Dynamics of scalar fields in the background of rotating black holes. II. A note on superradiance

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We analyze the amplification due to so-called superradiance from the scattering of pulses off rotating black holes as a numerical time evolution problem. We consider the ''worst possible case'' of scalar field pulses for which superradiance effects yield amplifications $<1\%$. We show that this small effect can be isolated by numerically evolving quasi-monochromatic, modulated pulses with a recently developed Teukolsky code. The results show that it is possible to study superradiance in the time domain, but only if the initial data is carefully tuned. This illustrates the intrinsic difficulties of detecting superradiance in more general evolution scenarios. [S0556-2821(98)04418-X]

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In the past few years, we have been involved in the development of a numerical code for the time evolution of perturbations of rotating black holes based on the Teukolsky equation [1,2]. There are several motivations for this work. One is the desire to reexamine problems that have previously only been approached in the frequency domain but now under a ''time-evolution'' point of view. More importantly, our ultimate goal is to provide a framework that will be used to extend the close-limit approximation of head-on black hole collisions to the case of inspiral black hole mergers [3]. Head-on, close-limit collisions [4] view the merger as perturbations of non-rotating black holes. In contrast, an inspiral close-limit approximation requires perturbations about a rotating black hole.

Our Teukolsky code project took us first to study the dynamics of scalar fields in Kerr geometry [1]. This work mainly concerned the late-time, power-law behavior of a scalar perturbation. The second installment concerned gravitational perturbations [2] and discussed the full dynamical response of a black hole to an external perturbation, namely the quasinormal mode ringing and the subsequent late-time tails. In Ref. [2], we also dealt briefly with superradiance. However, although the results in Ref. [2] indicated the presence of superradiance, we feel that our previous analysis was not completely satisfactory. Hence, the goal of this short paper is to return to the issue of superradiance in a setting that yields unequivocal evidence for the superradiance phenomenon.

The direct approach to measure superradiance from the time evolution of perturbations of rotating black holes is to compute the energy flux going ''down the hole.'' For perturbative fields that possess well-defined stress-energy tensors (e.g. scalar and electromagnetic fields), it is possible to construct such a conserved energy flux [5]. The case of gravitational perturbations is not that simple [5,6]. For this reason, we will concentrate our analysis on the ''simple'' case of scalar perturbations. The price to pay is that superradiant effects in this case are $<1\%$ [7], thus requiring a highly accurate evolution code.

For scalar perturbations, the Teukolsky equation in Boyer-Lindquist coordinates reads

$$\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\Phi}{\partial t^2}+\frac{4iMamr}{\Delta}\frac{\partial\Phi}{\partial t}-\frac{\partial}{\partial r}\left(\Delta\frac{\partial\Phi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)-m^2\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\Phi=0. \tag{1}$$

Above, M is the mass of the black hole, a is its angular momentum per unit mass and $\Delta\equiv r^2-2Mr+a^2$. The two horizons of the black hole follow from $\Delta=0$, and correspond to $r_\pm=M\pm\sqrt{M^2-a^2}$. Reference to the azimuthal angle φ has been removed by assuming $\Phi\propto e^{im\varphi}$.

Traditionally, solutions to the Teukolsky equation have been obtained via separation of variables by (i) assuming a harmonic time-dependence and (ii) using a suitable set of angular functions (standard spheroidal wave-functions [8] for scalar perturbations). That is,

$$\Phi=\int d\omega e^{-i\omega t}\sum_{l,m} e^{im\varphi}R_{lm}(r,\omega)S_{lm}(\theta,\omega). \tag{2}$$

It is important to notice that the angular functions depend explicitly on the frequency ω, namely time. Given the solution form (2), the problem reduces to a single ordinary differential equation (ODE) for $R_{lm}(r,\omega)$:

$$\frac{d^2R_{lm}}{dr_*^2}+\left[\frac{K^2+(2am\omega-a^2\omega^2-E)\Delta}{(r^2+a^2)^2}-\frac{dG}{dr_*}-G^2\right]R_{lm}=0, \tag{3}$$

where $K=(r^2+a^2)\omega-am$, $G=r\Delta/(r^2+a^2)^2$, and the tortoise coordinate r_* is defined from $dr_*=(r^2+a^2)/\Delta dr$. The variable E is the angular separation constant, real for real frequencies. When $a\to 0$, it reduces to $l(l+1)$, and, for nonzero a, it can be obtained from a power series in $a\omega$ [9].

The physical solution to Eq. (3) is defined by the asymptotic behavior

$$R_{lm}\sim\begin{cases}\mathcal{T}e^{-i(\omega-m\omega_+)r_*} & \text{as } r\to r_+,\\ \mathcal{R}e^{i\omega r_*}+e^{-i\omega r_*} & \text{as } r\to+\infty,\end{cases} \tag{4}$$

where $\omega_+\equiv a/2Mr_+$ is the angular velocity of the event horizon. $\mathcal{T}$ and $\mathcal{R}$ denote the transmission and reflection coefficients, respectively, satisfying $(1-m\omega_+/\omega)\mathcal{T}=1-\mathcal{R}$. Superradiance ($\mathcal{R}>1$) occurs then when $\omega<m\omega_+=ma/(2Mr_+)$. Alternatively, one can deduce that energy can be extracted from the black hole immediately from the boundary condition (4). If $\omega<m\omega_+$, the solution $\propto\exp[-i(\omega-m\omega_+)r_*]$, which behaves as ''ingoing'' into the horizon according to a local observer, will in fact correspond to waves coming out of the hole according to an observer at infinity. That is, for superradiant frequencies, one would expect to find energy flowing out from the horizon, cf. [10].

Figure 1 shows the reflection coefficient in the case when $l=m=2$, obtained by a straightforward integration of Eq. (3) and subsequent extraction of $\mathcal{R}$. The maximum amplification in this case is close to 0.2%, which agrees with the value of 0.3% obtained by Teukolsky and Press [5,7]. They also considered electromagnetic waves (4.4% max. amplification) and gravitational perturbations (138% max. amplification). The results in Fig. 1 agree with the standard conclusions regarding the apparent ''size'' of a rotating black hole as seen by different observers. It is well-known (e.g. [11]) that the black hole will appear larger to a particle moving around it in a retrograde orbit than to a particle in a prograde orbit. This is illustrated by the fact that the unstable circular photon orbit (at $r=3M$ in the non-rotating case) is located at $r=4M$ for a retrograde photon, while it lies at $r=M$ for a prograde photon. In our case, we have prograde motion when ω/m is positive and retrograde motion when ω/m is negative. The data in Fig. 1 correspond to $m=2$, and the enhanced reflection for positive frequencies as $a\to M$ has the effect that the black hole ''looks smaller'' to such waves. Conversely, the slightly decreased reflection for negative frequencies leads to the black hole appearing ''larger'' as $a\to M$.

To study superradiance in the time domain, we follow an idea introduced in Ref. [2] and set up a pulse containing mainly frequencies in the interval $0<\omega<m\omega_+$ with $m>0$. The analysis becomes easier and better suited to comparisons with frequency domain calculations if the pulse is ''almost monochromatic.'' This is achieved using as initial data an ingoing Gaussian pulse modulated by a monochromatic wave: $\Phi\propto\exp[-(r_*-r_o+t)^2/b^2-i\sigma(r_*-r_o+t)]$, where $r_o\gg M$ and σ the modulation frequency. The power spectrum of this pulse is $P(\omega)=P_{\max}e^{-(\omega-\sigma)^2b^2/4}$, so we ought to be able to detect superradiance if $0<\sigma<m\omega_+$. Furthermore, it is not enough for the peak of the power spectrum ($\omega=\sigma$) to lie within the superradiant frequency window. To maximize the effect, we need also to minimize the ''frequency overlap'' into the non-superradiant regime. This can be accomplished by a suitably choice of the parameter b. For instance, a non-superradiance frequency overlap $P(m\omega_+)/P_{\max}=\epsilon$, requires $b=2\sqrt{\ln(1/\epsilon)}/(m\omega_+-\sigma)$.

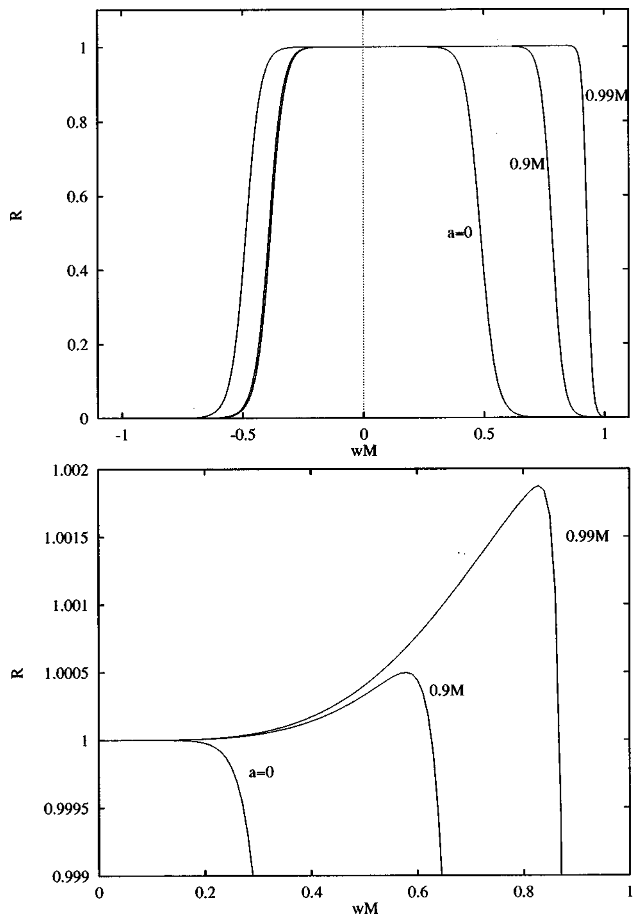

FIG. 1. Reflection coefficient ($\mathcal{R}$) for different values of the angular momentum parameter (a) with $l=m=2$. Superradiance is present in the interval $0<\omega M<ma/2r_+$. The bottom panel is a close-up of this superradiance regime.

Our Teukolsky code was described in detail in Ref. [1], but there is one specific issue that is important for the present study that has not yet been discussed. To avoid numerical difficulties, we perform a coordinate transformation and replace the azimuthal angle φ with the ''ingoing Kerr-coordinate'' $\tilde{\varphi}$ defined by $d\tilde{\varphi}=d\varphi+(a/\Delta)dr$. The replacement $\varphi\to\tilde{\varphi}$ changes the symmetry of the equations. While the original Teukolsky equation (1) is symmetric under the change $(m,\varphi)\to(-m,-\varphi)$, the Teukolsky equation in terms of $\tilde{\varphi}$ is not. That is, while evolutions for the same Gaussian pulse (unmodulated) should lead to the same emerging scalar waves for $\pm m$ in the original case, this will not happen when we use $\tilde{\varphi}$. To ensure that the anticipated symmetries are present in our results and that the pulse is centered in the superradiance window, we construct the modulated pulse in Boyer-Lindquist coordinates and then transform it to the $\tilde{\varphi}$ coordinate system.

To unveil the amplification due to superradiance, we will

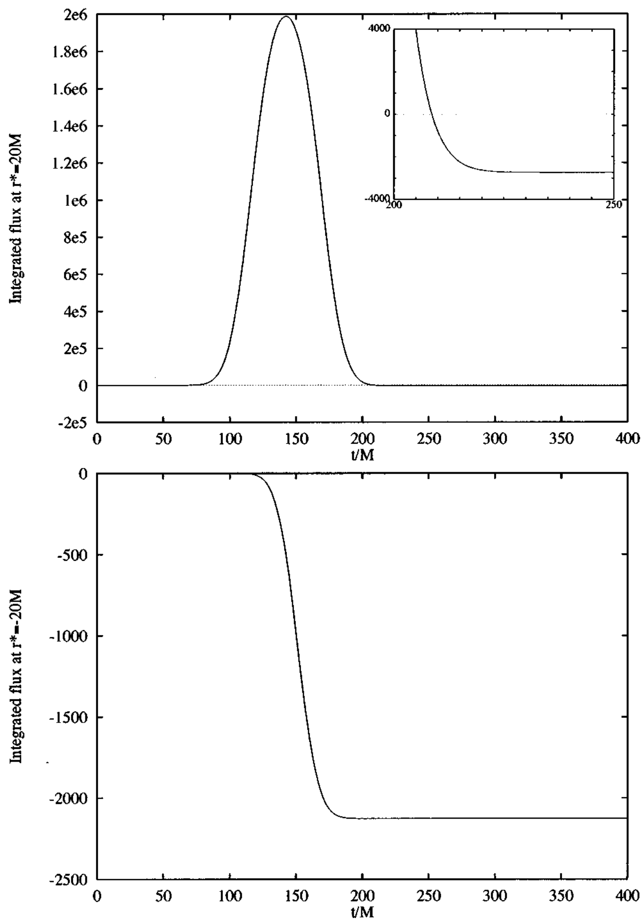

FIG. 2. Integrated energy flux of a superradiant evolution of a a modulated Gaussian pulse (with $\sigma = 0.8 m\omega_+$, $m=2$, $a=0.99M$ and $\epsilon = 0.01$). Surface normals are chosen such that energy flowing inwards into-the-hole across the outer surface is positive. At $r_* = 20M$ (upper panel), a total amplification of $\sim 0.14\%$ is achieved. At $r_* = -20M$ (the lower panel), energy mainly flows out of the horizon, approximately $\sim 0.11\%$ of the total energy falling onto the hole.

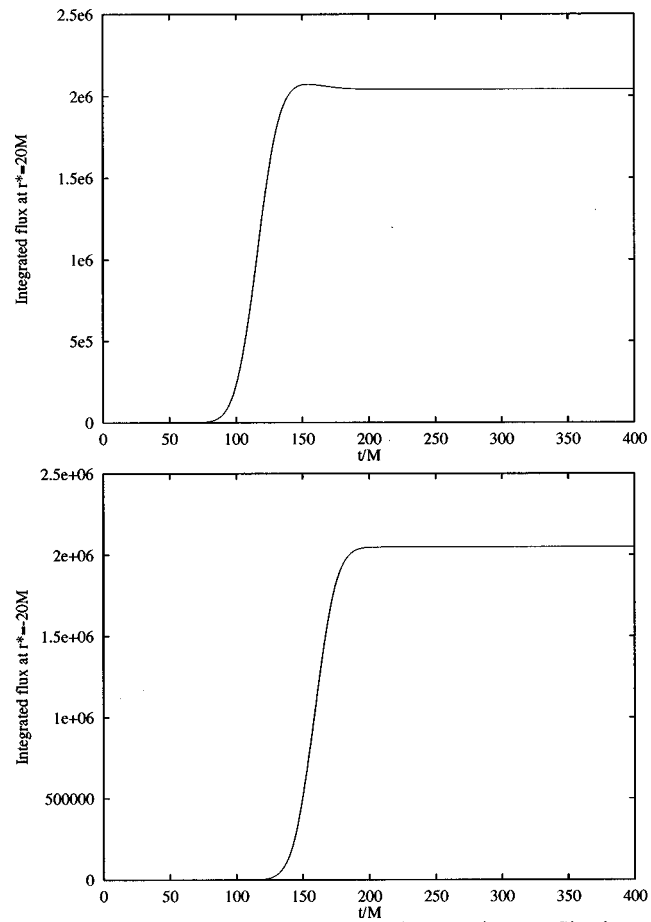

FIG. 3. An example of a non-superradiant evolution. The data are similar to that described in Fig. 2, but here the modulation frequency of the initial Gaussian is $\sigma = -0.8 m\omega_+$, and there is no sign of superradiance.

focus on the energy flux through various surfaces surrounding the black hole. Given a spacetime with a time Killing vector t^a and a perturbation with a well-defined stress-energy tensor T_{ab}, it is possible to define [5] a conserved energy flux vector $T^a{}_b t^b$. The flux of energy across a 3-dimensional time-like hypersurface with unit normal r^a is then given by $dE = T_{ab} t^a r^b dS$, where dS is the 3-surface element of the hypersurface. For a massless scalar field,

$$T_{ab} = \frac{1}{2}(\nabla_a \bar{\Phi} \nabla_b \Phi + \nabla_a \Phi \nabla_b \bar{\Phi}) - \frac{1}{2} g_{ab} \nabla_c \Phi \nabla^c \bar{\Phi}, \quad (5)$$

with over-bars denoting complex conjugation. For simplicity, we monitor the flux of energy through $r=$const surfaces in Boyer-Lindquist coordinates. This assumption together with $r^a r_a = 1$ yield $r^a = \pm(0, \sqrt{\Delta}/\rho, 0, 0)$, where as before $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2 \cos^2\theta$. Furthermore, the time Killing vector in this case reads $t^a = (1,0,0,0)$, and the surface element is given explicitly by $dS = \sqrt{-g^{(3)}} \sin\theta d\theta d\varphi dt = \sqrt{\Delta}\rho \sin\theta d\theta d\varphi dt$. Given dS and T_{ab} above, it is not difficult to show that

$$dE = \pm \pi(\partial_{r_*} \bar{\Phi} \partial_t \Phi + \partial_{r_*} \Phi \partial_t \bar{\Phi})(r^2 + a^2) \sin\theta d\theta dt. \quad (6)$$

We monitor the above energy flux through two surfaces located at $r_* = \pm 20M$. The outer surface is well away from the black hole while the inner one is reasonably close to the event horizon. The scattering of waves by the curved spacetime should be strongest in the region included between these surfaces. Figures 2 and 3 show the results of the superradiance ''experiment.'' The displayed data are for two qualitatively different situations. Both datasets correspond to evolutions with $m=2$ and a black hole rotation parameter $a=0.99M$. In both cases the pulse was initially centered around $r_o = 125M$, and the angular distribution of the initial data was chosen to be the standard spherical harmonic $Y_2^2(\theta,\varphi)$. The first case (see Fig. 2) shows a situation where one would expect to see superradiance. We have chosen the modulation frequency of the impinging Gaussian such that $\sigma = 0.8 m\omega_+$, and the width of the Gaussian corresponds to

$\epsilon = 0.01$. The second case (Fig. 3) corresponds to a Gaussian with the same width but now centered around $\sigma = -0.8 m \omega_+$.

As is obvious from Fig. 1, the scattering of the two pulses is quite different. In the superradiant case shown in Fig. 2, all the initial energy is reflected by the black hole. Superradiance is distinguished in two ways. By monitoring the energy flowing across the surface at $r_* = 20M$, we see that the reflected energy is slightly amplified after scattering (cf. the upper panel of Fig. 2). In this specific case, the amplification corresponds to 0.14%. It should be compared to the maximum single frequency amplification of 0.187% for $a = 0.99M$, deduced from the data in Fig. 1. That we are seeing superradiance is also clear from the fact that energy flows out through the surface at $r_* = -20M$ (cf. the lower panel of Fig. 2). The total energy flowing out through the inner surface corresponds to a superradiant amplification of 0.11%, in reasonable agreement with the result deduced at the outer surface.

The non-superradiant results ($\sigma = -0.8 m \omega_+$) are in clear contrast to the superradiant ones. Figure 3 shows no sign of amplification. In fact, as it can be seen from the upper panel of Fig. 3, the infalling pulse is almost entirely swallowed by the black hole. That there would be very little reflection in this case could, of course, be anticipated by comparing our chosen Gaussian pulse to the data in Fig. 1.

In summary, we have designed a numerical experiment that clearly exhibits the presence of superradiance phenomenon when waves of a certain character are scattered by a rotating black hole. One conclusion that can be drawn from the present work is that superradiance is perhaps best approached in the frequency domain. True, we have managed to extract the amplification due to superradiance in the ''worst possible case'' of scalar waves, but this was mainly due to having at our disposal a conserved flux and using ''almost monochromatic'' initial data. More than anything else this is direct evidence of the precision of our evolution code to solve the Teukolsky equation [1,2].

Our numerical experiment shows that superradiance can play a role in evolutions when the scattered pulse has support only in a restricted frequency domain. This is undoubtedly an interesting illustration, but what about superradiance in more general cases? It seems to us that the effect is easiest to isolate if one monitors different frequencies separately, i.e. works in the frequency domain. The main reason for this is that an amplification of a reflected signal with increasing a is not in itself an indication of superradiance. The results shown in Fig. 1 indicate that one would generally expect enhanced reflection of prograde moving waves as $a \to M$. This effect is likely to overwhelm the actual ''amplification'' of certain superradiant frequencies in an evolution of general initial data. This conclusion should also hold for the case of electromagnetic waves. However, the possibility that superradiance may play a distinctive role in an ''astrophysical'' evolution for gravitational waves cannot be ruled out. For gravitational waves, the amplitude of certain frequencies should be amplified by more than a factor of two [5]. A detailed study of that case could provide interesting results.

Finally, we have learned that the initial data require careful tuning in order that superradiance be observed. For general initial data, absorption of the non-superradiant frequencies will typically make the amplification due to superradiance difficult to distinguish. Moreover, superradiance should not be confused with the competing effect that the ''size'' of the black hole changes with the rate of rotation. As we have seen, this effect will generally lead to a much enhanced reflection of prograde waves which may confuse an attempt to distinguish superradiance.

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