Evolution equations for gravitating ideal fluid bodies in general relativity

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(Received 27 August 1997; published 16 January 1998)

We consider the Einstein-Euler equations for a simple ideal fluid in the domain where the speed of sound and the specific enthalpy are positive. Using a Lagrangian description of the fluid flow, we obtain evolution equations which are symmetric hyperbolic. Pressure free matter is discussed as a simple subcase. [S0556-2821(98)03904-6]

I. INTRODUCTION

It has been shown by Choquet-Bruhat [1] that the Cauchy problem for the Einstein-Euler equations, including the case of pressure free matter, is well posed. For this purpose a representation of the field equations has been derived in [1] which employed the harmonic gauge and used equations of third order for the metric coefficients such that the complete system is Leray hyperbolic. A discussion of this problem along similar lines has been given in [2].

Friedrichs, who introduced the idea of a symmetric hyperbolic system [3], pointed out that wave equations as well as (under certain assumptions) the Newtonian equations for an ideal fluid can be cast into the form of symmetric hyperbolic systems. Thus it is not surprising that later discussions of the Einstein-Euler equations made use of such systems after imposing the harmonicity condition (cf. [4]). It appeared difficult, however, to deduce a symmetric hyperbolic system for the Einstein equations with pressure free matter.

If in this article symmetric hyperbolic propagation equations will be extracted from the the Einstein-Euler equations, our intention is not so much to give just another version of such a system, but to cast the equations into a form which we expect to be useful in analyzing the evolution of ''gaseous stars.'' By this we mean solutions to the Einstein-Euler equations, where the fluid is restricted to a spatially compact region and the space-time satisfies the vacuum equations outside the fluid. Despite its interest for modelling physical systems, the ''general'' initial value problem for such solutions, even if considered only locally in time, is still waiting for a satisfactory treatment.

In the case of spherical symmetry, this problem has been analyzed successfully by Kind and Ehlers [5,6]. Extending a device developed by Makino [7] for the Newtonian case, Rendall [4] has shown the existence of solutions to our problem in cases without symmetry. However, as pointed out by the author, these solutions have the undesirable property that the boundary of the star is freely falling. Solutions close to static spherically symmetric standard cases cannot be obtained by this procedure.

From the point of view of the existence theory, the most conspicuous feature of the problem is the occurrence of a free boundary, where the Einstein-Euler equations pass into the vacuum equations. This plays a crucial role in almost all steps of a general discussion of the initial value problem: In the construction of initial data, a careful discussion of

smoothness properties and jumps near the boundary of the star is required. In the actual existence proof, the possible drop in smoothness at the boundary for various fields and the propagation behavior of jumps in the field along the boundary need to be controlled. A careful specification of the smoothness properties across the boundary is required for the uniqueness proof and for showing that gauge conditions and constraints are preserved.

Since we are dealing with a hyperbolic problem, it can be localized. If we are concerned with the problem locally in time, we need to analyze the situation only in a neighborhood of the boundary. What happens away from the boundary on the support of the fluid or in the vacuum region is well understood.

In the discussion of these various aspects it is clearly desirable to have firm control on the location of the boundary. However, if the field equations are written in a manifestly hyperbolic form by using harmonic coordinates, there appears to be no immediate way to gain the desired information. If the field equations, considered as a system of second order for the metric coefficients, are expressed in coordinates in which the Lagrangian description of the fluid flow is achieved, manifest hyperbolicity will be lost.

It is the purpose of this article to show that the Einstein-Euler equations imply evolution equations which combine symmetric hyperbolicity with a Lagrangian description of the fluid flow. This representation will be obtained by extending the technique discussed in [8]. The special case of dust follows as an aside.

Whether in the actual existence proofs one will deal with our system (or a variation of it) explicitly or if one will prefer to deal only with the type of energy estimates suggested by the system remains to be seen. In any case we consider the derivation of our system as a basic step in the discussion of the initial value problem for gaseous stars.

The problems near the boundary will in one way or another also turn up in numerical investigations of the evolution of general relativistic stars. A good understanding of the analytical aspects of the problem and the type of systems derived below is therefore likely to prove useful also in numerical calculations.

II. THE BASIC EQUATIONS

We shall study the Einstein equation

$$G_{ik} = \kappa T_{ik}, \qquad (2.1)$$

    <u>57</u>   2317

for a metric g with an energy momentum tensor of a simple ideal fluid,

$$T_{ik} = (\rho + p)U_i U_k - p\epsilon g_{ik}. \qquad (2.2)$$

Here ρ is the total energy density and p the pressure, as measured by an observer moving with the fluid, and U denotes the future directed, normalized, time-like flow vector field. For easier comparison with the discussion in [8], which makes use of the Bianchi identity in two different formalisms, we write the equations in a form which is independent of the signature. We set $U^i U_i = \epsilon$, where $\epsilon = 1$ or $\epsilon = -1$, depending on the signature of g.

Equations (2.1), (2.2) imply $\nabla^i T_{ij} = 0$, which is equivalent to the system consisting of the Euler equation

$$(\rho + p)U^i \nabla_i U_j + \{U_j U^i \nabla_i - \epsilon \nabla_j\}p = 0, \qquad (2.3)$$

and the equation

$$U^i \nabla_i \rho + (\rho + p)\nabla_i U^i = 0. \qquad (2.4)$$

We assume that the fluid is simple, i.e. that it consists of only one class of particles, and denote by n, s, T the number density of particles, the entropy per particle, and the absolute temperature as measured by an observer moving with the fluid. We shall assume the first law of equilibrium thermodynamics which has the familiar form $de = -p \, dv + T \, ds$ in terms of the volume $v = 1/n$ and the energy $e = \rho/n$ per particle. In terms of the variables above, we have

$$d\rho = \frac{\rho + p}{n} dn + nT \, ds. \qquad (2.5)$$

We assume an equation of state given in the form

$$\rho = f(n, s), \qquad (2.6)$$

with some suitable non-negative function f of the number density of particles and the entropy per particle. Using this in Eq. (2.5), we obtain

$$p = n\frac{\partial \rho}{\partial n} - \rho, \quad T = \frac{1}{n}\frac{\partial \rho}{\partial s}, \qquad (2.7)$$

as well as the speed of sound ν, given by

$$\nu^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{n}{\rho + p}\frac{\partial p}{\partial n}, \qquad (2.8)$$

as known functions of n and s. We require that the specific enthalpy and the speed of sound be positive: i.e.,

$$\frac{\rho + p}{n} > 0, \quad \frac{\partial p}{\partial n} = n\frac{\partial^2 \rho}{\partial n^2} > 0. \qquad (2.9)$$

Finally we assume the law of particle conservation,

$$U^i \nabla_i n + n\nabla_i U^i = 0. \qquad (2.10)$$

It implies together with Eqs. (2.4), (2.5) that the flow is adiabatic; i.e. the entropy per particle is conserved along the flow lines:

$$U^i \nabla_i s = 0. \qquad (2.11)$$

The system consisting of Eqs. (2.1), (2.2), (2.3), (2.10), (2.11), (2.6) is complete.

The case of an homentropic flow, where the entropy is constant in space and time, is of some interest. In this case the equation of state can be given in the form

$$p = h(\rho) \qquad (2.12)$$

with some suitable function h. A complete system is provided by Eqs. (2.1), (2.2), (2.3), (2.4), (2.12) and the resulting propagation equations will be somewhat simpler. As a special subcase we shall consider pressure free matter ("dust"), where $h \equiv 0$.

To derive our reduced system, we shall use a representation of the field equations which is different from Eqs. (2.1), (2.2), but includes these equations. One of the basic variables is a frame field $\{e_k\}_{k=0,\dots,3}$, satisfying $g_{ik} \equiv g(e_i, e_k) = \epsilon \, \mathrm{diag}(1, -1, -1, -1)$. In the following all tensor fields are given in terms of this frame.

The Levi-Cività covariant derivative in the direction of e_i is denoted by ∇_i. It defines (respectively is defined by) the connection coefficients satisfying the relations

$$\nabla_i e_k = \Gamma_i{}^j{}_k e_j, \quad \Gamma_i{}^j{}_k g_{jl} + \Gamma_i{}^j{}_l g_{jk} = 0,$$

and the first structure equation (torsion free condition)

$$[e_p, e_q] = (\Gamma_p{}^l{}_q - \Gamma_q{}^l{}_p)e_l. \qquad (2.13)$$

We shall consider the latter as an equation for the coefficients $e^\mu{}_k = e_k(x^\mu)$ of the frame e_k with respect to some suitably chosen coordinate system $\{x^\mu\}_{\mu=0,\dots,3}$.

Equations for the connection coefficients $\Gamma_i{}^j{}_k$ are given by the Ricci identity

$$e_p(\Gamma_q{}^i{}_j) - e_q(\Gamma_p{}^i{}_j) - \Gamma_k{}^i{}_j(\Gamma_p{}^k{}_q - \Gamma_q{}^k{}_p) + \Gamma_p{}^i{}_k \Gamma_q{}^k{}_j - \Gamma_q{}^i{}_k \Gamma_p{}^k{}_j$$
$$= R^i{}_{jpq}. \qquad (2.14)$$

We assume here the decomposition

$$R_{ijkl} = C_{ijkl} + \{g_{i[k}S_{l]j} - g_{j[k}S_{l]i}\} \qquad (2.15)$$

of the curvature tensor in terms of the conformal Weyl tensor C_{ijkl} and the tensor

$$S_{jk} = R_{jk} - \frac{1}{6}Rg_{jk}, \qquad (2.16)$$

given by the Ricci tensor R_{ij} and the Ricci scalar R.

To obtain equations for the conformal Weyl tensor, we consider the (once) contracted Bianchi identity. Using Eq. (2.15) we write it in the form

$$0 = F_{jkl} \equiv \nabla_i F^i{}_{jkl}, \qquad (2.17)$$

where we set

$$F_{ijkl} = C_{ijkl} - g_{i[k}S_{l]j}. \qquad (2.18)$$

Taking the dual of Eq. (2.17) (with respect to the index pair k, l), we obtain another equation of the form (2.17) with Eq. (2.18) replaced by

$$F_{ijkl} = C^*_{ijkl} + \frac{1}{2} S_{pj} \epsilon^p{}_{ikl}. \qquad (2.19)$$

III. REDUCED SYSTEM

For a further discussion of the equations above we need to introduce some notation. We write $N = e_0$ such that $N = N^i e_i$ with $N^i = \delta^i{}_0$. With N we associate ''spatial'' tensor fields, i.e. tensor fields $T_{i_1, \ldots, i_p}$ satisfying

$$T_{i_1, \ldots, i_l, \ldots, i_p} N^{i_l} = 0, \quad l = 1, \ldots, p.$$

The subspaces orthogonal to N inherit the metric $h_{ij} = g_{ij} - \epsilon N_i N_j$, and $h_i{}^j$ (indices being raised and lowered with g_{ij}) is the orthogonal projector onto these subspaces.

We shall have to consider for various tensor fields their projections with respect to N and its orthogonal subspaces. For a given tensor any contraction with N will be denoted by replacing the corresponding index by N and the projection with respect to $h_i{}^j$ will be indicated by a prime, such that for a tensor field T_{ijk} we write e.g.

$$T'_{iNk} = T_{mpq} h_i{}^m N^p h_k{}^q,$$

etc. Denoting by ϵ_{ijkl} the totally anti-symmetric tensor field with $\epsilon_{0123} = 1$ and setting $\epsilon_{jkl} = \epsilon'_{Njkl}$, we have the decomposition

$$\epsilon_{ijkl} = 2\epsilon(N_{[i}\epsilon_{j]kl} - \epsilon_{ij[k}N_{l]}).$$

Let C_{ijkl} be the conformal Weyl tensor of g, $C^*_{ijkl} = \frac{1}{2} C_{ijpq} \epsilon_{kl}{}^{pq}$ its dual, and denote by $E_{jl} = C'_{NjNl}$, $B_{jl} = C^{*'}_{NjNl}$ its N-electric and N-magnetic part. Writing $l_{jk} = h_{jk} - \epsilon N_j N_k$, we get the decompositions

$$C_{ijkl} = 2\epsilon(l_{j[k}E_{l]i} - l_{i[k}E_{l]j}) - 2(N_{[k}B_{l]p}\epsilon^p{}_{ij} + N_{[i}B_{j]p}\epsilon^p{}_{kl}), \qquad (3.1)$$

$$C^*_{ijkl} = 2N_{[i}E_{j]p}\epsilon^p{}_{kl} - 4E_{p[i}\epsilon_{j]}{}^p{}_{[k}N_{l]} - 4N_{[i}B_{j][k}N_{l]} - B_{pq}\epsilon^p{}_{ij}\epsilon^q{}_{kl}. \qquad (3.2)$$

We set

$$a^i = N^k \nabla_k N^i, \quad \chi_{ij} = h_i{}^k \nabla_k N_j, \quad \chi = h^{ij}\chi_{ij},$$

such that we have

$$\nabla_j N^i = \epsilon N_j a^i + \chi_j{}^i, \quad a^i = h_j{}^i \Gamma_0{}^j{}_0, \quad \chi_{ij} = -\epsilon h_i{}^k h_j{}^l \Gamma_k{}^0{}_l.$$

Since N is not required to be hypersurface orthogonal, the field χ_{ij} will in general not be symmetric.

If the tensor field T is spatial, i.e. $T = T'$, we define its spatial covariant differential by $\mathcal{D}T = (\nabla T)'$, i.e.

$$\mathcal{D}_i T_{i_1, \ldots, i_p} = \nabla_j T_{j_1, \ldots, j_p} h_i{}^j h_{i_1}{}^{j_1} \cdots h_{i_p}{}^{j_p}.$$

It follows that

$$\mathcal{D}_i h_{jk} = 0, \quad \mathcal{D}_i \epsilon_{jkl} = 0.$$

In the following a system of propagation equations for the unknowns

$$e^\mu{}_k, \quad \Gamma_i{}^j{}_k, \quad B_{jk}, \quad E_{jk}, \quad n, \quad s, \quad s_k$$

will be derived, where we introduced as an additional variable the spatial differential

$$s_k = \mathcal{D}_k s. \qquad (3.3)$$

We study first the Bianchi identity (2.17). For the various components of the decomposition

$$F_{jkl} = N_j(F'_{NkN}N_l - F'_{NlN}N_k) - 2\epsilon F'_{jN[k}N_{l]} + \epsilon N_j F'_{Nkl} + F'_{jkl},$$

we get the expressions

$$\epsilon F'_{NkN} = \mathcal{L}_N F'_{NNkN} + \epsilon \mathcal{D}^i F'_{iNkN} - a^i(F'_{NNki} + F'_{NikN} + F'_{iNkN}) - \epsilon \chi^{ij}(F'_{ijkN} + F'_{iNkj}) - \chi_k{}^j F'_{NNjN} + \chi F'_{NNkN}, \qquad (3.4)$$

$$\epsilon F'_{Nkl} = \mathcal{L}_N F'_{NNkl} + \epsilon \mathcal{D}^i F'_{iNkl} - a^i(F'_{Nikl} + F'_{iNkl}) + \epsilon(a_k F'_{NNNl} + a_l F'_{NNkN}) - \epsilon \chi^{ij} F'_{ijkl} - \chi_k{}^i F'_{NNil} - \chi_l{}^i F'_{NNki} + \chi^i{}_k F'_{iNNl} + \chi^i{}_l F'_{iNkN} + \chi F'_{NNkl}, \qquad (3.5)$$

$$\epsilon F'_{jNl} = \mathcal{L}_N F'_{NjNl} + \epsilon \mathcal{D}^i F'_{ijNl} - a^i(F'_{Njil} + F'_{ijNl}) + \epsilon a_j F'_{NNNl} - \epsilon \chi^{ik} F'_{ijkl} - \chi_j{}^i F'_{NiNl} - \chi_l{}^i F'_{NjNi} + \chi^i{}_j F'_{iNNl} + \chi F'_{NjNl}, \qquad (3.6)$$

$$\epsilon F'_{jkl} = \mathcal{L}_N F'_{Njkl} + \epsilon \mathcal{D}^i F'_{ijkl} - a^i F'_{ijkl} + \epsilon(a_j F'_{NNkl} + a_k F'_{NjNl} + a_l F'_{NjkN}) - \chi_j{}^i F'_{Nikl} - \chi_k{}^i F'_{Njil} - \chi_l{}^i F'_{Njki} + \chi^i{}_j F'_{iNkl} + \chi^i{}_k F'_{ijNl} + \chi^i{}_l F'_{ijkN} + \chi F'_{Njkl}, \qquad (3.7)$$

where $\mathcal{L}_N$ denotes the Lie derivative in the direction of N.

To relate the frame to the structure of the field equations (2.1), (2.2), we shall now make the specific choice

$$N = e_0 = U. \qquad (3.8)$$

By Eqs. (2.1), (2.2) we have

$$S_{ik} = \kappa\left(\frac{2}{3}\rho + p\right)U_i U_k - \frac{1}{3}\epsilon \kappa \rho h_{ik}. \qquad (3.9)$$

Inserting this together with Eq. (3.1) into Eq. (2.15) we get

$$R_{ijkl} = 2\epsilon(l_{j[k}E_{l]i} - l_{i[k}E_{l]j}) - 2(U_{[k}B_{l]p}\epsilon^p{}_{ij} + U_{[i}B_{j]p}\epsilon^p{}_{kl}) + \frac{1}{3}\epsilon \kappa \rho(l_{j[k}h_{l]i} - l_{i[k}h_{l]j}) + \kappa p\{h_{i[k}U_{l]}U_j - h_{j[k}U_{l]}U_i\}. \qquad (3.10)$$

It also follows from Eq. (3.9) that

$$F'_{NNkl} = F'_{NNkN} = 0$$

if either of the expressions (2.18), (2.19) for F_{ijkl} is used. Thus Eqs. (3.4), (3.5) should be regarded as constraint equations.

To obtain evolution equations, we consider Eq. (3.6). Using Eqs. (3.9), (2.19), and (2.18), we get from Eq. (2.17) the equations

$$0 = \epsilon F'_{(j|N|l)} \equiv \mathcal{L}_U B_{jl} - \mathcal{D}_i E_{k(j} \epsilon_{l)}{}^{ik} + 2\epsilon a_i \epsilon^{ik}{}_{(j} E_{l)k} - \chi^i{}_{(j} B_{l)i}$$

$$- 2\chi_{(j}{}^i B_{l)i} + \chi B_{jl} + \epsilon \chi_{ik} B_{pq} \epsilon^{pi}{}_{(j} \epsilon^{kq}{}_{l)}, \tag{3.11}$$

$$0 = \epsilon \left(F'_{(j|N|l)} - \frac{1}{3} h_{jl} h^{ik} F'_{iNk} \right) = \mathcal{L}_U E_{jl} + \mathcal{D}_i B_{k(j} \epsilon_{l)}{}^{ik}$$

$$- 2\epsilon a_i \epsilon^{ik}{}_{(j} B_{l)k} - 3\chi_{(j}{}^i E_{l)i} - 2\chi^i{}_{(j} E_{l)i} + h_{jl} \chi^{ik} E_{ik} + 2\chi E_{jl}$$

$$+ \frac{\kappa}{2}(\rho + p) \left(\chi_{(jl)} - \frac{1}{3} \chi h_{jl} \right), \tag{3.12}$$

$$0 = \epsilon \frac{2}{\kappa} h^{ik} F'_{(i|N|k)} = \mathcal{L}_U \rho + (\rho + p)\chi. \tag{3.13}$$

The last of these equations is just Eq. (2.4) while the first two are our evolution equations for E_{jk} and B_{jk}. Observing the symmetry of these tensor fields but ignoring the information about the trace, we find that Eqs. (3.11), (3.12) form a symmetric hyperbolic system for the unknowns E_{jk}, B_{jk}, $j \le k$, if the frame coefficients, the connection coefficients, and the functions ρ, p are given. By taking traces in Eqs. (3.11), (3.12), it is seen that the fields E_{kl} and B_{kl} remain trace free in the evolution if the data are given accordingly.

Propagation equations for the frame coefficients are obtained as follows. We assume that x^α, $\alpha = 1,2,3$, are local coordinates on some space-like initial hypersurface S and that they are dragged along with the vector field $e_0 = U$. Furthermore we choose $x^0 \equiv t$ to be a parameter of the flow lines of U, i.e. proper time for observers moving with the fluid. With this choice of coordinates we satisfy the Lagrange condition

$$U^\mu = e^\mu{}_0 = \delta^\mu{}_0. \tag{3.14}$$

Equation (2.13) then implies for the remaining frame coefficients the propagation equations

$$\partial_t e^\mu{}_a = (\Gamma_0{}^c{}_a - \Gamma_a{}^c{}_0) e^\mu{}_c + \Gamma_0{}^0{}_a \delta^\mu{}_0. \tag{3.15}$$

Here and in the following the indices a,b,c,d take values 1,2,3 and the summation rule applies.

To obtain evolution equations for the connection coefficients we have to impose gauge conditions on the vector fields e_a. We require that they be Fermi propagated in the direction of e_0, i.e. $\nabla_{e_0} e_k + \epsilon \{ g(e_k, \nabla_{e_0} e_0) e_0 - g(e_k, e_0) \nabla_{e_0} e_0 \} = 0$. In terms of the connection coefficients this equation reads

$$\Gamma_0{}^a{}_b = 0. \tag{3.16}$$

In the special case of pressure free matter we know more about the connection coefficients. Equation (2.3) reduces to the geodesic propagation law $U^i \nabla_i U^j = 0$. Thus the frame e_k is parallel propagated in the direction of U and we have

$$\Gamma_0{}^i{}_j = 0. \tag{3.17}$$

With this condition Eq. (2.14) implies the propagation equations

$$\partial_t \Gamma_q{}^i{}_j = -\Gamma_k{}^i{}_j \Gamma_q{}^k{}_0 + R^i{}_{j0q}, \tag{3.18}$$

with the curvature tensor given by Eq. (3.10). Consequently, in the case of pressure free matter the propagation equations for the unknowns

$$e^\mu{}_a, \quad \Gamma_a{}^j{}_k, \quad B_{jk}, \quad E_{jk}, \quad \rho$$

are given by Eqs. (3.15), (3.18) with Eqs. (3.10), (3.11), (3.12), (3.13), where we set $p = 0$ everywhere. This system is symmetric hyperbolic and incorporates the Lagrangian representation of the flow field by Eq. (3.14). Equation (2.3) is taken care of by Eq. (3.17).

We return to the general case. Equation (2.11) implies the propagation equation

$$\mathcal{L}_U s_k = 0. \tag{3.19}$$

Using Eqs. (2.10), (2.11), we get

$$U^i \nabla_i p = -(\rho + p)\nu^2 \nabla_i U^i,$$

which gives with Eq. (2.3) the relation

$$\epsilon \nabla_j p = (\rho + p)\{ U^i \nabla_i U_j - \nu^2 U_j \nabla_i U^i \}. \tag{3.20}$$

For this to be true the following integrability condition must be satisfied:

$$0 = J_{kj} \equiv \nabla_k \{ (\rho + p)(U^i \nabla_i U_j - \nu^2 U_j \nabla_i U^i) \} - \nabla_j \{ (\rho + p)$$

$$\times (U^i \nabla_i U_k - \nu^2 U_k \nabla_i U^i) \}.$$

Using again Eq. (3.20), we get, more explicitly,

$$0 = J_{kj} = (\rho + p) \left\{ U^i (\nabla_k \nabla_i U_j - \nabla_j \nabla_i U_k) - \nu^2 (U_j \nabla_k \nabla_i U^i - U_k \nabla_j \nabla_i U^i) - \nu^2 \nabla_l U^l (\nabla_k U_j - \nabla_j U_k) + \nabla_k U^i \nabla_i U_j - \nabla_j U^i \nabla_i U_k \right.$$

$$\left. + \epsilon \frac{\rho + p}{\nu^2} \left(\frac{\partial^2 p}{\partial \rho^2} \right)_s \nabla_l U^l (U_k U^i \nabla_i U_j - U_j U^i \nabla_i U_k) \right\} + (\alpha U_k - \beta U^i \nabla_i U_k) s_j - (\alpha U_j - \beta U^i \nabla_i U_j) s_k, \tag{3.21}$$

where we set

$$\alpha = (\rho+p)\frac{\partial\nu^2}{\partial s} - \left(1+\frac{n}{\nu^2}\frac{\partial\nu^2}{\partial n}\right)\frac{\partial p}{\partial s} + \nu^2 nT,$$

$$\beta = nT - \frac{1}{\nu^2}\frac{\partial p}{\partial s}.$$

Under our assumptions Eq. (3.21) is represented equivalently by the two equations

$$0 = e_0(\Gamma_0{}^0{}_a) - \nu^2 e_a(\Gamma_c{}^c{}_0) + \Gamma_0{}^0{}_c(\Gamma_a{}^c{}_0 - \Gamma_0{}^c{}_a)$$
$$+\left[\frac{\rho+p}{\nu^2}\left(\frac{\partial^2 p}{\partial\rho^2}\right)_s - \nu^2\right]\Gamma_c{}^c{}_0\Gamma_0{}^0{}_a - \frac{\alpha}{\rho+p}s_a \quad (3.22)$$

and

$$0 = e_a(\Gamma_0{}^0{}_b) - e_b(\Gamma_0{}^0{}_a) - \Gamma_0{}^0{}_c(\Gamma_a{}^c{}_b - \Gamma_b{}^c{}_a) - \nu^2\Gamma_c{}^c{}_0(\Gamma_a{}^0{}_b$$
$$-\Gamma_b{}^0{}_a) - \frac{\beta}{\rho+p}(\Gamma_0{}^0{}_a s_b - \Gamma_0{}^0{}_b s_a). \quad (3.23)$$

We can now complete our system of propagation equations. In view of Eq. (3.16) we get, from Eq. (2.14),

$$\partial_t\Gamma_a{}^b{}_c = -\Gamma_d{}^b{}_c\Gamma_a{}^d{}_0 - \Gamma_0{}^b{}_0\Gamma_a{}^0{}_c + \Gamma_a{}^b{}_0\Gamma_0{}^0{}_c + R^b{}_{c0a}. \quad (3.24)$$

From Eqs. (2.14) and (3.22) ensues

$$\partial_t\Gamma_0{}^0{}_a - \nu^2 e_c(\Gamma_a{}^c{}_0) = -\Gamma_0{}^0{}_c\Gamma_a{}^c{}_0 - \left[\frac{\rho+p}{\nu^2}\left(\frac{\partial^2 p}{\partial\rho^2}\right)_s - \nu^2\right]$$
$$\times\Gamma_c{}^c{}_0\Gamma_0{}^0{}_a + \frac{\alpha}{\rho+p}s_a + \nu^2[\Gamma_k{}^c{}_0(\Gamma_a{}^k{}_c$$
$$-\Gamma_c{}^k{}_a) - \Gamma_a{}^c{}_k\Gamma_c{}^k{}_0 + \Gamma_c{}^c{}_k\Gamma_a{}^k{}_0$$
$$+ R^c{}_{0ac}], \quad (3.25)$$

while Eqs. (2.14) and (3.23) give

$$\nu^2\partial_t\Gamma_a{}^0{}_b - \nu^2 e_b(\Gamma_0{}^0{}_a) = \nu^2\Bigg(\Gamma_k{}^0{}_b(\Gamma_0{}^k{}_a - \Gamma_a{}^k{}_0) - \Gamma_0{}^0{}_c\Gamma_a{}^c{}_b$$
$$+ R^0{}_{b0a} + \Gamma_0{}^0{}_c(\Gamma_a{}^c{}_b - \Gamma_b{}^c{}_a)$$
$$+ \nu^2\Gamma_c{}^c{}_0(\Gamma_a{}^0{}_b - \Gamma_b{}^0{}_a)$$
$$+ \frac{\beta}{\rho+p}(\Gamma_0{}^0{}_a s_b - \Gamma_0{}^0{}_b s_a)\Bigg). \quad (3.26)$$

Equations (3.15), (3.24), (3.25), (3.26) [where Eq. (3.10) is assumed], (3.11), (3.12), (2.10), (2.11), (3.19), and (2.6), (2.7) provide the desired symmetric hyperbolic system. The Lagrangian representation of the flow is again taken into account by Eq. (3.14). Where the operator $U^i\nabla_i$ occurs in these equations, it can be replaced by ∂_t. It should be no-

ticed, however, that in our formalism $\mathcal{L}_U s_k = \partial_t s_k - (\Gamma_0{}^j{}_k - \Gamma_k{}^j{}_0)s_j = e^\mu{}_k\partial_t s_\mu$, etc.

IV. CONCLUDING REMARKS

The propagation system we have developed here (as well as the otherwise completely different system considered in [1]) is of third order in the metric. As a consequence of this, Eq. (2.3) attains in our formalism the status of a constraint. But it should be observed that Eq. (2.3) has been used in the derivation of the integrability condition (3.21). Equations (3.25), (3.26) contribute the sound cone to the system of characteristics. If one wants to ensure that the sound characteristics are non-space-like, one has to require that $\nu\leq 1$. For our considerations there is no need to impose an upper limit on the speed of sound.

In our procedure the fluid equations serve two purposes. Their main role is, of course, to determine the motion of the fluid. In addition to that, we use them to remove the gauge freedom.

Since under our assumptions the equations for an ideal fluid can be written on any given background manifold as a symmetric hyperbolic system, they can be solved for arbitrary, sufficiently smooth data at least locally near any Cauchy hypersurface. On a given solution to the vacuum field equations they can thus be used as equations for the time-like vector field e_0 of an orthonormal frame e_k by setting $U = e_0$ (no coupling to the Einstein equations is intended here). Fixing the fields e_a by Fermi or some other suitable transport law and dragging along the coordinates with e_0, we obtain a way to fix the gauge in the frame formalism. This gives a new hyperbolic reduction of the Einstein vacuum equations. In this procedure any exotic (i.e. completely unphysical) ''equation of state'' may be prescribed, the sole objective being to obtain a useful, long-lived gauge.

It may facilitate the analysis near the boundary if such a gauge is used in the vacuum part of the problem, where the data for the field U, the density ρ, and the equation of state are extended in a suitable way beyond the boundary.

The system simplifies considerably in the homentropic case. Then $s_a\equiv 0$ in Eqs. (3.25), (3.26) and Eqs. (2.10), (2.11), (3.19), (2.6) are replaced by Eqs. (3.13), (2.12). In the resulting system the function $\rho+p$ neither occurs in the principle part nor in a denominator.

We do not demonstrate here that the gauge conditions and the constraints are preserved in the evolution defined by our system. The argument should closely follow the corresponding argument given in [8]. However, in the presence of a boundary such a demonstration will only be sensible if precise smoothness properties of the fields near and possible jumps travelling along the boundary have been worked out in the course of the existence proof.

We have given the equations in a form which displays those formal properties which are important for us. It is of interest to compare Eqs. (3.11), (3.12) with the way in which the Bianchi equations have been written in [9]. Here interpretations of some of the quantities entering the equations are provided.

There may exist variations of the system discussed in this paper which are of particular interest in numerical work. In [8] a hyperbolic reduction of the Einstein equations which is

based on the Arnowitt-Deser-Misner (ADM) representation of the metric has been given. The resulting system is symmetric hyperbolic and can be written in flux conservative form. It may be worthwile to attempt something similar in the present situation. One would then identify the time-like flow vector field inherent in the ADM representation with the flow vector field U and try to deduce from the integrability condition (3.21) propagation equations for the lapse function and the shift vector field.

Finally one may ask how our procedure relates to Newtonian theory. There appears to be no direct analogue in the Newtonian case. Nevertheless, it may be interesting to see what happens to our system if one tries to obtain a Newtonian limit.

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