On the consistency of the constraint algebra in spin network quantum gravity

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Abstract

We point out several features of the quantum Hamiltonian constraints recently introduced by Thiemann for Euclidean gravity. In particular we discuss the issue of the constraint algebra and of the quantum realization of the object $q^{ab}V_b$, which is classically the Poisson Bracket of two Hamiltonians.

I. INTRODUCTION

In a remarkable series of papers [1–4] by Thiemann, the loop approach to the quantization of general relativity reached a new level. Using ideas related to those of Rovelli and

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Smolin [8], Thiemann proposed a definition of the quantum Hamiltonian (Wheeler-DeWitt) constraint of Einstein-Hilbert gravity which is a densely defined operator on a certain Hilbert space and which is (in a certain sense [1]) anomaly free on diffeomorphism invariant states. The fact that the proposed constraints imply the existence of a self-consistent, well defined theory is very impressive. However, it is still not clear whether the resulting theory is connected with the physics of gravity. This question has been raised in [6,7], and this paper intends to probe it further.

We will address two issues. The first concerns the commutator of two Hamiltonian constraints. This topic has been partially considered by Thiemann [3] and in the preceding paper [7] by Lewandowski and Marolf. Thiemann argued [1] that the commutator of two Hamiltonians vanishes on diffeomorphism invariant states by commuting two 'regulated' constraints and showing that, for sufficiently small regulators, the result annihilates diffeomorphism invariant states. He also considered [3] the function appearing on the right hand side of the classical Poisson bracket of two Hamiltonians:

$$\{H(N), H(M)\} = \int d^3x (N\partial_b M - M\partial_b N)\tilde{E}_i^c F_{cb}^i q^{ab} \tag{1}$$

where the Hamiltonian constraint is

$$H(N) = \int d^3x N \epsilon_{ijk} E_i^a E_j^b F_{ab}^k. \tag{2}$$

Our notation is that E_i^a is the spatial triad while $\tilde{E}_i^a$ is the densitized triad, F_{ab} is the field strength of the pull back of the (Euclidean) self-dual curvature to a spatial hypersurface, and $\{,\}$ denotes the Poisson bracket. Thiemann was able to show that a quantum version of the right hand side of (1) annihilates diffeomorphism invariant states. This is the result one would expect from classical reasoning as

$$V(N) := \int N^b \tilde{E}_i^c F_{cb}^i =: \int N^b V_b \tag{3}$$

is the classical function that generates diffeomorphisms. Thus, modulo factor ordering issues and anomalies, the quantum commutator corresponding to (1) should annihilate diffeomorphism invariant states.

In summary, Thiemann showed that the algebra is consistent at this level: his quantum versions of both the left and right hand sides of (1) vanish on diffeomorphism invariant states. On the other hand, Lewandowski and Marolf [7] explored the commutator of two of Thiemann's quantum Hamiltonians on a larger space (called $\mathcal{T}'_*$) of states that are not necessarily diffeomorphism invariant. It was found that the commutator of the Hamiltonian constraints continues to vanish, even on this larger space. Such a result was also shown to hold for a large class of extensions [2,5] of the original proposal of [1]. This suggested that an inconsistency might appear, since the classical phase space function on the right hand side of (1) vanishes only when the diffeomorphism constraint is satisfied. Following Thiemann [3], let us denote this function by $\mathcal{O}(N, M)$:

$$\mathcal{O}(N, M) = \int d^3x (N\partial_a M - M\partial_a N)V_b q^{ab}. \tag{4}$$

We will discuss quantum versions of $\mathcal{O}(N, M)$ in this paper. In particular we will show that the original quantization of this function proposed by Thiemann also yields the zero

operator on the larger space $\mathcal{T}'_*$. As a result, one could consider the calculation consistent. However, the price for this consistency appears to be "setting $q^{ab} = 0$" as an operator on this space of states (more on this later). This is so in spite of the fact that $\mathcal{T}'_*$ contains both the diffeomorphism invariant states of [1,2] and, in particular, the 'physical' states that solve the Hamiltonian constraints. We will also show that there exist regularization ambiguities that allow us to propose quantum versions of q^{ab} and $\mathcal{O}(N, M)$ that are nonzero. However, the quantum commutator still vanishes, so the algebra is anomalous with this new proposal. This is discussed in section II.

The second issue concerns the computation of the commutator presented in the companion paper [7] in light of the previous point: would it be possible to redefine the constraints themselves in such a way as to get a non-zero result? And if so, would the result coincide with some nontrivial quantum version of $\mathcal{O}(N, M)$? From addressing these questions, section III will derive a rather general property that is necessary (but not necessarily sufficient) for operators of this type to reproduce the classical hypersurface deformation algebra.

The following treatment works within the loop approach to (Euclidean) quantum gravity and uses the now standard technology of spin networks and linear functionals thereon. We refer the reader to the companion paper [7] for a complete description of the context and conventions employed here. Some of the founding works in the field are [15,16]. For a thorough description of this approach, see [14,20]. For a description of spin networks, see [17] and [18].

II. COMMUTATOR OF TWO HAMILTONIANS

The classical Poisson bracket of two Hamiltonian constraints, equation (1), indicates that the commutator is proportional to a diffeomorphism generator. However, the proportionality is through an operator-valued quantity involving the metric. If one has a Hamiltonian constraint defined on diffeomorphism invariant states, its commutator should therefore vanish on such states if the factor ordering is such that the diffeomorphism acts on the state first. Thiemann showed from the outset [1] that the commutator of his (properly regulated) constraints annihilates such states. In fact, it is not difficult to see. The action of Thiemann's constraint on a spin network state $|\Gamma\rangle$ is to generate a number of terms, each of which is a single spin network resembling $|\Gamma\rangle$ but with an extra edge added near an appropriate intersection of the original spin network, and with the spins and contractor changed at that intersection. Two successive actions of such an operator add a total of two edges in each term and the order in which the operators are applied affects only the position occupied by the new lines. In particular, it does not effect the overall topology (or diffeomorphism class) of the new graph or the spins or contractors. Thus, the result is independent of which constraint is applied first if one is able to "slide the added lines" back and forth. Sliding lines in this way is allowed when we compute $\langle\psi|\hat{H}(N)|\Gamma\rangle$ for diffeomorphism invariant states $\langle\psi|$ in an appropriate dual space. As a result, the *dual* action of the commutator on 'bra' states annihilates all such diffeomorphism invariant states $\langle\psi|$. In this sense then, the constraints presented have a consistent algebra on these states.

The previous paper [7] considered a variation of this construction. We describe it only briefly here and refer the reader to the original work for further details. The basic idea was to consider an enlarged space of states, whose members need not be invariant under

diffeomorphisms. We now consider a large subspace of this space which will be sufficient for our purposes. This space is constructed as follows.

Suppose that we work on a three manifold Σ and consider a spin network $\Gamma = (\gamma, j, c)$ which we think of as embedded in Σ. In this notation, γ is graph, the label j assigns a representation of $SU(2)$ to each edge of the graph, and the contractors c assign a certain type of operator to each vertex in γ. Suppose that the associated graph γ has no symmetries in the sense of [7]. That is, the only diffeomorphisms of Σ that map γ to itself also map each edge of γ to itself without changing the edge's orientation. The reader may consult [7] for the more general case.

We will not just leave this spin network at some fixed location in Σ, but will transport it to other locations in the manifold. In fact, given any map σ which assigns a point of Σ to each vertex of γ, we introduce a linear functional $\langle\Gamma_\sigma|$ on the space $(\mathcal{T})$ spanned by spin networks. To define this functional, suppose that we have another spin network $\Gamma' = (\gamma', j', c')$ for which γ' is related to γ by a smooth diffeomorphism φ, $\gamma' = \varphi(\gamma)$, for some smooth diffeomorphism φ that takes the vertices of γ to just those points assigned by σ. Then we set $\langle\Gamma_\sigma|\Gamma'\rangle := \langle\Gamma|\mathcal{D}_{\varphi^{-1}}|\Gamma'\rangle$, where $\mathcal{D}_\phi$ denotes the unitary action of a diffeomorphism ϕ on spin network states and the matrix elements $\langle\Gamma|\mathcal{D}_{\varphi^{-1}}|\Gamma'\rangle$ are computed using the inner product of $L^2(\overline{A/G}, d\mu_0)$ for the Ashtekar-Lewandowski measure $d\mu_0$ of [19]. For any other Γ' (over the wrong sort of graph or in the wrong location) we set $\langle\Gamma_\sigma|\Gamma'\rangle := 0$.

The states of interest are those obtained by 'smearing out' these states with some smooth function f on the space $\Sigma^{V(\gamma)}$ of all maps from the vertices $V(\gamma)$ of γ to our three manifold Σ. That is, we are interested in states $\langle\Gamma, f|$ of the form

$$\langle\Gamma, f| = \sum_{\sigma\in\Sigma^{V(\Gamma)}} \langle\Gamma_\sigma|f(\sigma). \tag{5}$$

While this sum involves an (uncountable) infinity of terms, its action on spin network states $|\Gamma'\rangle$ is well-defined since only one term (the one in which σ maps the vertices of Γ to the vertices of Γ' in the proper way) can be nonzero. So, this sum basically says that $\langle\Gamma, f|$ acts on $|\Gamma'\rangle$ by moving Γ 'on top of' Γ' and then taking the inner product of $|\Gamma\rangle$ and $|\Gamma'\rangle$ in $L^2(\overline{A/G}, d\mu_0)$, weighted by the function f. In particular, $\langle\Gamma, f|$ is diffeomorphism invariant whenever f is a constant function. If the vertices of Γ are numbered from 1 to k, then we may rewrite the sum (5) as

$$\langle\Gamma, f| = \sum_{x_1, ..., x_k \in \Sigma} \langle\Gamma_{x_1, ... x_k}|f(x_1, ..., x_k) \tag{6}$$

with the obvious correspondence between σ and $(x_1, ..., x_k)$. Note that f is required to be a smooth function on the entire space Σ^k. The space $\mathcal{T}'_*$ of [7] consists of arbitrary superpositions of the $\langle\Gamma, f|$, together with the corresponding extensions to graphs with symmetries. Again, such superpositions may include an uncountable infinity of terms, so long as all but a finite number annihilate any give spin network. The constraints of [1] are well-defined on this space, as are the variations of [2,5,6]. Such constraints also map $\mathcal{T}'_*$ to itself, so that commutators can be calculated directly. Note that the generator of diffeomorphisms is well-defined and nonzero, and acts just by Lie dragging the function f.

At the intuitive level (and in the typical case), it is not difficult to see that Thiemann's constraint is well defined on this space of states and that it has a vanishing commutator with

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itself. For simplicity, suppose that we evaluate the 'matrix elements' $\langle \Gamma, f | [H(N), H(M)] | \Gamma' \rangle$ of the commutator between the state $\langle \Gamma, f |$, and the spin network $| \Gamma' \rangle$, where Γ has only one vertex x. The idea is that the successive action of two Hamiltonian constraints will add new lines at the intersection of the spin network, but the smearing functions N and M get evaluated at exactly the same point x, (the original vertex). Thus, one gets a single term with a pre-factor of the form $N(x)M(x) - M(x)N(x)$ and the commutator vanishes. When Γ' has multiple vertices, these vertices do not interact with each other and the commutator continues to vanish. This is true [7] even if one uses the version of the volume operator due to Rovelli and Smolin [12] in the definition of the Thiemann Hamiltonian. In that case, the second Hamiltonian acts not only at the original intersection but also at the newly added vertices (the Ashtekar–Lewandowski volume does not act on those vertices because they are planar, but the Rovelli–Smolin volume does not automatically vanish on planar vertices)[1]. In this case, one gets a contribution with a prefactor $N(x)M(y) - M(x)N(y)$ where x is the location of the first intersection and y the one of the second. However, one removes the regularization by taking $x \to y$ and the commutator vanishes for smooth N and M.

Clearly, this should be a source of worry. Why does the commutator vanish on non-diffeomorphism invariant states? However, it could be that the states considered are not general enough and therefore that the commutator just happens to vanish on these states. To analyze this question, one should look at a quantization of $\mathcal{O}(N, M)$. It is rather difficult to realize a quantum version of the right hand side in the spin network representation, due to the presence of the inverse metric. Nevertheless, it can be done, as discussed in detail by Thiemann [3]. The idea is to break up the metric into two un-densitized contravariant triads $q^{ab} = e_i^a e_i^b$, which in turn can be related to co-triads via,

$$e_i^a = \epsilon^{abc} \epsilon_{ijk} \frac{e_b^j e_c^k}{\sqrt{det(q)}} \tag{7}$$

and finally the co-triads can be expressed in terms of a commutator of a connection and a volume,

$$\frac{1}{2} \text{sgn}(det(e)) e_a^i = \{A_a^i, V\} \tag{8}$$

and $V = \int d^3 x \sqrt{\epsilon^{abc} \epsilon_{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c}$ is the volume of the three manifold.

We refer the reader to Thiemann's paper [3] for the details, but the main point is that the resulting quantum operator for the right-hand side of the constraint can be written as a finite well defined operator times a prefactor in which the smearing functions N and M are evaluated at the intersection point v where the Hamiltonian acts and at the end of an

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[1]It has been pointed out by Roberto De Pietri [9] that, because Thiemann proceeds by triangulating the manifold Σ with nondegenerate tetrahedrons, the planar vertices are explicitly removed from consideration. Strictly speaking, this represents a breakdown of the arguments of [1] when used with the Rovelli-Smolin volume operator. However, it is straightforward to generalize the argument so that the Rovelli-Smolin volume can be used and so that the Hamiltonian acts non-trivially at planar vertices.

outgoing link Δ from v; the exact form is $M(v)N(\Delta) - M(\Delta)N(v)$ in the notation of [3]. In the limit in which the regularization is removed, the length of the segment Δ shrinks to zero and the prefactor vanishes for continuous N, M. We have therefore shown that the quantization of $\mathcal{O}(N, M)$ proposed by Thiemann coincides with the commutator derived by Lewandowski and Marolf on the particular space of states $\mathcal{T}'_*$ defined above: both operators vanish. Therefore, there is no anomaly or inconsistency at this level. However, an inspection of eq. (3.7) from paper [3] shows that the $M(v)N(\Delta) - M(\Delta)N(v)$ term in fact arises in the regularization from the factor $(MN_{,a} - NM_{,a})q^{ab}$. Since the vector constraint factor V_b may be treated as simply the infinitesimal diffeomorphism generator, the procedure of [3] is tantamount to a regularization of q^{ab} which sets $q^{ab} = 0$ on $\mathcal{T}'_*$. Note that, while q^{ab} is not diffeomorphism invariant, this does not mean that it cannot be defined as an operator on some space of states (such at $\mathcal{T}$) whose members are not diffeomorphism invariant.

It should also be pointed out that Thiemann [10] has proposed a general scheme for regularizing a large class of operators that would include $\sqrt{\det q}\,q^{ab}$ but not q^{ab} itself. This procedure appears to give a nonzero $\sqrt{\det q}\,q^{ab}$ operator. However, the procedure of [10] differs from the procedure used to regularize $O(M, N)$ in [3]. In particular, if the procedure of [10] were applied to $O(M, N)$ it would not yield an intact factor of the vector constraint V_b on the left hand side of $O(M, N)$. As a result, the relationship between the operators resulting from these different schemes is unclear. Further study of this issue would be worthwhile.

Let us therefore attempt to understand just why the aforementioned quantization of q^{ab} yields the zero operator. The root of the issue seems to lie in the fact that this quantization involves the prefactor $M(v)N(\Delta) - M(\Delta)N(v)$, involving the smearing functions evaluated at "close" points v, Δ, but does not produce a corresponding denominator that goes to zero when the regulators are removed and $\Delta \to v$. Therefore one always fails to reconstruct the derivatives that appear in the classical expression (1) and one gets a vanishing result.

This feature can in turn be traced to the quantization of expressions of the form:

$$\{A_a^i(x), \sqrt{V(x,\epsilon)}\} = \frac{\{A_a^i(x), V(x,\epsilon)\}}{2\sqrt{V(x,\epsilon)}} \tag{9}$$

that were introduced through equations (7) and (8). Here,

$$V(x,\epsilon) = \int d^3x \sqrt{\epsilon^{abc}\epsilon_{ijk}\tilde{E}_i^a\tilde{E}_j^b\tilde{E}_k^c}\,\chi(\epsilon, x) \tag{10}$$

is the metric volume of a box with center x and size ϵ and χ is the characteristic function of this box. Factors of this type arise in the constructions of [3]. Classically, expression (9) diverges in the limit $\epsilon \to 0$: the denominator vanishes since it is the volume of a box of size ϵ, and the numerator is independent of ϵ because the Poisson bracket is local. This divergence is, of course, canceled by other factors of ϵ so that Thiemann's classical expression for $\mathcal{O}(N, M)$ is finite. However, Thiemann's quantization [3] of (9) has a *finite* action at intersections of spin networks. The point is that he quantizes this expression by promoting the Poisson Bracket on the left to a commutator (and replacing the connection with a holonomy). Since his operator $\hat{V}(x,\epsilon)$ is finite as $\epsilon \to 0$, the commutator is finite as well. As a result, the other factors of ϵ cause his quantum version of $\mathcal{O}(N, M)$ and the

corresponding version of q^{ab} to vanish. This suggests that the quantization is not faithfully recovering properties of the classical operator.

It turns out that, within the general quantization scheme described in [1–3] there is in fact enough freedom to alter this result. Consider again the classical functions (9). Since the Poisson bracket $\{A_a, V(x, \epsilon)\}$ is really ϵ independent, one could replace $V(x, \epsilon)$ with some $V(x, \delta)$ in the Poisson bracket $\{A_a^i(x), V(x, \epsilon)\}$, while keeping the same regulator ϵ. Furthermore, for small ϵ, $V(x, \epsilon)$ scales as ϵ^3 for smooth fields. If one now reworks the above argument keeping this in mind, one may rewrite equation (9) as

$$\frac{\{A, \widehat{V(x, \delta)}\}}{\sqrt{V(x, \epsilon)}} = 2 \left(\frac{\delta}{\epsilon}\right)^{\frac{3}{2}} \{\hat{A}, \hat{V}^{1/2}(x, \delta)\}. \tag{11}$$

The point is that, in the proposed quantization, $\lim_{\delta \to 0} \hat{V}(x, \delta)$ yields a well-defined finite operator, which is therefore independent of δ. It is then evident[2] that, by choosing $\delta = \epsilon^n$, one can generate extra powers of ϵ in the the construction of the quantum $\mathcal{O}(N, M)$. In particular, because four factors of the form (9) are used to write q^{ab}, the choice of $n = 5/6$ provides the single inverse power of epsilon needed to convert the difference $M(v)N(\Delta) - N(v)M(\Delta)$ into the structure that appears in (1). With such a choice one gets a non-zero operator q^{ab} which is well-defined on $\mathcal{T}'_*$, although it is clearly not unique.

One could imagine using similar tricks to rescale the quantum constraints so that their commutator is non-vanishing. We will consider the resulting expressions in the next section. However these manipulations are only formal in character. In order to yield a non-vanishing result, the commutator must be rescaled by $1/\epsilon$. This, however, requires each Hamiltonian to be rescaled by $1/\sqrt{\epsilon}$. Since the original constraints $\hat{H}(N)$ were nonzero and well-defined, the new constraints are roughly $\hat{H}(N)/\sqrt{\epsilon}$ and are divergent as $\epsilon \to 0$. Whether or not this can be dealt with through some sort of renormalization is unclear.

III. COMPUTING THE COMMUTATOR

Ignoring for the moment the issue raised in the last paragraph, one can ask what would be obtained if we would compute the commutator of two regulated Hamiltonians and rescale it by one power of $1/\epsilon$ before taking the limit $\epsilon \to 0$. We will not worry here about the details of how this is done. Although Thiemann does not explicitly use expressions of the form (9) in writing his constraints, they may be introduced through suitable manipulations. Modulo perhaps extra powers of the regulator ϵ, the result is again an 'RST-like' (Rovelli–Smolin–Thiemann-like) operator in the sense of [7]. As a result, if there were no extra ϵ^{-1}'s, the commutator of constraints would still be zero [7].

If the $\epsilon \to 0$ limit of the rescaled commutator is properly taken, one would expect it not to vanish. The point of such a computation is that one might think that extra powers of ϵ could be associated with some sort of renormalization. Thus, without studying the details, we address the question of whether any such renormalization is likely to yield the classical

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[2]One needs to choose dimensionless coordinates; if not one needs to add extra dimensional factors.

algebra. The importance of the calculation lies in general properties that one may recognize as contributing to reproducing —or failing to reproduce— the result expected from the classical theory. Exploring the theory in this way before removing regulators is reminiscent of manipulations performed in a lattice context [21].

Instead of beginning the calculation immediately, let us first discuss the *desired* result. We would hope to find some analogue of the classical function $\mathcal{O}(N, M)$. Now, if there are no anomalies, one would expect the quantum commutator also to resemble (4) and, what is more, to have the factor ordering implied by (4). This is because the constraints of [1] are to be applied on 'bra' states $\langle \Gamma, f|$ so that, in order for the algebra to close properly, the vector constraint V_b should appear on the left.

Recall now that V_b are the diffeomorphism generators and that the action of diffeomorphism generators are well defined on $\mathcal{T}'_*$. Given a vector field N^a, the action of an infinitesimal diffeomorphism along N on the state $\langle \Gamma, f|$ yields just the state $\langle \Gamma, \mathcal{L}_N f|$ with the same spin network Γ smeared against the Lie derivative $\mathcal{L}_N f$ of f along N. We note that this state takes the form

$$\langle \Gamma, \mathcal{L}_N f| = \sum_{\sigma \in \Sigma^{V(\Gamma)}} \left(\langle \Gamma_\sigma| \sum_{x \in V(\Gamma)} N^a(\sigma(x)) \frac{\partial f}{\partial(\sigma(x))^a}(\sigma) \right). \tag{12}$$

As a result, one would expect matrix elements of the quantum commutator to be of the form

$$\langle \Gamma, f|[\hat{H}(N), \hat{H}(M)]|\Gamma' \rangle =$$
$$\sum_{\sigma \in \Sigma^{V(\Gamma)}} \left(\sum_{x \in V(\Gamma)} (N(\sigma(x)) \partial_a M(\sigma(x)) - M(\sigma(x)) \partial_a(\sigma(x)) N) \langle \Gamma_\sigma| \widehat{q^{ab}}(x)|\Gamma' \rangle \frac{\partial f}{\partial(\sigma(x))^b}(\sigma) \right) \tag{13}$$

if in fact the action of the operator $\widehat{q^{ab}}(x)$ commutes with the sum over σ. Here, $\widehat{q^{ab}}(x)$ is some quantization of $q^{ab}(x)$. The details of this operator, and therefore of (13), will depend on the form of this operator. Still, one expects each term to have a prefactor $(N\partial_a M - M\partial_a N)$ and to involve a derivative of f. Also, for each vertex $x \in V(\Gamma)$, one would expect to find a term involving $\frac{\partial f}{\partial \sigma(x)}$.

For simplicity we will calculate the commutator only on a single state $\langle \Gamma, f|$, where Γ and a labeling of its vertices is shown in figure 1.

Since Γ has only trivalent vertices, we may fix the contractors simply by requiring them to be projectors. Also note that the edge labeled by the spin j_3 forms a trefoil knot, as does the loop formed by traveling from x to z to w along the straight edges and then closing the loop along the edge labeled by j_1. Furthermore, these trefoil knots are of opposite chiralities, so that neither the graph γ underlying Γ above nor the graph obtained from it by removing the straight edges connecting x, z, y, w have nontrivial symmetries in the sense of [7].

As we do not know what to expect for the matrix elements $\langle \Gamma_\sigma| \widehat{q^{ab}}|\Gamma' \rangle$ in (13), our calculation of the commutator will be only schematic. We will not concern ourselves with the details of spins and contractors, but merely note how the lapse functions N, M and the function f are evaluated so that we can look for the derivatives which appear in (13).

To make the action of the Hamiltonian more interesting, we will consider the case in which the volume operator is the Rovelli–Smolin one. The calculation for the Ashtekar–
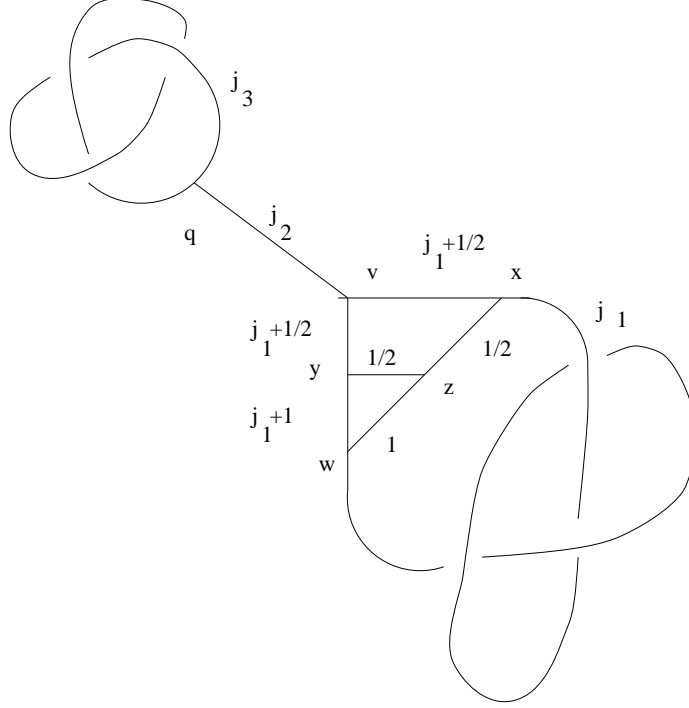
FIG. 1. The type of spin network state on which we act with the commutator of two Hamiltonians. Our notation is that $j_1, j_1 + 1/2, J_1 + 1, J_2, J_3, 1, 1/2$ are spins that label edges while q, v, w, x, y, z are vertices of the graph.

Lewandowski volume is similar, but yields only a subset of the terms we will obtain[3]. Let us consider a bra state of the form $\sum_{v,w,x,y,z,q} f(v, w, x, y, z, q) < \Gamma_{v,w,x,y,z,q}|$ where Γ is shown in figure 1. We now act with two Hamiltonians $\hat{H}^\alpha(N)$ and $\hat{H}^\beta(M)$. By "α" and "β" we mean two possibly distinct regularizations of the Hamiltonian. The action of the two Hamiltonians consists in fixing the vertices of the spin network w, x, y, z at points determined by the regularizations. The first constraint can act only at the vertex w and removes the spin $1/2$ edge between y and z. The second can only act at the vertex v, and removes the edges between x and z and between z and w. Since we act with regulated constraints, when the first edge is removed, the corresponding arguments y, z of f are evaluated at points y^β, z^β (or y^α, z^α) that depend on the regulator α (β) and of course implicitly on the positions of the remaining vertices. Similarly, when the second edge is removed, the arguments w, x are evaluated at some w^α, x^α (w^β, x^β).

The end result is,

$$\sum_{v,w,x,y,z,q} f(v, w, x, y, z, q) < \Gamma_{v,w,x,y,z,q}|(\hat{H}^\beta(N)\hat{H}^\alpha(M) - \hat{H}^\alpha(M)\hat{H}^\beta(N))$$
$$= A(j_1, j_2) \sum_{v,q} \left[f(v, w^\alpha, x^\alpha, y^\beta, z^\beta, q)N(w^\alpha)M(v) \right.$$

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[3]However, a small change in the the construction of [1] yields a constraint which acts at planar vertices, even though it is built from the Ashtekar-Lewandowski volume [7]. The results for this new operator are then schematically the same as for the Rovelli-Smolin case.
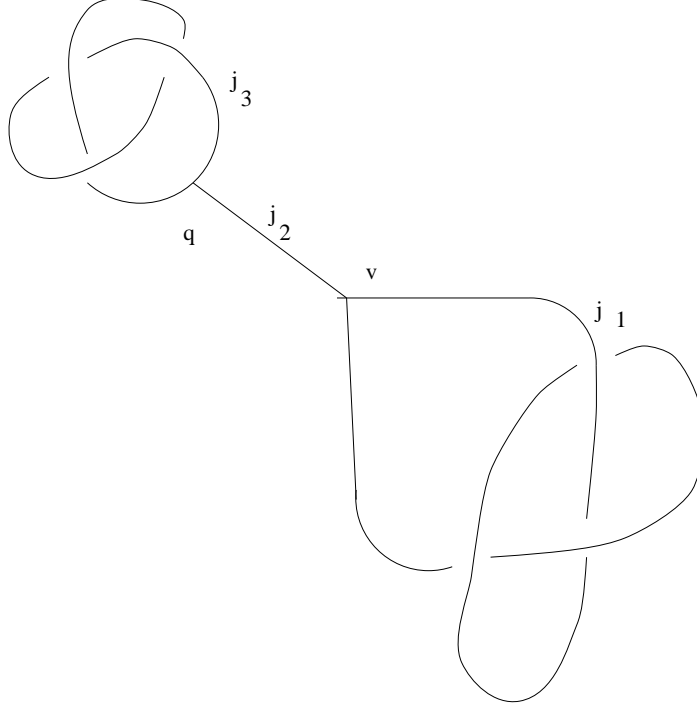
FIG. 2. The type of state that results from the action of two Hamiltonians on a state of the form shown in figure 1. As we see, the action of the Hamiltonians is to "eliminate" lines from the vertices of the graph.

$$- \; f(v, w^\beta, x^\beta, y^\alpha, z^\alpha, q)M(w^\beta)N(v)\Big] \langle \Gamma_{v,q}| \tag{14}$$

where $\langle \Gamma_{v,q}|$ is shown in figure 2 and $A(j_1, j_2)$ is some constant that depends on j_1 and j_2.

If we now take

$$
\begin{aligned}
|y^\beta - v|, |z^\beta - v| &\sim \epsilon^{\alpha,\beta}, \\
|w^\alpha - v|, |x^\alpha - v| &\sim \epsilon^{\alpha}, \\
|y^\alpha - v|, |z^\alpha - v| &\sim \epsilon^{\beta,\alpha}, \\
|w^\beta - v|, |x^\beta - v| &\sim \epsilon^{\beta},
\end{aligned}
\tag{15}
$$

for small parameters $\epsilon^{\alpha,\beta}$, ϵ^α, $\epsilon^{\beta,\alpha}$, and ϵ^β, the result is schematically of the form

$$\sum_v \left[(M\partial N\epsilon^\alpha - N\partial M\epsilon^\beta)f + MN\partial f(\epsilon^\alpha - \epsilon^\beta + \epsilon^{\beta,\alpha} - \epsilon^{\alpha,\beta}) \right] \langle \Gamma_v|. \tag{16}$$

In particular, because (16) involves evaluating fNM at only two sets of arguments, it corresponds only to a set of terms having only one derivative each. Thus, there are no terms like those found in (13), which contain products of ∂f with ∂N or ∂M and so involve two derivatives of f, N, M.

If one had used the Ashtekar–Lewandowski volume, then in fact $\hat{H}^\beta(N)\hat{H}^\alpha(N)$ annihilates $\langle \Gamma, f|$ for the particular spin network Γ shown in figure 1. However, in general the result is nonzero and the calculation proceeds along similar lines. This time, one finds only a contribution of the form $MN\partial f(\epsilon^\alpha - \epsilon^\beta)$; that is, the only terms that arise have both N and M evaluated at the same vertex v. The most general calculation for the Rovelli–Smolin volume generates only the terms in (16).

In all cases one sees that the Hamiltonian is lacking the ingredients to produce the correct commutator, even if one could somehow reregulate the commutator to produce a finite result. In particular, one notices the first argument of f in (16) is always evaluated at the original vertex v. Thus, a term of the form $\frac{\partial f}{\partial v}$, in which f is differentiated with respect to its *first* argument, never appears. It is clear that Hamiltonians of this general form which fail to move the vertex at which they originally act cannot reproduce the classical commutator, since one is missing the term involving $\partial_v f$. Thus, we see that the proposal of [6], which does not involve explicitly shrinking loops to points, also cannot reproduce this structure.

As we see from the previous calculation, the Hamiltonian's action on the proposed space of linear functionals is tantamount to "eliminating a line" between two trivalent vertices in which, at each vertex, two of the incident edges join to make a single smooth curve. This in particular implies that all spin networks without such pairs of vertices are annihilated by the action of a single Hamiltonian and therefore by the commutator of two Hamiltonians. To have consistency, a quantum version of $\mathcal{O}(N, M)$ would also have to annihilate such states. The point is again that we expect V_a to act as a Lie derivative, so that the only real freedom is the regularization of the operator corresponding to q^{ab}. Thus, it would appear that $\hat{q}^{ab}$ would have to annihilate such states as well. One might think that Hamiltonians which both add and remove edges would be better candidates to have a consistent algebra. However, we note that the 'symmetrized' Hamiltonians discussed in [7] do just this, but still do not yield an appropriate algebra [7]. In particular, one may repeat the calculation (14) for the symmetrized operators and find similar results.

IV. CONCLUSIONS

In this paper we have studied two issues concerning the quantum version of the Hamiltonian (Wheeler–DeWitt) constraint of gravity proposed by Thiemann. We have shown that, with the regulation scheme of [3], the commutator algebra of two constraints is consistent on the space of non-diffeomorphism invariant functions introduced by Lewandowski and Marolf, but at the price of representing the contravariant metric q^{ab} (contracted with the diffeomorphism generator) by the zero operator on a rather generic class of states which includes the familiar diffeomorphism invariant linear functionals. We found a way to regularize the object $\mathcal{O}(N, M)$ (4) (the classical right hand side of the Poisson bracket (1)) that avoids this problem, but then the commutator is anomalous. We then studied a rescaled version of the regulated commutator on non-diffeomorphism invariant states and found that, in this context, an algebra of the classical form will not be obtained unless the constraints somehow move the vertices on which they act.

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