Canonical Quantization of Cylindrical Gravitational Waves with Two Polarizations

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The canonical quantization of the minisuperspace model describing cylindrically symmetric gravitational waves with two polarizations is presented. A Fock space-type representation is constructed. It is based on a complete set of quantum observables. Physical expectation values may be calculated in arbitrary excitations of the vacuum. Our approach provides a nonlinear generalization of the quantization of the collinearly polarized Einstein-Rosen gravitational waves. [S0031-9007(97)04861-8]

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The quantization of dimensionally reduced models of 4D Einstein gravity serves as interesting testing ground for many issues of quantum gravity. The physical output of this approach to an understanding of characteristic features of the full theory, however, strongly depends on the complexity of the model under consideration.

Probably, the simplest and best understood examples are the minisuperspace models [1] which contain only a finite number of physical degrees of freedom and thus hide the field effects of quantum gravity. A more complicated example of steady interest is given by the minisuperspace model of cylindrically symmetric gravitational waves with one polarization [2,3]. This model already involves an infinite number of degrees of freedom. It becomes treatable with the methods of flat space quantum field theory, because the Einstein field equations essentially reduce to the axisymmetric 3D wave equation. This underlying linearity, on the other hand, may conceal typical nonlinear features of quantum gravity.

It is the purpose of this Letter to generalize the results of Refs. [2,3] to cylindrical gravitational waves with two polarizations, where the Einstein equations become nonlinear. We achieve a consistent canonical quantization in terms of a complete set of quantum observables. Creation and annihilation operators are constructed in a type of Fock space representation of these observables. The full space of physical quantum states is then built from excitations of the vacuum. This allows one to calculate, in principle, all physical expectation values in arbitrary quantum states.

We start from a general space-time with cylindrical symmetry, i.e., we assume the existence of two commuting Killing vector fields, one of which has closed orbits. We choose coordinates such that the Killing vector fields are given by ∂_z and ∂_φ associated with the axis of symmetry z and the azimuthal angle φ, respectively. Further gauge fixing brings the metric into the standard form [4],

$$ds^2 = e^{\Gamma(\rho,\tau)}(-d\tau^2 + d\rho^2) + \rho\tilde{g}_{ab}(\rho,\tau)dx^a dx^b, \quad a,b = 2,3, \tag{1}$$

with $x^2 \equiv z$, $x^3 \equiv \varphi$, radial coordinate ρ, and time τ. The symmetric 2×2 matrix $\tilde{g}$ is restricted by the condition $\det \tilde{g} = 1$.

The essential part of the Einstein field equations is given by the Ernst equation for the matrix $\tilde{g}$ or, equivalently, for the 2×2 matrix g which carries the dualized potentials of the Ernst picture [5];

$$\partial_\rho(\rho g^{-1}\partial_\rho g) - \partial_\tau(\rho g^{-1}\partial_\tau g) = 0\,. \tag{2}$$

The conformal factor $\Gamma(\rho,\tau)$ is then a functional of g. At spatial infinity, $\rho = \infty$ it is given by

$$\Gamma_\infty = \frac{1}{2}\int_0^\infty \rho\, d\rho\, \mathrm{tr}[(g^{-1}\partial_\rho g)^2 + (g^{-1}\partial_\tau g)^2]\,. \tag{3}$$

This factor generates evolution with respect to the time coordinate τ; its exponential measures the total energy per unit length in the z direction H^t and the deficit angle in the asymptotic region φ_0,

$$H^t = \frac{1}{\pi G}\varphi_0 = \frac{2}{G}(1 - e^{-\Gamma_\infty/2})\,. \tag{4}$$

The reduction of the metric to the form (1) can be performed within the canonical formalism, such that the Poisson bracket of the reduced model is the Dirac bracket of the original structure after appropriate gauge fixing [3]. The resulting canonical Poisson structure is easily extracted from the effectively two-dimensional Lagrangian density $\mathcal{L}^{(2)}$ that comes from reduction via Killing symmetries and gauge fixing of the original Lagrangian $\mathcal{L}_{EH}^{(4)} = (1/G)\sqrt{|g_{\mu\nu}|}R^{(4)}$:

$$\mathcal{L}^{(2)}(\rho,\tau) = \frac{1}{2G}\rho\,\mathrm{tr}[(g^{-1}\partial_\rho g)^2 - (g^{-1}\partial_\tau g)^2]\,.$$

In matrix components g_{ab}, the Poisson brackets read

$$\{g_{ab}(\rho), (g^{-1}\partial_\tau g g^{-1})_{cd}(\rho')\} = \frac{G}{\rho}\delta_{ad}\delta_{bc}\delta(\rho - \rho')\,.$$

The restrictions of symmetry and the unit determinant of g require some additional technical effort and have been taken into account in the derivation of the following results.

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Collinear polarizations.—Among the simplest nontrivial metrics of this model are the collinearly polarized gravitational waves discovered by Einstein and Rosen [6]. They correspond to a diagonal form of the matrix $g \equiv \mathrm{diag}(e^{\phi}, e^{-\phi})$, i.e., the number of degrees of freedom reduces to one. The Ernst equation (2) in this case reduces to the cylindrical wave equation,

$$-\partial_\tau^2 \phi + \rho^{-1}\partial_\rho \phi + \partial_\rho^2 \phi = 0\,,$$

with the general solution,

$$\phi(\rho,\tau) = \int_0^\infty d\lambda [A_+(\lambda) J_0(\lambda\rho) e^{i\lambda\tau} + A_-(\lambda) J_0(\lambda\rho) e^{-i\lambda\tau}]\,,$$

where J_0 denotes Bessel functions of the first kind. The coefficients $A_+ = \overline{A_-}$ build a complete set of observables with canonical Poisson brackets,

$$\{A_+(\lambda), A_-(\lambda')\} = G\delta(\lambda - \lambda')\,. \tag{5}$$

Thus, quantization of this structure is straightforward [3] and gives rise to a representation in terms of creation and annihilation operators,

$$A_-|0\rangle = 0 \quad \text{with } A_+ = A_-^\dagger\,. \tag{6}$$

In particular, coherent quantum states may be constructed in the same way as in flat space quantum field theory. Recent discussion, however, has shown that these states do not provide coherence of all essential physical quantities [7].

As the first step towards the general case, we cast the truncated model of collinear polarization into a form that will allow proper generalization. We introduce new variables,

$$T_\pm(w) \equiv \exp \int_0^\infty d\lambda A_\pm(\lambda) e^{\pm iw\lambda}, \tag{7}$$

which build an equivalent complete set of observables. In the Fock space representation (6), $T_-(w)$ is represented as identity, whereas $T_+(w)$ generates the coherent state associated with a classical field that, on the symmetry axis $\rho = 0$, is peaked as a δ function at $\tau_0 = w$. In terms of these new variables, the Poisson structure (5) becomes

$$\{T_-(v),\, T_+(w)\} = -\frac{G}{v - w}\, T_-(v) T_+(w)\,. \tag{8}$$

We shall see in the sequel that it is this quadratic form of Poisson brackets which generically appears in the case of two polarizations. Linearization to (5) is a special feature of the truncated model but not possible in the general case.

Two polarizations.—In general, the Ernst equation (2) does not admit explicit solution. However, it is possible to construct the analog of the quantities $T_\pm$ defined above. Inspired by the auxiliary linear system associated with the Ernst equation [8], we define

$$T_\pm(w,\tau) \equiv \mathcal{P} \exp \int_0^\infty d\rho 2\left(\frac{\gamma^2 g^{-1}\partial_\rho g}{1 - \gamma^2} - \frac{\gamma g^{-1}\partial_\tau g}{1 - \gamma^2}\right), \quad \text{for } w \in H_\pm\,, \tag{9}$$

where $H_\pm$ denote the upper and lower half of the complex plane, respectively and $\gamma \equiv -(1/\rho)[w - \tau + \sqrt{(w - \tau)^2 - \rho^2}]$. For diagonal g, this definition indeed reduces to (7) above. The variables $T_\pm$ are still constants of motion, i.e.,

$$\partial_\tau T_\pm(w,\tau) = 0\,. \tag{10}$$

They turn out to be holomorphic in $H_\pm$, respectively, and generically do not coincide on the real w-axis. Definition (9) further implies $\det T_\pm = 1$ and $T_+(w) = \overline{T_-(\overline{w})}$.

Interestingly, it may be shown that the matrix product $M = T_+ T_-^t$ on the real axis has a well-defined physical meaning, namely, it coincides with the values of the original matrix g on the symmetry axis:

$$M(w \in \mathbb{R}) \equiv T_+(w) T_-^t(w) = g(\rho = 0, \tau = w)\,, \tag{11}$$

In particular, it is symmetric and real:

$$M(w) = M^t(w) \quad \text{and} \quad M(w) = \overline{M(w)}\,. \tag{12}$$

Since the $T_\pm$ contain the initial values of the metric and the Ernst potential on the symmetry axis $\rho = 0$, they contain sufficient information to recover g everywhere by means of (2) [note that $\partial_\rho g(\rho = 0) = 0$ for solutions which are regular on the symmetry axis]. Thus, the set of $T_\pm(w)$ builds a *complete* set of observables for the Ernst equation.

Continuing the program of canonical quantization, we next calculate the Poisson algebra to subsequently quantize it. A direct but lengthy calculations reveals a *quadratic* Poisson algebra for the matrix entries $T_\pm^{ab}(w)$:

$$\{T_\pm^{ab}(v), T_\pm^{cd}(w)\} = \frac{G}{v - w}[T_\pm^{ad}(v) T_\pm^{cb}(w) - T_\pm^{cb}(v) T_\pm^{ad}(w)]\,, \tag{13}$$

$$\{T_-^{ab}(v), T_+^{cd}(w)\} = \frac{G}{v - w}[T_-^{ab}(v) T_+^{cd}(w) - T_-^{cb}(v) T_+^{ad}(w) - \delta^{bd} T_-^{am}(v) T_+^{cm}(w)]\,. \tag{14}$$

which consistently encloses the scalar algebra (8) in the components $T_\pm^{11}(w)$. Quantization of this quadratic structure is rather more subtle than that of a linear algebra, since *a priori* there appear ambiguities on the right-hand side (rhs.) due to different orderings of the quadratic expressions. Fortunately, the proper quantum analog of the Poisson brackets (13) is known in the theory of integrable systems [9] as the so-called SL(2)-Yangian algebra,

$$[T_\pm^{ab}(v), T_\pm^{cd}(w)] = \frac{i\hbar G}{v - w}\,[T_\pm^{cb}(w)T_\pm^{ad}(v) - T_\pm^{cb}(v)T_\pm^{ad}(w)]. \tag{15}$$

The problem is to translate the Poisson brackets (14) and the symmetry relation (12) into the quantum picture since both involve nonlinear expressions in the fields $T_\pm$. Their consistent quantization is uniquely given by the following set of mixed relations:

$$\begin{aligned}[T_-^{ab}(v), T_+^{cd}(w)] = {} & \frac{i\hbar G}{v - w + i\hbar G}\,T_+^{cd}(w)T_-^{ab}(v) \\ & - \frac{i\hbar G(v - w)}{(v - w + i\hbar G)(v - w - i\hbar G)}\,[T_+^{ad}(w)T_-^{cb}(v) + \delta^{bd}T_+^{cm}(w)T^{am}(v)] \\ & + \frac{(i\hbar G)^2}{(v - w + i\hbar G)(v - w - i\hbar G)}\,\delta^{bd}[T_+^{am}(w)T_-^{cm}(v) - T_+^{cm}(w)T_-^{am}(v)],\end{aligned} \tag{16}$$

and the symmetry condition

$$T_+(w)T_-^t(w) = [T_+(w)T_-^t(w)]^t. \tag{17}$$

Apart from the proper ordering of the quadratic expressions and the quantum corrections of order $\hbar^2$ in (16), the essential content of these equations is the shift of the denominator on the rhs. in (16). This provides a central extension of (14), which is required for consistency of the quantum model.

Classically, $M(w)$ contains the essential physical objects according to (11). In the quantum model, the definition $M(w) = T_+(w)T_-^t(w)$ ensures that the commutation relations (15) and (16) actually yield a *closed* commutator algebra of the matrix entries of $M(w)$. Moreover, these are Hermitian operators, provided that

$$T_+^{ab}(w) = [T_-^{ab}(\bar{w})]^\dagger, \tag{18}$$

in accordance with the classical relations. Finally, the classical condition of unit determinant $\det T_\pm(w) = 1$ requires quantum corrections because of the nonlinear terms and is substituted by the "quantum determinant" [10],

$$T_\pm^{11}(w + i\hbar G)T_\pm^{22}(w) - T_\pm^{12}(w + i\hbar G)T_\pm^{21}(w) = 1, \tag{19}$$

which is indeed compatible with the relations (15) and (16) and may as such be imposed as an operator identity.

In summary, we have formulated the consistent quantum algebra in terms of the operators $T_\pm^{ab}(w)$, subject to the commutation relations (15) and (16), as well as to unit quantum determinant (19), hermiticity (18), and symmetry (17). We are now in a position to introduce a Fock space-type representation of this algebra, inspired by the scalar case (6). Therefore, let $T_-(w)$ act trivially on the vacuum,

$$T_-^{ab}(w)\,|0\rangle = \delta^{ab}|0\rangle, \tag{20}$$

and $T_+(w)$ generate the Fock space spanned by the basic states,

$$\prod_{i=1}^{m} T_+^{a_i\,b_i}(w_i)\,|0\rangle, \qquad w_i \in H_+, \qquad m = 0, 1, \ldots. \tag{21}$$

These excitations are not independent but obey the relations (15), (17), and (19) for T_+. The intuitive idea that the $T_+^{ab}(w)$ generate the complete spectrum of states is not only supported by the exactly solved scalar case from above, but even stronger by the fact that the conserved charges $T_+(w)$ canonically Poisson generate the Geroch group [11] which, as a symmetry group, acts transitively among the classical solutions of the field equations [12].

It is straightforward to further extract the relevant physical information from the quantum model. The hermiticity relations (18), together with the commutation relations (16), allow one to calculate the expectation values of arbitrary polynomials in the $T_\pm^{ab}(w)$ in arbitrary excitations of the vacuum. Indeed, the commutation relations (16) show that the $T_-^{ab}(w)$ may be shuffled through to the right in any sequence of operators, where they finally "annihilate" the vacuum according to (20). The rhs. of (16) may be viewed accordingly as a normal ordering of the quadratic expressions.

We can also derive expectation values of the conformal factor Γ_∞ and its exponential e^{Γ_∞}, related to energy, deficit angle, and metric components at infinity (4). Namely, classically one may calculate the Poisson brackets

$$\{\Gamma_\infty, T_\pm(w)\} = G\,\partial_w T_\pm(w). \tag{22}$$

In the quantum theory, the conformal factor can thus be represented as derivation operator $i\hbar G\,\partial/\partial w$; its exponential e^{Γ_∞} becomes the shift operator $w \mapsto w + i\hbar G$. These operators may be shown to be Hermitian in the representations (18), (20), and (21). It is an elementary exercise to calculate their matrix elements between arbitrary quantum states. In particular, the conformal factor Γ_∞ exhibits a positive spectrum in accordance with its classical form (3); e.g., its eigenstates of the first level are of the form

$$\int_{\mathbb{R}} dw\,\exp(-i\lambda w/\hbar G)T_+(w)\,|0\rangle,$$

such that, due to holomorphy of $T_+(w)$ in H_+, the integral vanishes for negative λ.

The presented quantum model provides the exact quantization of a minisuperspace model of quantum gravity with nonlinear characteristics. The complete set of quantum observables and a complete spectrum of physical quantum states are at hand. The techniques are sufficiently developed to start exploring the properties of the spectrum and relevant observables.

It would be of great interest to identify some kind of coherent states in this model, i.e., quantum states with certain semiclassical properties. Because of the nonlinear setting, it is reasonable to suspect that not all of the standard properties of the usual coherent states can be satisfied. The insufficiency of the traditional framework of coherent states for the description of quantized gravitational waves is actually supported by recent observations in the simpler model of collinear polarizations [7].

Another exciting feature of our quantum model emerges from the quantum analog of the determinant (19): In view of the physical interpretation of $M(w)$ (11), which supplies the spectral parameter w with a spacetime meaning, it is tempting to consider (19) as a sign of arising discrete structures and nonlocality of the quantum operators on the Planck scale. Let us recall that we have already encountered two other discrete effects showing up in the quantum model: The classical singularities in the Poisson algebra (14) have been shifted away from the real line by an amount of $i\hbar G$ in (16). They may hence affect the holomorphy of the action of the quantum operators $T_{\pm}(w)$ in a corresponding domain of Planck size. Secondly, the Hamiltonian (4) has been shown to be represented by a discrete Planck length shift operator in the quantum theory.

Since the presented quantization employs mainly the group-theoretical properties of the model, it will allow natural generalization to other and more complicated models of dimensionally reduced gravity, including higher dimensional supergravity as well as Einstein-Maxwell systems [13]. Similarly, it should find application to the Gowdy model, where ρ becomes a timelike variable [14]. The weak field limit of the nonlinear Poisson structure in this case is isomorphic to the isomonodromic Poisson structure quantized in [15]. With a different norm of the reducing Killing vector fields, the whole scheme may futhermore be applied to stationary axisymmetric spacetimes, providing an exact quantization of the black hole solutions in a vast class of models. A detailed account of the presented results will follow.

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