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On Solutions of the Schlesinger Equations in Terms of Θ-functions

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1 Introduction

The Schlesinger equations (see [18]) arise in the context of the following Riemann-Hilbert (inverse monodromy) problem:

For an arbitrary $g \in \mathbb{N}$ and distinct $2g+2$ points $\lambda_j \in \mathbb{C}$, construct a function $\Psi(\lambda)$: $\mathbb{CP}^1 \setminus \{\lambda_1, \dots, \lambda_{2g+2}\} \to SL(2, \mathbb{C})$ which has the following properties:

(1) $\Psi(\infty) = I$;

(2) $\Psi(\lambda)$ is holomorphic for all $\lambda \in \mathbb{CP}^1 \setminus \{\lambda_1, \dots, \lambda_{2g+2}\}$;

(3) $\Psi(\lambda)$ has regular singular points at $\lambda = \lambda_j$, $j = 1, \dots, 2g+2$, with given monodromy matrices, $M_j \in SL(2, \mathbb{C})$.

In the case where the monodromy matrices are independent of the parameters $\lambda_1, \dots, \lambda_{2g+2}$, the function $\Psi \equiv \Psi(\lambda)$ solves the matrix differential equation

$$\frac{d\Psi}{d\lambda} = \sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} \Psi, \tag{1.1}$$

where the $sl(2, \mathbb{C})$-valued matrices A_j solve the system of Schlesinger equations

$$\frac{\partial A_j}{\partial \lambda_i} = \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad i \neq j \ , \qquad \frac{\partial A_i}{\partial \lambda_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}. \tag{1.2}$$

Obviously, the eigenvalues of A_j,which will be denoted by $t_j/2$ and $-(t_j/2)$ in the sequel, are integrals of motion of system (1.2).

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The important object associated with system (1.2) is the so-called τ-function—the function generating Hamiltonians of the Schlesinger system (see [17], [9], [8]); it can be defined as the solution to the system of equations

$$\frac{\partial \ln \tau}{\partial \lambda_j} \equiv \sum_{i \neq j} \frac{\mathrm{tr} A_j A_i}{\lambda_j - \lambda_i}$$

(see Sec. 2 for details).

For $g = 1$, the Schlesinger system may equivalently be rewritten in terms of a single function of one variable, the position $y(t)$ of the zero of the (12) matrix element of the function $A_1/\lambda + A_2/\lambda - 1 + A_3/\lambda - t$ in the λ-plane. The equation for $y(t)$ turns out to coincide with the sixth Painlevé equation,

$$\begin{aligned}\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right),\end{aligned} \tag{1.3}$$

where

$$\alpha \equiv \frac{(t_1 - 1)^2}{2}, \quad \beta \equiv -\frac{t_2^2}{2}, \quad \gamma \equiv \frac{t_3^2}{2}, \quad \delta \equiv \frac{1}{2} - \frac{t_4^2}{2}. \tag{1.4}$$

K. Okamoto showed in [16] that the general solution to the sixth Painlevé equation can be written explicitly in terms of elliptic functions, provided that the set of the parameters t_j satisfies one of the following conditions: $t_i \in \mathbb{Z}$, $t_1 + \cdots + t_4 \in 2\mathbb{Z}$, or $t_i + 1/2 \in \mathbb{Z}$. More recently, the algebro-geometric aspects of the sixth Painlevé equation have once again attracted attention; see the papers [6], [14] (some details which are relevant to our work are given in the Appendix).

Our interest in the problem of finding explicit solutions to the Schlesinger system in algebro-geometric terms was initiated, on one hand, by the work of Okamoto, and, on the other hand, by our papers [11], [13], [12], [10], devoted to the study of solutions to the Ernst equation arising as a partial case of the vacuum Einstein equations; in particular, it turns out that some of the elliptic solutions of the Ernst equation studied in [12] may also be described by the sixth Painlevé equation [10]: in fact, being rewritten in appropriate variables, these solutions give a certain one-parameter sub-family of Okamoto's solutions with $t_j = 1/2$.

In this paper we solve, in terms of theta-functions, the inverse monodromy problem formulated at the beginning of the Introduction for an arbitrary g and an arbitrary set of antidiagonal monodromy matrices. Our approach originated from the so-called finite-gap integration method for the integrable systems (see [4]). The solution of the inverse monodromy problem allows us, in turn, to express in terms of theta functions the

$2g$-parameter family of solutions to the Schlesinger system for $t_j = 1/2$ and calculate the corresponding τ-function. The τ-function (up to multiplication by an arbitrary constant) is given by the expression

$$\tau(\{\lambda_j\}) = \frac{\Theta[\mathbf{p},\mathbf{q}](0 \mid \mathbf{B})}{\sqrt{\det \mathcal{A}}} \prod_{j<k} (\lambda_j - \lambda_k)^{-(1/8)}, \tag{1.5}$$

where the vectors $\mathbf{p} \in \mathbb{C}^g$, $\mathbf{q} \in \mathbb{C}^g$ are parameters corresponding to parameters of the monodromy matrices, $\mathbf{B}$ is the matrix of b-periods of the hyperelliptic curve

$$w^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j),$$

and

$$\mathcal{A}_{kj} \equiv 2 \int_{\lambda_{2j+1}}^{\lambda_{2j+2}} \frac{\lambda^{k-1} d\lambda}{w}, \quad j, k = 1, \ldots, g.$$

For the elliptic case $g = 1$, applying a conformal transformation of the λ-plane, one can always map the points $\lambda_1, \ldots, \lambda_4$ to $0, 1, t$, and ∞, respectively (t is equal to the cross-ratio of the points $\lambda_1, \ldots, \lambda_4$). Then (again up to an arbitrary constant) the τ-function (1.5) can be rewritten in the form

$$\tau(t) = \frac{\theta_{p,q}(0 \mid \sigma)}{\sqrt[8]{t(t-1)}} \left[\int_0^1 \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}} \right]^{-(1/2)}, \tag{1.6}$$

where $\theta_{p,q}(0 \mid \sigma)$ is the elliptic theta-function with characteristic $[p, q]$: here, the module $\sigma(t)$ of the curve $w^2 = \lambda(\lambda-1)(\lambda-t)$ is chosen so that $t = \theta_4^4(0 \mid \sigma)/\theta_2^4(0 \mid \sigma)$.

The latter τ-function defines a new representation of the solution to the sixth Painlevé equation with the parameters $t_j = 1/2$; i.e.,

$$\alpha = \frac{1}{8}, \quad \beta = -\frac{1}{8}, \quad \gamma = \frac{1}{8}, \quad \delta = \frac{3}{8} \ : \tag{1.7}$$

$$y(t) = t - t(t-1) \left[D\left(\frac{\frac{d}{dt} D(\tau)}{\frac{d}{dt} D\left(\sqrt[8]{t(t-1)}\tau\right)} \right) + \frac{t(t-1)}{D^2\left(\sqrt[8]{t(t-1)}\tau\right)} \right]^{-1}, \tag{1.8}$$

where the operator D is defined as

$$D(\cdot) \equiv t(t-1) \frac{d}{dt} \ln(\cdot).$$

As a corollary of the sixth Painlevé equation (1.3) with coefficients (1.7), the function

$$\zeta(t) \equiv D(\tau),$$

where the τ-function $\tau(t)$ is given by (1.6), satisfies the equation

$$[t(t-1)\zeta'']^2 = \zeta'\left[\left(\zeta' + \frac{1}{4}\right)^2 - ((2t-1)\zeta' - \zeta)^2\right]. \tag{1.9}$$

One more form of solution (1.8), namely,

$$y(t) = \frac{tu(\frac{\sigma}{2} \mid \sigma)}{u\left(\frac{\sigma}{2}, \mid \sigma\right) + (1-t)u\left(\frac{1}{2} \mid \sigma\right)}, \quad \text{where} \quad u(z \mid \sigma) = \frac{\frac{\partial}{\partial z} \ln \frac{\partial}{\partial z} \ln \frac{\theta_{p,q}(z|\sigma)}{\theta_1(z|\sigma)}}{\frac{\partial}{\partial \sigma} \ln \frac{\theta_{p,q}(z|\sigma)}{\theta_1(z|\sigma)}}, \tag{1.10}$$

may be obtained from our construction by a straightforward calculation of the position of the zero of the (12) component of the matrix $\Psi_\lambda \Psi^{-1}$ in the λ-plane.

This paper is organized as follows. In Section 2, we recall some basic facts about isomonodromy deformations and Schlesinger equations. In Section 3, we begin with the solution of an inverse monodromy problem with an arbitrary even number of singular points and antidiagonal monodromy matrices. In Section 4, we find the related τ-function, and finally, in Section 5, we apply the results of the previous sections to the $g = 1$ case, i.e., to the sixth Painlevé equation.

It is also worth mentioning that the solution of some inverse monodromy problems, including singularities of regular and irregular type in the framework of the finite-gap integration technique, were given by M. Jimbo and T. Miwa [8]. However, their construction cannot be applied to solve the inverse monodromy problems considered here. In the case of 2×2 monodromy problems with only regular singularities, say, the construction by Jimbo and Miwa leads to an analytic function with $3g + 2$ regular singular points whose $2g + 2$ monodromy matrices, after a proper normalization (see Section 2), equal $i\sigma_1$, and g monodromy matrices are just equal to $-I$. Therefore, the solution of the Schlesinger system, which can be obtained from the construction of Jimbo and Miwa, does not contain any parameters in contrast to the construction presented in this paper.

Simultaneously with the present work, a solution of the same Riemann-Hilbert problem was given in the paper of P. Deift, A. Its, A. Kapaev, and X. Zhou [2] in rather different terms. The problem of calculating the corresponding τ-function (1.5) was not considered there.

2 The Schlesinger equations

In this section, we recall the basic notation and definitions related to isomonodromy deformations of the 2×2 matrix linear ordinary differential equation,

$$\frac{d}{d\lambda}\Psi = A(\lambda)\Psi, \tag{2.1}$$

where $A(\lambda) \in sl_2(\mathbb{C})$ is a rational function of λ with $2g+2$ poles of the first order,

$$A(\lambda) = \sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j}, \quad i \neq j \Rightarrow \lambda_i \neq \lambda_j, \quad \frac{d}{d\lambda} A_j = 0. \tag{2.2}$$

We suppose that $\lambda = \infty$ is not a pole, which means that the following condition is fulfilled:

$$\sum_{j=1}^{2g+2} A_j = 0. \tag{2.3}$$

To fix a fundamental solution of equation (2.1), choose a point $\lambda_0 \in \mathbb{CP}^1 \setminus \{\lambda_1, \dots, \lambda_{2g+2}\}$ and impose the following normalization condition:

$$\Psi(\lambda_0) = I. \tag{2.4}$$

Since $\mathrm{tr} A(\lambda) = 0$, this means that $\det \Psi(\lambda) = 1$ for $\lambda \in \mathbb{C}$. Now one defines the *monodromy matrices*,

$$M_j = \Psi(\lambda_0)\Big|_{\gamma_j}, \quad j = 1, \dots, 2g+2,$$

as analytic continuations of the fundamental solution normalized by condition (2.4) along the generators, l_k, of the fundamental group $\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \lambda_2, \dots, \lambda_{2g+2}\}, \lambda_0)$ defined in Figure 1. The monodromy matrices satisfy the cyclic relation

$$M_{2g+2} \cdot \ldots \cdot M_1 = I, \tag{2.5}$$

and generate a subgroup of $SL(2, \mathbb{C})$, i.e.,

$$\det M_j = 1, \quad j = 1, \dots, 2g+2. \tag{2.6}$$

Matrix elements of M_j and eigenvalues $\pm(t_j/2)$ of the matrices A_j, $j = 1, \dots, 2g+2$, are called the *monodromy data* of the function Ψ. The monodromy data are locally analytic functions of the variables $A_1, \dots, A_{2g+2}$ and $\lambda_0, \lambda_1, \dots, \lambda_{2g+2}$. The condition

$$\frac{dt_j}{d\lambda_l} = 0 \quad \text{and} \quad \frac{dM_j}{d\lambda_l} = 0, \quad \text{for } j,\ l = 1, \dots, 2g+2, \tag{2.7}$$

is called the *isomonodromy condition*. In the generic situation, when the numbers t_j are noninteger, the isomonodromy condition (2.7) is equivalent to the following system of linear differential equations for the function Ψ:

$$\frac{d\Psi}{d\lambda_j} = \left(\frac{A_j}{\lambda_0 - \lambda_j} - \frac{A_j}{\lambda - \lambda_j} \right) \Psi, \qquad j = 1, \dots, 2g+2. \tag{2.8}$$

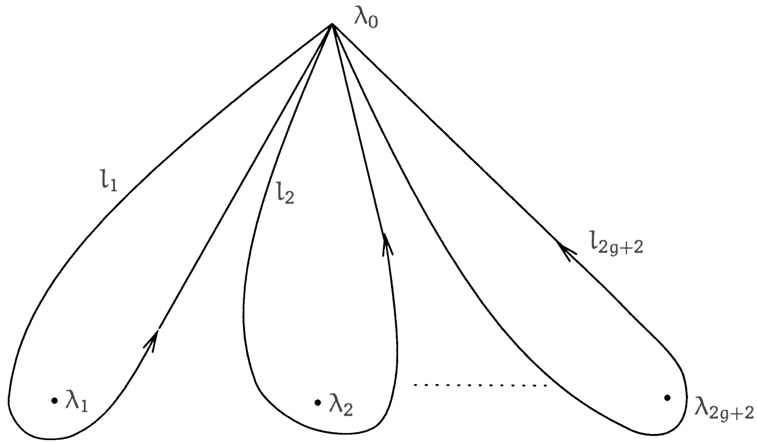

Figure 1 Generators of $\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \lambda_2, \dots, \lambda_{2g+2}\}, \lambda_0)$

Following [18], we choose the normalization point $\lambda_0 = \infty$ to exclude the nonessential parameter λ_0. In this case, the compatibility condition of system (2.8), (2.1) reads as the following system of nonlinear ODE's, the *Schlesinger equations*:

$$j \neq i: \ \frac{\partial A_j}{\partial \lambda_i} = \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \tag{2.9}$$

$$j = i: \ \frac{\partial A_i}{\partial \lambda_i} = -\sum_{\substack{j=1 \\ j \neq i}}^{2g+2} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}. \tag{2.10}$$

Solutions of these equations define *isomonodromy deformations* of the matrix elements of A_j. Note that system (2.9), (2.10) is equivalent to system (2.9), (2.3).

Proposition 2.1. If a set $\{A_1, \dots, A_{2g+2}\}$ is a solution of system (2.9), (2.10), then the monodromy data of the function Ψ, which solves equation (2.1) with the corresponding matrix $A(\lambda)$ given by equation (2.2), are independent of $\lambda_1, \dots, \lambda_{2g+2}$. □

The set of the monodromy data, $\{t_1, \dots, t_{2g+2}, M_1, \dots, M_{2g+2}\} \in \mathbb{C}^{2g+2} \times \mathcal{M}_{2g+2}$, where the variety $\mathcal{M}_{2g+2} \equiv \mathcal{M}_{2g+2}(t_1, \dots, t_{2g+2})$ is defined via equations (2.5) and (2.6), is known to be in one-to-one correspondence with the solutions of the system of Schlesinger equations (2.9), (2.10) in the generic case of noninteger t_j. The nontrivial part of this statement follows from the solvability of the inverse monodromy problem (see [1], [3]).

In this paper, we consider the case when all $t_j = 1/2$, so that the matrices A_j and M_j can be represented in the form

$$A_j = \frac{1}{4} G_j \sigma_3 G_j^{-1}, \quad M_j = i C_j^{-1} \sigma_3 C_j, \tag{2.11}$$

and λ-independent matrices G_j and C_j are defined via the asymptotic behavior of the function Ψ in the neighborhood of the points λ_j,

$$\Psi \underset{\lambda\to\lambda_j}{=} (G_j + \mathcal{O}(\lambda-\lambda_j))(\lambda-\lambda_j)^{(1/4)\sigma_3} C_j; \tag{2.12}$$

$$\det G_j = \det C_j = 1.$$

In the isomonodromy case, one can always choose C_j to be independent of $\lambda_1, \ldots, \lambda_{2g+2}$. Hereafter, we use the standard notation for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can formulate the following proposition.

Proposition 2.2. Let $\Psi^*(Q)$ be a holomorphic function on the universal covering, pr: $X \to \mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_{2g+2}\}$, which has the asymptotic behavior as $\lambda = \operatorname{pr} Q \to \lambda_j$ prescribed by equation (2.12) and normalized as $\Psi^*(Q_0) = I$ at some point Q_0, $\operatorname{pr} Q_0 = \lambda_0$. Then the function $\Psi(\lambda) = \Psi^*(Q)|_{\operatorname{pr} Q=\lambda}$ has the monodromy data corresponding to the variety $\mathcal{M}_{2g+2}(\pm(1/2), \ldots, \pm(1/2))$, with the matrices M_j defined via the second equation (2.11), and solves the system of differential equations (2.1), (2.8), where the matrix $A(\lambda)$ is defined by equations (2.1) and (2.2). □

If a set of matrices $\{A_1, \ldots, A_{2g+2}\}$ is a solution of system (2.9), (2.10), then for any matrix $K \in SL(2,\mathbb{C})$ independent of $\lambda_1, \ldots, \lambda_{2g+2}$, the new set $\{A_j^{\text{new}} = KA_jK^{-1}, \quad j = 1, \ldots, 2g+2\}$ is also a solution of the system. This gauge transformation on the set of the solutions of the Schlesinger system corresponds to the gauge transformation of the function $\Psi(\lambda)$,

$$\Psi^{\text{new}} = K\Psi K^{-1}, \tag{2.13}$$

which leaves the normalization condition (2.4) invariant and acts on $\mathcal{M}_{2g+2}$ in the same way as on the space of the solutions,

$$M_j^{\text{new}} = KM_jK^{-1}. \tag{2.14}$$

By choosing $K = C_0C_1$, where C_1 is given by (2.12) for $j = 1$ and $C_0 = (i/\sqrt{2})(\sigma_3 + \sigma_1)$, we use this gauge transformation to fix

$$M_1 = i\sigma_1. \tag{2.15}$$

Since we have one more parameter in our gauge transform, $C_0 \to C_0\kappa^{\sigma_3}$, we can use the remaining freedom to remove one more parameter from $\mathcal{M}_{2g+2}$. More exactly, by making one more gauge transform (2.13) with the matrix $K = C_0\kappa^{\sigma_3}C_0^{-1}$, we, by choosing appropriately the parameter κ, fix the next monodromy matrix M_2:
If $\operatorname{tr}(M_2\sigma_1)^2 \neq -2$, then

$$M_2 = \begin{pmatrix} 0 & m_2 \\ -m_2^{-1} & 0 \end{pmatrix}, \quad m_2 \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0, \infty\}; \tag{2.16}$$

if $\operatorname{tr}(M_2\sigma_1)^2 = -2$ but $M_2 \neq \pm i\sigma_1$, then $M_2 = \pm i(\sigma_3 + \sigma_1 + i\sigma_2)$; and, finally, if $M_2 = \pm i\sigma_1$, then we can use the parameter κ to fix analogously the structure of the next matrix, M_3.

The variety $\mathcal{M}_{2g+2}(\pm(1/2), \ldots, \pm(1/2))$ contains the following subvariety, $\mathbb{C}^*_{2g} \cong \mathbb{T}_{2g} \times \mathbb{R}^{2g}$:

$$M_j = \begin{pmatrix} 0 & m_j \\ -m_j^{-1} & 0 \end{pmatrix}, \quad j = 1, \ldots, 2g+2, \tag{2.17}$$

where

$$m_1 = i, \quad m_j \in \mathbb{C}^*, \quad j = 2, \ldots, 2g+2; \qquad \prod_{j=1}^{g+1} m_{2j} = (-1)^{g+1} \prod_{j=1}^{g+1} m_{2j-1}. \tag{2.18}$$

Note that if the matrices M_1 and M_2 are fixed by equations (2.15) and (2.16) correspondingly, then $\dim_{\mathbb{C}} \mathcal{M}_{2g+2}(\pm(1/2), \ldots, \pm(1/2)) = 4g-2$ and $\dim_{\mathbb{C}} \mathbb{C}^*_{2g} = 2g$; in fact, for $g = 1$, the subvariety $\mathbb{C}^*_{2g}$ constitutes almost all the variety $\mathcal{M}_{2g+2}(\pm(1/2), \ldots, \pm(1/2))$. More precisely, one can formulate the following proposition.

Proposition 2.3 ([6]). If $g = 1$, then the variety $\mathcal{M}_4(\pm(1/2), \ldots, \pm(1/2))$ coincides, up to the conjugation defined by equation (2.14) with arbitrary matrix $K \in SL(2, \mathbb{C})$, with the union of the following two sets of the monodromy matrices:

$$(1) \quad M_k = \begin{pmatrix} 0 & m_k \\ -\frac{1}{m_k} & 0 \end{pmatrix}, \; k = 1, \ldots 4, \quad m_1 = i, \;\; m_k \in \mathbb{C}^*, \quad m_4 m_2 = i m_3; \tag{2.19}$$

$$(2) \quad M_1 = -i\sigma_3, \;\; M_2 = i\epsilon_2 \begin{pmatrix} -1 & a-1 \\ 0 & 1 \end{pmatrix},$$

$$M_3 = i\epsilon_3 \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}, \;\; M_4 = i\epsilon_4 \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.20}$$

where $\epsilon_k = \pm 1, \;\; \epsilon_2\epsilon_3\epsilon_4 = 1, \;\; a \in \mathbb{C}$. □

Isomonodromy deformations of equation (2.1), in the case when the matrix $A(\lambda)$ has four poles, are governed by solutions to the sixth Painlevé equation (1.3). We rewrite the corresponding relation given by M. Jimbo and T. Miwa [8] in the notation which more suits our basic construction as follows.

Denote by $\vec{g}_j^{\,p}$ the p-th column of the matrix G_j from equation (2.11), and introduce new matrices $G_{ij}^{pq} \overset{\text{def}}{=} (\vec{g}_i^{\,p}\ \vec{g}_j^{\,q})$; in particular, $G_{jj}^{12} \equiv G_j$.

Proposition 2.4. The functions

$$\hat{A}_j^{12} = t_j \frac{\det G_{j1}^{12} \det G_{1j}^{22}}{\det G_{11}^{12} \det G_{jj}^{12}}, \qquad j = 1, \ldots, 4, \tag{2.21}$$

depend on the variables $\{\lambda_k\}$ only through their cross-ratio,

$$t = \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\,\frac{\lambda_4 - \lambda_2}{\lambda_4 - \lambda_1}. \tag{2.22}$$

Moreover, the function

$$y(t) = -\frac{t}{1 + (1-t)\hat{A}_4^{12}/\hat{A}_2^{12}} = -\frac{1}{1 - \frac{1-t}{t}\hat{A}_3^{12}/\hat{A}_2^{12}} \tag{2.23}$$

is the solution of the sixth Painlevé equation (1.3) with the parameters given by equation (1.7). □

Proof. If the set $\{A_j\}$ is a solution of the system (2.9), (2.3), then the monodromy data of the function Ψ, which solves the corresponding equation (2.1), are independent of $\{\lambda_j\}$ and λ. Define the new variable

$$\mu = \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\,\frac{\lambda - \lambda_2}{\lambda - \lambda_1} \tag{2.24}$$

and consider

$$\hat{\Psi} = G_1^{-1}\Psi C_1^{-1} \tag{2.25}$$

as a function of μ. In the complex μ-plane, the function Φ has singularities only at the points 0, 1, t, and ∞ with the behavior prescribed by equations (2.25) and (2.12): it is normalized at $\mu = \infty$ by the condition

$$\hat{\Psi} \underset{\mu\to\infty}{=} \left(I + \mathcal{O}\left(\mu^{-1}\right)\right)\mu^{(1/4)\sigma_3},$$

and its monodromy data are independent of $\{\lambda_j\}$. Such a function is uniquely defined and depends on $\{\lambda_j\}$ only via the cross-ratio t. It means that the logarithmic derivative,

$$\frac{d\hat{\Psi}}{d\mu}\hat{\Psi}^{-1} = \frac{\hat{A}_2}{\mu} + \frac{\hat{A}_3}{\mu - 1} + \frac{\hat{A}_4}{\mu - t} \stackrel{\text{def}}{=} \hat{A}(\mu), \tag{2.26}$$

and, in particular, the matrices

$$\hat{A}_j = \frac{t_j}{2} G_1^{-1} G_j \sigma_3 G_j^{-1} G_1,$$

also depend on $\{\lambda_j\}$ only via the variable t. The matrices $\hat{A}_j$ can be rewritten as

$$\hat{A}_j = -\frac{t_j}{4} \hat{G}_j^{-1} \sigma_3 \hat{G}_j,$$

where

$$\hat{G}_j = \begin{pmatrix} \det G_{j1}^{11} & \det G_{j1}^{12} \\ \det G_{j1}^{21} & \det G_{j1}^{22} \end{pmatrix}, \qquad \det \hat{G}_j = \det G_j \, \det G_1. \tag{2.27}$$

To complete the proof, one has to recall that according to [8], the function $y(t)$, which solves the equation $\hat{A}^{12}(y) = 0$, where $A^{12}(\cdot)$ is the corresponding matrix element of $\hat{A}(\cdot)$ (see (2.26)), is the solution of the sixth Painlevé equation. ■

Remark 2.1. Proposition 2.4 is valid not only for the present case, when all coefficients t_j equal $1/2$, but also in the case of arbitrary complex t_j (assuming that all matrices A_j are diagonalizable). In the latter case, the function $y(t)$ (1.3) solves the sixth Painlevé equation with the coefficients

$$\alpha = \frac{1}{2}(t_1 - 1)^2, \quad \beta = -\frac{1}{2}t_2^2, \quad \gamma = \frac{1}{2}t_3^2, \quad \delta = \frac{1}{2}(1 - t_4^2).$$

The object playing the important role in applications of isomonodromy deformations in differential geometry and mathematical physics is the so-called tau function $\tau(\{\lambda_j\})$. We recall here the definition of the τ-function given in [8], [17], [9].

The Schlesinger equations (2.9), (2.10) can be rewritten in the Hamiltonian form,

$$\frac{dA_j}{d\lambda_k} = \{H_k, A_j\}, \tag{2.28}$$

where the Poisson bracket is defined as

$$\{(A_i)_{ab}, (A_j)_{cd}\} = \delta_{ij}\left((A_i)_{ad}\delta_{cd} - (A_i)_{bc}\delta_{ad}\right), \tag{2.29}$$

and the Hamiltonians are given by

$$H_j = \frac{1}{2} \operatorname*{Res}_{\lambda=\lambda_j} \mathrm{Tr} A^2(\lambda) = -\operatorname*{Res}_{\lambda=\lambda_j} \det A(\lambda) \equiv \sum_{i \neq j}^{2g+2} \frac{\mathrm{tr} A_j A_i}{\lambda_j - \lambda_i}. \tag{2.30}$$

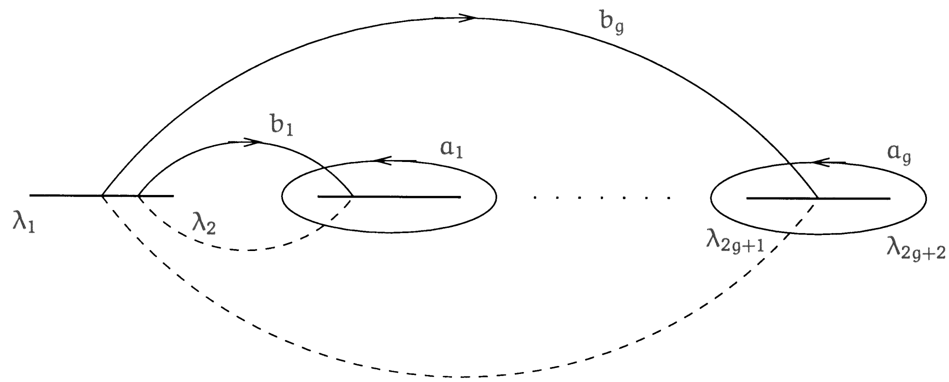

Figure 2 Branch cuts and canonical basis of cycles on the hyperelliptic curve, $\mathcal{L}$. Continuous and dashed paths lie on the first and the second sheet of $\mathcal{L}$, respectively.

One proves that

$$\{H_k, H_j\} = 0, \qquad \frac{\partial H_k}{\partial \lambda_j} = \frac{\partial H_j}{\partial \lambda_k}, \tag{2.31}$$

which implies the compatibility of system (2.28). Taking into account the previous equations, one can correctly define the τ-function $\tau \equiv \tau(\lambda_1, \dots, \lambda_{2g+2})$ generating Hamiltonians H_j by

$$\frac{d}{d\lambda_j} \ln \tau = H_j, \tag{2.32}$$

which is holomorphic outside of the hyperplanes $\lambda_j = \lambda_i$, $i,\ j = 1, \dots, 2g+2$.

3 Solutions of the Schlesinger system

Consider the hyperelliptic curve $\mathcal{L}$ of genus g defined by the equation

$$w^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j) \tag{3.1}$$

with arbitrary noncoinciding $\lambda_j \in \mathbb{C}$, and the basic cycles (a_j, b_j) chosen according to Figure 2.

Let us denote the fundamental polygon of $\mathcal{L}$ by $\hat{\mathcal{L}}$. The basic holomorphic 1-forms on $\mathcal{L}$ are given by

$$dU_k^0 = \frac{\lambda^{k-1} d\lambda}{w}, \qquad k = 1, \dots, g. \tag{3.2}$$

Let us define $g \times g$ matrices of a- and b-periods of these 1-forms by

$$\mathcal{A}_{kj} = \oint_{a_j} dU_k^0, \qquad \mathcal{B}_{kj} = \oint_{b_j} dU_k^0. \tag{3.3}$$

Then the holomorphic 1-forms

$$dU_k = \frac{1}{w} \sum_{j=1}^{g} (\mathcal{A}^{-1})_{kj} \lambda^{j-1} d\lambda \tag{3.4}$$

satisfy the normalization conditions $\oint_{a_j} dU_k = \delta_{jk}$.

The matrices $\mathcal{A}$ and $\mathcal{B}$ define the symmetric $g \times g$ matrix of b-periods of the curve $\mathcal{L}$:

$$\mathbf{B} = \mathcal{A}^{-1}\mathcal{B}.$$

Let us now introduce the theta function with characteristic $[\mathbf{p}, \mathbf{q}]$ ($\mathbf{p} \in \mathbb{C}^g$, $\mathbf{q} \in \mathbb{C}^g$) by the following series:

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} \mid \mathbf{B}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\{\pi i \langle \mathbf{B}(\mathbf{m} + \mathbf{p}), \mathbf{m} + \mathbf{p} \rangle + 2\pi i \langle \mathbf{z} + \mathbf{q}, \mathbf{m} + \mathbf{p} \rangle\}, \tag{3.5}$$

for any $\mathbf{z} \in \mathbb{C}^g$. It possesses the following periodicity properties:

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} + \mathbf{e}_j) = e^{2\pi i p_j} \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}), \tag{3.6}$$

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} + \mathbf{B}\mathbf{e}_j) = e^{-2\pi i q_j} e^{-\pi i \mathbf{B}_{jj} - 2\pi i \mathbf{z}_j} \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}), \tag{3.7}$$

where

$$\mathbf{e}_j \equiv (0, \ldots, 1, \ldots, 0) \tag{3.8}$$

(1 stands in the j-th place).

Denote the universal covering of $\mathcal{L}$ by Γ. The multi-valued on $\mathcal{L}$, and single-valued on Γ, map $U(P) \in \mathbb{C}^g$ is defined by the contour integral $U_j(P) = \int_{\lambda_1}^{P} dU_j$. The vector of Riemann constants corresponding to our choice of the initial point of the map reads as follows [5]:

$$K = \frac{1}{2}\mathbf{B}(\mathbf{e}_1 + \ldots + \mathbf{e}_g) + \frac{1}{2}(\mathbf{e}_1 + 2\mathbf{e}_2 \ldots + g\mathbf{e}_g). \tag{3.9}$$

The characteristic with components $\mathbf{p} \in \mathbb{C}^g/2\mathbb{C}^g$, $\mathbf{q} \in \mathbb{C}^g/2\mathbb{C}^g$ is called the half-integer characteristic: the half-integer characteristics are in one-to-one correspondence with the half-periods $\mathbf{Bp} + \mathbf{q}$. If the scalar product $4\langle \mathbf{p}, \mathbf{q} \rangle$ is odd, then the related theta function is odd with respect to its argument $\mathbf{z}$ and the characteristic $[\mathbf{p}, \mathbf{q}]$ is called odd. If this scalar product is even, then the theta function $\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z})$ is even with respect to $\mathbf{z}$ and the characteristic $[\mathbf{p}, \mathbf{q}]$ is called even.

The odd characteristics which will be of importance for us in the sequel correspond to any given subset $S = \{\lambda_{i_1}, \ldots, \lambda_{i_{g-1}}\}$ of $g - 1$ arbitrary noncoinciding branch points. The odd half-period associated to the subset S is given by

$$\mathbf{Bp}^S + \mathbf{q}^S = U(\lambda_{i_1}) + \ldots + U(\lambda_{i_{g-1}}) - K. \tag{3.10}$$

Analogously, we shall be interested in the even half-periods which may be represented as

$$\mathbf{Bp}^T + \mathbf{q}^T = U(\lambda_{i_1}) + \ldots + U(\lambda_{i_{g+1}}) - K, \tag{3.11}$$

where $T = \{\lambda_{i_1}, \ldots, \lambda_{i_{g+1}}\}$ is an arbitrary subset of $g + 1$ branch points.

Theorem 3.1. Let the 2×2 matrix-valued function $\Phi(P)$ be defined on the universal covering Γ of $\mathcal{L}$ by the formula

$$\Phi(P) = \begin{pmatrix} \varphi(P) & \varphi(P^*) \\ \psi(P) & \psi(P^*) \end{pmatrix}, \tag{3.12}$$

where

$$\varphi(P) = \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\varphi))\Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\varphi)), \tag{3.13}$$

$$\psi(P) = \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\psi))\Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\psi)), \tag{3.14}$$

with arbitrary (possibly $\{\lambda_j\}$-dependent) $P_{\varphi,\psi} \in \mathcal{L}$ and arbitrary constant characteristic $[\mathbf{p}, \mathbf{q}]$; $*$ is the involution on $\mathcal{L}$ interchanging the sheets. The odd theta characteristic $[\mathbf{p}^S, \mathbf{q}^S]$ corresponds to an arbitrary subset S of $g - 1$ branch points via equation (3.10).

Then the function $\Phi(P)$ is holomorphic and invertible outside of the branch points $\lambda_1, \ldots, \lambda_{2g+2}$ and transforms as follows with respect to the tracing along the basic cycles of $\mathcal{L}$:

$$T_{a_j}[\Phi(P)] = \Phi(P)e^{2\pi i(p_j + p_j^S)\sigma_3}, \qquad T_{b_j}[\Phi(P)] = \Phi(P)e^{-2\pi i(q_j + q_j^S)\sigma_3}e^{-2\pi i \mathbf{B}_{jj} - 4\pi i U(P)}, \tag{3.15}$$

where by T_l we denote the operator of analytic continuation along the contour l. Moreover, the function Φ has the following asymptotic behaviour in the neighborhood of point λ_j:

$$\Phi(P) \underset{\lambda\to\lambda_j}{=} \left\{F_j + O(\sqrt{\lambda-\lambda_j})\right\} \begin{pmatrix} (\lambda-\lambda_j)^{1/2+\delta_j} & 0 \\ 0 & (\lambda-\lambda_j)^{\delta_j} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{3.16}$$

with some λ-independent matrices F_j, $j = 1, \ldots, 2g+2$; $\delta_j = 1$ for $\lambda_j \in S$ and $\delta_j = 0$ for $\lambda_j \notin S$. □

Proof. Let us first check the announced monodromy properties of $\Phi(P)$ around the basic cycles of $\mathcal{L}$. From the periodicity properties of the theta function given by equations (3.6), (3.7), we deduce the following transformation laws for φ:

$$T_{a_j}[\varphi(P)] = e^{2\pi i(p_j+p_j^S)}\varphi(P), \tag{3.17}$$

$$T_{b_j}[\varphi(P)] = e^{-2\pi i(q_j+q_j^S)}e^{-2\pi i\mathbf{B}_{jj}-4\pi i U(P)}\varphi(P), \tag{3.18}$$

and we deduce the same transformation laws for ψ. Taking into account the action of the involution $*$ on the basic cycles and holomorphic differentials,

$$a_j^* = -a_j, \qquad b_j^* = -b_j, \qquad dU_j(P^*) = -dU_j(P); \tag{3.19}$$

we get the transformation laws for the function $\varphi(P^*)$,

$$T_{a_j}[\varphi(P^*)] = e^{-2\pi i(p_j+p_j^S)}\varphi(P^*), \tag{3.20}$$

$$T_{b_j}[\varphi(P^*)] = e^{2\pi i(q_j+q_j^S)}e^{-2\pi i\mathbf{B}_{jj}-4\pi i U(P)}\varphi(P^*), \tag{3.21}$$

which coincide with the transformation laws for the function $\psi(P^*)$. Altogether, this implies relations (3.15) for the function $\Phi(P)$.

The holomorphy of the function Φ follows from the holomorphy of the theta function. Let us show that $\det\Phi$ does not vanish outside of the branch points λ_j. Since the transformations (3.15) preserve the positions of the zeros of $\det\Phi$, it makes sense to speak about the positions of the zeros of $\det\Phi$ in the fundamental polygon $\hat{\mathcal{L}}$. First, notice that $\det\Phi(P)$ vanishes at the branch points λ_j, where two columns of the matrix Φ coincide. Moreover, $\det\Phi$ has, at the points $\lambda_j \in S$, zeros of order 3. This can be seen if we rewrite the second theta function in equation (3.13) up to a nonvanishing exponential factor as

$$\Theta(U(P) - U(P_\varphi) - \sum_S U(\lambda_j) - K).$$

Thus we know altogether $3(g-1)+g+3=4g$ zeros of $\det\Phi$, taking into account their multiplicities. To check that $\det\Phi$ does not vanish outside of λ_j, we integrate the function $\partial/\partial\lambda \ln\det\Phi(P)$ along the boundary of the fundamental polygon $\partial\hat{\mathcal{L}}$. From the transformation properties (3.15), we deduce

$$T_{a_j}[\det\Phi(P)] = \det\Phi(P), \qquad T_{b_j}[\det\Phi(P)] = e^{-4\pi i \mathbf{B}_{jj} - 8\pi i U_j(P)} \det\Phi(P). \tag{3.22}$$

Now one can check that this integral equals $4g$ in the same way as in the standard calculation of the number of zeros of the theta-function of dimension g (see [15]). Therefore, $\det\Phi(P)$ does not have any zeros outside of the branch points λ_j.

The form of the asymptotic expansion (3.16) is a direct consequence of the holomorphicity of φ and ψ, the structure (3.12) of the function Φ, and the previous discussion of the zeros of $\det\Phi$.

Starting from the function $\Phi(P)$ on Γ constructed in Theorem 3.1, we shall now define a new function $\Psi(Q)$ on the universal covering X of $\mathbb{C}\setminus\{\lambda_1,\dots,\lambda_{2g+2}\}$. Let us denote by $\Omega\subset\mathbb{C}$ an arbitrary neighborhood of ∞ on $\mathbb{C}$ which does not overlap with the points λ_j and the projections of all basic cycles of $\mathcal{L}$ on $\mathbb{C}$. Let us fix some sheet X_0 of X choosing the branch cuts between the points λ_j to lie outside of domain Ω. Let us also fix some sheet $\hat{\mathcal{L}}$ of the universal covering Γ of $\mathcal{L}$; then $\hat{\mathcal{L}}$ will contain two nonintersecting copies of Ω. Choose one of them and denote it by Ω_1. The domain Ω_1 contains the point at infinity, which we call ∞^1. Now we are in position to define

$$\Psi(\lambda\in\Omega) = \sqrt{\frac{\det\Phi(\infty^1)}{\det\Phi(\lambda)}}\,\Phi^{-1}(\infty^1)\Phi(\lambda) \tag{3.23}$$

(by λ, we denote the projection of $Q\in X$ as well as of $P\in\Gamma$ on $\mathbb{C}$). On the rest of X, the function $\Psi(Q)$ is defined via the analytic continuation along the contours l_j (Fig. 1).

Theorem 3.2. Let $\mathbf{p},\mathbf{q}\in\mathbb{C}^g$ be an arbitrary set of $2g$ constants such that $[\mathbf{p},\mathbf{q}]$ is not a half-integer characteristic. Then the function $\Psi(Q\in X)$ defined by (3.23), (3.12) is independent of the choice of the points $P_{\varphi,\psi}\in\mathcal{L}$ and the choice of the set $S=\{\lambda_{i_1},\dots,\lambda_{i_{g-1}}\}$. Moreover, Ψ is holomorphic outside of the branch points $\lambda_1,\dots,\lambda_{2g+2}$, satisfies the normalization conditions $\det\Psi(\lambda)=1$ and $\Psi(\lambda=\infty)=I$, and has the antidiagonal monodromies M_j given by equation (2.17) along the contours l_j (Fig. 1). The matrix elements of the monodromies (2.17) are given by the expressions

$$m_1 = i, \qquad m_2 = i(-1)^g \exp\left\{-2\pi i \sum_{k=1}^{g} p_k\right\},$$

$$m_{2j+1} = i(-1)^{g+1} \exp\left\{2\pi i q_j - 2\pi i \sum_{k=j}^{g} p_k\right\},$$

$$m_{2j+2} = i(-1)^g \exp\left\{2\pi i q_j - 2\pi i \sum_{k=j+1}^{g} p_k\right\}, \tag{3.24}$$

for $j = 1, \ldots, g$, where p_j and q_j are components of the vectors $\mathbf{p}$ and $\mathbf{q}$, respectively. The asymptotic expansion of $\Psi(Q)$ in the neighborhood of λ_j is of the form (2.12) with some G_j and

$$C_j = \frac{1}{\sqrt{2im_j}} \begin{pmatrix} 1 & im_j \\ -1 & im_j \end{pmatrix}. \tag{3.25}$$

□

Proof. The nontrivial part is to calculate the monodromies M_j of $\Psi(P)$ along the contours l_j.

Combining the transformations (3.15) of function Φ along the basic cycles of $\mathcal{L}$ with the jumps of Φ,

$$\Phi(P) \to \Phi(P)\sigma_1,$$

on the branch cuts $[\lambda_{2j+1}, \lambda_{2j+2}]$, which follow directly from definition (3.12), we come to the following relations:

$$\Psi(P)M_{2j+2}M_{2j+1} = \frac{T_{l_{2j+1}\circ l_{2j+2}}[\sqrt{\det \Phi(P)}]}{\sqrt{\det \Phi(P)}} \Psi(P) e^{2\pi i(p_j - p_j^S)\sigma_3}, \tag{3.26}$$

$$\Psi(P)M_{2j+1}M_{2j} = \frac{T_{l_{2j}\circ l_{2j+1}}[\sqrt{\det \Phi(P)}]}{\sqrt{\det \Phi(P)}} \Psi(P) e^{2\pi i(q_j - q_{j-1} + q_j^S - q_{j-1}^S)\sigma_3}, \tag{3.27}$$

$j = 1, \ldots, g$. Furthermore, taking into account that

$$U(\lambda_1) = 0, \qquad U(\lambda_2) = \frac{1}{2}\sum_{k=1}^{g} \mathbf{e}_k,$$

$$U(\lambda_{2j+1}) = \frac{1}{2}\mathbf{B}\mathbf{e}_j + \frac{1}{2}\sum_{k=j}^{g}\mathbf{e}_k, \qquad U(\lambda_{2j+2}) = \frac{1}{2}\mathbf{B}\mathbf{e}_j + \frac{1}{2}\sum_{k=j+1}^{g}\mathbf{e}_k, \qquad j = 1, \dots, g, \quad (3.28)$$

we get

$$p_j^S = \frac{1}{2}(\delta_{2j+1} + \delta_{2j+2} + 1), \qquad q_{j+1}^S - q_j^S = \frac{1}{2}(\delta_{2j+2} + \delta_{2j+3} + 1), \tag{3.29}$$

where δ_j are the same as in equation (3.16).

The function $\sqrt{\det \Phi(P)}$ transforms in the following way with respect to the tracing along the cycles l_j:

$$T_{l_{2j+1}\circ l_{2j+2}}[\sqrt{\det \Phi(P)}] = e^{\pi i(\delta_{2j+1}+\delta_{2j+2}+1)}\sqrt{\det \Phi(P)}, \tag{3.30}$$

$$T_{l_{2j}\circ l_{2j+1}}[\sqrt{\det \Phi(P)}] = e^{\pi i(\delta_{2j+2}+\delta_{2j+3}+1)}\sqrt{\det \Phi(P)}. \tag{3.31}$$

To prove relations (3.30), (3.31), it is enough to notice that in the λ-plane, the function $\sqrt{\det \Phi(P)}$ has at the point λ_j: a zero of degree $3/4$ if $\lambda_j \in S$; and a zero of degree $1/4$ if $\lambda_j \notin S$.

Altogether, we get

$$M_{2j+2}M_{2j+1} = \exp\{2\pi i p_j \sigma_3\},$$

$$M_{2j+1}M_{2j} = \exp\{2\pi i(q_j - q_{j-1})\sigma_3\},$$

which imply (3.24), taking into account that $m_1 = i$ and the monodromy around infinity is trivial (2.18).

Now the independence of the function Ψ of the choice of the divisor S and the points $P_{\varphi,\psi}$ follows from the uniqueness of the solution to the Riemann-Hilbert problem with fixed monodromy data.

Existence of the local expansion (2.12) of the function $\Psi(Q)$ at the points λ_j follows from the related statement (3.16) for the function Φ which was proved in Theorem 3.1. The form (3.25) of the matrices C_j follows from the relation (2.11) between the matrices M_j and C_j.

Remark 3.1. The assumption made in Theorem 3.2 that $[\mathbf{p}, \mathbf{q}]$ does not coincide with any half-integer characteristic is nothing but the nontriviality condition; namely, if $[\mathbf{p}, \mathbf{q}]$ is a half-integer characteristic, all monodromies M_j become proportional to σ_1: $M_j = \pm i\sigma_1$; therefore, they can be simultaneously diagonalized by the transformation

$$\Psi \to \tilde{\Psi} \equiv \Psi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The function $\tilde{\Psi}$ has diagonal monodromies $\pm i\sigma_3$, and, therefore, can be chosen to be diagonal itself. Thus, we are in the framework of the scalar Riemann-Hilbert problem: the related matrices A_j are diagonal, and, therefore, λ_j-independent, as follows from the Schlesinger equations.

By the special choices $P_\varphi = \infty^2$ and $P_\psi = \infty^1$ in the formulas of Theorem 3.1, we can simplify the previous expression for the function Ψ to get the following corollary.

Corollary 3.1. The function $\Psi(\lambda)$ defined by equation (3.23) may alternatively be represented as follows:

$$\Psi(\lambda \in \Omega) = \frac{1}{\sqrt{\det \Phi^\infty(\lambda)}} \Phi^\infty(\lambda), \tag{3.32}$$

where

$$\Phi^\infty(P) = \begin{pmatrix} \varphi^\infty(P) & \varphi^\infty(P^*) \\ \psi^\infty(P) & \psi^\infty(P^*) \end{pmatrix}, \tag{3.33}$$

$$\varphi^\infty(P) = \frac{\Theta[\mathbf{p},\mathbf{q}](U(P)+U(\infty^2))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(P)-U(\infty^2))}{\Theta[\mathbf{p},\mathbf{q}](0)\Theta[\mathbf{p}^S,\mathbf{q}^S](-2U(\infty^2))}, \tag{3.34}$$

$$\psi^\infty(P) = \frac{\Theta[\mathbf{p},\mathbf{q}](U(P)+U(\infty^1))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(P)-U(\infty^1))}{\Theta[\mathbf{p},\mathbf{q}](0)\Theta[\mathbf{p}^S,\mathbf{q}^S](-2U(\infty^1))}. \tag{3.35}$$

□

From the Taylor series of the function $\Phi^\infty(P)$ at the points λ_j, we can now construct solutions to the Schlesinger system.

Theorem 3.3. The solution to the Schlesinger system (2.9), (2.10) corresponding to the monodromy matrices (2.17), (3.24) is given by

$$A_j = \frac{1}{4} F_j^\infty \sigma_3 (F_j^\infty)^{-1}, \tag{3.36}$$

where

$$(F_j^\infty)^{11} = \frac{\Theta[\mathbf{p},\mathbf{q}](U(\lambda_j)+U(\infty^2))\Theta[\mathbf{p}^{S_j},\mathbf{q}^{S_j}](U(\lambda_j)-U(\infty^2))}{\Theta[\mathbf{p},\mathbf{q}](0)\Theta[\mathbf{p}^{S_j},\mathbf{q}^{S_j}](-2U(\infty^2))}, \tag{3.37}$$

$$\begin{aligned}(F_j^\infty)^{12} = {} & \sum_{k=1}^{g} \frac{\sum_{l=1}^{g} (\mathcal{A}^{-1})_{lk} \lambda_j^{l-1}}{\prod_{l\neq j}(\lambda_j - \lambda_l)^{1/2}} \\ & \times \frac{\partial}{\partial z_k} \left\{ \frac{\Theta[\mathbf{p},\mathbf{q}](\mathbf{z}+U(\infty^2))\Theta[\mathbf{p}^{S_j},\mathbf{q}^{S_j}](\mathbf{z}-U(\infty^2))}{\Theta[\mathbf{p},\mathbf{q}](0)\Theta[\mathbf{p}^{S_j},\mathbf{q}^{S_j}](-2U(\infty^2))} \right\} (\mathbf{z} = U(\lambda_j)),\end{aligned} \tag{3.38}$$

and $\partial/\partial z_k$ means the derivative of the theta function (3.5) with respect to its k-th variable; matrix $\mathcal{A}$ is given by equation (3.3); S_j are arbitrary $2g+2$ sets of $g-1$ branch points such that $\lambda_j \notin S_j$. The solution (3.36) is independent of the choice of the sets S_j as long as these conditions are fulfilled.

The formulas for the matrix elements $(F_j^\infty)^{21}$ and $(F_j^\infty)^{22}$ may be obtained from the formulas for $(F_j^\infty)^{11}$ and $(F_j^\infty)^{12}$, respectively, by interchanging ∞^1 and ∞^2. □

Proof. In the neighborhood of the point λ_j, we have

$$\varphi_j^\infty(P) = (F_j^\infty)^{11} + \sqrt{\lambda-\lambda_j}(F_j^\infty)^{12} + O(\lambda-\lambda_j), \tag{3.39}$$

$$\psi_j^\infty(P) = (F_j^\infty)^{21} + \sqrt{\lambda-\lambda_j}(F_j^\infty)^{22} + O(\lambda-\lambda_j), \tag{3.40}$$

with F_j given by equations (3.37), (3.38); the functions $\varphi_j^\infty(P)$ and $\psi_j^\infty(P)$ are defined by equations (3.13), (3.14), with $P_\varphi = \infty^2$, $P_\psi = \infty^1$, and $[\mathbf{p}^S, \mathbf{q}^S]$ substituted by $[\mathbf{p}^{S_j}, \mathbf{q}^{S_j}]$.

Therefore,

$$\det \Phi_j^\infty(P) = \sqrt{\lambda-\lambda_j}\{\det F_j^\infty + O(\lambda-\lambda_j)\}, \tag{3.41}$$

and

$$[\det \Phi_j^\infty(P)]^{-1/2}\varphi_j^\infty(P) = [\det F_j^\infty]^{-1}[(F_j^\infty)^{11} + \sqrt{\lambda-\lambda_j}(F_j^\infty)^{12} + O(\lambda-\lambda_j)],$$
$$[\det \Phi_j^\infty(P)]^{-1/2}\psi_j^\infty(P) = [\det F_j^\infty]^{-1}[(F_j^\infty)^{21} + \sqrt{\lambda-\lambda_j}(F_j^\infty)^{22} + O(\lambda-\lambda_j)].$$

We conclude that the matrices G_j, from the asymptotic expansions (2.12) of the function $\Psi(Q)$ at the points λ_j, are given by

$$G_j = (\det F_j^\infty)^{-1} F_j^\infty, \tag{3.42}$$

which proves equation (3.36). ■

Remark 3.2. The matrices F_j^∞ from Theorem 3.3 are related to the coefficients F_j of the Taylor series (3.16) of function $\Phi(P)$ at the points λ_j as follows:

$$F_j^\infty = \Phi^{-1}(\infty^1) F_j.$$

Therefore, using equation (3.42), we get the following relation between the matrices F_j from the Taylor series (3.16) of function $\Phi(P)$, and the matrices G_j from the Laurent series (2.12) of function $\Psi(Q)$:

$$F_k^{-1} F_j \sigma_3 F_j^{-1} F_k = G_k^{-1} G_j \sigma_3 G_j^{-1} G_k, \tag{3.43}$$

for any j and k.

4 Tau function for the Schlesinger system

Here we calculate the τ-function which corresponds to the solution (3.36), (3.37), (3.38) of the Schlesinger system. The remainder is devoted to the proof of the following main theorem.

Theorem 4.1. The τ-function corresponding to the solution (3.36), (3.37), (3.38) of the Schlesinger system (with arbitrary $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ such that $[\mathbf{p}, \mathbf{q}]$ is not a half-integer characteristic) is given by

$$\tau = \Theta[\mathbf{p}, \mathbf{q}](0)(\det \mathcal{A})^{-1/2} \prod_{j<k} (\lambda_j - \lambda_k)^{-1/8}, \tag{4.1}$$

where the $g \times g$ matrix $\mathcal{A}$ of a-periods of holomorphic 1-forms on $\mathcal{L}$ is defined by equation (3.3). □

Proof. According to the definition of the τ-function (2.32), (2.30), let us first calculate $(1/2)\mathrm{tr}(\Psi_\lambda \Psi^{-1})^2$ for the function Ψ given by equation (3.23). We have

$$\frac{1}{2}\mathrm{tr}(\Psi_\lambda \Psi^{-1})^2 \equiv -\det(\Psi_\lambda \Psi^{-1}) = -\frac{\det(\Phi_\lambda)}{\det \Phi} + \frac{1}{4}\left(\frac{(\det \Phi)_\lambda}{\det \Phi}\right)^2. \tag{4.2}$$

Together with the function Ψ, the function $\det(\Psi_\lambda \Psi^{-1})$ is independent of P_φ and P_ψ; moreover, function Ψ does not undergo any modification if we multiply $\psi(P)$ with an arbitrary λ-independent factor C_ψ. So, we can choose the parameters P_φ, P_ψ, and C_ψ at our disposal to simplify the calculation. Our choice will be the following: first we put $C_\psi = \lambda_\psi - \lambda_\varphi$ (λ_φ denotes the projection of the point P_φ in the λ-plane), and then take the limit $P_\psi \to P_\varphi$. We get

$$\psi(P) = \varphi(P) + \frac{\partial \varphi(P)}{\partial \lambda_\varphi}. \tag{4.3}$$

Since the function $\Psi(P)$ is independent of the remaining parameter P_φ, we can calculate $\det(\Psi_\lambda \Psi^{-1})$ assuming $P_\varphi = P$. Intermediate results of this calculation are as follows:

$$\frac{(\det \Phi)_\lambda}{\det \Phi} = 2\frac{\partial}{\partial \lambda}\left\{\ln \Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(P))\right\},$$

and

$$\begin{aligned}\frac{\det(\Phi)_\lambda}{\det \Phi} &= \frac{1}{\Theta[\mathbf{p}, \mathbf{q}](0)} \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \{\Theta[\mathbf{p}, \mathbf{q}](U(P) - U(P_\varphi))\}_{P_\varphi = P} \\ &\quad + \frac{1}{\Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(P))} \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \left\{\Theta[\mathbf{p}^S, \mathbf{q}^S]((-U(P) - U(P_\varphi))\right\}_{P_\varphi = P};\end{aligned}$$

therefore,

$$\frac{1}{2}\mathrm{tr}(\Psi_\lambda\Psi^{-1})^2(\lambda) = -\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\left\{\ln\Theta[\mathbf{p}^S,\mathbf{q}^S]((-U(P)-U(P_\varphi))\right\}_{P_\varphi=P}$$
$$-\frac{1}{\Theta[\mathbf{p},\mathbf{q}](0)}\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\{\Theta[\mathbf{p},\mathbf{q}](-U(P)+U(P_\varphi))\}_{P_\varphi=P}\,. \tag{4.4}$$

To find the asymptotic behaviour of this expression as $\lambda\to\lambda_j$, we shall use the well-known asymptotic behaviour which is valid for any odd theta-characteristic $[\mathbf{p}^S,\mathbf{q}^S]$:

$$\frac{\partial^2}{\partial x(P_1)\partial x(P_2)}\left\{\ln\Theta[\mathbf{p}^S,\mathbf{q}^S](U(P_1)-U(P_2))\right\} = \frac{1}{(x(P_1)-x(P_2))^2}+F(P)+o(1) \tag{4.5}$$

as $P_1, P_2\to P$, where x is a local parameter in the neighborhood of P. The function $F(P)$ is independent of the choice of the set S; it is given by the expression (see [5, p. 20])

$$F(P) \equiv \frac{1}{6}\{\lambda,x\}(P)+\frac{1}{16}\left(\frac{d}{dx}\ln\prod_{k=1}^{g+1}\frac{\lambda-\lambda_{i_k}}{\lambda-\lambda_{j_k}}\right)^2(P)$$
$$-\sum_{i,j=1}^{g}\frac{\partial^2}{\partial z_i\partial z_j}\Theta[\mathbf{p}^T,\mathbf{q}^T](0)\frac{dU_i}{dx}(P)\frac{dU_j}{dx}(P), \tag{4.6}$$

where $\{\lambda,x\}$ denotes the Schwarzian derivative of λ with respect to x,

$$\frac{\lambda'''}{\lambda'}-\frac{3}{2}\left(\frac{\lambda''}{\lambda'}\right)^2\,,$$

and $[\mathbf{p}^T,\mathbf{q}^T]$ is an even characteristic corresponding to an arbitrary set $T\equiv\{\lambda_{i_1},\dots,\lambda_{i_{g+1}}\}$ of $g+1$ branch points via equation (3.11). The remaining $g+1$ branch points are denoted by $\lambda_{j_1},\dots,\lambda_{j_{g+1}}$. Expression (4.6) is independent of the choice of the set T.

Applying equation (4.6) for $P=\lambda_j$, we get the following asymptotic behaviour:

$$\frac{1}{2}\mathrm{tr}(\Psi_\lambda\Psi^{-1})^2(\lambda) \underset{\lambda\to\lambda_j}{=} \frac{1}{16(\lambda-\lambda_j)^2}+\frac{H_j}{\lambda-\lambda_j}+O(1), \tag{4.7}$$

where

$$H_j = \frac{1}{8}\sum_{k\neq j}\frac{n_j n_k}{\lambda_j-\lambda_k}-\frac{1}{4\Theta[\mathbf{p}^T,\mathbf{q}^T](0)}\sum_{l,k=1}^{g}\frac{\partial^2\Theta[\mathbf{p}^T,\mathbf{q}^T]}{\partial z_l\partial z_k}(0)\frac{dU_l}{dx_j}(\lambda_j)\frac{dU_k}{dx_j}(\lambda_j)$$
$$+\frac{1}{4\Theta[\mathbf{p},\mathbf{q}](0)}\sum_{l,k=1}^{g}\frac{\partial^2\Theta[\mathbf{p},\mathbf{q}]}{\partial z_l\partial z_k}(0)\frac{dU_l}{dx_j}(\lambda_j)\frac{dU_k}{dx_j}(\lambda_j), \tag{4.8}$$

and $x_j\equiv\sqrt{\lambda-\lambda_j}$; $n_k=1$ for $\lambda_k\in T$ and $n_k=-1$ for $\lambda_k\notin T$. Now, to integrate equations (2.32), we have to use the heat equations

$$\frac{\partial^2\Theta[\mathbf{p},\mathbf{q}](\mathbf{z}\mid\mathbf{B})}{\partial z_l\partial z_k} = 4\pi i\frac{\partial\Theta[\mathbf{p},\mathbf{q}](\mathbf{z}\mid\mathbf{B})}{\partial\mathbf{B}_{lk}} \tag{4.9}$$

which are valid for theta functions with arbitrary characteristic $[\mathbf{p},\mathbf{q}]$, and the following lemma.

Lemma 4.1. The dependence of the matrix of b-periods on the branch points is described by the equations

$$\frac{\partial \mathbf{B}_{kl}}{\partial \lambda_j} = \pi i \frac{dU_l}{dx_j}(\lambda_j)\frac{dU_k}{dx_j}(\lambda_j). \tag{4.10}$$

□

Proof. The dependence of the nonnormalized 1-forms dU_k^0 (3.2) on λ_j is

$$\frac{\partial}{\partial \lambda_j}\{dU_k^0(\lambda)\} = \frac{1}{2(\lambda-\lambda_j)}dU_k^0(\lambda).$$

Now, calculation of the integral

$$\oint_{\partial\hat{\mathcal{L}}} U_l^0(\lambda)\frac{\partial}{\partial\lambda_j}dU_k^0(\lambda) = \oint_{\partial\hat{\mathcal{L}}}\frac{1}{2(\lambda-\lambda_j)}U_l^0(\lambda)dU_k^0(\lambda)$$

by means of the residue theorem gives the following result:

$$\pi i\frac{dU_l^0}{dx_j}(\lambda_j)\frac{dU_k^0}{dx_j}(\lambda_j) \equiv \pi i\left[\mathcal{A}\frac{dU_l}{dx_j}(\lambda_j)\frac{dU_k}{dx_j}(\lambda_j)\mathcal{A}^t\right]_{kl}.$$

On the other hand, standard arguments used, for example, in the proof of the Riemann bilinear identities (see [7]), show that the same integral equals

$$\sum_{m=1}^{g}\mathcal{A}_{lm}\frac{\partial\mathcal{B}_{km}}{\partial\lambda_j} - \frac{\partial\mathcal{A}_{km}}{\partial\lambda_j}\mathcal{B}_{lm};$$

therefore,

$$\frac{\partial\mathcal{B}}{\partial\lambda_j}\mathcal{A}^t - \frac{\partial\mathcal{A}}{\partial\lambda_j}\mathcal{B}^t = \pi i\mathcal{A}\frac{dU_l}{dx_j}(\lambda_j)\frac{dU_k}{dx_j}(\lambda_j)\mathcal{A}^t,$$

which leads to the statement of the lemma after taking into account the symmetry of the matrix $\mathbf{B} \equiv \mathcal{A}^{-1}\mathcal{B}$. ■

Now, using equations (4.8), (4.9), and (4.10), we can rewrite the Hamiltonians H_j as follows:

$$H_j \equiv \frac{\partial}{\partial\lambda_j}\ln\tau = \frac{1}{8}\sum_{k\neq j}\frac{n_j n_k}{\lambda_j-\lambda_k} + \frac{\partial}{\partial\lambda_j}\ln\left\{\frac{\Theta[\mathbf{p},\mathbf{q}](0)}{\Theta[\mathbf{p}^T,\mathbf{q}^T](0)}\right\}.$$

Finally, applying the classical Thomae formula (see [19], [15])

$$\Theta^4[\mathbf{p}^T,\mathbf{q}^T](0) = \pm\frac{(\det\mathcal{A})^2}{(2\pi i)^{2g}}\prod_{l<k,\ l,k=1}^{g+1}(\lambda_{i_l}-\lambda_{i_k})\prod_{l<k\ l,k=1}^{g+1}(\lambda_{j_l}-\lambda_{j_k}),$$

we get the τ-function in the form (4.1) up to multiplication by an arbitrary $\{\lambda_j\}$-independent constant of integration. The ambiguity in the choice of this constant allows, in particular, to arbitrarily choose the branch cuts in the formula (4.1).

5 The elliptic case and the sixth Painlevé equation

In this section, we show how the solution of the sixth Painlevé equation in terms of elliptic functions can be derived from the results of the previous sections.

Put $g = 1$. Then the equation of the curve $\mathcal{L}$ is given by

$$w^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4). \tag{5.1}$$

The matrix of b-periods, $\mathbf{B}$, turns into the module σ, and $\Theta[\mathbf{p}^S, \mathbf{q}^S]$ becomes the Jacobi theta-function ϑ_1; to shorten all the formulas, we shall denote $\Theta[\mathbf{p}, \mathbf{q}]$ by $\vartheta_{p,q}$.

Parameters m_j of the monodromy matrices are, according to (3.24), given by

$$m_1 = i, \quad m_2 = -ie^{-2\pi ip}, \quad m_3 = ie^{2\pi i(q-p)}, \quad m_4 = -ie^{2\pi iq}.$$

The formulas (3.13) and (3.14) now read as follows:

$$\varphi(P) = \vartheta_{p,q}(U(P) + u_\varphi)\vartheta_1(U(P) - u_\varphi), \tag{5.2}$$

$$\psi(P) = c_\psi \vartheta_{p,q}(U(P) + u_\psi)\vartheta_1(U(P) - u_\psi), \tag{5.3}$$

where $u_{\varphi,\psi} \equiv U(P_{\varphi,\psi}) \in \mathbb{C}$ are arbitrary parameters, and, in analogy to the previous section, we introduced an arbitrary multiplier $c_\psi(\{\lambda_j\})$ which obviously does not influence the function $\Psi(\lambda)$.

Again, since the function $\Psi(\lambda)$ does not depend on c_ψ, u_φ, and u_ψ, we can freely fix these parameters to simplify our calculations. First, it is convenient to put $u_\varphi = 0$ (i.e., $P_\varphi = \lambda_1$), which leads to

$$\varphi(P) = \vartheta_{p,q}(U(P))\vartheta_1(U(P)). \tag{5.4}$$

The most convenient choice for the parameters of the function ψ is the following: We put $c_\psi = u_\psi^{-1}$ and take the limit $u_\psi \to 0$. Then we get

$$\psi(P) = \varphi(P) + \frac{\partial \varphi(P)}{\partial u_\varphi}(u_\varphi = 0), \tag{5.5}$$

and the components of matrices F_j from equation (3.16) are given by

$$F_j^{11} = \vartheta_{p,q}(u_j)\vartheta_1(u_j),$$

$$F_j^{12} = f_j\{\vartheta'_{p,q}(u_j)\vartheta_1(u_j) + \vartheta_{p,q}(u_j)\vartheta'_1(u_j)\},$$

$$F_j^{21} = F_j^{11} + \vartheta'_{p,q}(u_j)\vartheta_1(u_j) - \vartheta_{p,q}(u_j)\vartheta'_1(u_j),$$

$$F_j^{22} = F_j^{12} + f_j\{\vartheta''_{p,q}(u_j)\vartheta_1(u_j) - \vartheta_{p,q}(u_j)\vartheta''_1(u_j)\}. \tag{5.6}$$

Here

$$f_j \equiv \left\{ \prod_{l\neq j}(\lambda_j - \lambda_l)^{1/2} \oint_a \frac{d\lambda}{\sqrt{(\lambda-\lambda_1)\dots(\lambda-\lambda_4)}} \right\}^{-1}, \tag{5.7}$$

and

$$u_1 = 0, \quad u_2 = \frac{1}{2}, \quad u_3 = \frac{1}{2} + \frac{\sigma}{2}, \quad u_4 = \frac{\sigma}{2}. \tag{5.8}$$

In particular, for $j = 1$ we have

$$F_1^{11} = 0, \qquad F_1^{21} = \vartheta_{p,q}(0)\vartheta'_1(0), \qquad F_1^{12} = F_1^{22} = f_1 F_1^{21}. \tag{5.9}$$

In accordance with equations (3.43), (2.23), to obtain the solution of the sixth Painlevé equation, we have to calculate the (12) elements of the matrices

$$\hat{A}_j = \frac{1}{4} F_1^{-1} F_j \sigma_3 F_j^{-1} F_1, \qquad j = 2, 3, 4 \tag{5.10}$$

(obviously $\hat{A}_1 = I$). Substitution of the matrix elements (5.6) into equation (5.10) leads to the following result:

$$\hat{A}_j^{12} = -f_1 \frac{((\ln\vartheta_{p,q})' - (\ln\vartheta_1)')(\vartheta''_{p,q}/\vartheta_{p,q} - \vartheta''_1/\vartheta_1)}{(\ln\vartheta_{p,q})'' - (\ln\vartheta_1)''}(z = u_j), \tag{5.11}$$

where ϑ' denotes $\partial\vartheta(z \mid \sigma)/\partial z$. Finally, choosing $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, and $\lambda_4 = t$, and making use of the "heat" equation for the theta-function,

$$\frac{\partial\vartheta_{p,q}(z,\sigma)}{\partial\sigma} = \frac{1}{4\pi i}\frac{\partial^2\vartheta_{p,q}(z,\sigma)}{\partial z^2},$$

we get, according to equation (2.23), the following theorem.

Theorem 5.1. The function

$$y = -\frac{t}{1 + (1-t)y_1}, \tag{5.12}$$

where t is the cross-ratio of the points $\{\lambda_j\}$ given by equation (2.22), and

$$y_1 = \frac{\frac{\partial}{\partial z} \ln \frac{\partial}{\partial z} \ln\{\vartheta_{p,q}/\vartheta_1\}\left(\frac{1}{2}\right) \frac{\partial}{\partial \sigma} \ln\{\vartheta_{p,q}/\vartheta_1\}\left(\frac{\sigma}{2}\right)}{\frac{\partial}{\partial z} \ln \frac{\partial}{\partial z} \ln\{\vartheta_{p,q}/\vartheta_1\}\left(\frac{\sigma}{2}\right) \frac{\partial}{\partial \sigma} \ln\{\vartheta_{p,q}/\vartheta_1\}\left(\frac{1}{2}\right)}\,, \tag{5.13}$$

where $p, q \in \mathbb{C}$ are arbitrary constants such that $[p, q] \neq [1/2, 0]$ and $[p, q] \neq [0, 1/2]$, solves the sixth Painlevé equation (1.3), with coefficients (1.7). Here the module σ of elliptic curve $\mathcal{L}$ is chosen such that $t = \theta_4^4(0 \mid \sigma)/\theta_2^4(0 \mid \sigma)$. □

Expression (5.13) is a combination of derivatives of the function $\ln(\vartheta_{p,q}/\vartheta_1)$ with respect to both arguments of the theta functions.

One more representation for solution (5.12) of the sixth Painlevé equation may be obtained by using the following relation between $y(t)$ and the τ-function, $\tau(t)$, valid for $t_j = 1/2$:

$$y(t) = t - t(t-1)\left[D\left(\frac{\frac{d}{dt}D(\tau)}{\frac{d}{dt}D\left(\sqrt[8]{t(t-1)}\tau\right)}\right) + \frac{t(t-1)}{D^2\left(\sqrt[8]{t(t-1)}\tau\right)}\right]^{-1}, \tag{5.14}$$

where operator D acts on functions $f(t)$ as follows: $D(f) \equiv (d/dt)\ln f$. The τ-function for the $g = 1$ case can be obtained from the general formula (4.1) simply by assuming that $\lambda_1, \ldots, \lambda_4$ coincide with $0, 1, t$, and ∞, respectively. Then, up to an arbitrary constant, we get

$$\tau(t) = \frac{\theta_{p,q}(0 \mid \sigma)}{\sqrt[8]{t(t-1)}}\left[\int_0^1 \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}}\right]^{-(1/2)}.$$

Remark 5.1. It seems that it is not easy to check directly (by applying appropriate identities for the theta functions) the coincidence of the different forms of the same solution (5.13), (5.14). It is also not easy to check directly the coincidence of our formulas to other forms of this solution given by Okamoto (A.6) and Hitchin (A.7). However, we can explicitly see the relationship of our construction to the construction by Hitchin on the level of the functions φ and ψ from Theorem 3.1; namely, the choice of the rows of the function Φ made in [6] corresponds to the choice $u_\varphi \equiv -(1/2)(p\sigma + q) + (\sigma + 1)/4$. The variable c from [6] is given by $-u_\varphi w_1$, where w_1 is the first full elliptic integral on $\mathcal{L}$. The parameter u_ψ is fixed in [6] to coincide with one of the zeros of the Weierstrass $\wp$-function, $\wp[w_1(U(P)+u_\varphi)]$, with the periods w_1 and $w_2 = w_1\sigma$. Constants c_1 and c_2 from [6] are related to our p and q as follows: $c_1 = p + 1/2$, $c_2 = q + 1/2$.

Remark 5.2. Here we discuss only the generic two-parametric family of elliptic solutions of the sixth Painlevé equation with coefficients (1.7), which corresponds to monodromy matrices (2.19). An additional one-parametric family of solutions corresponding to monodromy matrices (2.20) was constructed in [6].

Appendix: Elliptic solutions of the sixth Painlevé equation

In his studies of the Painlevé equations, K. Okamoto has shown in [16] that the function $y = y(t)$, the general solution of the sixth Painlevé equation, (1.3), can be explicitly written in terms of the elliptic functions, provided that the set of the parameters satisfies one of the following conditions:

$$t_i \in \mathbb{Z}, \qquad t_1 + \ldots + t_4 \in 2\mathbb{Z}, \tag{A.1}$$

or

$$t_i + \frac{1}{2} \in \mathbb{Z}. \tag{A.2}$$

The major ingredients of Okamoto's construction are:

(1) the so-called Picard solution,

$$y_0(t) = \tilde{\wp}(c_1\omega_1(t) + c_2\omega_2(t)), \tag{A.3}$$

of equation (1.3) with the coefficients

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \frac{1}{2}. \tag{A.4}$$

In equation (A.3), $\tilde{\wp}(\cdot)$ is the elliptic function satisfying the equation $\tilde{\wp}'^2 = 4\tilde{\wp}(\tilde{\wp} - 1)(\tilde{\wp} - t)$, with the primitive periods $2\omega_1(t)$ and $2\omega_2(t)$; c_1, $c_2 \in \mathbb{C}$ are the constants of integration, so that the function $y(t)$ is the general solution;

(2) the subgroup of transformations of solutions of equation (1.3) which acts on the space of coefficients $\{t_j\}$ as: (a) reflections: for any $j = 1, \ldots, 4$ there is a transformation which transforms $t_j \to -t_j$ and leaves all t_k for $k \neq j$ unchanged; (b) permutations of the set $\{t_j\}$; (c) the shifts: $t_j \mapsto t_j + n_j$, where $\sum_{j=1}^4 n_j = 0(\mathrm{mod}\ 2)$;

(3) more nontrivial transformation,

$$\mathbf{O}\colon (t_1, t_2, t_3, t_4) \leftrightarrow \left(\frac{t_1 + t_2 - t_3 - t_4}{2}, \frac{t_1 + t_2 + t_3 + t_4}{2}, \right. \\ \left. \frac{-t_1 + t_2 + t_3 - t_4}{2}, \frac{-t_1 + t_2 - t_3 + t_4}{2}\right). \tag{A.5}$$

It is important to mention that all the transformations described above, as well as their inversions, are given by explicit formulas, so that "new" solutions can be explicitly written in terms of the "old" ones as rational functions of the "old" solution and its derivative (see [16]). In particular, the solution of equation (1.3) with the coefficients (1.7) obtained via Okamoto's transformations reads

$$y(t) = y_0 + \frac{y_0^2(y_0 - 1)(y_0 - t)}{t(t-1)y_0' - y_0(y_0 - 1)}, \tag{A.6}$$

where $y_0 = y_0(t)$ is given by equation (A.3).

N. Hitchin, in the work [6] devoted to the study of SU(2)-invariant anti-self-dual Einstein metrics, rediscovered the case (1.7) of integrability of equation (1.3) in elliptic functions. He got the following representation for the solution (A.6) in the parametric form:

$$\begin{aligned} y_1(\sigma) &= \frac{\theta_1'''(0)}{3\pi^2\theta_4^4(0)\theta_1'(0)} + \frac{1}{3}\left(1 + \frac{\theta_3^4(0)}{\theta_4^4(0)}\right) \\ &\quad + \frac{\theta_1'''(\nu)\theta_1(\nu) - 2\theta_1''(\nu)\theta_1'(\nu) + 2\pi i c_1(\theta_1''(\nu)\theta_1(\nu) - \theta_1'^{\,2}(\nu))}{2\pi^2\theta_4^4(0)\theta_1(\nu)(\theta_1'(\nu) + \pi i c_1\theta_1(\nu))}, \\ t(\sigma) &= \frac{\theta_3^4(0)}{\theta_4^4(0)}, \qquad \nu = c_1\sigma + c_2, \end{aligned} \tag{A.7}$$

where $\theta_k(\cdot) = \theta(\cdot \mid \sigma),\ k = 1, \dots, 4$, are the Jacobi theta functions (see [20]).

Yu. I. Manin [14] noticed that the well-known uniformization of equation (1.3) in terms of the Weierstrass $\wp$-function can be further converted into the beautiful form:

$$\begin{aligned} &y(\sigma) = \frac{\wp(z(\sigma),\sigma) - e_1(\sigma)}{e_2(\sigma) - e_1(\sigma)}, && t(\sigma) = \frac{e_3(\sigma) - e_1(\sigma)}{e_2(\sigma) - e_1(\sigma)}, \\ &e_j(\sigma) = \wp\left(\frac{1}{2}T_j, \sigma\right), && (T_1, T_2, T_3, T_4) \equiv (0, 1, \sigma, 1+\sigma), \\ &\frac{d^2 z}{d\sigma^2} = \frac{1}{(2\pi i)^2}\sum_{j=1}^{4}\alpha_j\wp'\left(z + \frac{T_j}{2}, \sigma\right), && (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv \left(\alpha, -\beta, \gamma, \frac{1}{2} - \delta\right), \end{aligned} \tag{A.8}$$

where $\wp(\cdot, \sigma)$ is the Weierstrass elliptic function with the primitive periods 2 and 2σ; $\wp'(\cdot, \sigma)$ denotes the partial derivative of the $\wp$-function with respect to its first argument. By applying to equation (A.8) the Landin transform for the Weierstrass elliptic functions, Manin found a new transformation for solutions of equation (1.3). In terms of the Manin variables, z and σ, this transformation reads: if $z(\sigma)$ is any solution of equation (A.8) with the coefficients $\alpha_1 = \alpha_3,\ \alpha_2 = \alpha_4$, then $z(2\sigma)$ is the solution of equation (A.8) for $\alpha_1^{new} = 4\alpha_1,\ \alpha_2^{new} = 4\alpha_2,\ \alpha_3^{new} = \alpha_4^{new} = 0$. The converse statement is, of course, also true. Schematically, for the constants, t_j (1.4), we can write

$$\mathbf{M}\colon \quad (t_1,\ t_2,\ t_3 = t_1 - 1,\ t_4 = t_2) \leftrightarrow (2t_1 - 1,\ 2t_2,\ 0, 0). \tag{A.9}$$

In the case (A.4), the Manin form of the sixth Painlevé equation (A.8) immediately reproduces the Picard solution (A.3). In terms of the parameters t_j, equations (A.4) read: $t_1 = 1$, $t_2 = 0$, $t_3 = 0$, and $t_4 = 0$. After the permutation, we get the set $t_1 = 0$, $t_2 = 1$, $t_3 = 0$, and $t_4 = 0$, therefore, by setting the formal monodromies $t_1 = 1/2$, $t_2 = -(1/2)$ in the right-hand side of (A.9); and, choosing the left arrow in the Manin transformation **M**, one finds the second basic case of the integrability (1.7). The corresponding explicit formula can be written as the composition of the transformation corresponding to the permutation (see [16]) and **M**.

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