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Loop constraints: A habitat and their algebra

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Abstract

This work introduces a new space $\mathcal{T}'_*$ of 'vertex-smooth' states for use in the loop approach to quantum gravity. Such states provide a natural domain for Euclidean Hamiltonian constraint operators of the type introduced by Thiemann (and using certain ideas of Rovelli and Smolin). In particular, such operators map $\mathcal{T}'_*$ into itself, and so are actual operators *in* this space. Their commutator can be computed on $\mathcal{T}'_*$ and compared with the classical hypersurface deformation algebra. Although the classical Poisson bracket of Hamiltonian constraints yields an inverse metric times an infinitesimal diffeomorphism generator, and despite the fact that the diffeomorphism generator has a well-defined non-trivial action on $\mathcal{T}'_*$, the commutator of quantum constraints vanishes identically for a large class of proposals.

I. INTRODUCTION.

Within the loop-based approach to quantum gravity, there are now a number of proposals for the Hamiltonian constraint [1,2,4,7]. Most of these are modifications of Thiemann's proposal [1], and in particular make use of an observation by Rovelli and Smolin [11] that certain limits of operators can be taken on diffeomorphism invariant states. One would like to test any proposal for the quantum constraints of gravity in a variety of ways. Below, we consider the proposals for Euclidean quantum gravity, computing the constraint algebras for each and comparing them to the classical hypersurface deformation algebra of [6,8].

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Although the constraint algebra has been studied at a more heuristic level and in a less well-defined context [20,21], an actual computation has until now been impossible for the proposals of [1,2,7] due to the subtle way in which these works construct their constraints. The proposals follow the ideas of [11] and define the constraints only when acting on a space $\mathcal{T}'_{Diff}$ of 'diffeomorphism invariant' states. This is because they employ a limiting procedure which does not converge on a general state. However, because a typical Hamiltonian constraint has nonvanishing commutator with the diffeomorphism constraint, the action of such a constraint takes a diffeomorphism invariant state to a state that is *not* diffeomorphism invariant. The ranges of the proposed constraint operators are therefore not contained in their domains and it is not possible to apply two of them in succession, or to directly compute a commutator. It is important to note that what we have in mind differs from the "anomaly-free" calculation of [1] in that we wish to commute the so-called 'unregulated' or 'regulator independent' operators, whereas [1] studied the commutator of regulator dependent constraints.

The main result of this paper is that the limiting procedures of [1,11] in fact converge on a larger space $\mathcal{T}'_* \supset \mathcal{T}'_{Diff}$, which we shall call the space of 'vertex-smooth states.' Thus, the proposed operators extend naturally to $\mathcal{T}'_*$. Furthermore, $\mathcal{T}'_*$ is mapped into itself by all of the proposed constraints. As a result, the proposals define constraint operators *within* the space $\mathcal{T}'_*$, and products and commutators of such operators are well defined in this space.

Let us recall that, classically, the Poisson bracket of two Euclidean Hamiltonian constraints is an inverse metric q^{ab} times an infinitesimal diffeomorphism generator C_b:

$$\{H(N), H(M)\} = \int (MN_a - NM_a) C_b q^{ab}. \tag{1.1}$$

We will see that a generic element of $\mathcal{T}'_*$ is not annihilated by the diffeomorphism generator[1] Indeed, the action of the diffeomorphism group on $\mathcal{T}'_*$ provides a faithful representation. In addition, $\mathcal{T}'_*$ contains the entire space of solutions to the constraints discussed in [1,2] – presumably, the entire space of physical states in these proposals. It would therefore be a great surprise if the inverse metric was degenerate on this space. Nevertheless, we find that the commutator of two Hamiltonian constraints vanishes identically on $\mathcal{T}'_*$ for a large class of proposals. More will be said about quantum versions of $\int N_a C_b q^{ab}$ in the accompanying paper [18].

There is in fact a general difficulty in constructing a quantum version of 1.1 using operators that act on (and preserve) some subspace of a Hilbert space which contains diffeomorphism invariant states. With a few natural assumptions, we shown in the Appendix that, in such a case, every diffeomorphism invariant state in domain of the Hamiltonian operators must be annihilated by the Hamiltonians. It is interesting to note that our argument breaks down if the constraints are rescaled and made into minus-half-densities – a case never considered in canonical gravity to our knowledge.

The plan of this paper is as follows. Section II first establishes the context and conventions for our work and then describes the new space $\mathcal{T}'_*$. Section III then describes a general

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[1]The infinitesimal diffeomorphism generators are in fact well defined on $\mathcal{T}'_*$, a fact first pointed out to the authors by José Mourão.

class of 'RST-like' operators on $\mathcal{T}'_*$ which includes many of the (so-called 'non-symmetric') proposals for the Euclidean Hamiltonian constraints. It also shows that the commutator of such operators vanishes in general. In section IV, we discuss various 'symmetrized' operators that have been proposed. Here, the commutator again tends to vanish and, when it does not, it also fails to annihilate diffeomorphism invariant states. This accounts for all existing proposals except that of [4], which will be considered in [18]. We end with a brief discussion in section V.

II. A NEW SPACE: THE VERTEX-SMOOTH STATES

This section introduces the new space $\mathcal{T}'_*$ of 'vertex-smooth states' which will allow us to compute constraint algebras. Section IIA sets the framework for our discussion and establishes notation and conventions. Section IIB then describes the vertex-smooth states. We save the demonstration that $\mathcal{T}'_*$ provides a natural habitat for RST-like constraints for a later section, after the constraints themselves have been introduced.

A. Preliminaries

We now take a few moments to fix our context and conventions before introducing the new space. We recall that standard constructions [12–15,17] of the space of generalized connections make use of an analytic structure on the three manifold Σ. They were generalized by Baez and Sawin [9,10] to the smooth category, however the notion of the spin-network has not been completely successfully defined in that case. On the one hand, the definition of Hamiltonian constraints given in [1] requires the action of smooth, rather then analytic diffeomorphisms but, on the other hand, the construction of the diffeomorphism invariant states of [17] makes use of the spin-networks. Merging these two features requires some care. The Hilbert space we desire is constructed without invoking an analytic structure but it is only a subspace of that of [9]; in fact, it is the subspace studied in [10].

In [12], a space of 'generalized connections' was constructed using the C^* algebra defined by the traces of holonomies of a connection along piecewise smooth closed curves in Σ. The spectrum of this algebra is the Ashtekar-Isham space $\overline{\mathcal{A}/\mathcal{G}}$. The elements of this space can be thought of as 'distributional' connections for which the holonomy around any closed curve is well-defined, but for which such holonomies satisfy no continuity properties [14]. This is to be the 'quantum configuration space,' and quantum states are to be functions on this space. Following [16] we consider a special set of such functions associated with graphs embedded in Σ. By a graph γ we mean a finite set of 'edges' (1-dimensional, smooth oriented submanifolds of Σ with a 2-point boundary called 'the ends' of an edge) such that any two of them intersect, if at all, at only one or both ends. We denote the set of edges of γ by $E(\gamma)$, and the particular subset with at least one end at v by $E(\gamma, v)$. Also associated with a graph γ is a set of vertices $V(\gamma)$; the vertices $V(\gamma)$ are the end points of the edges.

We will say that a function on $\overline{\mathcal{A}/\mathcal{G}}$ is 'cylindrical over a graph γ' if it depends only on the holonomies of the generalized connection along curves that lie in that particular graph. As the graphs we consider are smoothly embedded, every cylindrical function over a graph belongs to the Hilbert space described by Baez and Sawin [9]. As a result, there is an inner product on these smooth cylindrical functions, and they can be completed to form a Hilbert space $\mathcal{H}$ which is a proper subspace of the Hilbert space of [9]. The construction of [9] is more general and allows curves to intersect an infinite number of times, but such cases were not considered in [1,2,4,5] so we will also exclude them here (see [36] for an extension of the theory to such cases). The natural action of smooth diffeomorphisms of Σ in $\mathcal{H}$ is unitary. For $\varphi \in Diff(\Sigma)$, the action will be denoted $\mathcal{D}_\varphi$, with $\mathcal{D}_\varphi|\Gamma\rangle = |\varphi(\Gamma)\rangle$.

To each graph γ one associates a certain subspace $\mathcal{H}_\gamma \subset \mathcal{H}$ in such a manner that $\mathcal{H}_\gamma$ is orthogonal to $\mathcal{H}'_\gamma$ whenever the ranges of the graphs differ from each other, $R(\gamma) \neq R(\gamma')$. The Hilbert space $\mathcal{H}$ has the property that

$$\mathcal{H} \;=\; \oplus_{R(\gamma)}\mathcal{H}_\gamma \tag{2.1}$$

where $\oplus$ denotes the direct sum of Hilbert spaces and implies that that the result should be completed to obtain another Hilbert space.

Given a graph γ, the space $\mathcal{H}_\gamma$ can be formed from the associated to γ 'spin-network functions' [31,32] Recall that spin networks Γ are smooth cylindrical functions which are parameterized by triples (γ, j, c) where γ ranges over all graphs embedded in Σ and j, c range over certain lists of 'spins' and 'contractors' associated with the graph γ. The label j assigns a representation of $SU(2)$ to each edge of γ, while a contractor c assigns an 'intertwinor' to each vertex v in γ which ensures that Γ is invariant with respect to the gauge transformations. The intertwinors are linear operators which act in a space determined by the spins assigned by j to the edges that intersect at v; the reader should consult [31,32] for details. Now, $\mathcal{H}_\gamma$ is the Hilbert completion of the space spanned by all the spin-network functions given by all the labels (γ, j, c) such that for every edge $e \in E(\gamma)$, $j(e) \neq 0$. It is convenient to use the symbol $V(\Gamma)$ to denote the vertex set of the underlying graph γ, and to refer to the vertices of γ as vertices of the spin network Γ. Also, given a graph γ or a spin-network Γ, by $R(\gamma)$ and $R(\Gamma)$ respectively we denote the range of the graph.

If the list of possible contractors is properly chosen, then the states $\{|\Gamma\rangle = |\gamma, j, c\rangle\}$ form an orthonormal basis of $\mathcal{H}$. Let us choose once and for all a particular such orthonormal basis $\mathcal{B}$. An important point is that, for $|\gamma, j, c\rangle \in \mathcal{B}$, the set of allowed contractors c is finite for a fixed pair (γ, j). This means that any spin network is a finite linear combination of states in $\mathcal{B}$. It follows that the space $\mathcal{T}$ of finite linear combinations of spin networks is also the space of finite linear combinations of states in $\mathcal{B}$.

In order to remove the regulators, [1] required the constraints to act on 'diffeomorphism-invariant' states. While no state in $\mathcal{H}$ is invariant under all diffeomorphisms, a space of diffeomorphism invariant states was constructed in [17]. This was done by working in a larger space which consists of linear functionals on some dense subspace of $\mathcal{H}$. We will take this dense subspace to be $\mathcal{T}$ and consider the space of all linear functionals on $\mathcal{T}$, the dual $\mathcal{T}'$ of $\mathcal{T}$. Because the elements of $\mathcal{T}'$ are linear functions on $\mathcal{T}$, they will be denoted by 'bra' vectors $\langle\psi| \in \mathcal{T}'$. Note that if one chooses the topology on $\mathcal{T}$ to be just that due to its linear structure, the algebraic and topological duals of $\mathcal{T}$ coincide. Our spaces satisfy the relation

$$\mathcal{T}' \supset \mathcal{H} \supset \mathcal{T} \tag{2.2}$$

and are analogous to a rigged Hilbert triple. Since smooth diffeomorphisms of Σ act on $\mathcal{T}$, they have a natural (dual) action on $\mathcal{T}'$. The space $\mathcal{T}'$ is quite large, and in particular contains many linear functionals which are invariant under the action of all such diffeomorphisms. We use $\mathcal{T}'_{Diff}$ to denote the space of such diffeomorphism invariant functionals.

It is on this space that the *un*regulated constraints $\hat{H}(N)$ of [1] were defined[2]. However, because a given constraint $\hat{H}(N)$ depends on a choice of lapse function N, the constraints themselves are not diffeomorphism invariant. Thus, the action of $\hat{H}(N)$ in general yields a state that is *not* diffeomorphism invariant. Products such as $\hat{H}(N)\hat{H}(M)$ are therefore not, a priori, defined. If one wishes to compute commutators, one needs a space larger than $\mathcal{T}'_{Diff}$ in which to work.

Because the constraints of gravity enforce diffeomorphism invariance, $\mathcal{T}'_{Diff}$ may be expected to contain any 'physical' states (in the sense of Dirac [19]). However, in the current work we are interested in the constraint algebra, which must vanish on physical states. In fact, the classical commutator [6,8] of two Hamiltonian constraints becomes trivial when just the diffeomorphism constraint is satisfied, so we again see that $\mathcal{T}'_{Diff}$ is too small for our purposes. We now introduce a larger space $\mathcal{T}'_*$ of 'vertex-smooth' states with $\mathcal{T}'_{Diff} \subset \mathcal{T}'_* \subset \mathcal{T}'$.

B. The vertex-smooth states

We seek a space which carries a well-defined action of the constraints of [1] and which is preserved by that action. The fact [1] that the constraints are 'anomaly-free' (in the sense defined in [1]) on diffeomorphism invariant states may be taken as a hint that such a space should exist. Furthermore, we would like the natural action of the diffeomorphism group to give a faithful representation on this new space. That is to say, only the identity diffeomorphism should be represented trivially.

Readers who are already familiar with the constraints introduced in [1] will recall that those constraints were defined only on diffeomorphism invariant states. Specifically it was important that the action of the (dual) state $\langle\psi| \in \mathcal{T}'_{Diff}$ on a spin network over a graph γ does not depend on the exact placement of the edges of γ. This is true for any diffeomorphism invariant state, as its action remains the same when an edge is moved by a small diffeomorphism. The key point concerning our new space is that its states, too, will not care about the exact placement of edges, yet they will care about the placement of *vertices* As a result, the space $\mathcal{T}'_*$ will carry a faithful representation of the diffeomorphism group. The careful reader may object that moving an edge generally involves moving vertices as well, but this will be dealt with in section III.

Our definition is as follows. Let $\mathcal{T}'_* \subset \mathcal{T}'$ contain those $\langle\psi|$ such that:

- A) if two spin networks Γ_1 and Γ_2 are related by a smooth diffeomorphism which is the identity on their vertices, then

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[2]For the reader familiar with [1–3] we should caution that these constraints were denoted $\hat{H}'(N)$ in those works. In order to reduce the already formidable amount of notation present in this paper, we will not explicitly differentiate between the action of an operator on a space $\mathcal{T}$ and the dual action of the operator on the space of linear functionals on $\mathcal{T}'$. In addition, we will explicitly display the regulators for regulated constraints so that $\hat{H}(N)$, with no regulator, will always denote an operator that acts in the dual space.

$$\langle \psi | \Gamma_1 \rangle = \langle \psi | \Gamma_2 \rangle. \tag{2.3}$$

Thus, if we fix some 'reference' spin network Γ^0 with k vertices $v_1, ..., v_k$, then, as φ ranges over $Diff^\infty(\Sigma)$, $\langle \psi | \mathcal{D}_\varphi | \Gamma^0 \rangle$ is some function of the k-tuple $(\varphi(v_1), ..., \varphi(v_k))$ of vertices of $\Gamma^0 \circ \varphi$. That is to say that $\langle \psi | \mathcal{D}_\varphi | \Gamma^0 \rangle$ is described by a function $\tilde{\psi}_{\Gamma_0}$ on the space of maps $\varphi|_{V(\Gamma)} : V(\Gamma) \to \Sigma$ given by restricting diffeomorphisms φ to $V(\Gamma)$.

- B) Each function $\tilde{\psi}_{\Gamma^0}$ as above extends to a *smooth*[3] function $\psi_{\Gamma^0} : \Sigma^{V(\Gamma)} \to \mathbf{C}$ on the *entire* space $\Sigma^{V(\Gamma)}$ of maps $\{\sigma : V(\Gamma) \to \Sigma\}$ from $V(\Gamma)$ to Σ. In particular, ψ_{Γ^0} must be smooth at points where two or more vertices are mapped to the same point in Σ, despite the fact that $\tilde{\psi}_{\Gamma_0}$ was only defined on maps σ that take distinct vertices to distinct points.

As a result, a state $\langle \psi | \in \mathcal{T}'_*$ can be characterized by a family of smooth functions $\psi_\Gamma : \Sigma^{V(\Gamma)} \to \mathbf{C}$, one for each equivalence class of spin networks under smooth diffeomorphisms. The diffeomorphism invariant elements of $\mathcal{T}'$ are just those states $\langle \psi |$ for which each ψ_Γ is a constant function. Thus, $\mathcal{T}'_* \supset \mathcal{T}'_{Diff}$.

We will see below that the Euclidean constraints of [1,2,7,4] are well-defined on this space and that they map this space into itself, allowing us to compute their algebra.

III. RST-LIKE OPERATORS AND THEIR COMMUTATOR

In this section we discuss a general class of operator families which we call the the 'Rovelli-Smolin-Thiemann-like' operators or the 'RST-like' operators. Such a family is labeled by a lapse function $N : \Sigma \to \mathbf{C}$, as are the Hamiltonian constraints of gravity. This class will include the (so-called 'nonsymmetric') constraints introduced in [1]. We show below that all such operators are defined on $\mathcal{T}'_*$ and map $\mathcal{T}'_*$ into itself. We will also show that any two operators $\hat{H}(N)$ and $\hat{H}(M)$ in the same family commute.

A. The Regulated operators

RST-like operators are based on the notion of a 'loop assignment scheme' α, which takes a vertex v of a graph γ and an ordered pair (I, J) of edges in γ and assigns to (γ, v, I, J) a smooth loop $\alpha(\gamma, v, I, J) : [0, 1] \to \Sigma$. Below, we use the symbol $\alpha(\gamma, v, I, J)$ to denote either the map from $[0, 1]$ to Σ or its orientation preserving reparametrization invariance class; the meaning should be clear from the context. For the purposes of this paper, we require a loop assignment scheme to have the following properties. 1) Each loop $\alpha(\gamma, v, I, J)$ must begin and end at v and be such that $R(\gamma) \cup R(\alpha(\gamma, v, I, J))$ is a subset of the range $R(\gamma')$ some other graph γ'; for example, this excludes loops with infinitely many self-intersections. 2)

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[3] For many purposes, it would in fact be sufficient to use continuous functions f. However, requiring ψ_{Γ_0} to be differentiable allows infinitesimal diffeomorphisms to act on $\mathcal{T}'_*$, and taking ψ_{Γ_0} to be smooth allows $\mathcal{T}'_*$ to be preserved under this action.

The loop $\alpha(\gamma, v, I, J)$ is also required to span a nontrivial area and to be tangent to the plane defined by (I, J) at its beginning and its end. 3) Finally, a loop assignment scheme must be 'locally diffeomorphism covariant,' in the sense that if (γ, v, I, J) restricted to a neighborhood W of v is related to (γ', v', I', J') restricted to a neighborhood W' of v' by a smooth diffeomorphism φ, then $\alpha(\gamma', v', I', J') = \varphi' \circ \alpha(\gamma, v, I, J)$ where φ' is some (possibly different) smooth diffeomorphism which coincides with φ on the restriction of (γ, v, I, J) to W. Here, the symbol $\circ$ denotes the composition of functions.

The loop assignment scheme will play the role of a 'regulator' for the quantum operator with the idea that, as the regulator is removed, one should pass through a series of loop assignment schemes in which the loops shrink to points. This limit will be discussed in more detail shortly.

Having chosen a loop assignment scheme α, a *regulated* RST-like operator is constructed from a family of operators

$$\hat{h}^{\alpha}(x) : \mathcal{T} \to \mathcal{T} \tag{3.1}$$

associated with the points $x \in \Sigma$. The action of $\hat{h}^{\alpha}(x)$ on a spin network $|\Gamma\rangle = |\gamma, j, c\rangle$ vanishes when x is not a vertex of γ, and otherwise can be written in the form

$$\hat{h}^{\alpha}(x)|\gamma, j, c\rangle = \sum_{I,J \in E(\gamma,v)} U^i[\alpha(\gamma, x, I, J)]|\gamma, j, h_i(\gamma, j, x, I, J)c\rangle, \tag{3.2}$$

where I, J are members of the set $E(\gamma, v)$ of edges of γ incident at v and there is a vector of linear operators

$$h_i(\gamma, j, x, I, J) : c \mapsto h_i(\gamma, j, x, I, J)c, \tag{3.3}$$

on the space of contractors for γ, j associated to every pair of edges (I, J) intersecting at the point x. The repeated index i is summed over $i \in 1, 2, 3$ and $U^i[\alpha]$ is the traceless part of the holonomy $U[\alpha]$ defined by

$$U[\alpha] = U^0[\alpha]\mathbb{1} + U^i[\alpha]\tau_i, \tag{3.4}$$

where τ_i are the generators of $SU(2)$. The operator $h_i(x, I, J)$ transforms according to the the adjoint representation of $SU(2)$ under gauge transformations at x, is antisymmetric in (I, J), and carries a gauge invariant intertwinor c into a vector of intertwinors $h_i(x, I, J)c$ by changing only the linear operators assigned by c to the particular vertex x. These operators must again satisfy a 'local diffeomorphism covariance' condition in the sense that if (γ, j, x, I, J) restricted to W is related to $(\gamma', j', x', I', J')$ restricted to W' by a diffeomorphism $\varphi \in Diff(\Sigma)$ with $\varphi(W) = W'$ for open sets $W \ni x$, $W' \ni x'$, then $h_i(\gamma, j, x, I, J)$ and $h_i(\gamma', j', x', I', J')$ are related by the same diffeomorphism; specifically,

$$\mathcal{D}_{\varphi}\left(U^i[\alpha(\gamma, x, I, J)]|\gamma, j, h_i(\gamma, j, x, I, J)c\rangle\right) = U^i[\varphi \circ \alpha(\gamma, x, I, J)]|\gamma', j', h_i(\gamma', j', x', I', J')c\rangle. \tag{3.5}$$

Given a loop assignment scheme α and a smooth lapse function N, the regulated constraint $\hat{H}^{\alpha}(N)$ is defined by:

$$\hat{H}^\alpha(N) = \sum_{x \in \Sigma} N(x) \hat{h}^\alpha(x). \tag{3.6}$$

The (uncountably infinite) sum is well defined when acting on an element $|\phi\rangle$ of $\mathcal{T}$ as all but a finite number of terms annihilate any given such $|\phi\rangle$.

It is clear from [1] that the regulated 'non-symmetric' constraints proposed in that work are of the form (3.2) and define regulated RST-like operators. The same is true of the constraints discussed in [7] (which are related to those of [1] by 'changing the factor ordering'). For these particular proposals, the loop assigned to any vertex v and edge pair (I, J) first runs along I, then crosses over to J without intersecting any other edges, and returns to v along J. The details of $h_i(\gamma, j, x, I, J)$ for the proposal of [1] depend on the choice of volume operator and on the particular interpretation of the regularization scheme[4].

B. Removing the regulator

Having defined the regulated operators, the regulator α is now to be 'removed' by considering sequences $\{\alpha_n : n \in \mathbf{Z}, n \geq 0\}$ in which, as $n \to \infty$, the loops $\alpha_n(\gamma, v, I, J)$ shrink to the vertex v, and such that loops $\alpha_n(\gamma, v, I, J)$ which correspond to the same graph, vertex, and edges but to different values of n are related by diffeomorphisms which map the graph γ to itself. That is to say that α_n should satisfy $\alpha_n(\gamma, v, I, J) = \varphi_n \circ \alpha_0(\Gamma, v, I, J)$ for some $\varphi_n \in Diff(\Sigma)$ such that φ_n preserves the edges of γ (and their orientations) and $\varphi_n(v) = v$ for all $v \in V(\gamma)$. The sequence should also be such that, given (γ, v, I, J) and an open set $W \ni v$, there is some $\tilde{n}$ for which, for all $n \geq \tilde{n}$, we have $\alpha_n(\gamma, v, I, J) \subset W$ and $\varphi_n \circ \varphi_{\tilde{n}}^{-1}$ is the identity outside of W. The 'unregulated' constraint operator is to be defined through

$$\hat{H}(N)|\psi\rangle = \lim_{n \to \infty} \hat{H}^{\alpha_n}(N)|\psi\rangle. \tag{3.7}$$

We may schematically denote this limit by $\hat{H}(N) = \lim_{\alpha \to 0} \hat{H}^\alpha(N)$, though the final object $\hat{H}$ will depend on the particular sequence of loop assignment schemes chosen. Such an object $\hat{H}(N)$ will be called an (unregulated) RST-like operator.

Note that, when acting on $\mathcal{H}$, this limit does not converge at all: typically, for a spin network $|\Gamma\rangle$, $H^{\alpha_n}|\Gamma\rangle$ is orthogonal to $H^{\alpha_m}|\Gamma\rangle$ for $n \neq m$ because the two states are supported on graphs occupying different positions in Σ. It is interesting to note, however, that (as remarked in [1]) if cylindrical functions are viewed as functions on *continuous* (i.e., nondistributional) connections, then the limit (3.7) does converge when acting on such functions, but the result is just the zero operator. This follows from the fact that, as the loops shrink to a point, the holonomies $U[\alpha_n(\gamma, v, I, J)]$ become $\mathbb{1}$ so that $U^i[\alpha_n]$ goes to zero. Nonetheless, a well-defined non-zero limit will be obtained by considering the dual operator induced by $\hat{H}^\alpha$ in the space $\mathcal{T}'_*$ of linear functionals defined above.

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[4]Di Pietri has pointed out [28] that the construction given in [1] explicitly excludes the possibility of the constraints acting at planar vertices, due to its reliance on (nondegenerate) tetrahedra. This limitation is easily removed, and our discussion includes both cases, with either the volume operator of [23] or that of [25].

To show this, let us consider $\langle\psi|\hat{H}^{\alpha_n}(N)|\Gamma\rangle$ and take $|\Gamma\rangle = |\gamma, j, c\rangle$ to be a spin network. For each n, the functions $U^i[\alpha_n(\gamma, v, I, J)]|\gamma, j, h^i(\gamma, j, x, I, J)c\rangle$ are all cylindrical over some graph γ_n, and γ_n can be chosen such that $\gamma_n = \varphi_n(\gamma_0)$ where $\{\varphi_n\}$ is the sequence of diffeomorphisms described in the definition of an RST-like operator above.

We would like to decompose the function $U^i[\alpha_n(\gamma, v, I, J)]|\gamma, j, h^i(\gamma, j, x, I, J)c\rangle$ as a sum of spin networks in our basis $\mathcal{B}$. The important point is that only a finite number of spin network states can appear in this decomposition. This is because the allowed spin is bounded by the sum of the maximum spin in the list j and (1/2 times) the maximum number of times the loop $\alpha_n(\gamma, v, I, J)$ retraces itself. As a result, we may write

$$\sum_{I,J\in E(\gamma,v)} U^i[\alpha_n(\gamma, v, I, J)]|\gamma, j, h^i(\gamma, j, x, I, J)c\rangle = \sum_{k=1}^{K^v} a^k_{v,n}|\Gamma^k_{v,n}\rangle \tag{3.8}$$

for some $K^v \in \mathbf{Z}$, where we have explicitly indicated that the coefficients $a^k_{v,n}$, the spin networks $\Gamma^k_{v,n}$, and the integer K^v will depend on the vertex v. Note that the $|\Gamma^k_{v,n}\rangle$ can be chosen so that $|\Gamma^k_{v,n}\rangle = \mathcal{D}_{\varphi_n}|\Gamma^k_{v,0}\rangle$, in which case the local diffeomorphism covariance of the loop assignment guarantees that the coefficients $a^k_{v,n}$ are in fact independent of n. We will assume that such a choice has been made and write $a^k_v := a^k_{v,n}$. It then follows that the action of $\langle\psi| \in \mathcal{T}'_*$ on such a state is

$$\sum_{I,J\in E(\gamma,v)} \langle\psi|U^i[\alpha_n(\gamma, v, I, J)]|\gamma, j, h_i(\gamma, v, I, J)c\rangle = \sum_{k=1}^{K^v} a^k_v \psi_{\Gamma^k_{v,0}}(\varphi_n|_{V(\Gamma^k_{v,0})}), \tag{3.9}$$

where $\varphi_n|_{V(\Gamma^k_{v,0})} : V(\Gamma^k_{v,0}) \to \Sigma$ is just the map obtained by restricting φ_n to $V(\Gamma^k_{v,0})$.

Taking the limit $n \to \infty$ amounts to simply moving around the vertices $V(\Gamma^k_{v,0})$. We note that a vertex $v' \in V(\Gamma^k_{v,0})$ is either a vertex of the original graph γ (in which case it is mapped to itself by φ_n), or a point on one of the curves $\alpha_0(\gamma, v, I, J)$. Since the sequence $\{\varphi_n\}$ contracts all points of $\alpha_0(\gamma, v, I, J)$ to v and the collection of such curves is finite, the limit of (3.9) as $n \to \infty$ is given by replacing φ_n on the right-hand side of (3.9) with φ_∞, where $\varphi_\infty(v') = \lim_{n\to\infty}\varphi_n(v')$ is well-defined for $v' \in V(\Gamma^k_{v,0})$. Thus, $\langle\psi|\hat{H}(N) = \lim_{n\to\infty}\langle\psi|H^{\alpha_n}(N)$ is a well-defined element of $\mathcal{T}'$.

Note that in fact, for $v' \in V(\Gamma^k_{v,0})$, $\varphi_\infty(v')$ is a vertex of the graph Γ; this reminds us of the implicit dependence of φ_n on $V(\Gamma)$. Due to the local diffeomorphism covariance of the loop assignments, (3.10) is clearly unchanged when the spin-network Γ is replaced by $\phi(\Gamma)$ for $\phi \in Diff^\infty(\Sigma)$ such that ϕ is the identity on $V(\Gamma)$. For a ϕ that does act nontrivially on $V(\Gamma)$, it changes only the points to which φ_∞ contracts the vertices of $V(\Gamma^k_{v,0})$. As a result, we may rewrite the $n \to \infty$ limit of (3.9) by introducing a map $\eta^{k,v} : \Sigma^{V(\Gamma)} \to \Sigma^{V(\Gamma^k_{v,0})}$ which takes an assignment $\sigma : V(\Gamma) \to \Sigma$ of points in Σ to vertices of Γ and generates a new assignment $\eta^{k,v}(\sigma) : V(\Gamma^k_{v,0}) \to \Sigma$ of points Σ to vertices of $\Gamma^k_{v,0}$. The new assignment will in general send many vertices to the same point of Σ and is given by $[\eta^{v,k}(\sigma)](v') = \sigma(\varphi_\infty(v'))$. Thus,

$$\lim_{n\to\infty}\langle\psi|H^{\alpha_n}(N)\mathcal{D}_\phi|\Gamma\rangle = \sum_{v\in V(\Gamma)} N(v) \sum_{k=1}^{K^v} a^k_v \psi_{\Gamma^k_{v,0}} \circ \eta^{k,v}(\phi|_{V(\Gamma)}) \tag{3.10}$$

for any $\phi \in Diff(\Sigma)$. Finally, each $\psi_{\Gamma^k_{v,0}} \circ \eta^{k,v}$ depends smoothly on the map $\phi|_{V(\Gamma)}$, and extends to a smooth function on all of $\Sigma^{V(\Gamma)}$.. As a result, $\langle\psi|\hat{H}(N)$ is an element of $\mathcal{T}'_*$ and $\hat{H}(N)$ maps $\mathcal{T}'_*$ into itself, as desired. Note that, because of the covariance of the loop assignments and the operators $h_i(\gamma, j, x, I, J)$, the operator $\hat{H}(N)$ is also covariant in the sense that

$$\mathcal{D}_\varphi \hat{H}(N) \mathcal{D}_\varphi^{-1} = \hat{H}(N \circ \varphi). \tag{3.11}$$

It is clear that the space $\mathcal{T}'_*$ could in fact be extended even further. Just as it was sufficient for our states to depend smoothly on the positions of the vertices, it is not necessary for the states to be completely independent of the placement of the edges. Thus, one could replace requirement A in the definition of $\mathcal{T}'_*$ with a condition more like that of B, requiring that the states depend sufficiently smoothly on the positions of edges so that the limit defining the unregulated RST-like operators can still be taken. However, the space $\mathcal{T}'_*$ defined in section II B is enough for our purposes and we will not discuss further generalizations here.

C. The commutator.

We are now in a position to study the commutator $[\hat{H}(N), \hat{H}(M)]$ as an operator on $\mathcal{T}'_*$. Although we work on a larger space $\mathcal{T}'_*$ and with the unregulated operators (which are of a more general form than in [1]), the following argument is much like the anomaly-free calculation of [1]. We will proceed by choosing some $\langle\psi| \in \mathcal{T}'_*$ and some spin network Γ. We wish to evaluate $\langle\psi|[\hat{H}(N), \hat{H}(M)]|\Gamma\rangle$ for all $N, M \in C^\infty(\Sigma)$. It will be easiest to compute the answer in the special case where N vanishes at all vertices of Γ except v_N, and M vanishes at all vertices of Γ except v_M. The general case can then be reconstructed using the fact that $\hat{H}(N)$ is linear in N.

It is clear from (3.10) that we may write

$$\langle\psi|\hat{H}(N)\hat{H}(M)|\Gamma\rangle = N(v_N)M(v_M)\psi^{v_N,v_M} \tag{3.12}$$

for some complicated function ψ^{v_N,v_M}. Note that when $v_N = v_M$, the right hand side is symmetric in N and M. As a result, for this case we have $\langle\psi|[\hat{H}(N), \hat{H}(M)]|\Gamma\rangle = 0$.

Let us therefore consider $v_N \neq v_M$. The case where $N(v_N) = 0$ or $M(v_M) = 0$ is trivial, so we will assume $N(v_N) \neq 0$ and $M(v_M) \neq 0$. Since (3.12) depends on the values of N and M only at v_N and v_M respectively, the result (3.12) is unchanged if N, M are replaced by smooth functions $\tilde{N}, \tilde{M}$, such that $\tilde{N}(v) = N(v)$, $\tilde{M}(v) = M(v)$ for all $v \in V(\Gamma)$ but for which the support (supp$\tilde{N}$) of $\tilde{N}$ does not intersect the support (supp$\tilde{M}$) of $\tilde{M}$.

The action of $\hat{H}^{\alpha_m}(\tilde{M})$ on $|\Gamma\rangle$ is,

$$\hat{H}^{\alpha_m}(\tilde{M})|\Gamma\rangle = \tilde{M}(v_M) \sum_{I,J \in E(\gamma,v)} U^i[\alpha_m(\gamma, v_M, I, J)]|\gamma, j, h_i(\gamma, j, v_M, I, J)c\rangle. \tag{3.13}$$

Since the support of $\tilde{M}$ is an open set containing v_M, for m greater than or equal to some $\tilde{m}$ we must have $R(\alpha_m(v_M, \gamma, I, J)) \subset \text{supp}\tilde{M}$ for all I, J. Choosing $m \geq \tilde{m}$, let us now act on (3.13) with $\hat{H}^{\alpha_n}(\tilde{N})$. Note that because $R(\alpha_m(\gamma, v, I, J)) \cap \text{supp}\tilde{N}$ is empty, $\hat{H}^{\alpha_n}(\tilde{N})$ will act only at vertices of the original graph γ.[5] In particular, it will act only at v_N. Thus, it can be shown that for sufficiently large n and m, we have

$$\hat{H}^{\alpha_n}(\tilde{N})\hat{H}^{\alpha_m}(\tilde{M})|\Gamma\rangle \sim \tilde{N}(v_N)\tilde{M}(v_M) \sum_{\substack{I_1,J_1\in E(\gamma,v_N)\\ I_2,J_2\in E(\gamma,v_M)}} U^i[\alpha_n(\gamma,v_N,I_1,J_1)]U^j[\alpha_m(\gamma,v_M,I_2,J_2)] \times|\gamma,j,h_i(\gamma,j,v_N,I_1,J_1)h_j(\gamma,j,v_M,I_2,J_2)c\rangle \tag{3.14}$$

where $\sim$ denotes equality modulo terms which are annihilated by any $\langle\psi| \in \mathcal{T}'_*$ in the limit $n,m \to \infty$. Indeed, due to local diffeomorphism covariance of the linear operators $h_i(\gamma,j,v,I,J)$, the operator $h_i(\gamma,j,v_N,I_1,J_1)$ above acts in the same way, independent of whether we first attach the loop $\alpha_m(\gamma,v_M,I_2,J_2)$ or not. For the analogous reason, the loop $\alpha_n(\gamma,v_N,I_1,J_1)$ is independent, modulo appropriate diffeomorphisms, of whether we first attach the loop $\alpha_m(\gamma,v_M,I_2,J_2)$ or not. The rest is assured by properties A and B of the vertex-smooth states $\langle\psi| \in \mathcal{T}'_*$.

Since the above result is symmetric with respect to $(N,M) \to (M,N)$ we find

$$\langle\psi|[\hat{H}(\tilde{N}),\hat{H}(\tilde{M})]|\Gamma\rangle = 0. \tag{3.15}$$

From our general considerations at the beginning of this subsection, since (3.15) holds for arbitrary $\langle\psi|$ and $|\Gamma\rangle$, the commutator of $\hat{H}(N)$ and $\hat{H}(M)$ must vanish identically for all smooth N and M. We again stress that this holds for any RST-like operator, whether it acts on planar vertices or not. This is our main result.

IV. SYMMETRIZED OPERATORS

Let us now turn to the issue of 'symmetrizing' the Hamiltonian operators, in the sense of adding some kind of 'hermitian conjugate operator' $\hat{H}^\dagger(N)$ to $\hat{H}(N)$. Note that we have not defined an inner product on $\mathcal{T}'_*$; indeed, appendix A shows that a fully satisfactory such inner product does not exist either on $\mathcal{T}'_*$ or on any subspace that both A) contains at least one diffeomorphism invariant state $\langle\psi|_{Diff}$ for which $\langle\psi|_{Diff}\hat{H}(N) \neq 0$ (for some N) and B) is preserved by the action of a family of Hamiltonian constraints. Thus, the operators $\hat{H}(N)$ do not act in a Hilbert space and there is no canonical notion of whether $\hat{H}(N)$ is 'symmetric' or of how to make it so. However, for the special case of constant lapse, $\hat{H}(1)$ is invariant under diffeomorphisms and maps diffeomorphism invariant states into diffeomorphism invariant states ($\hat{H}(1) : \mathcal{T}'_{Diff} \to \mathcal{T}'_{Diff}$). As described in [17], a family of natural Hermitian inner products can be introduced on a subspace (which we shall call $\tilde{\mathcal{T}}'_{Diff}$) of $\mathcal{T}'_{Diff}$, and this subspace may then be completed to a Hilbert space $\mathcal{H}_{Diff}$. The precise Hilbert space obtained depends on which member of the family of hermitian inner products was chosen, but we will not indicate this dependence explicitly. We note that a

[5]That easy observation is crucial here; the reader familiar with [1] knows that before, it was thought that an action of the second Hamiltonian on the vertices produced by the first one was relevant for the result.

particular choice was made in [3], but we will instead leave the relevant parameters arbitrary. For a generic loop assignment (such that the attached loop $\alpha(\gamma, v, I, J)$ never overlaps γ), $\hat{H}(1)$ is a densely defined operator in $\mathcal{H}_{Diff}$. For other loop assignments, $\hat{H}$ may only be defined on some smaller domain (say, Φ) as states in $\mathcal{H}_{Diff} \setminus \Phi$ are carried out of $\mathcal{H}_{Diff}$ by $\hat{H}(1)$. In any case, one can see if $\hat{H}(1)$ is symmetric in the sense of a bilinear form on its domain. Because $\hat{H}(1)$ typically 'destroys' edges of graphs and usually does not 'create' edges, $\hat{H}(1)$ is not symmetric in any of the Hilbert spaces $\mathcal{H}_{Diff}$ unless the inner product is chosen so degenerate that $\hat{H}(1)$ is just the zero operator in $\mathcal{H}_{Diff}$. In this sense then, no family of RST-like constraints is symmetric.

The view has been expressed [1,2,7,5] that one might like to have a family of Hamiltonian constraints that *are* symmetric in some sense. A minimal requirement might be that $\hat{H}(1)$ defines a symmetric operator on some natural domain in $\mathcal{H}_{Diff}$. The status of this view is not completely clear, as there are general arguments [26] that, due to the structure of the classical constraint algebra, the Hamiltonian constraints should not be self-adjoint. Since, however, the classical commutator of Hamiltonian constraints vanishes on the surface in phase space satisfying the diffeomorphism constraints, this argument need not apply to $\hat{H}(1)$ on $\mathcal{H}_{Diff}$. See also the commentary of [1] on this issue. We therefore wish to consider a 'symmetrization' of our family of constraints which gives a self-adjoint $\hat{H}(1)$ on some $\mathcal{H}_{Diff}$ and compute the algebra of the resulting constraints. We will refer to a family of constraints which satisfies this property as being 'constant lapse symmetric.'

A. Review of Symmetrization Proposals

The set of proposals [2,7,5] for 'symmetrizing' the constraints is in fact quite diverse. In this subsection, we quickly review the proposals for unregulated symmetric constraints which have appeared in the literature before discussing a new (and, we believe, more satisfying) definition of 'symmetrization' in section IV B. However, we then show (in section IV C) that even this new definition of symmetrization leads to either commuting constraints or to constraints which are anomalous in the sense that their commutator does not even vanish on diffeomorphism invariant states. It is in this sense that we use the terms 'anomaly' or 'anomalous' in the rest of this work.

At first, it may seem natural to use the Hilbert space structure of $\mathcal{H}$ to symmetrize the regulated constraints $H^{\alpha_n}(N)$ before taking the limit $n \to \infty$. This possibility was mentioned in [1] and was used as a basis for [5]. After all, for appropriate loop assignment schemes, one can arrange for a given spin network $|\Gamma'\rangle$ to appear on the right hand side of the decomposition (3.8) for only a finite number of spin networks $|\Gamma\rangle$ and, with this arrangement, it is true that $[H^{\alpha_n}(N)]^\dagger$ maps $\mathcal{T}$ to $\mathcal{T}$ and so has a dual action $\mathcal{T}' \to \mathcal{T}'$. Unfortunately, this method of 'symmetrization' fails to define a constant lapse symmetric family of constraints.

To see this, consider, for example, $\langle\psi|[H^{\alpha_n}(N)]^\dagger|\Gamma\rangle = \sum_{\Gamma'}\langle\psi|\Gamma'\rangle\overline{\langle\Gamma|\hat{H}^{\alpha_n}(N)|\Gamma'\rangle}$ for $\langle\psi| \in \mathcal{T}'_*$ and $|\Gamma\rangle = |\gamma, j, c\rangle$, where the sum is over an orthonormal basis of spin networks Γ' and the matrix elements $\langle\Gamma|\hat{H}^{\alpha_n}(N)|\Gamma'\rangle$ are taken in $\mathcal{H}$. Recall that $\hat{H}^{\alpha_n}(N)$ basically adds edges (due to the $U[\alpha_n]$ factor) to Γ'. Thus, $\langle\Gamma|\hat{H}^{\alpha_n}(N)|\gamma', j', c',\rangle$ typically vanishes unless γ' is a subgraph of γ and $\alpha_n(\gamma', v, I, J)$ supplies exactly the missing edges (for some (v, I, J)).

However, the required edges of γ lie at a finite separation from the vertices of γ', while the edges added by $U_i[\alpha_n(\gamma', v, I, J)]$ approach v as $n \to \infty$. Since there are only a finite number of subgraphs of γ' and one may use the fact that the $\alpha_n(\gamma', v, I, J)$'s for various n are related by diffeomorphisms to bound the change in spin, for fixed $|\Gamma\rangle$ there is only a finite set of Γ' for which $\langle\Gamma|H^{\alpha_n}|\Gamma'\rangle$ can be nonzero, independent of the value of n. As a result, for n greater than some $\tilde{n}$, $\langle\Gamma|H^{\alpha_n}(N)|\Gamma'\rangle = 0$ for all Γ'. Thus, adding $\lim_{n\to\infty}[\hat{H}^{\alpha_n}(N)]^\dagger$ to $\hat{H}^\alpha(N)$ has, in general, no effect whatsoever[6].

Another proposal was made by De Pietri, Rovelli, and Borrisov in [7] and corresponds to 'changing the factor ordering' of the regulated constraints of [1]. However, in our notation their proposal amounts to simply using a different set of operators $h_i(\gamma, j, v, I, J)$ than the original proposal of Thiemann. In fact, under their proposal, the fully 'symmetrized' operator is still an RST-like operator. Thus, such operators are not constant-lapse symmetric in the above sense. In addition, the calculation of section III C applies and the commutator of two such constraints vanishes on $\mathcal{T}'_*$.

Finally, we remark that another kind of symmetrization was considered in [2], and involved 'marking' various edges. However, this method applied only to the regulated constraints. See we are interested in the unregulated constraints, we will not discuss this proposal here.

B. A new definition of the Hermitian Conjugate

It appears that the sort of 'symmetrization' which is desired [2,5,29] is something that does not involve marking special edges and which is somehow closer to the symmetrization of $\hat{H}(1)$ induced by the inner product on $\mathcal{H}_{Diff}$. A step in this direction was suggested to us by Thiemann [33] and will be described below together with the resulting definition of the hermitian conjugates $H^\dagger(N)$. The idea is to rewrite the definition of the Hermitian conjugate of $\hat{H}(1)$ induced by the inner product on $\mathcal{H}_{Diff}$ in a suggestive form, which can then be refined to define a family of operators $\hat{H}^\dagger(N)$ which we will refer to as the 'hermitian conjugates' of $\hat{H}(N)$. It is important to note that our definition of the hermitian conjugate will make use of special properties of $\hat{H}(N)$, and will not be applicable to a general family of operators $\hat{A}(N)$ on $\mathcal{T}'_*$ labeled by lapse functions. In particular, it does not directly provide a definition of the hermitian conjugate of $\hat{H}^\dagger(N)$, or of the 'symmetrized' operator $\hat{H}^S(N) = \hat{H}(N) + \hat{H}^\dagger(N)$. As a result, this structure in no ways runs counter to the arguments of [26]. On the other hand, the family $H^S(N)$ will be constant lapse symmetric, as desired.

Let us begin by reviewing the inner product defined by [17] on diffeomorphism invariant states. In fact, [17] considered a space of spin network states based on analytic graphs and

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[6]For some choices of loop assignment α and operators $h_i(\gamma, v, I, J)$a, $\hat{H}^\alpha(N)$ can occasionally 'destroy' an edge (see [27]). In that case, due to the fact that we have not required the loops to shrink in a uniform (γ independent) way, the limit of $[H^{\alpha_n}(N)]^\dagger$ may be nonzero, or may not even by defined on all of $\mathcal{T}'_{Diff}$. Still, $\lim_{n\to}[\hat{H}^{\alpha_n}(N)]^\dagger$ annihilates 'most' of $\mathcal{T}'_*$, and certainly does not lead to symmetry of $\hat{H}(1)$.

invariance only under analytic diffeomorphisms. What we need here is an extension to our smooth case. The construction is similar to that of [17], and in fact simpler [10]. For example there is no issue of 'type I' vs. 'type II' graphs (see [17]); in our smooth case, all graphs may be studied together. Since the treatment of our case is direct (given the methods of [17]), we simply state the results below; the reader may consult [10] for details.

We will first need a bit of notation. Recall that an important notion in [17] was the group $GS(\gamma)$ of 'graph symmetries' of a graph γ. Roughly speaking, this is the group of all embeddings of γ into itself. We define it as follows: consider the 'isotropy' group $\mathrm{Iso}(\gamma) \subset Diff(\Sigma)$ of diffeomorphisms φ that map γ to itself. Also, let $\mathrm{TA}(\gamma) \subset \mathrm{Iso}(\gamma)$ (the 'trivial action' subgroup) consist of those $\varphi \in \mathrm{Iso}(\gamma)$ that map every *edge* of the graph γ to itself and preserve every edge's orientation. The trivial action subgroup is normal in $\mathrm{Iso}(\gamma)$, and the graph symmetry group is defined to be the quotient: $\mathrm{GS}(\gamma) = \mathrm{Iso}(\gamma)/\mathrm{TA}(\gamma)$. Given a graph γ, the symmetry group $\mathrm{GS}(\gamma)$ acts naturally in the linear space spanned by the spin-networks over γ.

Now, for a spin network Γ over a graph γ, we will define a linear functional $\langle \Gamma, 1| \in \mathcal{T}'_{Diff}$. Suppose that $|\Gamma'\rangle$ is a spin network over a graph γ' which is diffeomorphic to γ (so that $\phi_{\gamma'\gamma}(\gamma) = \gamma'$ for some $\phi_{\gamma'\gamma} \in Diff(\Sigma)$. Then, the action of $\langle \Gamma, 1|$ on $|\Gamma'\rangle$ is[7]

$$\langle \Gamma, 1|\Gamma'\rangle \ := \sum_{s\in \mathrm{GS}(\gamma)} \langle (\phi_{\gamma'\gamma} \circ s)(\Gamma)|\Gamma'\rangle. \tag{4.1}$$

When γ' is not diffeomorphic to γ, the result is just zero. If we let $\tilde{\mathcal{T}}'_{Diff}$ be the space spanned by (finite) linear combinations of the $\langle \Gamma, 1|$, then a natural family of inner products [17] on $\tilde{\mathcal{T}}'_{Diff}$ is given by

$$\langle \Gamma, 1|\Gamma', 1\rangle \ = \ a_{[\gamma]} \langle \Gamma, 1|\Gamma'\rangle \tag{4.2}$$

for any set of positive real constants $a_{[\gamma]}$ which may depend on the diffeomorphism class of γ. The Hilbert space $\mathcal{H}_{Diff}$ is just the completion of $\tilde{\mathcal{T}}'_{Diff}$ in one of the inner products (4.2).

As a result, the hermitian conjugate of $\hat{H}(1)$ is defined on an appropriate domain by $\langle \Gamma, 1|\hat{H}^\dagger(1)|\Gamma'\rangle = \overline{\langle \Gamma', 1|H(1)|\Gamma\rangle}$ where the overline denotes complex conjugation. This led Thiemann to suggest [33] that a 'Hermitian conjugate family' $\hat{H}^\dagger(N)$ be defined on diffeomorphism invariant states of the form $\langle \Gamma, 1|$ by some sort of expression of the form:

$$\langle \Gamma, 1|H^\dagger(N)|\Gamma'\rangle = \overline{\langle \Gamma', 1|H(N)\mathcal{D}_{\varphi_{\Gamma,\Gamma'}}|\Gamma\rangle} \tag{4.3}$$

for an appropriate diffeomorphism $\varphi_{\Gamma,\Gamma'}$ that, in some sense, moves Γ to the location in Σ occupied by Γ'. Our task it to make this suggestion precise, and in fact we will simultaneously extend it so that $\hat{H}^\dagger(N)$ can act on states which are not necessarily diffeomorphism invariant. Nevertheless, we take (4.3) as our moral inspiration and link to the inner product (4.2).

To proceed, we will first need to introduce some new notation. For example, for every graph γ we define the pointed symmetry group $\mathrm{GS}_*(\gamma)$ by replacing, at every stage in the

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[7] Such a definition may be obtained by 'averaging $\langle \Gamma|$ with respect to the action of the group of diffeomorphisms' [17].

the definition of GS(γ), the groups $Iso(\gamma)$ and $TA(\gamma)$ with their subgroups $Iso_*(\gamma)$ and $TA_*(\gamma)$ of diffeomorphisms which are the identity on the vertices of γ. Next, given any spin network Γ and any map $\sigma : V(\Gamma) \to \Sigma$, we define the state $\langle \Gamma_\sigma |$ to be the linear functional for which

$$\langle \Gamma_\sigma | \Gamma' \rangle \;=\; \sum_{s \in \mathrm{GS}_*(\gamma)} \langle \phi^\sigma_{\gamma'\gamma} \circ s(\Gamma) | \Gamma' \rangle \tag{4.4}$$

for $|\Gamma'\rangle$ such that there is a diffeomorphism $\phi^\sigma_{\gamma'\gamma}$ satisfying $\phi^\sigma_{\gamma'\gamma}(\gamma) = \gamma'$ and which, when restricted to the vertices of γ, coincides with σ. For other $|\Gamma'\rangle$, we set $\langle \Gamma_\sigma | \Gamma' \rangle = 0$. Note that if σ happens to map two distinct vertices of Γ to the same point, then $\langle \Gamma_\sigma |$ is just the zero functional. The states $\langle \Gamma_\sigma |$ do not lie in $\mathcal{T}'_*$, because their action does not depend smoothly on the location of the vertices of Γ'; it vanishes unless the vertices of Γ' occupy exactly the positions assigned by the map σ. However, these states can be used to build a large class of states in $\mathcal{T}'_*$ which we will label $\langle \Gamma, f|$ with Γ a spin network and $f : \Sigma^{V(\Gamma)} \to \mathbf{C}$ a smooth function. These are the states defined by

$$\langle \Gamma, f| \;:=\; \sum_{\sigma \in \Sigma^{V(\gamma)}} f(\sigma) \langle \Gamma_\sigma | \tag{4.5}$$

Recall that such (uncountably infinite) sums are well defined in $\mathcal{T}'$ as, when acting on a given spin network state, only a finite number of the terms contribute. For example, given spin networks Γ and Γ' associated to graphs γ and γ' respectively, if there is some $\phi_{\gamma'\gamma} \in Diff(\Sigma)$ such that $\phi_{\gamma'\gamma}(\gamma) = \gamma'$ then we have

$$\langle \Gamma, f | \Gamma' \rangle = \sum_{s \in GS(\gamma)} \langle (\phi_{\gamma'\gamma} \circ s)(\Gamma) | \Gamma' \rangle f((\phi_{\gamma'\gamma} \circ s)|_{V(\Gamma)}). \tag{4.6}$$

Otherwise, we have $\langle \Gamma, f | \Gamma' \rangle = 0$. It follows from the definition, that every state $\langle \Gamma, f|$ is uniquely determined by the sum

$$sum_{s \in GS(\gamma)} \bar{f}(s_{|V(\gamma)}) | s(\Gamma) \rangle \tag{4.7}$$

Before introducing the new definition of the Hermitian conjugate of a family of RST-like operators, we will need one more bit of notation. For any subset $X \subset V(\Gamma)$ of vertices of a spin-network Γ, we define a linear functional on $\mathcal{T}$ corresponding to the notion of $\langle \Gamma |$ 'averaged with respect to the diffeomorphisms acting trivially on X',

$$\mathcal{T}' \;\ni\; \langle \Gamma |^X \;:=\; \sum_{\sigma_{|X} = \mathrm{id}} \langle \Gamma, \sigma |. \tag{4.8}$$

The functional $\langle \Gamma |$ is again not an element of $\mathcal{T}'_*$ but the action of $\hat{H}(N)$ is naturally defined on such states by (3.10).

For any RST-like operator $\hat{H}(N)$, we now define a 'conjugate' operator $\hat{H}^\dagger(N)$ through the following procedure. When Γ' has at least as many vertices as Γ, we set

$$\langle \Gamma, f | H^\dagger(N) | \Gamma' \rangle \;=\; \sum_{\sigma \in \Sigma^{V(\gamma)}} f(\sigma) \sum_{X \in V_{|V(\Gamma)|}(\Gamma')} \left(\overline{\langle \Gamma' |^X | \hat{H}(N) | \phi_\sigma(\Gamma) \rangle} \right), \tag{4.9}$$

where ϕ_σ is any diffeomorphism of Σ that coincides with σ on $V(\Gamma)$ and $V_n(\Gamma')$ is the set of all n-element subsets of $V(\Gamma')$ (so that $V_{|V(\Gamma)|}(\Gamma')$ is the set of all sets of vertices of Γ' which have the same number of elements as the set of the vertices of Γ). When Γ' has less vertices than Γ, we set $\langle\Gamma, f|H^\dagger(N)|\Gamma'\rangle = 0$. The result is a well-defined family of operators $\hat{H}^\dagger(N) : \tilde{\mathcal{T}}'_* \to \mathcal{T}'$.

Implicit in our definition is the fact that a (regulated) RST-like operator always adds vertices when acting on a spin network $|\Gamma\rangle$. Thus, the action of an (unregulated) RST-like operator on the (dual) state $\langle\Gamma, f|$ is of the form $\sum_{\Gamma'}\langle\Gamma', f_{\Gamma'}|$ where each Γ' has *less* vertices than Γ. As a result, we may expect that $\langle\Gamma, f|H^\dagger(N) = \sum_{\Gamma'}\langle\Gamma', f'|$ where now each Γ' has *more* vertices than Γ. It is this idea which motivates the definition of $\hat{H}^\dagger(N)$ given above.

There are several properties of (4.9) that we would like to point out. For example, the reader may readily verify that the hermitian conjugate family of operators (4.9) is consistent with the Hilbert space inner product on $\mathcal{H}_{diff}$: when the domains are properly chosen, the restriction of $\hat{H}^\dagger(N)$ to $\mathcal{H}_{diff}$ for $N = 1$ is the hermitian conjugate of the associated restriction of $\hat{H}(1)$ with respect to the Hilbert product of $\mathcal{H}_{diff}$.

In addition, one sees that when Γ' has at least as many vertices as Γ, the only nonzero contribution in (4.9) comes from terms in which $\sigma(V(\Gamma)) = X$. Thus (4.9) depends on the values of the lapse N only at the vertices of Γ'. In this sense ϕ_σ moves Γ to the position occupied by Γ'. The details of (4.9) are complicated due to the desire to deal carefully with graphs with symmetries; we recall that the action (4.6) of $\langle\Gamma, f|$ also has several terms when $GS(\gamma)$ is not empty. In this sense then, (4.9) can be seen as a precise implementation of the ideas of (4.3). It would also appear that (4.9) is a precise implementation of the ideas of [34].

Finally we note that, as expected, $H^\dagger(N)$ generally 'adds edges': when the spins of Γ are large enough, $\langle\Gamma, f|H^\dagger(N) = \sum_{\Gamma'}\langle\Gamma', f^{\Gamma'}|$ such that γ is diffeomorphic to a subgraph of each γ'. The reader may also wish to investigate other properties of this expression for himself.

Now, a priori, the action of $H^\dagger(N)$ is only defined on the vector space (which we shall call $\tilde{\mathcal{T}}'_*$) spanned by the states $\langle\Gamma, f|$. This space is in fact sufficient for our study of the commutator, but we may also attempt to extend the definition of $H^\dagger(N)$ to all of $\mathcal{T}'_*$ using a kind of 'super linearity.' The point is that a general state $\langle\psi| \in \mathcal{T}'_*$ can be written as a certain sum of states of the form $\langle\Gamma, f|$. Indeed, for a state $\langle\psi| = \langle\Gamma, f|$ such that the graph symmetry group of the underlying graph γ is trivial, the function f is just the function ψ_Γ from the definition of $\mathcal{T}'_*$ in section II B, while $\psi_{\Gamma'}$ for all Γ' not diffeomorphic to Γ is zero. Thus, a general state $\langle\psi| \in \mathcal{T}'_*$ can be written as

$$\langle\psi| = \sum_{\Gamma \in S^*} \langle\Gamma, \tilde{\psi}_\Gamma|, \tag{4.10}$$

where the sum is over a set S^* of orthonormal spin networks such that $\mathcal{T}$ is spanned by states for the form $\mathcal{D}_\varphi|\Gamma\rangle$ for $\varphi \in Diff(\Sigma)$ and $\Gamma \in S^*$. For spin networks Γ over graphs which have no symmetries we have $\tilde{\psi}_\Gamma = \psi_\Gamma$, while for graphs with nontrivial graph symmetry groups ψ_Γ and $\tilde{\psi}_\Gamma$ are related by a certain symmetrization procedure induced by (4.5). As a result, we may attempt to extend $\hat{H}^\dagger(N)$ to an operator on $\mathcal{T}'_*$ by defining

$$\langle\psi|H^\dagger(N)|\Gamma\rangle = \sum_{\Gamma' \in S^*} \langle\Gamma', \tilde{\psi}_{\Gamma'}|H^\dagger(N)|\Gamma\rangle. \tag{4.11}$$

Such a sum will make sense if, for each state $|\Gamma\rangle$, only a finite number of terms contribute. This is the case if one uses a loop assignment such that none of the loops $\alpha(\gamma, v, I, J)$ overlaps γ.

However, for the assignment used in [1,2,7,11], we must distinguish two cases in the work of Thiemann. Due to the phenomenon of 'disappearing edges' [27], $H^\dagger(N)$ as defined by the original constraints $\hat{H}(N)$ of [1] does not satisfy the above requirement. That is, for certain choices of $|\Gamma\rangle$, an (uncountably) infinite number of terms in the sum (4.11) may be nonzero. Thus, we say that the action of $\hat{H}^\dagger(N)$ on a general element of $\mathcal{T}'_*$ is divergent. Unfortunately, this is true even if we attempt to extend the definition only to the image $\hat{H}(M)[\tilde{\mathcal{T}}'_*]$ of $\tilde{\mathcal{T}}'_*$ under $\hat{H}(M)$. (Again, because of the disappearing edges phenomenon, the smaller space $\tilde{\mathcal{T}}'_*$ is not preserved by $\hat{H}^\dagger(N)$). Thus, it is not possible to define $\hat{H}(N)\hat{H}^\dagger(M)$ even on $\tilde{\mathcal{T}}'_*$, or to compute a commutator on this space. Furthermore, there is no subspace on which $\hat{H}(N)$ and $\hat{H}^\dagger(N)$ are both defined and which is preserved by both of these operators. Similarly, for this case, $\hat{H}(1) + \hat{H}^\dagger(1)$ is only defined on a subspace $\tilde{\mathcal{H}}$ of $\mathcal{H}_{Diff}$ which is not dense in $\mathcal{H}_{Diff}$. However, there are subspaces $\mathcal{T}'_{*,n}$ of $\mathcal{T}'$ on which the n-fold action of $\hat{H}(N)$ or $\hat{H}^\dagger(N)$ are defined. Thus, we might calculate the commutator on $\mathcal{T}'_{*,2}$. More will be said about this shortly.

The other case considered by Thiemann [1,2] is when certain additional 'projection' operators are introduced to remove the offending terms generated by $\hat{H}(N)$. As a result, we must now consider the class of 'projected RST-like operators' defined in the same way as the RST-like operators, but from regulated operators of the form

$$\hat{h}^\alpha(x)|\gamma, j, c\rangle \;=\; p_\gamma \sum_{I,J\in e(\gamma,v)} U^i[\alpha(\gamma, x, I, J)]|\gamma, j, h_i(\gamma, j, x, I, J)c\rangle. \tag{4.12}$$

Here, all is as with the RST-like operators, except for the addition of the projection p_γ. This p_γ is just the projection onto the space of all spin networks $|\gamma', j', c'\rangle$ over graphs γ' such that the graph γ is a subgraph of γ'. These projections clearly satisfy their own version of 'diffeomorphism covariance' and, as a result, the arguments of sections III B and III C can be repeated for the projected RST-like operators, showing that they yield well defined operators $\hat{H}(N) : \mathcal{T}'_* \to \mathcal{T}'_*$ and that any two such constraints commute. As before, we do not comment on the motivations behind this construction, but simply use it to calculate the commutator of the symmetrized constraints.

With the projectors in place, the associated Hermitian conjugate operators $H^\dagger(N)$ defined by (4.9) are well-defined on all of $\mathcal{T}'_*$. We also note that for a state $\langle\psi|$ in $\mathcal{T}'_{*,2}$, the action of a projected operator on $\langle\psi|$ is *identical* to the action of the corresponding unprojected operator on $\langle\psi|$. As a result, our discussion of the projected case below includes much of the information about the unprojected commutator In fact, an argument similar to the one given below also holds for the unprojected operator on $\mathcal{T}'_{*,2}$ (and produces similar results), so long as appropriate care is taken with the convergence of various 'superlinear' expressions. However, we will not explicitly deal with this more subtle case.

C. A commutator, again

We would now like to compute the commutator of the symmetrized Hamiltonian constraints $H^S(N) = H(N) + H^\dagger(N)$ on $\mathcal{T}'_*$, with $H^\dagger(N)$ given by (4.9). We restrict ourselves to the particular constraints proposed by Thiemann, and use the projected form (4.12). As stated above, an argument similar to the one below also applies to Thiemann's unprojected operators on $\mathcal{T}'_{*,2}$, so long as appropriate care is taken with superlinear expressions. In addition, we consider only forms of Thiemann's constraint which do not act at planar vertices – it is straightforward to show than any other case is anomalous in the sense that $[H^S(N), H^S(M)]$ does not annihilate diffeomorphism invariant states. We will not review the details of Thiemann's proposal here, but we remind the reader of what for us is the most important property of this proposal: that for a spin network $|\Gamma\rangle$, the graphs $\gamma^k_{v,0}$ associated with the spin networks $|\Gamma^k_{v,0}\rangle$ appearing in (3.8) differ from γ only in having a single extra edge attached to two edges that intersect at v as shown below:

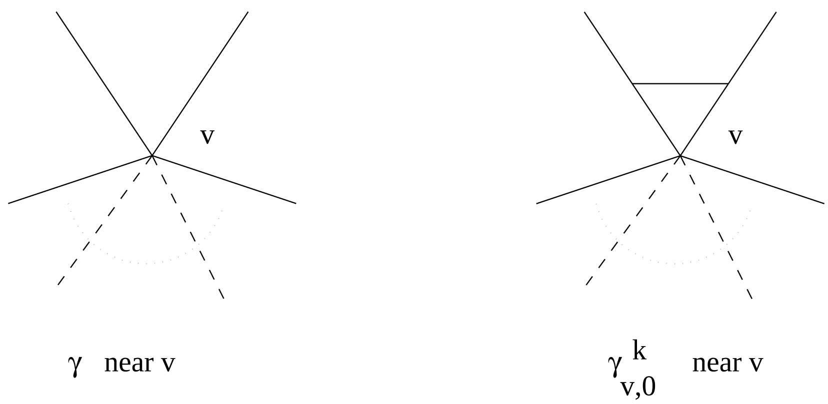

Fig. 1

As a result, $\langle \Gamma, f|\hat{H}(N) = \sum_{\Gamma'} \langle \Gamma', f^{\Gamma'}|$ where each γ' differs from γ by the *removal* of such an edge and for $\langle \Gamma, f|\hat{H}^\dagger(N) = \sum_{\Gamma'} \langle \Gamma', f^{\Gamma'}|$ each γ' differs from γ only by the *addition* of such an edge.

There are in fact several types of terms to consider in calculating the commutator and it is useful to dispense with some of them right at the beginning. Suppose that we define the 'symmetrized' Hamiltonian $\hat{H}^S(N) = \hat{H}(N) + \hat{H}^\dagger(N)$. The commutator of two such operators can be expanded in four terms:

$$[\hat{H}^S(N), \hat{H}^S(M)] = [\hat{H}(N), \hat{H}(M)] + [\hat{H}^\dagger(N), \hat{H}(M)] + [\hat{H}(N), H^\dagger(M)] + [\hat{H}^\dagger(N), \hat{H}^\dagger(M)]. \tag{4.13}$$

Since we have already shown that the first term is zero, we shall concentrate on the last three. In fact, it is the treatment of the middle two that will be most complicated. It is quite easy to outline the proof that these terms also vanish. Given a state $\langle f, \Gamma|$ and a probe spin-network function $|\Gamma'\rangle$ the quantity $\langle f, \Gamma|[\hat{H}^S(N), \hat{H}^S(M)]|\Gamma'\rangle$ is the sum of contributions coming, roughly speaking, from the action of the two operators either at a same vertex v, say, of Γ' or at disjoint vertices v, w, say. For the first kind of contribution, the only dependence on the lapse functions is an overall factor $N(v)M(v)$ which is obviously symmetric with respect to the change $(N, M) \mapsto (M, N)$. For the second, the action at each of the vertices depends only on the characteristics of the spin-network Γ' in an appropriate neighborhood of that vertex. In most cases the neighborhoods are disjoint so that the order in which the operators act is irrelevant. The only special case is when an edge added by the operator at v_1 can be annihilated at v_2 [35]. On such spin-networks, the commutator is not zero. Thus, in computing the commutator of the constraints, our general approach will be quite similar to (but dual to) that of section III C. That is, we will attempt to decompose the action of $\hat{H}(N)$ and $\hat{H}^\dagger(N)$ on $\langle \Gamma, f|$ as a sum over actions that we may describe as 'localized at the various vertices of Γ' and then use local diffeomorphism invariance (and properties of Thiemann's proposal) to compute the commutator. However, before attacking the problem directly, it will be useful to recall some general facts about spin networks and cylindrical functions on $\overline{A/G}$.

We recall that the space $\overline{A/G}$ can in fact be constructed as a quotient space. In the notation of [13] (see also a later work [16]), $\overline{A/G} = \overline{A}/\overline{G}$. We will not dwell on the details here, but simply mention that $\overline{A}$ is a space of distributional connections much like $\overline{A/G}$, except that holonomies themselves (and not just their traces) are defined along *all* piecewise smooth curves, open as well as closed. A certain gauge group $\overline{G}$ acts on the space $\overline{A}$, and $\overline{A/G}$ is the quotient $\overline{A}/\overline{G}$. Thus, the space of functions on $\overline{A/G}$ is just the space of functions on $\overline{A}$ which are invariant under the gauge group $\overline{G}$. In particular, the gauge invariant spin network functions can be constructed as sums and products of simpler functions on $\overline{A}$, each of which is not separately gauge invariant. For example, one may think of the objects $U^i[\alpha]$ and $|\gamma, j, h_i(\gamma, j, v, I, J)c\rangle$ of section III for fixed i as being functions on $\overline{A}$[8]. It will be useful below to 'take apart' a spin network function into a product of several functions that are not separately gauge invariant. To this end, suppose that we have a spin network function $\Gamma = (\gamma, j, c)$ and an open set $\mathcal{U} \subset \Sigma$ such that the boundary of $\mathcal{U}$ does not contain any vertices of Γ. Then there is a (gauge dependent) spin network function Γ_U defined by the triple (γ_U, j_U, c_U) such that the graph γ_U is $\overline{U} \cap R(\gamma)$ where $\overline{U}$ is the closure of U. If an edge e_U of γ_U is part of an edge e of γ, then $j_U(e_U) = j(e)$ while if v is a vertex of γ_U, then $c_U(v) = c(v)$. Furthermore, such edges as oriented in a manner consistent with the orientation of edges in γ. Here, we will call a point v a vertex of γ_u only if it was a vertex of the original graph v. Thus, this process does not create new vertices on the boundary of U and no contractors need be assigned to points on this boundary. Note also that the graph underlying a gauge dependent spin network is open. The spin network Γ_U will also be denoted $R(\Gamma) \cap \overline{U}$.

Strictly speaking, Γ_U is not just a function on $\overline{A}$, but a set of functions labeled by one gauge index i in the spin j representation (or its complex conjugate) for every initial (final) end of a spin j edge of Γ_U which is not at a vertex of Γ_U. We will call such an end a 'virtual vertex' of Γ_U. Given two gauge dependent spin networks Γ_1 and Γ_2, we will say that Γ_1 is 'consistent' with Γ_2 if, for every initial virtual vertex of Γ_1 of spin j, it is either a *final* virtual vertex of Γ_2 of spin j, or it is not a point on the graph γ_2. Similarly, final virtual vertices of Γ_1 which lie in γ_2 should be initial virtual vertices, and the virtual vertices of Γ_2 should satisfy a similar condition with respect to γ_1 so that this condition is symmetric. When Γ_1 is consistent with Γ_2, there is a naturally defined product spin network $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$ given by multiplying the associated functions on $\overline{A}$ and contracting any indices that correspond to the same virtual vertex. Note that when U is related to a (gauge invariant) spin network Γ as above, we have $\Gamma = \Gamma_U \Gamma_{(\overline{U})^c}$, where ^{c} denotes the complement of a set.

We now commence the proof by finding a more convenient expression for $\langle \Gamma, f| H^\dagger(N)$.

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[8]There is a natural generalization of the spin-networks to the 'extended spin-networks' defined in [24].

In particular, note that if $\langle\Gamma', f|H(\tilde{N})|\Gamma\rangle$ is nonzero, then $|\Gamma'\rangle$ is diffeomorphic to a spin network over a graph γ_0 of which γ is a subgraph ($\gamma_0 \supset \gamma$). In fact, γ_0 differs from γ only by having a single extra edge. We would like to fix the position of this extra edge in some sense, but to keep it close to v.

As a result, we introduce a special set $S^{\Gamma_U,\dagger}_{v,U}$. Consider a spin network Γ and an open set U as above, such that U contains exactly one vertex $v \in V(\Gamma)$. Furthermore, the set $\overline{U} \cap R(\gamma)$ should be connected, where again $\overline{U}$ denotes the closure of U. Now, choose an arbitrary function $\tilde{N}$ whose support includes v but includes no other vertex of Γ.

As indicated, the set $S^{\Gamma_U,\dagger}_{v,U}$ will depend only on the vertex v, the set U, and the spin network Γ_U defined by the part of Γ inside $\overline{U}$. This is to be some fixed set of (gauge dependent) spin networks $\Gamma'_U = (\gamma'_U, j'_U, c'_U)$ such that A) each spin network is entirely contained in $\overline{U}$: $\gamma'_U \subset \overline{U}$, B) for and Γ_1 with $R(\Gamma_1) \cap \overline{U} = \Gamma_U$, $\langle\Gamma_0, 1|H(\tilde{N})|\Gamma_1\rangle$ is nonzero only if Γ_0 is diffeomorphic to a state Γ'_0 such that $R(\Gamma'_0) \cap \overline{U}$ is a nontrivial member of the span $\overline{S}^{\Gamma_U,\dagger}_{v,U}$ of $S^{\Gamma_U,\dagger}_{v,U}$; $\varphi(R(\Gamma_0)) \cap \overline{U} \in \overline{S}^{\Gamma_U,\dagger}_{v,U}$ for some $\varphi \in Diff(\Sigma)$. We may also choose $S^{\Gamma_U,\dagger}_{v,U}$ to be independent of $\tilde{N}$. Note that, the spin assignments of Γ_1 and Γ_0 can only differ by a bounded amount. Therefore, since a finite set of graphs suffices, we may choose the set $S^{\Gamma_U,\dagger}_{v,U}$ to be finite.

Consider now the action of the hermitian conjugate operator $H^\dagger(1)$ on a state $\langle\Gamma, f|$. This can be expanded as

$$\langle\Gamma, f|H^\dagger(1) = \sum_{v\in V(\Gamma)} \sum_{\Gamma'_U \in S^{\Gamma_U,\dagger}_{v,U}} \langle\Gamma'_U \Gamma_{(\overline{U})^c}, f \circ i^*_{\Gamma,U,\Gamma'_U}| a^{\Gamma_U,\Gamma'_U} \tag{4.14}$$

for an appropriate set of coefficients a^{Γ_U,Γ'_U} and where $i^*_{\Gamma,U,\Gamma'_U} : \Sigma^{V(\Gamma'_U \Gamma_{U^c})} \to \Sigma^{V(\Gamma)}$ is the map on assignments of points to vertices induced by the embedding of $V(\Gamma)$ in $V(\Gamma'_U \Gamma_{(\overline{U})^c})$. That is, for $\sigma \in \Sigma^{V(\Gamma'_U \Gamma_{(\overline{U})^c})}$, we have $i^*_{\Gamma,U,\Gamma'_U}(\sigma) = \sigma|_{V(\Gamma)}$. Note that $V(\Gamma)$ is always a subset of $V(\Gamma'_U \Gamma_{(\overline{U})^c})$. To fully define the coefficients a^{Γ_U,Γ'_U}, suppose that Γ_0 is consistent with Γ_U and Γ'_U, satisfies $\gamma_0 \subset U^c$, that both $\Gamma_U\Gamma_0$ and $\Gamma'_U\Gamma_0$ are gauge invariant spin networks over graphs with trivial symmetry groups, and that $\Gamma_U\Gamma_0$ has only one vertex. Such a Γ_0 always exists, and we define

$$a^{\Gamma_U,\Gamma'_U} := \langle\Gamma_U, \Gamma_0|H^\dagger(1)|\Gamma'_U\Gamma_0\rangle. \tag{4.15}$$

The result is independent of the choice of Γ_0.

To include a nontrivial lapse function, we need just a bit more notation. For a function $f : \Sigma^{V(\Gamma)} \to \mathbf{C}$, a vertex $v \in V(\Gamma)$, and a function $N : \Sigma \to \mathbf{C}$, let the product $f \star_v N : \Sigma^{V(\Gamma)} \to \mathbf{C}$ be the function

$$(f \star_v N)(\sigma) = f(\sigma)N(\sigma(v)). \tag{4.16}$$

This product is commutative in the sense that, given another function $M : \Sigma \to \mathbf{C}$, we have

$$f \star_v N \star_{v'} M = f \star_{v'} M \star_v N. \tag{4.17}$$

The action of a general $\hat{H}(N)$ on $\langle\Gamma, f|$ is then

$$\langle\Gamma, f|H^\dagger(N) = \sum_{v\in V(\Gamma)} \sum_{\Gamma'_{U_v}\in S^{\Gamma_{U_v},\dagger}_{v,U_v}} \langle\Gamma'_{U_v}\Gamma_{(\overline{U_v})^c}, f\circ i^*_{\Gamma,U_v,\Gamma'_{U_v}} \star_v N|a^{\Gamma_{U_v},\Gamma'_{U_v}}. \tag{4.18}$$

where appropriate open sets U_v have been chosen. We now condense this slightly by defining, for each $v \in V(\Gamma)$, operators $H^\dagger_v(N)$ given by

$$\langle\Gamma, f|H^\dagger_v(N) := \sum_{\Gamma'_U\in S^{\Gamma_U,\dagger}_{v,U}} \langle\Gamma'_U\Gamma_{\overline{(U)^c}}, f\circ i^*_{\Gamma,U,\Gamma'_U} \star_v N|a^{\Gamma_U,\Gamma'_u}. \tag{4.19}$$

Note that definition of $H^\dagger_v(N)$ does not depend on the open set U.

Unfortunately, it is not really correct to call $H^\dagger_v(N)$ an operator on states: if the same state is parameterized in two different ways ($\langle\Gamma, f| = \langle\Gamma', f'|$), then (4.19) in general gives different results. The object $H^\dagger_v$ is more properly considered as an operator on certain lists of pairs (Γ, f). Nonetheless, it will be convenient to treat $H^\dagger_v$ as a operator and to not introduce further notation to treat it properly. So long as the true character of this object is understood, this should not cause any problems.

Another useful property of $\hat{H}^\dagger_v(N)$ for $v \in V(\Gamma)$ is that $\hat{H}^\dagger_v(N)$ annihilates the state $\langle\Gamma, f|$ unless the vertex v in Γ has at least three incident edges with linearly independent tangents; we shall call such vertices 'eternal' since they are neither added nor removed by the action of the Hamiltonian constraints (see figure 1). Furthermore, we may note that, for $\Gamma'_U \in S^{\Gamma_U,\dagger}_{v,U}$ we have $V(\Gamma'_U) \supset V(\Gamma_U)$ and that any vertex present in any Γ'_U which is not in Γ_U fails to have three incident edges with independent tangents. Thus, if we act again with another $\hat{H}^\dagger(M)$ and expand the result as above, only terms involving $\hat{H}^\dagger_{v'}(M)$ where v' is a vertex of the original graph Γ are nonzero.

This notation can be now used to calculate the commutator $[\hat{H}^\dagger(N), \hat{H}^\dagger(M)]$ very directly. As described above,

$$\langle\Gamma, f|[\hat{H}^\dagger(N), \hat{H}^\dagger(M)] = \sum_{v,v'\in V(\Gamma)} \langle\Gamma, f|(H^\dagger_v(N)H^\dagger_{v'}(M) - H^\dagger_{v'}(M)H^\dagger_v(N)). \tag{4.20}$$

Now, let $U \ni v$ and $U \ni v'$ be disjoint open sets such that the closures $\overline{U}$, $\overline{U'}$ each contain exactly one vertex (v or v') of Γ and such that each graph $R(\gamma)\cap\overline{U}$ and $R(\gamma)\cap\overline{U'}$ is connected. Then, for each spin network $\Gamma'_U\Gamma_{(\overline{U})^c}$ in (4.19), $R(\Gamma'_U)R(\Gamma_{(\overline{U})^c}) \cap \overline{U'} = R(\Gamma) \cap \overline{U'} =: R(\Gamma_{U'})$ is independent of Γ'_U. Thus, the operator $H^\dagger(M)$ acts on each such term in essentially the same way and we have:

$$\langle\Gamma, f|H^\dagger_v(N)H^\dagger_{v'}(M) = \sum_{\Gamma'_U\in S^{\Gamma_U,\dagger}_{v,U}} \sum_{\Gamma'_{U'}\in S^{\Gamma_{U'},\dagger}_{v',U'}} \langle\Gamma'_U\Gamma'_{U'}\Gamma_{(\overline{U\cup U'})^c}, f\circ i^* \star_v N \star_{v'} M|a^{\Gamma_U,\Gamma'_U}a^{\Gamma_{U'},\Gamma'_{U'}}. \tag{4.21}$$

Here, we have used i^* to denote each of the maps $i^*_{\Gamma,U,\Gamma'_U} \circ i^*_{\Gamma'_U\Gamma_{(\overline{U})^c},U',\Gamma'_{U'}}$ which should appear in (4.21). The point is that all of these maps act in essentially the same way: they simply restrict an assignment $\sigma : V(\Gamma'_U\Gamma'_{U'}\Gamma_{(U\cup U')^c}) \to \Sigma$ to $V(\Gamma)$. In writing (4.21), we have used the fact that composition with i^* and the star product over v commute when $v \in V(\Gamma)$; that is,

$$(f \star_v N) \circ i^* = (f \circ i^*) \star_v N \tag{4.22}$$

since $[i^*(\sigma)](v) = \sigma(v)$. Due to the commutativity (4.17) of the star product, it is clear that $\langle \Gamma, f|\hat{H}^\dagger_v(N)\hat{H}^\dagger_{v'}(M) = \langle \Gamma, f|\hat{H}^\dagger_{v'}(M)\hat{H}^\dagger_v(N)$. Using (4.20) above, we find that $[\hat{H}^\dagger(N), \hat{H}^\dagger(M)] = 0$.

We would now like to define 'localized' versions of the operators $\hat{H}(N)$ in analogy with the definition of $\hat{H}^\dagger_v(N)$. This will allow us to compute the terms $[\hat{H}^\dagger(N), \hat{H}(M)]$ and $[\hat{H}(N), \hat{H}^\dagger(M)]$. This time, however, there will be certain differences. Recall that the action of $\hat{H}(N)$ on a state $\langle \Gamma, f|$ generates a series of terms $\langle \Gamma', f'|$ such that each Γ' sits over some graph γ' given by *removing* some edge from γ. As a result, it will be useful to decompose the action of the constraint operator not only with respect to the 'vertex at which it acts,' but also with respect to the 'edge that is removed'. We denote the set of edges of Γ by $E(\gamma)$.

Let us begin with the following observation. Consider any edge e of Γ and let U be any open set which contains e and v, as well as any edges which connect v to the endpoints of e. However, the closure $\overline{U}$ of U is not to contain any other vertices and $\gamma \cap \overline{U}$ is to be connected. Now, consider the set $\tilde{S}^{e,\Gamma_U}_{v,U}$ of (gauge dependent) spin networks such that for $\Gamma'_U \in S^{e,\Gamma_U}_{v,U}$ we have A) the spin network is fully contained in $\overline{U}$: $\Gamma'_U \cap \overline{U} = \Gamma'_U$, B) the associated graph γ'_U is the subgraph of γ_U obtained by removing the edge e and C) $\langle \Gamma, 1|\hat{H}(\tilde{N})|\Gamma'_U \Gamma'_{(\overline{U})^c}\rangle$ is nonzero for some $\Gamma'_{(\overline{U})^c}$ consistent with Γ'_U and some function $\tilde{N}$ whose support supp$\tilde{N}$ satisfies supp$\tilde{N} \subset U$ and supp$\tilde{N} \cap V(\Gamma_U) = \{v\}$. Let $\overline{S}^{e,\Gamma_U}_{v,U}$ be the linear space spanned by $\tilde{S}^{e,\Gamma_U}_{v,U}$ and let $S^{e,\Gamma_U}_{v,U}$ be a basis for that space. Because the change in spins caused by $\hat{H}(\tilde{N})$ is bounded, and because there are a finite number of subgraphs of γ, the set $S^{e,\Gamma_U}_{v,U}$ is finite. Furthermore, if $\langle \Gamma, 1|\hat{H}(\tilde{N})|\Gamma_0\rangle$ is nonzero, then for some $\varphi \in Diff(\Sigma)$, $\varphi(\Gamma) \cap \overline{U}$ is a nontrivial element of the space $\overline{S}^{e,\Gamma}_{v,U}$ for some edge e.

For each vertex v and spin networks Γ', Γ for which $V(\Gamma) \supset V(\Gamma') \ni v$, let us introduce the maps $\eta^{*,\Gamma,\Gamma'}_v : \Sigma^{V(\Gamma')} \to \Sigma^{V(\Gamma)}$ and $\eta^{\Gamma',\Gamma}_v : V(\Gamma) \to V(\Gamma')$ defined by

$$\eta^{\Gamma',\Gamma}_v(v') = \begin{cases} v \text{ if } v' \notin V(\Gamma') \\ v' \text{ if } v' \in V(\Gamma') \end{cases} \quad \text{and}$$
$$\eta^{*,\Gamma',\Gamma}_v(\sigma) = \sigma \circ \eta^{\Gamma',\Gamma}_v. \tag{4.23}$$

The action of $\hat{H}(N)$ on $\langle \Gamma, f|$ may then be written

$$\langle \Gamma, f|\hat{H}(N) = \sum_{e\in E(\gamma)} \sum_{v\in V(\Gamma)} \sum_{\Gamma'_{U_{e,v}} \in S^{e,\Gamma_{U_{e,v}}}_{v,U_{e,v}}} \langle \Gamma'_{U_{e,v}} \Gamma_{U^c_{e,v}}, f \circ \eta^*_v \star_v N| b^{\Gamma'_{U_{e,v}},\Gamma_{U_{e,v}}} \tag{4.24}$$

where $E(\gamma)$ are the edges of γ and an appropriate collection of open sets $U_{e,v}$ has been chosen and we have used η^*_v as an abbreviation for $\eta_v^{*,\Gamma'_{U_{e,v}}\Gamma_{U^c_{e,v}},\Gamma}$. The coefficients $b^{\Gamma'_{U_{e,v}},\Gamma_{U_{e,v}}}$ are given by

$$b^{\Gamma'_{U_{e,v}},\Gamma_{U_{e,v}}} = \langle \Gamma_0 \Gamma_{U_{e,v}}|H(1)|\Gamma_0 \Gamma'_{U_{e,v}}\rangle \tag{4.25}$$

for some gauge dependent spin network Γ_0 contained in $(\overline{U_{e,v}})^c$ consistent with $\Gamma_{U_{e,v}}$ and $\Gamma'_{U_{e,v}}$ such that $\Gamma_0 \Gamma_{U_{e,v}}$ and $\Gamma_0 \Gamma'_{U_{e,v}}$ are gauge invariant spin networks over graphs with no symmetries, and such that the only eternal vertex of $\Gamma_0 \Gamma_{U_{e,v}}$ is v. Such a Γ_0 always exists, and (4.25) does not depend on the particular choice of Γ_0.

In analogy with $H^\dagger_v(M)$, we now introduce an object $H_{e,v}(N)$. Given $\langle\Gamma, f|$ with $e \in E(\Gamma)$ and $v \in V(\Gamma)$ and given an open set U as in the definition of $S^{e,\Gamma_U}_{v,U}$ above, we define

$$\langle\Gamma, f|\hat{H}_{e,v}(N) := \sum_{\Gamma'_U \in S^{e,\Gamma_U}_{v,U}} \langle\Gamma'_U \Gamma_{(\overline{U})^c}, f \circ \eta^*_v \star_v N| b^{\Gamma'_U, \Gamma_U} \tag{4.26}$$

so that

$$\langle\Gamma, f|\hat{H}(N) = \sum_{e\in E(\Gamma)} \sum_{v \in V(\Gamma)} \langle\Gamma, f|\hat{H}_{e,v}(N). \tag{4.27}$$

as before, the state (4.27) does not depend on the choice of the open set U.

Suppose that we wish to act on (4.27) with $H^\dagger_{v'}$ for some $v' \in V(\Gamma)$, $v \neq v'$. Recall that we need only act at eternal vertices v' which were present in the original graph γ. Since $v \neq v'$, this means that $v' \in V(\Gamma_{(\overline{U})^c})$ and that the action of $\hat{H}^\dagger_{v'}(M)$ on each term is well-defined. Choosing $U' \cap U = \emptyset$ and U' as in the definition of $S^{\Gamma_{U'},\dagger}_{v,U'}$, it is clear that the action of $\hat{H}^\dagger_v(M)$ is decoupled from that of $\hat{H}_{e,v}(N)$ and we have

$$\langle\Gamma, f|H_{e,v}(N) H^\dagger_{v'}(M) = \sum_{\Gamma'_U \in S^{e,\Gamma_U}_{v,U}} \sum_{\Gamma'_U \in S^{\Gamma_{U'},\dagger}_{v,U'}} \langle\Gamma'_U \Gamma'_{U'} \Gamma_{(\overline{U \cup U'})^c}, (f \circ \eta^*_v \star_v N) \circ i^* \star_{v'} M|. \tag{4.28}$$

On the other hand, for $e \in E(\Gamma)$, $v \in V(\Gamma)$, we may act on (4.19) with $\hat{H}_{e,v}(N)$ and the result is

$$\langle\Gamma, f|H^\dagger_{v'}(M) H_{e,v}(N) = \sum_{\Gamma'_{U'} \in S^{\Gamma_{U'},\dagger}_{v,U'}} \sum_{\Gamma'_U \in S^{e,\Gamma_U}_{v,U}} \langle\Gamma'_U \Gamma'_{U'} \Gamma_{(\overline{U \cup U'})^c}, (f \circ i^* \star_{v'} M) \circ \eta^*_v \star_v N| \tag{4.29}$$

where we again used the available freedom to choose $U' \cap U = \emptyset$.

Recall (4.22) that composition with i^* commutes with the star product over $v' \in V(\Gamma)$. Also, so long as v, v' are 'eternal' vertices that are neither added nor removed by the Hamiltonian constraints $f \circ \eta^*_v \star_{v'} M = (f \star_{v'} M) \circ \eta^*_v$ (whether $v = v'$ or not) since $[\eta^*_v(\sigma)](v') = \sigma(v')$. Finally, we note that since

$$\eta_v^{(\Gamma'_{U'}\Gamma'_U \Gamma_{(U\cup U')^c}),(\Gamma'_{U'}\Gamma_{(U')^c})}|_{V(\Gamma)} = \eta_v^{\Gamma'_U \Gamma_{U^c}, \Gamma}, \tag{4.30}$$

we may drop the superscripts on η_v and write

$$\begin{aligned}(i^* \circ \eta^*_v)(\sigma) &= \sigma \circ \eta_v \\ &= \sigma|_{V(\Gamma)} \circ \eta_v = (\eta^*_v \circ i^*)(\sigma).\end{aligned} \tag{4.31}$$

Combining this with commutativity (4.17) of the star product, we see that (4.28) and (4.29) are identical for eternal vertices v and v'.

However, this does *not* imply that $[\hat{H}(N), \hat{H}^\dagger(M)] = 0$. The point is that, while summing (4.28) over $v, v' \in V(\Gamma)$ and $e \in E(\Gamma)$ gives $\langle\Gamma, f|\hat{H}(N)\hat{H}^\dagger(M)$, it is not true that summing (4.29) over this set gives $\langle\Gamma, f|\hat{H}^\dagger(M)\hat{H}(N)$. This is because, when $\hat{H}(N)$ acts on (4.19), it generates terms corresponding not only to edges of the original graph Γ, but also to the edges (that we will call $e(\Gamma'_U, \Gamma_U)$) which were created by $\hat{H}^\dagger(M)$. Thus, there are still terms of the form

$$\sum_{\Gamma'_{\overline{U'}} \in S^{\Gamma_U,\dagger}_{U,v}} \langle \Gamma'_{U'} \Gamma_{(\overline{U'})^c}, f \circ i^* \star_{v'} M \star_v N | a^{\Gamma_{U'}, \Gamma'_{U'}} \hat{H}_{v', e(\Gamma_U, \Gamma'_U)} \tag{4.32}$$

to consider. For the special case $v = v'$, terms of this type can give no net contribution as it is clear that each such term is symmetric in M and N, whereas the commutator $[\hat{H}^S(N), \hat{H}^S(M)]$ is antisymmetric. The important question is therefore whether there are any terms of this kind for $v \neq v'$.

Let us therefore examine the requirements for the existence of such a term. The operator $\hat{H}^\dagger(M)$ creates an edge which is such that it can be 'slid' to the vertex v; that is, the new vertices created are on edges e_1, e_2 which are incident at v, and there are no vertices along e_1, e_2 between v and the new vertices (see figure 1). In the terms of interest, this newly created edge is then removed by the action of $\hat{H}_{v'}(N)$ acting at some *other* vertex v'. However, $\hat{H}_{v'}(N)$ can only remove edges that may be 'slid' to v'. Thus, such a term can only arise when the graph γ has a subgraph (Simon's subgraph [35]) of the form

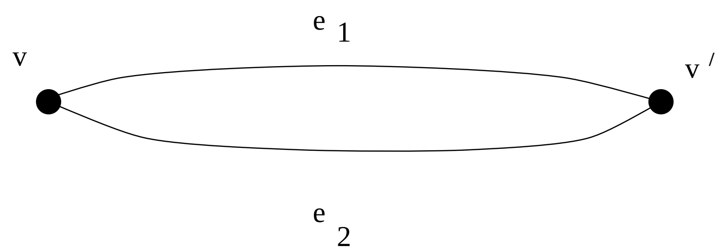

Fig. 2

with no additional edges linking with the above subgraph.

Explicit computations which we have made using the detailed form of the coefficients of Thiemann's Hamiltonian [7] show that, when γ is of this type, $\langle \Gamma, f | [H^S(N), H^S(M)]$ does not in general vanish. However, this has nothing to do with whether or not the function f is diffeomorphism invariant: even for constant functions such as $f = 1$, the action of the commutator on a diffeomorphism invariant state $\langle \Gamma, 1|$ is nonzero when Γ is based on a graph γ containing Simon's subgraph. Thus, the commutator of the symmetrized Hamiltonians may be called anomalous on such states.

In summary then, the space $\mathcal{T}'_*$ splits into two subspaces $\mathcal{T}'_* = \mathcal{T}'_0 \oplus \mathcal{T}'_A$ where $\mathcal{T}'_0$ is built from states $\langle \Gamma, f|$ such that γ has no subgraph of the form described by figure 2, while $\mathcal{T}'_A$ is built from those that do. It is worth mentioning that there are many subspaces of $\mathcal{T}'_0$ that are invariant under the action of H^S, so that it would be possible to simply restrict the definition of H^S to such a subspace. The commutator $[\hat{H}^S(N), \hat{H}^S(M)]$ annihilates all states in $\mathcal{T}'_0$, but does not annihilate the general state in $\mathcal{T}'_A$. For the case of the unprojected constraints on $\mathcal{T}'_{*,2}$, the result is similar except that the corresponding $\mathcal{T}'_A$ is larger and in the associated $\mathcal{T}'_0$ the only subspaces invariant under the action of $\hat{H}(N)$ and $\hat{H}^\dagger(N)$ are rather small.

Finally, we note that the need to split $\mathcal{T}'_*$ into the spaces $\mathcal{T}'_0$ and $\mathcal{T}'_A$ is a consequence of the particular loop assignment chosen above. If, on the other hand, only loops $\alpha(\gamma, v, I, J)$ that did not overlap the original graph γ (and therefore the intersect γ only at the vertex

v) were used instead, an analogous argument could be made but, this time, there would be no terms of the form (4.32). Instead, we would have $[\hat{H}^S(N), \hat{H}^S(M)] = 0$ on all of $\mathcal{T}'_*$.

V. DISCUSSION

In this work, we considered 'RST-like' proposals for the the Euclidean Hamiltonian constraints of quantum gravity in a loop representation which follow the suggestions of [11] and define the operators through a particular kind of limiting construction. We have shown that this limit converges not only on the diffeomorphism invariant states $\mathcal{T}'_{Diff}$ where it was first introduced, but in fact on a much larger space of 'vertex-smooth' states, $\mathcal{T}'_* \supset \mathcal{T}'_{Diff}$. In particular, $\mathcal{T}'_*$ contains the space in which the 'physical states' were sought in [1]. One might also have liked to introduce an inner product on $\mathcal{T}'_*$. We have not done so, and in fact the appendix shows that a fully satisfactory inner product cannot be introduced on *any* space in which Hamiltonian-like constraints are well defined and which contains a diffeomorphism invariant state not annihilated by the constraints. The results of the appendix may have consequences for the more general idea of first solving the diffeomorphism constraint and then defining the Euclidean Hamiltonian constraint on the resulting solutions.

We have also computed the algebra of RST-like constraints. For any two members of a family of such operators, their commutator is identically zero on $\mathcal{T}'_*$. Furthermore, we have addressed suggestions for 'symmetrizing' constraints of the type described in [1,2,7], with the result that their commutator is again identically zero. However, because the proposed 'symmetrizations' are considered unsatisfactory for other reasons, we also introduced another symmetrization which seems to be of the type desired by many researchers. With this latter definition, we find that $\mathcal{T}'_*$ decomposes as $\mathcal{T}'_0 \oplus \mathcal{T}'_A$ with the commutator vanishing on $\mathcal{T}'_0$ and being anomalous (in the sense defined above) on $\mathcal{T}'_A$. Thus, we have addressed all existing proposals for the Hamiltonian constraints in loop quantum gravity which involve a limiting procedure in which loops are shrunk to a point. Such results are in agreement with the prediction of [20] that, if a procedure for removing the regulators in a loop representation would be found, the resulting operators would not satisfy the classical algebra. The only proposal known to us which is not of the type considered here is that of [4], and this will be addressed in [18].

Although we have discussed only the Euclidean constraints here, an extension of the results of section II to the Lorentzian case is straightforward. The only definition of the Lorentzian constraints is that given in [1], and the proposed Lorentzian constraints are constructed in much the same way as the RST-like operators. By rewriting the 'anomaly-free' calculation of [1] along the lines of our section III C, it is clear that the commutator of two such Lorentzian Hamiltonian constraints vanishes as well.

This calculation was intended as a test of the extent to which the proposed quantum constraints capture the classical structure of general relativity. We found that their algebra is correct as long as the commutators are applied to diffeomorphism invariant states. At the level of diffeomorphism non-invariant states, the answer was found to be 'rather little.' It would appear that three interpretations are possible:

The first would be to note that the constraint algebra is not actually a 'physical' object since the constraints should simply annihilate any physical states. Thus, a consistent interpretation is to state that point of view to state that any meaningful comparison of a quantum theory with general relativity must take the form of examining the classical limit of gauge invariant quantities and physical states. Since our analysis does not achieve this, it is not truly meaningful.

As a first response, let us consider a theory of gravity coupled to matter. Then the classical phase space functions corresponding to the operators studied here no longer annihilate physical states and neither does their commutator. Instead, while that commutator is not an observable, it does generate 'diffeomorphisms of the gravitational degrees of freedom relative to the matter degrees of freedom' even on physical states. However, if our results carry over to such a setting and if the physical states lie in a space corresponding to the one studied here, then the analogous quantum commutator is still the zero operator. It is true that even this would not address the action of a physical operator on a physical state. Nonetheless, it appears close enough that our work may be taken as a caution that, when confronted with a a proposed quantum object arising from a complicated regularization procedure, it is important to find a nontrivial check that this object does in fact have some relation to the desired physics.

The next interpretation would be to suggest that the space $\mathcal{T}'_*$ is somehow too small to see a nontrivial commutator; that the commutator just 'happens to vanish' on $\mathcal{T}'_*$. In argument against this interpretation we note that $\mathcal{T}'_*$ is quite large and contains both the spaces $\mathcal{T}$ and $\mathcal{T}'_{Diff}$, which were supposed to capture much of the physics of general relativity [22–25]. However, this question merits a more thorough discussion of the 'right-hand side' of the classical commutator. Recall that the classical Poisson bracket of two Hamiltonian constraints is

$$\{H(N), H(M)\} = \int N_a C_b q^{ab}, \tag{5.1}$$

where q^{ab} is the inverse three-metric and C_b is the vector constraint. The question is then of how $\int N_a C_b q^{ab}$ should be represented on $\mathcal{T}$ or $\mathcal{T}'_*$ and whether or not it should, in general, vanish.[9]. One study of this operator was performed in [3] and it will be addressed further in [18], but here we content ourselves with two observations. We recall that C_b generates diffeomorphisms and that the generator of diffeomorphisms is well-defined and nonvanishing on $\mathcal{T}'_*$. Also, q^{ab} is invertible classically, so we would be surprised if it vanished on a large set of quantum states.

The third interpretation is that the quantum constraint proposals studied here fail to capture the physics of general relativity. Taking that point of view the question remains, however, of whether one can single out a particular aspect of these proposals as being responsible for the difficulties. The use of diffeomorphism invariant (or partially diffeomorphism invariant) states in defining the limit in which the regulators are removed seems a likely culprit. This question will be examined in more detail in [18], which will study certain variations on that theme.

It is, however, important to note that even if these constraints by themselves fail to capture the physics of gravity, this does not necessarily mean that they are *incompatible* with that physics and cannot be used as a starting point. An important example in this

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[9]This issue was first raised with us by Jorge Pullin.

regard is the work of Thiemann [37] on 2+1 gravity. We expect that all of the results given here in the 3+1 context also hold for the 2+1 constraints used there. Nevertheless, Thiemann showed that the solutions space of these constraints contains (as a small subspace) the usual physical states of the Witten formulation [38] of 2+1 gravity. He was then able to pick out these states by using various elements of the Witten formulation. Something similar may be possible in the 3+1 case as well, though the question remains of what additional structure would then play the role of the 2+1 Witten formulation.

It would appear that our work is related to the commentary of Smolin in [4]. His work notes that constraints defined by the RST-like limiting procedure (and their Hermitian conjugates) are 'too local' at an intuitive level and do not seem to generate structures that resemble the features of general relativity. It then suggests a number of difficulties which may be expected to follow, but unfortunately it is difficult to make these arguments conclusive. While the algebra was not a specific concern of [4], at least for the RST-like operators themselves (if not for the 'symmetrized' versions of section IV), it is the intuitive 'locality' which is responsible for the vanishing of the commutator. Thus, although our analysis does not deal directly with observables, one could regard our work in section III as a precise statement of the type desired in [4].

One might hope [5,30] to achieve better results by abandoning the canonical framework completely and defining the quantum theory via some sort of path integral. While this may be possible, we take the results derived here as a general warning that, until some well-defined and nontrivial calculation can be done to connect the quantum theory with general relativity, it remains unclear to what extent the proposed quantization captures the desired physics.

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APPENDIX A: ON AN INNER PRODUCT

In this appendix, we derive the following theorem:

Theorem: *Suppose that there is a Hilbert space $\mathcal{H}$ on which the diffeomorphism group $Diff(\Sigma)$ acts unitarily through operators $\mathcal{D}_\varphi$ for $\varphi \in Diff(\Sigma)$. Here, Σ is some compact* *manifold. Furthermore, suppose that there is a family of operators $H(N)$ labeled by smooth real-valued functions on Σ such that $H(N)$ is linear in N, and such that $D_\varphi H(N) D_\varphi^{-1} = H(N \circ \varphi)$. Then a state $|\psi_{Diff}\rangle$ can belong to the common domain of all operators $H(N)$ only if it is annihilated by them.*

Note that we have not required $H(N)$ to be symmetric or self-adjoint and that by definition laps functions have density weight zero. For a non-compact manifold Σ the same conclusion holds provided $H(N)$ is 'super linear' with respect to a sum given by a partition of the unity. That assumption is satisfied by the RST-Hamiltonian operators.

Since we have a large class of operators (the RST-like constraints) which are well-defined on $\mathcal{T}'_{Diff} \subset \mathcal{T}'_*$ and not all of $\mathcal{T}'_{Diff}$ is annihilated by the constraints, this theorem will show that there can be no Hermitian inner product on $\mathcal{T}'_*$ such that the action of $Diff(\Sigma)$ is unitary, unless it is very degenerate.

To prove the theorem, consider the inner product $(H(N)|\psi_{Diff}\rangle, H(M)|\psi_{Diff}\rangle) \equiv (N, M)$ and note that (N, M) is a positive definite *real* bilinear product on smooth real functions N and M. This product is diffeomorphism invariant in the sense that $(N, M) = (N \circ \varphi, M \circ \varphi)$ for any diffeomorphism $\varphi \in Diff(\Sigma)$.

Let us begin by covering Σ with coordinate charts $\mathcal{U}_\rangle$. It is enough to show that $H(N)|\psi_{Diff}\rangle = 0$ for any N supported in some fixed coordinate chart $\mathcal{U}$, since, due to the compactness of Σ, the general case then follows from diffeomorphism invariance and linearity.

Let N be supported in a chart $\mathcal{U}$. Then, there exists a function M supported in $\mathcal{U}$ such that, for sufficiently large value of a parameter λ, the function $M + \frac{N}{\lambda}$ is diffeomorphism equivalent to M. Indeed let M be any smooth function which coincides with x_1 on U. Since N is smooth and Σ is compact, the first derivative $\frac{\partial N}{\partial x_1}$ must be bounded by some positive real number λ'_0. Thus, for $\lambda > \lambda'_0$, $x_1 + N/\lambda$ is a smooth monotonically increasing function of x_1. Since N must vanish at the boundary of $\mathcal{U}$, $x + N/\lambda = x_1$ on the boundary of $\mathcal{U}$ and the map $\varphi : (x_1, x_2, ..., x_k) \mapsto (x_1 + N/\lambda, x_2, ..., x_k)$ is a diffeomorphism. From the diffeomorphism invariance of the product $(\ ,\ )$, for $\lambda > \lambda_0$, we must have $(M, M) = (M_\lambda, M_\lambda) = (M, M) + \lambda^{-1}[(M, N) + (N, M)] + \lambda^{-2}(N, N)$. Thus, $(N, M) + (M, N)$ and (N, N) must vanish. Since (N, N) is just the norm of $H(N)|\psi_{Diff}\rangle$ in $\mathcal{H}$, we find that $H(N)$ annihilates $|\psi_{Diff}\rangle$ which completes the proof.

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