of η in every term.
Now consider the remaining elliptic term; call it E_2. We have:

$$\begin{aligned} E_2 &= \textstyle\int_0^\infty e^{-2at} A_t(\Delta_h^k u, \eta^{2p+2}\Delta_h^k u)\, dt \\ &= \textstyle\int_0^\infty \int_{M^n} e^{-2at} \sum_{|I|,|J| \le p} A^{IJ} \nabla_I(\Delta_h^k u) \nabla_J(\eta^{2p+2}\Delta_h^k u)\, d\mu \otimes dt,\end{aligned}$$

and whatever else may happen, every term in the sum must contain at least $p+1$ powers of η. Bearing this in mind, and isolating the term of highest order in u,

$$E_2 \geq \int_0^\infty \int_{M^n} e^{-2at} \left(A^{I_p J_p} \nabla_{I_p} f \nabla_{J_p} f \cdot \eta^{2p+2} - C\eta^{p+1} \sum_{i,j \leq p; i+j<2p} |\nabla^i f| \cdot |\nabla^j f| \right) d\mu \otimes dt$$

where now $f = \Delta_h^k u$, and so using the ellipticity of A, Young's inequality and the estimates assumed from the outset for u,

$$E_2 \geq \frac{\lambda^p}{2} \left|\left| \eta^{p+1} \nabla^p (\Delta_h^k u) \right|\right|_{LL_a}^2 - C\,||u||_{LW^{k+p-1}}^2 . \tag{7.11}$$

This controls the terms in A_t in (7.10); now consider those which remain. The first term is of the form handled in (7.9):

$$\left\langle D_t(\eta^{p+1}\Delta_h^k u),\ \eta^{p+1}\Delta_h^k u \right\rangle = a \left|\left| \eta^{p+1}\Delta_h^k u \right|\right|_{LL_a}^2 - \frac{1}{2} \left|\left| \eta^{p+1}\Delta_h^k u_0 \right|\right|_{L^2(M^n)}^2 \tag{7.12}$$

while the forcing term is easily estimated if we now assume $k \leq p$:

$$\begin{aligned} \left\langle g,\ \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u) \right\rangle_{LL_a} &\leq ||g||_{LL_a} \cdot \left|\left| \Delta_{-h}^k(\eta^{2p+2}\Delta_h^k u) \right|\right|_{LL_a} & (7.13)\\ &\leq C\,||g||_{LL_a} \cdot \left|\left| \eta^{2p+2}\Delta_h^k u \right|\right|_{LW_a^k} & (7.14)\\ &\leq C\,||g||_{LL_a} \cdot \left|\left| \eta^{2p+2}\Delta_h^k u \right|\right|_{LW_a^p} . & (7.15) \end{aligned}$$

Combining (7.10), (7.10), (7.11), (7.12) and (7.15), we have

$$\left|\left| \eta^{p+1}\nabla^p(\Delta_h^k u) \right|\right|_{LL_a}^2 \leq C \left(\left|\left| \eta^{p+1}\Delta_h^k u_0 \right|\right|_{L^2(M^n)}^2 + ||u||_{LW_a^{k+p-1}}^2 + ||g||_{LL_a}^2 \right).$$

Now assume that u_0 is in the class $W^{p,2}(M^n)$. This isn't the same as assuming the initial data are taken on in $W^{p,2}$ — indeed, in the above calculation, we only make use of the fact that u, and hence $\Delta_h^k u$, assume their initial values in L^2. Nonetheless, if u_0 is (by chance) $W^{p,2}$, then we have

$$\left|\left| \eta^{p+1}\nabla^p(\Delta_h^k u) \right|\right|_{LL_a}^2 \leq C \left(||u_0||_{W^{p,2}(M^n)}^2 + ||u||_{LW_a^{k+p-1}}^2 + ||g||_{LL_a}^2 \right).$$

To keep the notation from becoming needlessly complicated, this estimate was written for the 'pure' k-th difference quotients; though it is clearly just as true for the mixed difference quotients as well. Thus, we have uniform $W^{p,2}$-bounds in h for all the difference quotients of k-th order in the ball B_i^*, which therefore converge weakly to genuine weak derivatives satisfying the estimate

$$\left|\left| \nabla^{k+p} u \right|\right|_{LL_a(B_i^*)}^2 \leq C \left(||u_0||_{W^{p,2}(M^n)}^2 + ||u||_{LW_a^{k+p-1}}^2 + ||g||_{LL_a}^2 \right),$$

and summing now over all co-ordinate charts, we have

$$\left|\left| \nabla^{k+p} u \right|\right|_{LL_a}^2 \leq C \left(||u_0||_{W^{p,2}(M^n)}^2 + ||u||_{LW_a^{k+p-1}}^2 + ||g||_{LL_a}^2 \right). \tag{7.16}$$

Starting from the energy estimate (7.10), then, and iterating (7.16) over $k = 0, 1, 2, \ldots, p$, this proves the lemma.

Once this much is settled, u is smooth enough that we no longer need represent $A_t u$ as an operator — indeed, we have $D_t u = g - A_t u$ pointwise almost everywhere in $M^n \times [0, \infty)$. This means in turn that $D_t u \in LL_a$ — well, this much we knew already — but with a concrete estimate. Combining this bound with that of the previous lemma, we have then

Lemma 7.12 *If u is a solution to (7.7), where $g \in P_a^0$ and $u_0 \in W^{p,2}(M^n)$, and if a is chosen larger than some constant which depends only on A_t, then $u \in P_a^1$, with the estimate*

$$||u||_{P_a^1}^2 \le Q\left(||u_0||_{W^{p,2}(M^n)}^2 + ||g||_{P_a^0}^2\right).$$

The constant Q depends only on A_t and M^n.

Higher Regularity: The above proof made use only of the minimum of regularity needed to make sense of the defining equation. Now suppose that $g \in P_a^m$ and that $u_0 \in W^{2p(m+\frac{1}{2}),2}(M^n)$. In this case, we choose $\phi = \Delta_{-h}^{2mp+k}(\eta^{2p+2}\Delta_h^{2mp+k}u)$ as test function in (7.7) and proceed almost as before; the one difference is in the bound for the integral which comes from the forcing term. In this case, we exploit the greater smoothness of g:

$$\left\langle g,\ \Delta_{-h}^{2mp+k}(\eta^{2p+2}\Delta_h^{2mp+k}u)\right\rangle_{LL_a} \le ||g||_{LW_a^{2mp}} \cdot ||\eta^{2p+2}\Delta_h^{2mp+k}u||_{LW_a^k}^2$$

and again, the term in u is subsumed by the ellipticity term $(\frac{1}{2}\lambda^p||\eta^{2p+2}\nabla^p(\Delta_h^{2mp+k})||_{LL_a}^2)$ provided $k \le p$. In this case, then, we can iterate (7.16) as far as

$$||u||_{LW_a^{2p(m+1)}}^2 \le Q\left(||u_0||_{W^{2p(m+\frac{1}{2}),2}(M^n)}^2 + ||g||_{LW_a^{2mp}}^2\right). \tag{7.17}$$

So much for the spatial regularity; now consider the time derivatives. Here again we argue incrementally. Assume that, for each $s \le j$, $D_t^s u$ exists and is in the class $LW_a^{2p(m-s)}$. Since $D_t u = g - A_t u$ pointwise almost everywhere, this means that u is $(j+1)$ times differentiable in time; more to this, this gives us the estimate

$$||D_t^{j+1}u||_{LW_a^{2p(m-j)}}^2 \le Q\left(||D_t^j g||_{LW_a^{2p(m-j)}}^2 + \sum_{l \le j}\left|\left|D_t^l u\right|\right|_{LW_a^{2p(m-l)}}^2\right).$$

Starting from (7.17) then, and iterating up to $j = m$, this proves

Lemma 7.13 *If u is a solution to (7.7), where now $g \in P_a^m$ and $u_0 \in W^{2p(m+\frac{1}{2}),2}(M^n)$, then $u \in P_a^{m+1}$, with the estimate*

$$||u||_{P_a^{m+1}}^2 \le Q\left(||u_0||_{W^{2p(m+\frac{1}{2}),2}(M^n)}^2 + ||g||_{P_a^m}^2\right).$$

These estimates are the foundations for the theorem foreshadowed at the start of this section:

Theorem 7.14 *The map* $F : P_a^{m+1} \to W^{2p(m+\frac{1}{2}),2}(M^n) \times P_a^m$ *defined by*

$$F(u) = (u_0, g(u)) = (u_0, D_t u - A_t u) \tag{7.18}$$

is a Banach space isomorphism.

Proof. First we show that F is continuous. This much is clear for the second component; the interesting part is the continuity of the initial values.
Let $u \in P_a^{m+1}$. Then u has, at the very least, a weak time derivative which is in LL_a. This ensures that $\|u_t\|_{L^2(M^n)}$ is a Lipschitz function of time; in particular, u_t converges in L^2 to u_0 as $t \to 0$.
To prove u_0 is better than merely an L^2 function, we argue again with difference quotients. Recycling the notation used to derive the estimates above, we have

$$\left\|\eta^2 \Delta_h^{(2m+1)p} u_0\right\|^2_{L^2(M^n)} = 2a \left\|\eta^2 \Delta_h^{(2m+1)p} u\right\|^2_{LL_a} - 2\left\langle D_t \Delta_h^{(2m+1)p} u,\ \eta^2 \Delta_h^{(2m+1)p} u\right\rangle_{LL_a}$$

(which is simply equation (7.9) rearranged). The first term on the right is clearly controlled by the LW_a^{2mp+p}-norm of u, and in turn by the P_a^{m+1}-norm. To handle the second, we shift p difference operators from one factor in the inner product to the other, giving

$$\left\|\eta^2 \Delta_h^{(2m+1)p} u_0\right\|^2_{L^2(M^n)} \leq C \|u\|^2_{P_a^{m+1}} - 2(-1)^p \left\langle D_t \Delta_h^{2mp} u,\ \Delta_{-h}^p(\eta^2 \Delta_h^{(2m+1)p} u)\right\rangle_{LL_a}.$$

The second term is now easily estimated using the Cauchy-Schwarz inequality: the first factor by the LW_a^{2mp}-norm of $D_t u$, the second by the LW_a^{2mp+2p}-norm of u itself. Again, both of these are contained within the P_a^{m+1} norm, so we have

$$\left\|\eta^2 \Delta_h^{(2m+1)p} u_0\right\|^2_{L^2(M^n)} \leq C \|u\|^2_{P_a^{m+1}}$$

Since this is independent of h, we may pass to the weak limit, and infer that u_0 has weak derivatives of order $(2m+1)p$, with the estimate

$$\|u_0\|^2_{W^{2p(m+\frac{1}{2}),2}(M^n)} \leq C \|u\|^2_{P_a^{m+1}},$$

and with this, the mapping $u \to u_0$ is continuous between the given spaces.
The next step is to show that F has an inverse. This means showing that the equation $Fu = (u_0, g)$ for given u_0 and g is uniquely solvable in the appropriate class. This follows from the existence theorem 7.9 together with the regularity theory above. The one remaining loophole is that the initial data in the existence theorem were assumed to be at least $W^{2p,2}$. We now close this.
Let $u_0 \in W^{p,2}(M^n)$, and let u_0^i be a sequence of smooth functions on M^n converging to u_0 in the $W^{p,2}$ norm. For each i, the existence result returns a function u^i in P_a^1 which solves the equation $Fu^i = (u_0^i, g)$. Because of the P_a^1 estimate above, these converge in the P_a^1 norm to a limit u. By the continuity of F, $Fu = (u_0, g)$ and we have our solution. The uniqueness claim is an immediate consequence of the P_a^1 estimate.

Lastly, we wish to show that the inverse of F is continuous. But this is exactly the content of the estimates above. ∎

7.4 The Quasilinear Equation

The isomorphism estimates of the previous section are the means by which we proceed from the linear equation to the quasilinear.
Consider the quasilinear problem of $2p$-th order

$$D_t u = Q(u) = A^{i_1 j_1 i_2 j_2 \dots i_p j_p} D_{i_1 i_2 \dots i_p j_1 \dots j_p} u + b, \qquad u(\bullet, 0) = u_0, \tag{7.19}$$

where the functions $A = A(x, u, \nabla u, \nabla^2 u, \dots, \nabla^{2p-1} u)$ and $b = b(x, u, \nabla u, \dots, \nabla^{2p-1} u)$ are smooth in all their arguments and where A is elliptic, at least in a neighbourhood of the initial data. By 'elliptic', we mean still that A factors as a power of a positive tensor of second rank.

Theorem 7.15 *For any smooth initial data u_0 for which $A(u_0)$ is ellipic, the quasilinear problem (7.19) has a smooth solution defined on some finite time interval $[0,T)$. The solution is unique and depends continuously on u_0.*

Proof. Define an operator $F : P_a^{m+1} \longrightarrow W^{2p(m+\frac{1}{2}),2}(M^n) \times P_a^m$ thus:

$$F(u) = (u_0, D_t u - Q(u)).$$

F is Fréchet differentiable, and, by the results of the previous section, the linearised operator $DF(z)$ is an isomorphism provided $A(z_t)$ is elliptic for each t.
Let w be the solution to the 'frozen' linear problem,

$$D_t w - A(u_0) \cdot D^{2p} w = b(u_0), \qquad w_0 = u_0.$$

Since u_0 is smooth, the same is true of w, and in particular, w_t converges smoothly to u_0 as $t \longrightarrow 0$. It follows that $A(w_t)$ is elliptic for sufficiently small times.
Now linearise F around w. To simplify the following argument, we shall assume that $A(w_t)$ is in fact elliptic for *all* times; if this is not the case by nature, we simply tamper with w outside some interval $0 \leq t < \delta$. In the following analysis, we are in any case only concerned with the properties of w in a neighbourhood of $t = 0$.
The Implicit Function Theorem for mappings between Banach spaces (see, for instance, [15]) ensures that F is locally a diffeomorphism from a neighbourhood of w to a neighbourhood U of $F(w)$. However, $F(w) = (u_0, D_t w - Q(w))$, and hence, bearing in mind the definition of w,

$$F(w) = (u_0, (b(u_0) - b(w)) - (A(u_0) - A(w)) \cdot D^{2p} w),$$

and the second component here converges smoothly to zero for $t \longrightarrow 0$. In particular, then, we can choose $q \in W^{(2m+1)p,2}(M^n) \times P_a^m$ approximating $F(w)$ with second component vanishing in some whole interval of time $[0, \epsilon)$. For appropriately small ϵ, this q will fall inside the neigbourhood U.

Since F is a local diffeomorphism, this means there is some $u \in P_a^{m+1}$ such that $F(u) = (u_0, q)$. In particular, then, the second component of $F(u)$ vanishes for small time: $D_t u = Q(u)$ for $t < \epsilon$. With that, we have a solution to the quasilinear problem in P_a^{m+1}.
However, we may conclude more. The initial values here need not be precisely u_0; they may be taken from some neighbourhood of u_0 in $W^{(2m+1)p,2}(M^n)$. So we have a solution for all nearby initial data; and more to this, the Implicit Function Theorem guarantees that the correspondence between solution and initial data is continuous in the appropriate norms.
All this does not yet suffice to prove the claim of the theorem. The argument above gives solutions of increasing smoothness, but possibly defined on intervals of decreasing duration. To prove the existence of a C^∞ solution on a definite time interval, we need a further property from the regularity theory.
Let u be a solution now to (7.19) in $[0, \epsilon)$ which is uniformly bounded in C^m. For m sufficiently large, this will imply that u is in fact absolutely smooth.
Precisely, we consider u as a fixed function now, and derive the evolution equation for $w_j = \nabla^j u$. If u is uniformly bounded in C^{2p-1+a}, then this may be considered as a *linear* equation for w with coefficients which depend on u and its derivatives of up to $(2p-1)$-th order. These are therefore uniformly bounded in C^a. This limits the growth of w_j to exponential in time; in particular, each w_j remains bounded in the interval $[0, \epsilon)$, and this means that u is in fact C^∞ for $t < \epsilon$. This detail completes the proof of the theorem.

7.5 Short-time Existence for Geometric Equations

Now at last we are in a position to prove the existence of short-time solutions for a class of geometric evolution equations. Consider the deformation processes

$$\frac{\partial g}{\partial t} = \Big(-(-\Delta)^p R + \phi(g, \mathrm{Riem}, \nabla\mathrm{Riem}, \nabla^2\mathrm{Riem}, \ldots, \nabla^{2p-1}\mathrm{Riem})\Big)\, g \tag{7.20}$$

for metrics, and

$$\frac{\partial F}{\partial t} = \Big(-(-\Delta)^p H + \phi(F, n, A, \nabla A, \nabla^2 A, \ldots, \nabla^{2p-1} A)\Big)\, n \tag{7.21}$$

for immersions, where ϕ is in each case smooth in all its arguments, but otherwise arbitrary. This is the structure of the total-curvature problems considered in subsequent chapters; it also appears in a number of equations unrelated to variational problems. The cases $n = 1, \phi = 0$ in the above equations fall into this last category; they arise in general relativity [13] and crystal formation [10].
The derived equations of lemmas 7.3 and 2.1.6 describe the change in the curvature which follows from the above defomations. To highest order,

$$\frac{\partial R}{\partial t} = -(-\Delta)^{p+1} R + \cdots, \qquad \frac{\partial H}{\partial t} = -(-\Delta)^{p+1} H + \cdots$$

and these are parabolic, in the sense that the linearisations are parabolic, even using the very restrictive notion of ellipticity introduced in section 2.2.
It is not surprising, then, that (7.20) and (7.21) themselves are, properly interpreted, also parabolic equations. In this section, we show how they may be reduced to strictly

parabolic, quasilinear scalar equations, and hence that they have short-time solutions under appropriate initial conditions.

Theorem 7.16 *Let M^n $(n \geq 2)$ be a smooth manifold with metric g_0. There exists a solution to (7.20) defined for some period of time $0 \leq t < T$ which takes g_0 as its initial values. It is unique.*

Proof. If there is a solution at all, it may be represented as in Lemma 7.1 in the form

$$g_t(x) = \exp 2u(x,t) \cdot g_0(x)$$

where the conformal factor u evolves according to

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Big(-(-\Delta)^p R + \phi(g, \mathrm{Riem}, \nabla\mathrm{Riem}, \nabla^2\mathrm{Riem}, \ldots, \nabla^{2p-1}\mathrm{Riem})\Big). \tag{7.22}$$

Lemma 7.2 shows how the Laplace operator and Ricci curvature of g_t may be expressed in terms of those of g_0 and the conformal factor. The same is clearly true of the Riemann curvature of g and its covariant derivatives. So one can rewrite (7) in the form

$$\frac{\partial u}{\partial t} = -(p-1)\exp(-2(p+1)u)\cdot(-\Delta^0)^{p+1}u + \phi\left(x, u, \nabla^0 u, (\nabla^0)^2 u, \ldots, (\nabla^0)^{2p+1}u\right),$$

where ϕ is smooth in all its arguments. This, however, is now a quasilinear scalar equation on (M^n, g_0); it is parabolic in the sense of Theorem 7.15, and it therefore has a unique solution with initial values $u_0 \equiv 0$ for some period of time $0 \leq t < T$. Starting from this solution, we now take (7.22) as the *definition* of g_t and this gives a solution to the geometric problem.

When it comes to hypersurfaces, life is not quite so simple. In this case, the reduction to a quasilinear problem is not as clear-cut. The idea is to represent the the hypersurface at time t as a graph in Fermi co-ordinates over the initial hypersurface and consider the deformation process as a quasilinear scalar equation for the height function.

Consider again an isometric immersion $F_0 : (M^n, g) \hookrightarrow (N^{n+1}, \bar{g})$. Let $M = M^n \times (-\epsilon, \epsilon)$, where ϵ is chosen so small that the map

$$F : M \to N^{n+1} : (x, h) \to \exp_{F_0(x)}(hn(x))$$

is itself an immersion. If F_0 is an embedding, then F is simply the map which generates Fermi co-ordinates on a tubular neighbourhood of $F_0(M^n)$. The tangent space to M is spanned by the vectors $\frac{\partial}{\partial x^i}$, $i = 1, 2, \ldots n$ and $\frac{\partial}{\partial h}$.

Let G be the metric induced on M by F. It follows from the Gauß Lemma that the geodesics $\{x\text{=constant}\}$ are orthogonal to the parallel surfaces $\{h\text{=constant}\}$, and hence that G may be broken into a sum:

$$G(x, h) = G_h(x) + dh \otimes dh,$$

where G_h is the metric on the surface $M^n \times \{h\}$. In particular, $G_0 = F_0^* \bar{g} = g$.

Now let $u : M^n \longrightarrow (-\epsilon, \epsilon)$ be a smooth function, and consider the graph of u in M, parametrised by $\psi : M^n \longrightarrow M : x \longrightarrow (x, u(x))$. The tangent space to graph u, considered as a submanifold of M, is spanned at a point $\psi(x)$ by the vectors

$$\psi_* \frac{\partial}{\partial x^i} = \frac{\partial \psi}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{\partial u}{\partial x^i}\frac{\partial}{\partial h},$$

and the induced metric M^n inherits from graph u is given by

$$\gamma_{ij}(x) = \psi^* G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = G\left(\frac{\partial \psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j}\right) = (G_{u(x)})_{ij} + \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}.$$

The normal to graph u — one sees this simply by computing the product with each of the basis vectors above — is given by

$$n_{\mathrm{graph}\, u}(x) = \frac{1}{N}\left\{\frac{\partial}{\partial h} - G^{pq}_{u(x)}\frac{\partial u}{\partial x^p}\frac{\partial}{\partial x^q}\right\},$$

where $N = \left\{1 + G_{u(x)}(\nabla^{G_u} u, \nabla^{G_u} u)\right\}^{\frac{1}{2}}$. From this, we derive an expression for the second fundamental form of the graph:

$$\begin{aligned} h_{ij}(x) = G\left(\nabla^M_{\frac{\partial}{\partial x^i}\psi} n_{\mathrm{graph}\, u}, \frac{\partial \psi}{\partial x^j}\right) &= G\left(\nabla^M_{\frac{\partial}{\partial x^i}\psi}\left(\frac{1}{N}\left\{\frac{\partial}{\partial h} - G^{pq}_{u(x)}\frac{\partial u}{\partial x^p}\frac{\partial}{\partial x^q}\right\}\right), \frac{\partial \psi}{\partial x^j}\right) \\ &= -\frac{1}{N} G\left(\nabla^M_{\frac{\partial}{\partial x^i}\psi}\left\{G^{pq}_{u(x)}\frac{\partial u}{\partial x^p}\frac{\partial}{\partial x^q}\right\}, \frac{\partial \psi}{\partial x^j}\right) \\ &= -\frac{1}{N}\frac{\partial^2 u}{\partial x^i \partial x^j} + \mathrm{terms in} x, u \mathrm{and} \nabla u \\ &= -\frac{1}{N}\nabla^{G_0}_i \nabla^{G_0}_j u + \sigma_{ij}(x, u, \nabla u) \end{aligned}$$

and in the last equation, σ is a smooth tensor field. Its exact form need not concern us. Tracing with the metric on graph u, this yields the mean curvature:

$$H = -\frac{1}{N}\gamma^{ij}\nabla^{G_0}_i \nabla^{G_0}_j u + (\gamma^{ij}\sigma_{ij})(x, u, \nabla u), \tag{7.23}$$

nothing here that γ^{ij} too depends only on x, and u and its first derivatives.
Since γ depends on ∇u, its Christoffel symbols depend in turn upon derivatives of u of second order. The same is then true of covariant derivatives on graph u; while q-th covariant derivatives depend on derivatives of u up to and including the $(q+1)$-th.
In particular, if f is a function graph $u \longrightarrow \mathbb{R}$, then

$$\Delta^\gamma f = \gamma^{ij}\nabla^{G_0}_i \nabla^{G_0}_j f + \mathrm{terms in} x, u, \cdots, \nabla^3 u, f \mathrm{and} \nabla f. \tag{7.24}$$

Thus, the leading term in the mean curvature of graph u could be rewritten (and more naturally so) in terms of the laplacian on graph u; this, however, is precisely what we

don't want for the purposes of the short-time existence theory. We wish to repackage our equations for a fixed manifold with a fixed metric.
It follows from (7.23), (7.23) and (7.24), together with the preceding remarks about covariant derivatives, that the instantaneous speed of graph u under (7.21) is given by

$$(-1)^p \frac{1}{N} \gamma^{i_1 j_1} \gamma^{i_2 j_2} \ldots \gamma^{i_{p+1} j_{p+1}} \nabla_{i_1}^{G_0} \nabla_{j_1}^{G_0} \cdots \nabla_{i_{p+1}}^{G_0} \nabla_{j_{p+1}}^{G_0} u + \tilde{\phi}(x, u, \nabla u, \ldots, \nabla^{2p+1} u),$$

where $\tilde{\phi}$ is another function which is smooth in all its arguments but whose precise form is left unstated.

Theorem 7.17 *For any smooth hypersurface immersion $F_0 : M^n \longrightarrow N^{n+1}$, there exists a unique solution to the flow problem (7.21) defined on some interval $0 \leq t < T$ and taking F_0 as its initial values.*

Proof. The quasilinear equation

$$\frac{\partial u}{\partial t} = (-1)^n \gamma^{i_1 j_1} \gamma^{i_2 j_2} \ldots \gamma^{i_{n+1} j_{n+1}} \nabla_{i_1}^{G_0} \nabla_{j_1}^{G_0} \cdots \nabla_{i_{n+1}}^{G_0} \nabla_{j_{n+1}}^{G_0} u + N\tilde{\phi}(x, u, \nabla u, \ldots, \nabla^{2n+1} u)$$

is parabolic at $u = 0$, and therefore has a unique smooth solution with zero initial values defined for some period of time $0 \leq t < T$.
Consider now the family of surfaces graph u_t, parametrised by $\psi_t(x) = (x, u(x,t))$. The initial surface $u \equiv 0$ is simply M^n itself; and the family develops according to the equation

$$\frac{\partial \psi}{\partial t} = \frac{\partial u}{\partial t} \cdot \frac{\partial}{\partial h}.$$

In view of the preceding discussion, $\frac{\partial}{\partial t} u$ is the speed graph u_t would have under the flow (7.21), rescaled by a factor of N. So, ψ is a solution to the geometric equation

$$\frac{\partial \psi}{\partial t}(x) = N\omega \cdot \frac{\partial}{\partial h},$$

where ω is the speed in (7.21).
However, decomposing $\frac{\partial}{\partial h}$ into tangential and normal parts,

$$\frac{\partial}{\partial h} = \frac{1}{N} n_{\mathrm{graph}\, u_t} + \lambda$$

where $\lambda(x)$ is a tangential vector field. Since ψ is smooth, so is λ. Thus, the evolution of ψ can be rewritten,

$$\frac{\partial \psi}{\partial t}(x) = \omega \cdot n_{\mathrm{graph}\, u_t}(x) + N\lambda,$$

and so ψ is a solution to the original geometric problem, give or take a tangential motion. Tangential motion has no effect on the solution hypersurfaces, considered as sets; it affects only their parametrisations. In crude terms, it is the movement of a bicycle chain: each individual link is in motion, but the shape of the chain taken as a whole does not change.

Let $\alpha : M^n \times [0,T) \to M^n$ be a smoothly varying family of diffeomorphisms of M^n, with α_0 being the identity map and

$$\frac{\partial \alpha}{\partial t} = -\psi_t^* \lambda.$$

Since λ and ψ are completely smooth up to time T, this may be integrated directly to a unique solution for α. Now define

$$\Psi(x,t) = \psi(\alpha(x,t),t).$$

Then,

$$\begin{aligned}\frac{\partial \Psi}{\partial t}(x,t) &= \frac{\partial \psi}{\partial t}(\alpha(x,t),t) + \psi_*(\frac{\partial \alpha}{\partial t}(x,t),t) \\ &= (\omega \cdot n(\Psi(x,t)) + \lambda(\Psi(x,t))) - \lambda(\Psi(x,t)),\end{aligned}$$

and so Ψ now solves the geometric problem, at least in M, and because F is an isometry, $F \circ \Psi$ now satisfies the original geometric equation in N^{n+1}.
These two theorems provide the short-time existence results for the flows which appear in later chapters; as another special case, Theorem 7.17 incorporates the L^2-gradient flow for the Willmore energy of a surface immersed in a Riemannian manifold.

References

[1] B. Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. **2** (1994), 151–171.

[2] B. Andrews, *Gauss curvature flow: The fate of the rolling stones*, preprint ANU Canberra (1998), pp10.

[3] B. Andrews, *Contraction of convex hypersurfaces by their affine normal*, J. Diff. Geom. **43** (1996), 207–230.

[4] B. Andrews, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Diff. Geom. **39** (1994), 407–431.

[5] B. Andrews, *Monotone quantities and unique limits for evolving convex hypersurfaces*, IMRN **20** (1997), 1001–1031.

[6] B. Andrews, private communication.

[7] S.B. Angenent, J.J.L. Velazques, *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math. **482** (1997), 15–66.

[8] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, Springer-Verlag, New York.

[9] K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Mathematical Notes, Princeton University Press, Princeton.

[10] J. W. Cahn, W. C. Carter, A. R. Roosen and J. E. Taylor, *Shape Evolution by Surface Diffusion and Surface Attachment Limited Kinetics on Completely Faceted Surfaces* available over *http://www.ctcms.nist.gov/~roosen/SD_SALK/.*

[11] Y.G. Chen, Y. Giga, and S. Goto, *Uniqueness and Existence of Viscosity Solutions of Generalized Mean Curvature Flow Equations*, J Diff.Geom. **33** (1991), 749–786.

[12] B. Chow, *Deforming convex hypersurfaces by the nth root of the Guassian curvature*, J.Diff.Geom. **23** (1985), 117–138.

[13] P. T. Chruściel, *On Robinson-Trautman Space-Times,* Centre for Mathematical Analysis Research Report CMA-R23-90, The Australian National University, Canberra.

[14] D.M.DeTurck, *Deforming metrics in direction of their Ricci tensors,* J.Diff.Geom. **18** (1983), 157–162.

[15] J. Dieudonné, *Foundations of Modern Analysis,* Academic Press, New York and London.

[16] K. Ecker, G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent.Math. **105** (1991), 547–569.

[17] K. Ecker, G. Huisken, *Mean curvature evolution of entire graphs*, Ann.of Math. **130** (1989), 453–471.

[18] S. D. Eidel'man, *Parabolic Equations,* in *Partial Differential Equations VI,* Encyclopaedia of Mathematical Sciences, Volume 63, editors Yu. V. Egerov and M. A. Shubin, Springer-Verlag, Berlin, Heidelberg, New York.

[19] L. C. Evans, J. Spruck, *Motion of Level Sets by Mean Curvature I,* J. Diff.Geom. **33** (1991), 635–681.

[20] W.J. Firey, *Shapes of worn stones*, Mathematica **21** (1974), 1–11.

[21] Avner F. Friedman, *Partial Differential Equations of Parabolic Type,* Prentice-Hall, Englewood Cliffs, N.J.

[22] M. E. Gage and R. S. Hamilton, *The Heat Equation Shrinking Convex Plane Curves,* J. Diff. Geom. **23**, 285–314.

[23] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J.Diff.Geom. **32** (1990), 299–314.

[24] R. Geroch, *Energy Extraction*, Ann. New York Acad. Sci. **224** (1973), 108–17.

[25] M. Grayson, *The heat equation shrinks embedded plane curves to points*, J. Diff. Geom. **26** (1987), 285–314.

[26] M. Grayson, *Shortening embedded curves*, Annals Math. **129** (1989), 71–111.

[27] R. S. Hamilton, *The Ricci Flow on Surfaces,* Contemporary Mathematics **71**, 237–262.

[28] R. S. Hamilton, *The formation of singularities in the Ricci Flow*, Surveys in Differential Geometry Vol. II, International Press, Cambridge MA (1993), 7–136.

[29] R. S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Diff. Geom. **41** (1995), 215-226.

[30] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.

[31] R. S. Hamilton, *Four manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), 1-92.

[32] R. S. Hamilton, *Isoperimetric estimates for the curve shrinking flow in the plane*, Modern Methods in Compl. Anal., Princeton Univ.Press (1992), 201-222.

[33] R. S. Hamilton, *CBMS Conference Notes,* Hawaii.

[34] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geometry **20** (1984), 237-266.

[35] G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), 463–480.

[36] G. Huisken, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Diff. Geometry **31** (1990), 285-299.

[37] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Proceedings of Symposia in Pure Mathematics **54** (1993), 175–191.

[38] G. Huisken, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), 127–134.

[39] G. Huisken, T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, preprint http://poincare.mathematik.uni-tuebingen.de, to appear.

[40] G. Huisken, T. Ilmanen, *The Riemannian Penrose inequality*, IMRN **20** (1997), 1045–1058.

[41] G. Huisken, C. Sinestrari, *Mean curvature flow singularities for mean convex surfaces*, Calc. Variations, to appear.

[42] G. Huisken, C. Sinestrari, *Convexity estimates for mean curvature flow and singularities for mean convex surfaces*, preprint, to appear.

[43] T. Ilmanen, *Elliptic regularisation and partial regularity for motion by mean curvature*, Memoirs AMS **108** (1994), pp90.

[44] J. L. Lions, *Sur les problèmes mixtes pour certaines systèmes paraboliques dans des ouverts non cylindriques,* Ann. l'Insitute Fourier **7**, 143–182.

[45] W. W. Mullins, *Two-dimensional Motion of Idealised Grain Boundaries,* J. Appl. Phys. **27**/8 (August 1956), 900–904.

[46] G. Sapiro, A. Tannenbaum, *On affine plane curve evolution*, J. Funct. Anal. **119** (1994), 79–120.

[47] R. Schoen, L. Simon, S. T. Yau, *Curvature estimates for minimal hypersurfaces,* Acta Math. **134** (1975), 276–288.

[48] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann.of Math. **88** (1968), 62–105.

[49] K. Smoczyk, *Starshaped hypersurfaces and the mean curvature flow*, Preprint (1997), 13pp.

[50] V. A. Solonnikov, *Green matrices for parabolic boundary value problems,* Zap. Nauch. Sem. Leningr. Otd. Math. Inst. Steklova **14** 256–287, translated in Semin. Math. Steklova Math. Inst. Leningrad (1972), 109–121.

[51] F. Trèves, *Relations de domination entre opérateurs différentiels,* Acta. Math. **101**, 1–139.

[52] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985), 867–882.

[53] J. Urbas, *On the Expansion of Starshaped Hypersurfaces by Symmetric Functions of Their Principal Curvatures*, Math. Z. **205** (1990), 355–372.

[54] B. White, *Partial Regularity of Mean Convex Hypersurfaces Flowing by Mean Curvature*, IMRN **4** (1994), 185–192.