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On the diffeomorphism commutators of lattice quantum gravity

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Abstract. We show that the algebra of discretized spatial diffeomorphism constraints in Hamiltonian lattice quantum gravity closes without anomalies in the limit of small lattice spacing. The result holds for arbitrary factor-ordering and for a variety of different discretizations of the continuum constraints, and thus generalizes an earlier calculation by Renteln.

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Motivation

For the most part, our understanding of quantum gravity in four dimensions is not sufficiently developed to provide unambiguous tests for the consistency and appropriateness of any candidate theory. In the case of usual gauge field theories, there are well known criteria to ensure their consistency at the quantum level, for example, the validity of the Slavnov–Taylor identities. However, the Fock space methods on which they are based are only of limited use for gravitational theories in more than two spacetime dimensions.

Attempts to construct non-perturbative path integral formulations of 4D gravity have so far left the question of their continuation to Lorentzian signature unanswered. Even disregarding this problem, it is difficult to get a mathematical handle on the structure of the path integral measure. Canonical approaches circumvent the signature problem and allow one to pose (part of) the quantum consistency problem in a seemingly clear-cut form: is there an anomaly in the Dirac algebra of the Hamiltonian and diffeomorphism constraints? In the presence of anomalous terms, the quantum Dirac conditions overconstrain the space of physical states, which is physically unacceptable. One difficulty in the canonical formulation lies in finding appropriate regularized versions of the constraint operators, without which the question is known to be ill-posed [1]. A potential drawback of the Hamiltonian approaches is the fact that efficient computational techniques for a numerical study of their quantum properties have yet to be developed.

Schemes that proceed by a direct discretization of spacetime or of the space of all field configurations in order to achieve a regularization tend to break the diffeomorphism invariance present in the continuum theory. This, however, is not a fundamental objection, if one considers the smooth structure of the classical theory only as a semi-classical property, which does not continue to hold down to the smallest length scales. Still it leaves one with the technical inconvenience of having to work with an 'approximately diffeomorphism-invariant' regularized theory.

Considerable effort has gone into trying to construct a canonical quantization of $3+1$ gravity in terms of connection variables, without resorting to a discretization, and with

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the full spatial diffeomorphism group still acting on a suitably defined space of quantum states [2–5]. However, this approach seems to be running into some difficulties, among which are a 'non-interacting' property of typical realizations of the quantum Hamiltonian, which leads, for example, to the existence of unexpected local quantum observables [6, 7], vanishing commutation relations between two such Hamiltonians [8] (even in situations where the diffeomorphism group acts non-trivially), and commutation relations between geometric operators that unexpectedly do not vanish [9, 10].

This provides an additional incentive for studying a truly discretized version of this formulation, which does not share any of these features. On the other hand, like most discretized models, it does not carry any obvious, non-trivial representation of the classical invariance group of general relativity. The diffeomorphism group is only recovered in the limit as the regulator is taken to zero.

In this paper we will generalize a result obtained earlier by Renteln [11] in the framework of a lattice discretization of $3+1$ gravity in a connection formulation [11–17]. Various aspects of this version of lattice quantum gravity have changed since these earlier investigations, as a result of new developments both in the continuum loop representations of quantum gravity mentioned above, and in lattice gravity proper. The kinematic resemblance to Hamiltonian lattice gauge theory has become even closer since the theory was rephrased in terms of real $su(2)$-valued connections [18], instead of the original $sl(2,\mathbb{C})$-valued Ashtekar potentials [19], in order to avoid complications related to the reality conditions that appear in the complex case. As a by-product, there is now a well defined scalar product and a complete set of square-integrable states for each finite-size lattice. Although this may not give rise to a scalar product on the sector of physical states (satisfying all of the constraints), it is nevertheless remarkable that it should exist at all, given that inner products are usually hard to come by in quantum gravity.

We will show that *independent of the factor-ordering* and for a variety of operator symmetrizations, there are no anomalous terms in the commutator algebra of the lattice analogues of the spatial diffeomorphism generators in the limit of vanishing lattice spacing. The computations are already substantial for this subalgebra of the entire quantum constraint algebra. We will also spell out some details of the calculations, that may be of interest in attempts to compute the commutator in alternative regularization schemes. This result shows that no inconsistencies arise in the lattice discretization at this level, and paves the way for the calculation of commutators also involving the Hamiltonian constraint.

1. Introduction

A prominent feature of the lattice discretization we are considering is its resemblance to canonical lattice gauge theory. Like Hamiltonian gauge field theory, $3+1$ continuum gravity can be formulated in terms of Yang–Mills conjugate variable pairs (A, E) of 'Ashtekar-type', with Poisson brackets $\{A_a^i(x), E_j^b(y)\} = \delta_j^i \delta_a^b \delta^3(x, y)$. The variable A_a^i is a spatial $su(2,\mathbb{R})$-valued gauge potential.

Up to terms proportional to the Gauss law constraint, the Hamiltonian constraint in this formulation is the Barbero Hamiltonian [18] (rescaled by a factor $(\det E)^{-1/2}$ to make it a density of weight one), plus—for the sake of generality—a cosmological constant term,

$$\mathcal{H}(x) = \frac{1}{\sqrt{\det E}} \epsilon^{ijk} E_i^a E_j^b F_{ab}^k - \frac{1}{G} (\det E)^{-5/2} \eta_{a_1 a_3 a_4} \eta_{b_1 b_3 b_4} (E_k^{a_3} E_l^{a_4} E_m^{b_3} E_n^{b_4} - 2 E_m^{a_3} E_n^{a_4} E_k^{b_3} E_l^{b_4}) E_k^{a_2} E_m^{b_2} (\nabla_{a_2} E_l^{a_1})(\nabla_{b_2} E_n^{b_1}) + \lambda \sqrt{\det E} = 0. \tag{1.1}$$

We have chosen the dimensions of the basic variables and constants to be $[A] = L^{-3}$, $[E] = L^0$, $[G] = L^2$, $[\lambda] = L^{-4}$, where G is Newton's constant. Also recall that E transforms as a one-density under spatial diffeomorphisms. In (1.1), $\mathcal{H}$ has already been brought into the form of a polynomial modulo powers of $\sqrt{\det E}$, in order to make its discretization and quantization straightforward. The spatial diffeomorphism constraints are given by

$$\mathcal{V}_b(x) = E_i^a(x) F_{ab}^i(x) = 0. \tag{1.2}$$

Smearing out these four constraints with the lapse and shift functions $N(x)$ and $N^a(x)$, one arrives at the expressions

$$\begin{aligned} H[N] &:= \int \mathrm{d}^3x\, N(x)\mathcal{H}(x) \\ V[N^a] &:= \int \mathrm{d}^3x\, N^b(x)\mathcal{V}_b(x), \end{aligned} \tag{1.3}$$

which satisfy the usual Dirac Poisson algebra. In particular, for the spatial diffeomorphism generators one derives

$$\begin{aligned} \{V[N^a], V[M^a]\} &= \int \mathrm{d}^3x\, (N^b\partial_b M^a - M^b\partial_b N^a) F_{ac}^i E_i^c - \int \mathrm{d}^3x\, F_{ab}^i N^a M^b (\nabla_c E_i^c) \\ &\equiv V[L_N M^a] - G[F_{ab}^i N^a M^b], \end{aligned} \tag{1.4}$$

which coincides with the diffeomorphism Lie algebra on the subspace of phase space given by the vanishing of the integrated Gauss law constraints $G[\Lambda^i] = \int \mathrm{d}^3x\, \Lambda^i(\nabla_a E_i^a)$. One may also redefine the generators $V[N^a]$ by adding a suitable term proportional to the Gauss law constraint in order to get rid of the extra phase-space dependent term on the right-hand side of (1.4) (see, for example, [20]).

2. Discretization

Before setting up the discretization of operators relevant for lattice gravity, we start with a brief summary of the basic ingredients of Hamiltonian lattice gauge theory [21]. For computational simplicity, we take the lattice Λ to be a cubic N^3-lattice with periodic boundary conditions. The cubic symmetry is convenient but not strictly necessary. The basic variables and quantum operators may as well be defined on lattices where the valence of the intersections is not fixed to be 6.

The basic quantum operators associated with each lattice link l are a group-valued $SU(2)$-link holonomy $\hat{U}$ (represented by multiplication by U), together with its inverse $\hat{U}^{-1}$, and a pair of canonical momentum operators $\hat{p}_i^+$ and $\hat{p}_i^-$, where i is an adjoint index. The operator $\hat{p}_i^+(n, \hat{a})$ is based at the vertex n, and is associated with the link l oriented in the positive $\hat{a}$-direction. By contrast, $\hat{p}_i^-(n + \hat{1}_{\hat{a}}, \hat{a})$ is based at the vertex displaced by one lattice unit in the $\hat{a}$-direction, and associated with the inverse link $l^{-1}(\hat{a}) = l(-\hat{a})$. In mathematical terms, the momenta $\hat{p}^+$ and $\hat{p}^-$ correspond to the left- and right-invariant vector fields on the group manifold associated with a given link. The wavefunctions are elements of $\times_l L^2(SU(2), \mathrm{d}g)$, with the product taken over all links, and the canonical Haar measure $\mathrm{d}g$ on each copy of the group $SU(2)$. The basic commutators are

$$
\begin{aligned}
&[\hat{U}_A{}^B(n,\hat{a}),\hat{U}_C{}^D(m,\hat{b})]=0,\\
&[\hat{p}_i^+(n,\hat{a}),\hat{U}_A{}^C(m,\hat{b})]=-\tfrac{1}{2}\mathrm{i}\,\delta_{nm}\delta_{\hat{a}\hat{b}}\,\tau_{iA}{}^B\hat{U}_B{}^C(n,\hat{a}),\\
&[\hat{p}_i^-(n,\hat{a}),\hat{U}_A{}^C(m,\hat{b})]=-\tfrac{1}{2}\mathrm{i}\,\delta_{n,m+1}\delta_{\hat{a}\hat{b}}\,\hat{U}_A{}^B(n,\hat{a})\tau_{iB}{}^C,\\
&[\hat{p}_i^\pm(n,\hat{a}),\hat{p}_j^\pm(m,\hat{b})]=\pm\mathrm{i}\,\delta_{nm}\delta_{\hat{a}\hat{b}}\,\epsilon_{ijk}\,\hat{p}_k^\pm(n,\hat{a}),\\
&[\hat{p}_i^+(n,\hat{a}),\hat{p}_j^-(m,\hat{b})]=0,
\end{aligned}
\tag{2.1}
$$

where ϵ_{ijk} are the structure constants of $SU(2)$. (There are analogous Poisson bracket relations for the corresponding classical lattice variables.) The commutation relations for the inverse holonomy operators can be easily deduced from (2.1). The normalization for the $SU(2)$ generators τ_i is such that $[\tau_i,\tau_j]=2\,\epsilon_{ijk}\tau_k$, $\mathrm{Tr}\,\tau_i\tau_j=-2\delta_{ij}$ and $A_a=\frac{1}{2}A_a^i\tau_i$.

In order to relate discrete lattice expressions to their continuum counterparts, one uses power series expansions in the so-called lattice spacing a, which is an unphysical parameter with dimension of length. For the basic classical lattice variables, these are

$$
\begin{aligned}
&U_A{}^B(\hat{b})=1_A{}^B+a\,G\,A_{bA}{}^B+\tfrac{1}{2}a^2G(\partial_bA_b+GA_b^2)_A{}^B+\mathrm{O}(a^3),\\
&p_i^\pm(\hat{b})=a^2\,G^{-1}\,E_i^b\pm a^3\,G^{-1}\,\nabla_bE_i^b+\mathrm{O}(a^4).
\end{aligned}
\tag{2.2}
$$

We will assume that similar expansions continue to be valid in the quantum theory. Note that Newton's constant G appears in (2.2) since the dimensions of the basic gravitational variables A and E differ from those of the corresponding Yang–Mills phase space variables.

Using the expansions (2.2), one obtains unambiguous continuum limits of composite classical lattice expressions by extracting the coefficient of the lowest-order term in the a-expansion. The converse is not true: there is no unique lattice discretization of a continuum expression, since one may always add to the lattice version terms of higher order in a, which do not contribute in the continuum limit. We will consider different symmetrizations for the lattice diffeomorphism generators in the rest of this paper.

3. Further preparations

As a first step in the commutator calculation of two lattice diffeomorphisms, one needs to expand the classical link holonomies in a neighbourhood of the lattice vertex n_0 at which the local commutator will be computed. For simplicity we choose a local lattice coordinate system such that the vertex n_0 coincides with its origin, that is, $n_0=(0,0,0)$. There are six nearest neighbours, which are one lattice unit away, for example, the vertices $n=(\pm1,0,0)$ in the positive and negative $\hat{1}$-direction. Because of the non-locality of the diffeomorphism generators, all relevant link holonomies lie on a cube of edge length 2 about the origin.

Using the defining formula for the holonomy along a parametrized path γ between two points x_1 and x_2, with $\gamma(0)=x_1$ and $\gamma(1)=x_2$,

$$
U(x_1,x_2)=\mathbb{1}+\sum_{n=1}^{\infty}\int_0^1\mathrm{d}s_1\int_{s_1}^1\mathrm{d}s_2\cdots\int_{s_{n-1}}^1\mathrm{d}s_n\,A(\gamma(s_1))A(\gamma(s_2))\ldots A(\gamma(s_n)),
\tag{3.1}
$$

one expands this into a power series in the lattice spacing a for the special choice $x_1=(0,0,0)$, $x_2=(a,0,0)$, say. We will need terms up to power a^3 for our computation.

For the holonomy in the positive $\hat{b}$-direction based at $(0,0,0)$, one thus obtains

$$U\big((0,0,0),\hat{b}\big) = \mathbb{1} + aA_b + \frac{a^2}{2}(\partial_b A_b + A_b^2) \\ + \frac{a^3}{3!}\big(\partial_b^2 A_b + (\partial_b A_b)A_b + 2A_b(\partial_b A_b) + A_b^3\big) + \mathrm{O}(a^4), \tag{3.2}$$

where we now have set $G = 1$. (Note that at order a^3 this differs from the expression given in [11].) The other positively-oriented link holonomies are computed similarly, as Taylor expansions with respect to the local Cartesian coordinate system spanned by the lattice. From these, the inverse holonomies (associated with the opposite link orientation) are computed order by order using $UU^{-1} = \mathbb{1}$. Expansions for the link momenta are obtained in an analogous manner, using (2.2).

Two useful checks for possible errors are given by calculating holonomies $U_\Box$ around plaquettes, i.e. shortest closed lattice paths of edge length four. Denoting by $U_{\Box_{12}}$ the plaquette that starts with a link in the positive $\hat{1}$-direction, followed by links in directions $\hat{2}$, $-\hat{1}$ and $-\hat{2}$, one has the expansions

$$\mathrm{Tr}\, U_{\Box_{12}} = 2 + \mathrm{O}(a^4), \qquad \mathrm{Tr}(U_{\Box_{12}}\tau_i) = a^2 F_{12}^i + \mathrm{O}(a^3). \tag{3.3}$$

A natural discretization for the smeared spatial diffeomorphism generators, (the second equation in (1.3)), is schematically given by

$$\sum_n N^{\mathrm{latt}}(n,\hat{a})\,\mathrm{Tr}(U_{\Box_{ab}}\tau_i)\,p_i(n,\hat{b}) =: \sum_n \mathcal{V}^{\mathrm{latt}}[N,n], \tag{3.4}$$

where we have not yet specified the details of the implicit summation over the indices $\hat{a}$ and $\hat{b}$. We will consider four distinct ways of symmetrizing the local lattice expressions $N^{\mathrm{latt}}(n,\hat{a})\,\mathrm{Tr}(U_{\Box_{ab}}\tau_i)\,p_i(n,\hat{b})$. Let us introduce the notation $N^\pm(n,\hat{a})$ for the lattice shift functions at the vertex n, associated with the link in positive and negative $\hat{a}$-direction, respectively. This notation conforms with the one for the momenta $p^\pm(n,\hat{a})$. In order for (3.4) to have the correct continuum limit as $a \to 0$, we require that

$$\begin{aligned} N^+(n,\hat{b}) &\to \frac{1}{a}N^b(x) + \tilde{N}^b(x) + \mathrm{O}(a^1) \\ N^-(n,\hat{b}) &\to \frac{1}{a}N^b(x) + \tilde{\tilde{N}}^b(x) + \mathrm{O}(a^1). \end{aligned} \tag{3.5}$$

We leave the zeroth-order functions $\tilde{N}^b(x)$ and $\tilde{\tilde{N}}^b(x)$ unspecified; the final result will not depend on this choice. For the local continuum expression of the form $N^2 F_{23}^i E_i^3$, say, we will consider four different lattice representations:

(i) no symmetrization:

$$N^+(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{23}})\,p_i^+(n,\hat{3});$$

(ii) symmetrization over shift functions:

$$\tfrac{1}{2}\big(N^+(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{23}})\,p_i^+(n,\hat{3}) + N^-(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{3,-2}})\,p_i^+(n,\hat{3})\big);$$

(iii) symmetrization over momenta:

$$\tfrac{1}{2}\big(N^+(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{23}})\,p_i^+(n,\hat{3}) + N^+(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{-3,2}})\,p_i^-(n,\hat{3})\big);$$

(iv) symmetrization over both shift functions and momenta:

$$\tfrac{1}{4}\big(N^+(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{23}})\,p_i^+(n,\hat{3}) + N^-(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{3,-2}})\,p_i^+(n,\hat{3})$$
$$+ N^+(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{-3,2}})\,p_i^-(n,\hat{3}) + N^-(n,\hat{2})\,\mathrm{Tr}(U_{\Box_{-2,-3}})\,p_i^-(n,\hat{3})\big).$$

Note that in each case we sum over terms that are maximally localized on the lattice, i.e. consist only of holonomies and momenta located on a single lattice plaquette. Choice (iv) was the one considered in [11].

4. The commutator

We now come to the actual computation of the commutator of two lattice-regularized diffeomorphism constraints. As is well known, the algebra of the discretized constraints (3.4) does not close, not even at the level of the classical Poisson algebra, because the result is a sum of terms, where each term may extend over up to two plaquettes. That is, the Poisson commutator is not a linear combination of terms of the form (3.4). However, if the discretization is consistent, the commutator should yield the continuum result to lowest order in the lattice spacing a as $a \to 0$.

We will perform the corresponding quantum computation, and for definiteness will choose initially the factor-ordering for the $\mathrm{Tr}(\hat{U}_{\Box_{ab}}\tau_i)\,\hat{p}_i(n,\hat{b})$-terms with all momenta $\hat{p}$ to the right.

To obtain the contribution at a given lattice vertex n_0 to the commutator of two smeared lattice diffeomorphisms $\big[\sum_n \hat{\mathcal{V}}[N,n], \sum_m \hat{\mathcal{V}}[M,m]\big]$, one has to collect all terms where a momentum based at n_0 acts on either a holonomy or another momentum based on the same link. One quickly realizes that there are not only contributions from terms in $[\hat{\mathcal{V}}[N,n_0], \hat{\mathcal{V}}[M,n_0]]$, but also contributions from vertices close by, either from nearest neighbours (e.g. $n_0 \pm (1,0,0)$—there are 6 such vertices) or from vertices across a plaquette diagonal (e.g. $n_0 \pm (1,1,0)$—there are 12 such vertices). We will see below that these non-local contributions are indeed necessary for obtaining the correct result.

Without loss of generality, we will consider local lattice smearing functions $N(n_0,\hat{a}) = (N^1,0,0)$ and $M(n_0,\hat{a}) = (0,M^2,0)$. For the fully symmetrized version of the local constraints $\hat{\mathcal{V}}[N,n]$, there are 36 non-vanishing commutators between terms based at n_0 and 72 non-vanishing commutators also involving terms based at neighbouring vertices. The commutators are evaluated using the basic commutation relations (2.1) and various epsilon-function identities. The most time-consuming task is the expansion and subsequent simplification of the resulting expressions in powers of a. It turns out that the holonomies have to be expanded to third order, i.e. up to terms cubic in the local connection form $A(x)$, and there are rather a lot of terms at this order.

In the appendix we have collected the separate contributions from both the local commutators and those involving neighbouring vertices, for all four types of symmetrization. It is instructive to see what kind of terms arise in the different parts of the calculation and how they cancel. This may be relevant to attempts in the continuum loop representation of quantum gravity to reproduce similar quantum commutators. Note also that in order to derive (4.1) below from the formulae given in the appendix, some partial integrations had to be performed.

After a lot of algebra one finds that, independent of the symmetrization, the final result is given by

$$\lim_{a\to 0}\left[\sum_n \hat{\mathcal{V}}[N,n), \sum_m \hat{\mathcal{V}}[M,m)\right]\Big|_{n_0}$$

$$\begin{aligned}
&= a^3(N^1(\partial_1 M^2)(\hat{F}^i_{23}\hat{E}^3_i - \hat{F}^i_{12}\hat{E}^1_i) + M^2(\partial_2 N^1)(\hat{F}^i_{31}\hat{E}^3_i - \hat{F}^i_{12}\hat{E}^2_i) \\
&\quad - N^1 M^2 \hat{F}^i_{12}(\hat{\nabla}_1\hat{E}^1_i + \hat{\nabla}_2\hat{E}^2_i + \hat{\nabla}_3\hat{E}^3_i) \\
&\quad + N^1 M^2(\hat{\nabla}_1\hat{F}^i_{23} + \hat{\nabla}_2\hat{F}^i_{31} + \hat{\nabla}_3\hat{F}^i_{12})\hat{E}^3_i) + \mathrm{O}(a^4).
\end{aligned} \tag{4.1}$$

Comparing with equation (1.4), this is the expected answer, without any anomalies, and up to a term proportional to the Bianchi identity. It shows that at this level both the classical discretization and the quantization of the diffeomorphism constraints are consistent. The independence of the symmetrization suggests that there is a chance that the evaluation of more complicated commutators involving also the Hamiltonian constraint may already yield the right result if done in terms of the unsymmetrized lattice constraints. This is potentially important since the calculations become even more complicated.

Let us now turn to the issue of factor-ordering. We have already proven the absence of anomalies with momenta ordered to the right and will now use this result to deduce what happens for arbitrary factor-ordering. Let us adopt the notation $(\mathrm{Tr}\,U\tau_i)\hat{p}_i$ for a typical term of the quantized diffeomorphism constraint, where U denotes some plaquette loop. We have

$$\begin{aligned}
[(\mathrm{Tr}\,U\tau_i)\hat{p}_i, (\mathrm{Tr}\,V\tau_j)\hat{p}_j] &= (\mathrm{Tr}\,U\tau_i)[\hat{p}_i, \mathrm{Tr}\,V\tau_j]\hat{p}_j \\
&\quad - (\mathrm{Tr}\,V\tau_j)[\hat{p}_j, \mathrm{Tr}\,U\tau_i]\hat{p}_i + (\mathrm{Tr}\,U\tau_i)(\mathrm{Tr}\,V\tau_j)[\hat{p}_i, \hat{p}_j]
\end{aligned} \tag{4.2}$$

and want to investigate what happens when the momenta are ordered to the left, i.e. whether

$$\begin{aligned}
&[\hat{p}_i(\mathrm{Tr}\,U\tau_i), \hat{p}_j(\mathrm{Tr}\,V\tau_j)] - \hat{p}_j(\mathrm{Tr}\,U\tau_i)[\hat{p}_i, \mathrm{Tr}\,V\tau_j] \\
&\quad + \hat{p}_i(\mathrm{Tr}\,V\tau_j)[\hat{p}_j, \mathrm{Tr}\,U\tau_i] - [\hat{p}_i, \hat{p}_j](\mathrm{Tr}\,U\tau_i)(\mathrm{Tr}\,V\tau_j)
\end{aligned} \tag{4.3}$$

vanishes or yields new terms of order $\hbar$. Rewriting $\hat{p}_i(\mathrm{Tr}\,U\tau_i) = (\mathrm{Tr}\,U\tau_i)\hat{p}_i + [\hat{p}_i, \mathrm{Tr}\,U\tau_i]$ and using equation (4.2), expression (4.3) becomes

$$\begin{aligned}
&\big[(\mathrm{Tr}\,U\tau_i)\hat{p}_i, [\hat{p}_j, \mathrm{Tr}\,V\tau_j]\big] + \big[[\hat{p}_i, \mathrm{Tr}\,U\tau_i], (\mathrm{Tr}\,V\tau_j)\hat{p}_j\big] - \big[\hat{p}_j, (\mathrm{Tr}\,U\tau_i)[\hat{p}_i, \mathrm{Tr}\,V\tau_j]\big] \\
&\quad - \big[\hat{p}_i, (\mathrm{Tr}\,V\tau_j)[\mathrm{Tr}\,U\tau_i, \hat{p}_j]\big] - \big[[\hat{p}_i, \hat{p}_j], (\mathrm{Tr}\,U\tau_i)(\mathrm{Tr}\,V\tau_j)\big] \\
&= (\mathrm{Tr}\,U\tau_i)\big[\hat{p}_i, [\hat{p}_j, \mathrm{Tr}\,V\tau_j]\big] + (\mathrm{Tr}\,V\tau_j)\big[[\hat{p}_i, \mathrm{Tr}\,U\tau_i], \hat{p}_j\big] \\
&\quad - (\mathrm{Tr}\,U\tau_i)\big[\hat{p}_j, [\hat{p}_i, \mathrm{Tr}\,V\tau_j]\big] - (\mathrm{Tr}\,V\tau_j)\big[\hat{p}_i, [\mathrm{Tr}\,U\tau_i, \hat{p}_j]\big] \\
&\quad - \big[[\hat{p}_i, \hat{p}_j], \mathrm{Tr}\,U\tau_i\big](\mathrm{Tr}\,V\tau_j) - (\mathrm{Tr}\,U\tau_i)\big[[\hat{p}_i, \hat{p}_j], \mathrm{Tr}\,V\tau_j\big] = 0
\end{aligned} \tag{4.4}$$

where we have made repeated use of the basic commutators (2.1). The last equality in (4.4) holds by virtue of the Jacobi identity satisfied by the basic quantum operators. It follows immediately that for an arbitrary factor-ordering of the lattice constraints, $\alpha(\mathrm{Tr}\,U\tau_i)\hat{p}_i + (1-\alpha)\hat{p}_i(\mathrm{Tr}\,U\tau_i)$, $0 \leqslant \alpha \leqslant 1$, no anomalies appear. This is, in particular, true for the case $\alpha = \frac{1}{2}$, where the regularized constraint operators are self-adjoint.

5. Summary

We have performed a computation of the commutator of two regularized diffeomorphism constraint operators in lattice gravity, and found that their algebra closes without anomalies in the limit of vanishing lattice spacing. The discretization and quantization of the classical diffeomorphism phase space functions is not unique, but our result is independent of the choice of a local symmetrization and factor-ordering. Independence of factor-ordering followed from the simple structure of the constraints (linearity in momenta) and the fact that the basic lattice operators satisfy the Jacobi identity. We do not expect a similar behaviour for commutators involving also the Hamiltonian constraint.

There are not many regularization schemes for full four-dimensional quantum gravity in which a computation of this type could be performed. Within our lattice formulation, it would in principle be preferable to have an exact remnant of the diffeomorphism symmetry at each stage of the discretization. This would enable one to study invariant measures and quantum states *before* the continuum limit is taken. One way of proceeding is to try to identify a discrete subgroup of the diffeomorphism group (rather than using the discretized 'infinitesimal' generators) for each finite lattice, an issue we are currently considering. Transformations of this type will presumably be non-local in terms of the lattice variables.

Appendix

In this appendix we present some intermediate results of the commutator calculation, depending on the symmetrization chosen. This will be helpful to anybody attempting a similar calculation. We split up the contributions into those that arise from contributions based at the same lattice point (the origin in our case) and those involving contributions based at neighbouring points on the lattice. It is understood that A, F and E are operators.

(i) *No symmetrization.* Contributions at the origin:

$$\begin{aligned}-a^2\big(&N^1M^2F^i_{12}(E^1_i+E^2_i+E^3_i)+N^1M^2(F^i_{12}+F^i_{23}+F^i_{31})E^3_i\big)\\ &-a^3\big((M^2\tilde N^1+N^1\tilde M^2)F^i_{12}(E^1_i+E^2_i+E^3_i)\\ &+(M^2\tilde N^1+N^1\tilde M^2)(F^i_{12}+F^i_{23}+F^i_{31})E^3_i\\ &+\tfrac12 N^1M^2(\nabla_1+\nabla_2)F^i_{12}(E^1_i+E^2_i+E^3_i)\\ &+\tfrac12 N^1M^2\big((\nabla_1+\nabla_2)F^i_{12}+(\nabla_2+\nabla_3)F^i_{23}+(\nabla_1+\nabla_3)F^i_{31}\big)E^3_i\\ &+N^1M^2F^i_{12}(\nabla_1E^1_i+\nabla_2E^2_i+\nabla_3E^3_i)\\ &+N^1M^2(F^i_{12}+F^i_{23}+F^i_{31})\nabla_3E^3_i\big)+\mathrm{O}(a^4).\end{aligned}\tag{A.1}$$

Contributions involving neighbouring vertices:

$$\begin{aligned}a^2\big(&N^1M^2F^i_{12}(E^1_i+E^2_i+E^3_i)+N^1M^2(F^i_{12}+F^i_{23}+F^i_{31})E^3_i\big)\\ &+a^3\big((M^2\tilde N^1+N^1\tilde M^2)F^i_{12}(E^1_i+E^2_i+E^3_i)+(M^2\tilde N^1\\ &+N^1\tilde M^2)(F^i_{12}+F^i_{23}+F^i_{31})E^3_i\\ &+\tfrac12 N^1M^2(\partial_1F^i_{12}+3\epsilon^i{}_{jk}A^j_1F^k_{12}+\nabla_2F^i_{12})E^1_i\\ &+\tfrac12 N^1M^2(\partial_2F^i_{12}+3\epsilon^i{}_{jk}A^j_2F^k_{12}+\nabla_1F^i_{12})E^2_i\\ &+\tfrac12 N^1M^2\big(2(\nabla_1+\nabla_2)F^i_{12}+(\nabla_2+\nabla_3)F^i_{23}+(\nabla_1+\nabla_3)F^i_{31}\big)E^3_i\end{aligned}$$

$$
\begin{aligned}
&- N^1M^2F^i_{12}(\nabla_1E^1_i + \nabla_2E^2_i + \nabla_3E^3_i)\\
&- N^1M^2F^i_{12}(\partial_1E^1_i + \partial_2E^2_i + \partial_3E^3_i) + N^1M^2(F^i_{12} + F^i_{23} + F^i_{31})\nabla_3E^3_i\\
&- N^1M^2(F^i_{12}\partial_3E^3_i + F^i_{23}\partial_1E^3_i + F^i_{31}\partial_2E^3_i) - N^1(\partial_1M^2)F^i_{12}E^1_i\\
&- M^2(\partial_2N^1)F^i_{12}E^2_i - N^1(\partial_3M^2)F^i_{12}E^3_i - M^2(\partial_3N^1)F^i_{12}E^3_i\\
&- M^2(\partial_1N^1)F^i_{23}E^3_i - N^1(\partial_2M^2)F^i_{31}E^3_i\\
&+ N^1M^2\epsilon_{ijk}(2A^j_3F^k_{12} + A^j_2F^k_{31} + A^j_1F^k_{23})E^{3i}\big) + \mathrm{O}(a^4).
\end{aligned}
\tag{A.2}
$$

(ii) *Symmetrization over shift functions.* Contributions at the origin:

$$
\begin{aligned}
-a^2\big(&\tfrac{1}{2}N^1M^2F^i_{12}(E^1_i + E^2_i + 2E^3_i)\big) - a^3\big(\tfrac{1}{4}(M^2\tilde{N}^1 + M^2\tilde{\tilde{N}}^1 + 2N^1\tilde{M}^2)F^i_{12}E^1_i\\
&+ \tfrac{1}{4}(2M^2\tilde{N}^1 + N^1\tilde{M}^2 + N^1\tilde{\tilde{M}}^2)F^i_{12}E^2_i + \tfrac{1}{4}(3M^2\tilde{N}^1 + M^2\tilde{\tilde{N}}^1\\
&+ 3N^1\tilde{M}^2 + N^1\tilde{\tilde{M}}^2)F^i_{12}E^3_i + \tfrac{1}{2}M^2(\tilde{N}^1 - \tilde{\tilde{N}}^1)F^i_{23}E^3_i\\
&+ \tfrac{1}{2}N^1(\tilde{M}^2 - \tilde{\tilde{M}}^2)F^i_{31}E^3_i + \tfrac{1}{4}N^1M^2\big((\nabla_2F^i_{12})E^1_i + (\nabla_1F^i_{12})E^2_i\\
&+ ((\nabla_1 + \nabla_2)F^i_{12})E^3_i\big) + \tfrac{1}{2}F^i_{12}(\nabla_1E^1_i + \nabla_2E^2_i + 2\nabla_3E^3_i)\big) + \mathrm{O}(a^4).
\end{aligned}
\tag{A.3}
$$

Contributions involving neighbouring vertices:

$$
\begin{aligned}
a^2\big(&\tfrac{1}{2}N^1M^2F^i_{12}(E^1_i + E^2_i + 2E^3_i)\big) + a^3\big(\tfrac{1}{4}(M^2\tilde{N}^1 + M^2\tilde{\tilde{N}}^1 + 2N^1\tilde{M}^2)F^i_{12}E^1_i\\
&+ \tfrac{1}{4}(2M^2\tilde{N}^1 + N^1\tilde{M}^2 + N^1\tilde{\tilde{M}}^2)F^i_{12}E^2_i\\
&+ \tfrac{1}{4}(3M^2\tilde{N}^1 + M^2\tilde{\tilde{N}}^1 + 3N^1\tilde{M}^2 + N^1\tilde{\tilde{M}}^2)F^i_{12}E^3_i\\
&+ \tfrac{1}{2}M^2(\tilde{N}^1 - \tilde{\tilde{N}}^1)F^i_{23}E^3_i + \tfrac{1}{2}N^1(\tilde{M}^2 - \tilde{\tilde{M}}^2)F^i_{31}E^3_i\\
&+ \tfrac{1}{4}N^1M^2\big((\nabla_2F^i_{12})E^1_i + (\nabla_1F^i_{12})E^2_i + ((\nabla_1 + \nabla_2)F^i_{12})E^3_i\big)\\
&+ N^1M^2\epsilon_{ijk}(A^j_1F^k_{12}E^{1i} + A^j_2F^k_{12}E^{2i})\\
&+ \tfrac{1}{2}F^i_{12}(\nabla_1E^1_i + \nabla_2E^2_i + 2\nabla_3E^3_i) - N^1M^2F^i_{12}(\partial_1E^1_i + \partial_2E^2_i + \partial_3E^3_i)\\
&- N^1M^2(F^i_{12}\partial_3E^3_i + F^i_{23}\partial_1E^3_i + F^i_{31}\partial_2E^3_i) - N^1(\partial_1M^2)F^i_{12}E^1_i\\
&- M^2(\partial_2N^1)F^i_{12}E^2_i - N^1(\partial_3M^2)F^i_{12}E^3_i - M^2(\partial_3N^1)F^i_{12}E^3_i\\
&- M^2(\partial_1N^1)F^i_{23}E^3_i - N^1(\partial_2M^2)F^i_{31}E^3_i\\
&+ N^1M^2\epsilon_{ijk}(2A^j_3F^k_{12} + A^j_2F^k_{31} + A^j_1F^k_{23})E^{3i}\big) + \mathrm{O}(a^4).
\end{aligned}
\tag{A.4}
$$

(iii) *Symmetrization over momenta.* Contributions at the origin:

$$
\begin{aligned}
a^2\big(&-\tfrac{1}{2}N^1M^2E^3_i(F^i_{23} + F^i_{31})\big) + a^3\big(-\tfrac{1}{2}(M^2\tilde{N}^1 + N^1\tilde{M}^2)E^3_i(F^i_{23} + F^i_{31})\\
&- \tfrac{1}{4}N^1M^2E^3_i(\nabla_2F^i_{23} + \nabla_1F^i_{31})\\
&- \tfrac{1}{2}F^i_{12}(\nabla_1E^1_i + \nabla_2E^2_i + 2\nabla_3E^3_i)\big) + \mathrm{O}(a^4).
\end{aligned}
\tag{A.5}
$$

Contributions involving neighbouring vertices:

$$
\begin{aligned}
&a^2\big(\tfrac{1}{2}N^1M^2(F^i_{23}+F^i_{31})E^3_i\big)+a^3\big(\tfrac{1}{2}(M^2\tilde{N}^1+N^1\tilde{M}^2)(F^i_{23}+F^i_{31})E^3_i\\
&\qquad+N^1M^2\epsilon_{ijk}(A^j_1E^{1i}+A^j_2)F^k_{12}E^{2i}\big)+\tfrac{1}{4}N^1M^2(\nabla_2F^i_{23}+\nabla_1F^i_{31})E^3_i\\
&\qquad+\tfrac{1}{2}F^i_{12}(\nabla_1E^1_i+\nabla_2E^2_i+2\nabla_3E^3_i)-N^1M^2F^i_{12}(\partial_1E^1_i+\partial_2E^2_i+\partial_3E^3_i)\\
&\qquad-N^1M^2(F^i_{12}\partial_3E^3_i+F^i_{23}\partial_1E^3_i+F^i_{31}\partial_2E^3_i)-N^1(\partial_1M^2)F^i_{12}E^1_i\\
&\qquad-M^2(\partial_2N^1)F^i_{12}E^2_i-N^1(\partial_3M^2)F^i_{12}E^3_i-M^2(\partial_3N^1)F^i_{12}E^3_i\\
&\qquad-M^2(\partial_1N^1)F^i_{23}E^3_i-N^1(\partial_2M^2)F^i_{31}E^3_i\\
&\qquad+N^1M^2\epsilon_{ijk}(2A^j_3F^k_{12}+A^j_2F^k_{31}+A^j_1F^k_{23})E^{3i}\,\big)+\mathrm{O}(a^4).
\end{aligned}
\tag{A.6}
$$

(iv) *Symmetrization over both shift functions and momenta.* Contributions at the origin:

$$
\begin{aligned}
&-a^3\big(\tfrac{1}{4}M^2(\tilde{N}^1-\tilde{\tilde{N}}^1)F^i_{23}E^3_i+\tfrac{1}{4}N^1(\tilde{M}^2-\tilde{\tilde{M}}^2)F^i_{31}E^3_i\\
&\qquad+\tfrac{1}{2}F^i_{12}(\nabla_1E^1_i+\nabla_2E^2_i+2\nabla_3E^3_i)\big)+\mathrm{O}(a^4).
\end{aligned}
\tag{A.7}
$$

Contributions involving neighbouring vertices:

$$
\begin{aligned}
&a^3\big(\tfrac{1}{4}M^2(\tilde{N}^1-\tilde{\tilde{N}}^1)F^i_{23}E^3_i+\tfrac{1}{4}N^1(\tilde{M}^2-\tilde{\tilde{M}}^2)F^i_{31}E^3_i+N^1M^2\epsilon_{ijk}(A^j_1E^{1i}+A^j_2)F^k_{12}E^{2i}\\
&\qquad+\tfrac{1}{2}F^i_{12}(\nabla_1E^1_i+\nabla_2E^2_i+2\nabla_3E^3_i)-N^1M^2F^i_{12}(\partial_1E^1_i+\partial_2E^2_i+\partial_3E^3_i)\\
&\qquad-N^1M^2(F^i_{12}\partial_3E^3_i+F^i_{23}\partial_1E^3_i+F^i_{31}\partial_2E^3_i)-N^1(\partial_1M^2)F^i_{12}E^1_i\\
&\qquad-M^2(\partial_2N^1)F^i_{12}E^2_i-N^1(\partial_3M^2)F^i_{12}E^3_i-M^2(\partial_3N^1)F^i_{12}E^3_i\\
&\qquad-M^2(\partial_1N^1)F^i_{23}E^3_i-N^1(\partial_2M^2)F^i_{31}E^3_i\\
&\qquad+N^1M^2\epsilon_{ijk}(2A^j_3F^k_{12}+A^j_2F^k_{31}+A^j_1F^k_{23})E^{3i}\,\big)+\mathrm{O}(a^4).
\end{aligned}
\tag{A.8}
$$

The more one symmetrizes, the more cancellations occur individually among terms based at the origin and among terms coming from the more non-local commutators. For example, terms of order a^2 still appear in cases (i)–(iii), but disappear when the totally symmetrized lattice operators are used. This is in line with the general rule that for lattice discretizations the convergence properties in the continuum limit are improved when one symmetrizes the lattice expressions over positive and negative lattice directions. Note also that terms containing partial derivatives of the shift functions M and N (that are crucial in obtaining the correct commutator) come from the commutators involving neighbouring sites.

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