ELSEVIER

Nuclear Physics B 533 (1998) 210–242

NUCLEAR PHYSICS B

Integrability and canonical structure of $d = 2$, $N = 16$ supergravity

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Received 27 April 1998; accepted 23 June 1998

Abstract

The canonical formulation of $d = 2$, $N = 16$ supergravity is presented. We work out the supersymmetry generators (including all higher order spinor terms) and the $N = 16$ superconformal constraint algebra. We then describe the construction of the conserved non-local charges associated with the affine $E_{9(+9)}$ symmetry of the classical equations of motion. These charges are shown to commute weakly with the supersymmetry constraints, and hence with all other constraints. Under commutation, they close into a quadratic algebra of Yangian type, which is formally the same as that of the bosonic theory. The Lie–Poisson action of $E_{9(+9)}$ on the classical solutions is exhibited explicitly. Further implications of our results are discussed. © 1998 Elsevier Science B.V.

PACS: 04.65.+e; 04.20.Fy; 04.60.Kz; 02.20.Tw
Keywords: Supergravity; Superconformal algebras; Integrable systems; Yangian symmetry; Geroch group

1. Introduction

Maximal $d = 2$, $N = 16$ supergravity is the most symmetric of all known field theories in two dimensions, and therefore of special interest in many ways. It is classically integrable in the sense that its equations of motion can be obtained from a linear system [1–3]. As first argued in [4] on the basis of a general analysis of the hidden symmetries arising in the dimensional reduction of $d = 11$ supergravity to lower dimensions [5,6], the space of associated classical solutions admits an $E_{9(+9)}$ symmetry generalizing the

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PII S0550-3213(98)00496-9

Geroch group of general relativity.[3] Its high degree of symmetry, the emergence of a maximally extended superconformal structure and the natural appearance of exceptional groups of E-type indicate that this theory is destined to play a prominent role in the search for a non-perturbative and unified theory of quantum gravity encompassing superstring theory and $d = 11$ supergravity.

In this paper we present the canonical formulation of $d = 2$, $N = 16$ supergravity and analyze its symmetry structure within the canonical framework. After a brief description of the Lagrangian and equations of motion, we set up the canonical formalism and derive the complete expressions for the $N = 16$ constraint generators of local supersymmetry. These results allow us in particular to complete the proof that the linear system of Refs. [1,2] – which is at most quadratic in the fermionic fields – generates all the necessary higher order fermionic terms in the equations of motion. That is, the integrable structure of the model extends through all fermionic orders. We then proceed to work out the $N = 16$ superconformal constraint algebra, which is not of the standard type.

In the second part of the paper, we analyze the integrable structure of the model on the basis of an infinite set of conserved charges and their algebra. These non-local charges are determined via the transition matrices of the linear system. They are shown to commute weakly with the supersymmetry constraints, and hence the full gauge algebra; therefore they yield an infinite set of observables. We examine the canonical algebra that is generated by the non-local charges, exploiting the fact that the final result of this calculation is unambiguous – unlike the corresponding calculation for flat space integrable models of this type, which is plagued by irresolvable ambiguities (see e.g. Ref. [7]). It is quite remarkable that the Yangian algebra which one obtains turns out to be the same as that of the purely bosonic model. This enables us to take over the analysis of Ref. [8] entirely.

The connection of our results with previous ones on the affine $E_{9(+9)}$ symmetry of the classical solutions is finally established by defining the Lie–Poisson action of the non-local charges on the physical fields. This coincides with the known symmetry action on the fields and their associated dual potentials. The main advantage of the canonical realization of the affine symmetry is that in this way we gain complete control over the deviations of the symmetry action from a symplectic action. This issue is of particular importance when one studies the quantum mechanical realization of the symmetry. We believe that our results open new and promising perspectives for the exact quantization of $d = 2$, $N = 16$ supergravity, as it is known at least in principle how to quantize Lie–Poisson actions [9,10]. In particular, they confirm the relevance of Yangian-type deformations of $\mathfrak{e}_9$ for the classification of physical states in the quantum theory.

Our results could also be relevant in connection with recent developments in non-perturbative string theory. There are fascinating topics for future research in this direction, such as the search for possible analogs of D-branes, which would require reconciling the action of the Geroch group with boundary conditions other than asymptotically

[3] Following standard usage, we will designate by $E_{9(+9)}$ the relevant non-compact version of the affine Lie group E_9. However, most of our results will concern the associated Lie algebra, which we denote by $\mathfrak{e}_9$.

flat ones, or the investigation of the possible relevance of the symmetry structures found here for duality symmetries in non-perturbative string theory.

2. $N = 16$ supergravity in two dimensions

The Lagrangian and equations of motion of $d = 2$, $N = 16$ supergravity are most conveniently derived by dimensional reduction of $N = 16$ supergravity in three dimensions [11] as described in [1–3]. Let us first recall its field content. In the gravitational sector, we have the zweibein $e_\mu^{\ \alpha}$ and the dilaton ρ, both of which originate from the dreibein of $d = 3, N = 16$ supergravity. The Kaluza–Klein–Maxwell field A_μ, which is also part of the dreibein, is conventionally set to zero. It does not carry propagating degrees of freedom, but may have non-trivial holonomies on a topologically non-trivial world-sheet, and could give rise to a cosmological constant in two dimensions. Although we neglect such effects here, this field cannot be completely ignored because its elimination gives rise to extra quartic spinor terms which do contribute to our complete expression for supersymmetry constraint below.

The dilaton ρ satisfies a free field equation (in the gravitational background provided by $e_\mu^{\ \alpha}$), and this permits us to introduce its dual "axion" $\tilde{\rho}$

$$\partial_\mu \rho + \epsilon_{\mu\nu}\partial^\nu \tilde{\rho} = 0\,. \tag{2.1}$$

The partner of the dreibein, the $d = 3$ gravitino, gives rise to a gravitino in two dimensions and a "dilatino" according to the decomposition $\psi_a^I = (\psi_\alpha^I, \psi_2^I)$ (in flat indices), both of which transform as the $\mathbf{16}_v$ representation of $SO(16)$.

In addition to these non-propagating fields, there are 128 physical scalar fields and 128 physical fermions transforming in the left and right handed spinor representation of $SO(16)$, labeled by indices A and $\dot{A}$, respectively. The chiral components associated with the 128 propagating fermionic degrees of freedom are designated by $\chi_\pm^{\dot{A}}$ (see Appendix A.1 for our spinor conventions). The scalar sector is governed by a non-linear $E_{8(+8)}/SO(16)$ σ-model. I.e. the scalar fields are described by a matrix $\mathcal{V}(x) \in E_{8(+8)}/SO(16)$ representing all 128 propagating bosonic degrees of freedom.

For the Lagrangian and the equations of motion we need the decomposition

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = \tfrac{1}{2}Q_\mu^{IJ}X^{IJ} + P_\mu^A Y^A\,, \tag{2.2}$$

where X^{IJ} and Y^A are the algebra generators of $\mathfrak{e}_8$, see Appendix A.2.

The composite gauge field Q_μ^{IJ} serves to define the $SO(16)$ covariant derivatives by

$$\begin{aligned}
D_\mu \psi^I &= \partial_\mu \psi^I + Q_\mu^{IJ}\psi^J\,,\\
D_\mu \chi^{\dot{A}} &= \partial_\mu \chi^{\dot{A}} + \tfrac{1}{4}Q_\mu^{IJ}\Gamma^{IJ}_{\dot{A}\dot{B}}\chi^{\dot{B}}\,,\\
D_\mu P_\nu^A &= \partial_\mu P_\nu^A + \tfrac{1}{4}Q_\mu^{IJ}\Gamma^{IJ}_{AB}P_\nu^B\,,
\end{aligned} \tag{2.3}$$

and the field strength

$$Q_{\mu\nu}^{IJ} := \partial_\mu Q_\nu^{IJ} - \partial_\nu Q_\mu^{IJ} + Q_\mu^{IK}Q_\nu^{KJ} - Q_\mu^{JK}Q_\nu^{KI}\,. \tag{2.4}$$

From (2.2) one reads off the integrability relations (valid in any dimension)

$$D_{[\mu}P^A_{\nu]} = 0\,, \qquad Q^{IJ}_{\mu\nu} + \tfrac{1}{2}\Gamma^{IJ}_{AB}P^A_\mu P^B_\nu = 0\,. \tag{2.5}$$

The Lagrangian of $d = 2$, $N = 16$ supergravity can now be directly obtained from the one given in [11] and reads

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}\rho e R + \rho\epsilon^{\mu\nu}\overline{\psi}^I_2 D_\mu\psi^I_\nu - \tfrac{1}{2}\mathrm{i}\rho e\overline{\chi}^{\dot A}\gamma^\mu D_\mu\chi^{\dot A} + \tfrac{1}{4}\rho e P^{\mu A}P^A_\mu \\ &-\tfrac{1}{2}\rho e\overline{\chi}^{\dot A}\gamma^\nu\gamma^\mu\psi^I_\nu\Gamma^I_{A\dot A}P^A_\mu - \tfrac{1}{2}\mathrm{i}\rho e\overline{\chi}^{\dot A}\gamma^3\gamma^\mu\psi^I_2\Gamma^I_{A\dot A}P^A_\mu\,.\end{aligned} \tag{2.6}$$

up to higher order fermionic terms. The associated action is manifestly invariant under general coordinate transformations in two dimensions, as well as local $SO(16)$ transformations

$$\begin{aligned}\delta_\omega\, Q^{IJ}_\pm &= D_\pm\omega^{IJ} = \partial_\mu\omega^{IJ} + Q^{IK}_\mu\omega^{KJ} - Q^{JK}_\mu\omega^{KI}\,,\\ \delta_\omega\, P^A_\pm &= \tfrac{1}{4}\Gamma^{IJ}_{AB}\omega^{IJ}P^B_\pm\,,\\ \delta_\omega\,\psi^I &= \omega^{IJ}\psi^J\,,\\ \delta_\omega\,\chi^{\dot A} &= \tfrac{1}{4}\Gamma^{IJ}_{\dot A\dot B}\omega^{IJ}\chi^{\dot B}\,,\end{aligned} \tag{2.7}$$

with the $SO(16)$ parameter $\omega^{IJ}(x) = -\omega^{JI}(x)$.

For our further considerations we employ the superconformal gauge

$$e_\mu^{\ \alpha} = \lambda\delta^\alpha_\mu\,, \qquad \psi^I_\mu = \mathrm{i}\gamma_\mu\psi^I\,. \tag{2.8}$$

We will also make use of the fields

$$\sigma := \log\lambda\,, \qquad \widehat{\sigma} := \sigma - \tfrac{1}{2}\log(\partial_+\rho\partial_-\rho)\,. \tag{2.9}$$

The redefined field $\widehat{\sigma}$ transforms as a genuine scalar under conformal diffeomorphisms, whereas λ itself is a density. The gauge choice (2.8) must be accompanied by the following rescaling of the fermion fields:

$$\chi^{\dot A} \to \lambda^{1/2}\chi^{\dot A}\,, \qquad \psi^I_\mu \to \lambda^{1/2}\psi^I_\mu\,, \qquad \psi^I_2 \to \lambda^{1/2}\psi^I_2\,. \tag{2.10}$$

Then the conformal factor λ disappears almost entirely from the Lagrangian, except for the Einstein term which in the conformal gauge becomes

$$-\tfrac{1}{4}\rho e R = -\tfrac{1}{2}\partial_\mu\rho\partial^\mu\sigma\,. \tag{2.11}$$

As a consequence the theory would be conformally invariant if it were not for the remaining σ dependence of this term. Similarly, the superpartner ψ^I of σ does not completely decouple in the gauge (2.8), as it would in a superconformally invariant theory. Still, we can from now on drop the distinction between flat and curved indices $\mu, \ldots$ and $\alpha, \ldots$ We remark that there is no problem of principle in keeping the dependence on topological degrees of freedom which are eliminated by (2.8) (i.e. the moduli and supermoduli on the world-sheet); the requisite formalism has been set up in [3], building on earlier results in [12,13].

We next list the equations of motion in the superconformal gauge (2.8). Utilizing one-component spinors (see Appendix A.1), the fermionic equations of motion read

$$\begin{aligned} D_\pm(\rho^{1/2}\chi^{\dot A}_\mp) &= \mp\tfrac12\rho^{1/2}\psi^I_{2\mp}\Gamma^I_{A\dot A}P^A_\pm\,,\\ D_\pm\psi^I_\mp &= -\tfrac12\chi^{\dot A}_\mp\Gamma^I_{A\dot A}P^A_\pm\,,\\ D_\pm(\rho\psi^I_{2\mp}) &= 0\,, \end{aligned} \tag{2.12}$$

modulo cubic spinor terms. The equations of motion for the physical scalar fields are

$$\begin{aligned} D_+(\rho P^A_-) + D_-(\rho P^A_+) &= 2\mathrm{i}\Gamma^I_{A\dot A}D_-(\rho\psi^I_{2+}\chi^{\dot A}_+) - 2\mathrm{i}\Gamma^I_{A\dot A}D_+(\rho\psi^I_{2-}\chi^{\dot A}_-)\\ &\quad +2\mathrm{i}\rho\Gamma^{IJ}_{AB}P^B_-\psi^I_{2+}\psi^J_+ - 2\mathrm{i}\rho\Gamma^{IJ}_{AB}P^B_+\psi^I_{2-}\psi^J_-\\ &\quad +\tfrac14\mathrm{i}\rho\Gamma^{IJ}_{AB}P^B_-\,\chi^{\dot A}_+\Gamma^{IJ}_{\dot A\dot B}\chi^{\dot B}_+ + \tfrac14\mathrm{i}\rho\Gamma^{IJ}_{AB}P^B_+\,\chi^{\dot A}_-\Gamma^{IJ}_{\dot A\dot B}\chi^{\dot B}_-\,,\\ \partial_+\partial_-\widehat\sigma = \partial_+\partial_-\sigma &= -\tfrac12P^A_+P^A_- - \mathrm{i}\left(\chi^{\dot A}_+D_-\chi^{\dot A}_+ + \chi^{\dot A}_-D_+\chi^{\dot A}_-\right)\,, \end{aligned} \tag{2.13}$$

modulo quartic spinor terms.

The equations listed so far originate from the variation of those fields which survive the superconformal gauge fixing. There are, however, two more equations that follow by variation of the traceless modes of the metric and the gravitino, both of which are put to zero in the superconformal gauge. These are the constraints

$$\begin{aligned} T_{\pm\pm} &= \tfrac12\rho P^A_\pm P^A_\pm - \partial_\pm\rho\,\partial_\pm\hat\sigma \mp \mathrm{i}\rho P^A_\pm\Gamma^I_{A\dot B}\psi^I_{2\pm}\chi^{\dot B}_\pm - \mathrm{i}\rho\chi^{\dot A}_\pm D_\pm\chi^{\dot A}_\pm\\ &\quad \mp\mathrm{i}\psi^I_\pm D_\pm(\rho\psi^I_{2\pm}) \mp \mathrm{i}\rho\psi^I_{2\pm}D_\pm\psi^I_\pm \approx 0\,, \end{aligned} \tag{2.14}$$

$$S^I_\pm = \pm D_1(\rho\psi^I_{2\pm}) - \rho\partial_\pm\sigma\,\psi^I_{2\pm} \mp \rho\chi^{\dot A}_\pm\Gamma^I_{A\dot A}P^A_\pm \pm \partial_\pm\rho\,\psi^I_\pm \approx 0\,, \tag{2.15}$$

which will be seen to generate conformal and superconformal transformations, respectively. For this reason, they enjoy a somewhat different status from the previous equations: the equations setting them to zero must be interpreted as weak equalities in the sense of Dirac (indicated by the symbol "$\approx$"). Like in superconformal field theories the derivatives of these constraints vanish strongly:

$$\partial_\mp T_{\pm\pm} = D_\mp S^I_\pm = 0\,. \tag{2.16}$$

While we do not include the cubic spinor terms in the above equations, we will give the complete terms below in the canonical expression for the supersymmetry constraint.

The superconformal gauge (2.8) is preserved by local supersymmetry transformations with chiral parameters $\epsilon^I_\pm$ obeying

$$D_\pm\epsilon^I_\mp = 0\,, \tag{2.17}$$

again modulo cubic spinor terms. Up to such terms, the supersymmetry variations are given by

$$\begin{aligned} \mathcal V^{-1}\delta_\pm\mathcal V &= \mp2\mathrm{i}\epsilon^I_\pm\chi^{\dot A}_\pm\Gamma^I_{A\dot A}Y^A\,, & \delta_\pm\chi^{\dot A}_\pm &= \mp\epsilon^I_\pm\Gamma^I_{A\dot A}P^A_\pm\,,\\ \delta_\pm\rho &= 2\mathrm{i}\rho\epsilon^I_\pm\psi^I_{2\pm}\,, & \delta_\pm\psi^I_{2\pm} &= \rho^{-1}\partial_\pm\rho\epsilon^I_\pm\,,\\ \delta_\pm\sigma &= \mp2\mathrm{i}\epsilon^I_\pm\psi^I_\pm\,, & \delta_\pm\psi_\pm &= \mp\left(D^I_\pm\epsilon^I_\pm + \partial_\pm\sigma\epsilon^I_\pm\right)\,. \end{aligned} \tag{2.18}$$

The superalgebra generated by these transformations can be regarded as an $N = 16$ superconformal algebra, but it is distinguished from the standard superconformal algebras (which stop at $N = 4$) by the fact that it is a *soft* algebra (i.e. it has field-dependent structure "constants"). This is one of the reasons why, despite the evident similarities with superconformal field theories, $N = 16$ supergravity belongs to a different class of models. There is no immediate analog of the left–right (holomorphic) factorization of conformal field theory because $N = 16$ is not a free (or even "quasi-free") theory.[4] We will recognize this feature again when analyzing the constraint algebra.

We conclude this section with some comments on the physical interpretation of the fields ρ and $\tilde{\rho}$, which play a special role. As is well known, one can invoke the residual conformal invariance of (2.8) to identify these fields with the world-sheet coordinates at least locally. This gauge, which is the precise analog of the light-cone gauge in string theory, is complemented on the fermionic side by the elimination of the dilatino ψ_2^I from the equations of motion by means of the residual superconformal transformations (2.18) [1]. For stationary axisymmetric solutions of Einstein's equations (Euclidean signature of the world-sheet), one traditionally takes $\rho \geqslant 0$ as the radial variable and $\tilde{\rho}$ as the coordinate along the symmetry axis [15]. For the Minkowskian signature considered here, more and physically distinct choices are possible. Depending on whether the vector $\partial_\mu \rho$ is spacelike or timelike, we can either identify ρ with the radial and $\tilde{\rho}$ with the time coordinate (Einstein–Rosen gravitational waves [16]), or ρ with the time and $\tilde{\rho}$ with the space coordinate (in which case there is a "big bang" singularity at $\rho = 0$ [17]). However, one can also envisage more general situations with alternating signatures, as well as world-sheets of non-trivial topology.

From a "stringy" perspective, on the other hand, ρ and $\tilde{\rho}$ should be treated as (quantum) fields living on the world-sheet, whose vacuum expectation values would be associated with coupling constants of the theory. The Möbius subgroup of the hidden Witt–Virasoro symmetry of the theory [18–20] is then analogous to the strong-weak coupling duality in string theory.

3. Canonical brackets

The derivation of the of the canonical brackets from the action (2.6) is straightforward, except perhaps for the technical complication that the presence of second class constraints necessitates a Dirac procedure.

In the gravitational sector we have the canonical momenta

$$2\pi_\sigma = \partial_0 \rho\,, \qquad 2\pi_\rho = \partial_0 \sigma\,.$$

In order not to overburden the notation, we will set

$$\partial_\pm \sigma \equiv \pi_\rho \pm \tfrac{1}{2}\partial_1 \sigma\,, \qquad \partial_\pm \rho \equiv \pi_\sigma \pm \tfrac{1}{2}\partial_1 \rho\,, \tag{3.1}$$

[4] However, there is an analog of holomorphic factorization within the framework of isomonodromic solutions [14].

in the formulas below. Furthermore, we will be exclusively concerned with equal time brackets at a fixed but arbitrary time $t \equiv x^0$, and will therefore not explicitly indicate the full coordinate dependence, but only spell out the dependence on the space coordinates $x \equiv x^1, y \equiv y^1$.

The equal time brackets for the dilaton ρ and the conformal factor σ are given by

$$\{\rho(x)\,,\,\pi_\rho(y)\} = \{\sigma(x)\,,\,\pi_\sigma(y)\} = \delta(x-y)\,. \tag{3.2}$$

Duality implies that

$$\{\tilde\rho(x)\,,\,\partial_1\sigma(y)\} = 2\,\delta(x-y)\,, \tag{3.3}$$

and

$$\{\partial_\pm\rho(x)\,,\partial_\pm\sigma(y)\} = \pm\delta'(x-y), \qquad \{\partial_\pm\rho(x)\,,\partial_\mp\sigma(y)\} = 0, \tag{3.4}$$

where derivatives on the δ function are always understood to act on the first argument. Later, we will also introduce a spectral parameter γ (see (5.3)) depending on both ρ and $\tilde\rho$. In the canonical framework, this spectral parameter becomes a canonical variable, and thus an operator upon quantization.

To obtain the canonical bosonic Poisson brackets in the σ-model sector, we introduce conjugate momenta Π to the canonical variables Q_1 and P_1,

$$\Pi^{IJ} \equiv \frac{\delta S}{\delta(\partial_0 Q_1^{IJ})}\,, \qquad \Pi^A \equiv \frac{\delta S}{\delta(\partial_0 P_1^A)}\,.$$

with

$$\{Q_1^{IJ}(x), \Pi^{KL}(y)\} = \delta^{IJ}_{KL}\,\delta(x-y)\,, \qquad \{P_1^A(x), \Pi^B(y)\} = \delta^{AB}\,\delta(x-y)\,. \tag{3.5}$$

Taking into account the zero curvature condition (2.5), the Lagrangian (2.6) yields

$$\begin{aligned}\partial_1\Pi + \left[\mathcal{V}^{-1}\partial_1\mathcal{V}\,,\,\Pi\right] = {}& \tfrac12\,\rho P_0^A Y^A + \tfrac12\,\mathrm{i}\rho\,\Gamma^I_{A\dot A}\overline{\chi}^{\dot A}\gamma_1\psi_2^I\,Y^A + \mathrm{i}\rho\overline{\psi}^I\gamma_1\psi_2^J\,X^{IJ} \\ & -\tfrac18\,\mathrm{i}\rho\,\overline{\chi}^{\dot A}\gamma_0\chi^{\dot B}\,\Gamma^{IJ}_{\dot A\dot B}X^{IJ}\,,\end{aligned}$$

where Π is given by

$$\Pi \equiv -\Pi^{IJ}X^{IJ} + \Pi^A Y^A\,.$$

Solving the above relations for P_0^A, we arrive at

$$P_0^A = \frac{2}{\rho}\left(D_1\Pi^A + \tfrac14\Pi^{IJ}\Gamma^{IJ}_{AB}P_1^B\right) + 2\mathrm{i}\Gamma^I_{A\dot A}\psi^I_{2+}\chi^{\dot A}_+Y^A - 2\mathrm{i}\Gamma^I_{A\dot A}\psi^I_{2-}\chi^{\dot A}_-Y^A\,. \tag{3.6}$$

Furthermore, we deduce the (first class) $SO(16)$ constraint:

$$\begin{aligned}\Phi^{IJ} = {}& D_1\Pi^{IJ} + \tfrac14\Gamma^{IJ}_{AB}P_1^A\Pi^B \\ & -2\mathrm{i}\rho\left(\psi_+^{[I}\psi_{2+}^{J]} - \psi_-^{[I}\psi_{2-}^{J]}\right) - \tfrac14\mathrm{i}\rho\Gamma^{IJ}_{\dot A\dot B}\left(\chi_+^{\dot A}\chi_+^{\dot B} + \chi_-^{\dot A}\chi_-^{\dot B}\right) \approx 0\,.\end{aligned} \tag{3.7}$$

which generates the gauge transformations (2.7). From (3.6) we obtain the bosonic brackets in the σ-model sector:

$$
\begin{aligned}
\{P^A_\pm(x)\,,\mathcal{V}(y)\} &= \frac{1}{\rho}\mathcal{V}(x)Y^A\,\delta(x-y)\,,\\
\{P^A_\pm(x)\,,Q^{IJ}_1(y)\} &= -\frac{1}{4\rho}\,\Gamma^{IJ}_{AB}P^B_1\,\delta(x-y)\,,\\
\{P^A_\pm(x)\,,P^B_\pm(y)\} &= \mp\frac{1}{4\rho}\,\Gamma^{IJ}_{AB}Q^{IJ}_1\,\delta(x-y) \mp \frac{1}{2}\left(\frac{1}{\rho(x)}+\frac{1}{\rho(y)}\right)\,\delta^{AB}\,\delta'(x-y)\\
&\quad -\frac{\mathrm{i}}{2\rho}\,\Gamma^{IJ}_{AB}\left(2\psi^I_+\psi^J_{2+} - 2\psi^I_-\psi^J_{2-} + \psi^I_{2+}\psi^J_{2+} + \psi^I_{2-}\psi^J_{2-}\right)\,\delta(x-y)\\
&\quad -\frac{\mathrm{i}}{8\rho}\,\Gamma^{IJ}_{AB}\Gamma^{IJ}_{\dot A\dot B}\left(\chi^{\dot A}_+\chi^{\dot B}_+ + \chi^{\dot A}_-\chi^{\dot B}_-\right)\,\delta(x-y)\\
&\quad +\frac{1}{4\rho^2}\,\Gamma^{IJ}_{AB}\Phi^{IJ}\,\delta(x-y)\,,\\
\{P^A_\pm(x)\,,P^B_\mp(y)\} &= \pm\frac{1}{2\rho^2}\,\partial_1\rho\,\delta^{AB}\,\delta(x-y) + \frac{1}{4\rho^2}\,\Gamma^{IJ}_{AB}\Phi^{IJ}\,\delta(x-y)\\
&\quad -\frac{\mathrm{i}}{2\rho}\,\Gamma^{IJ}_{AB}\left(2\psi^I_+\psi^J_{2+} - 2\psi^I_-\psi^J_{2-} + \psi^I_{2+}\psi^J_{2+} + \psi^I_{2-}\psi^J_{2-}\right)\,\delta(x-y)\\
&\quad -\frac{\mathrm{i}}{8\rho}\,\Gamma^{IJ}_{AB}\Gamma^{IJ}_{\dot A\dot B}\left(\chi^{\dot A}_+\chi^{\dot B}_+ + \chi^{\dot A}_-\chi^{\dot B}_-\right)\,\delta(x-y)\,,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
\{P^A_\pm(x)\,,\partial_\pm\sigma(y)\} &= -\frac{1}{2\rho}\left(P^A_0 + \mathrm{i}\Gamma^I_{A\dot B}\left(\psi^I_{2+}\chi^{\dot B}_+ - \psi^I_{2-}\chi^{\dot B}_-\right)\right)\delta(x-y)\,,\\
\{P^A_\pm(x)\,,\partial_\mp\sigma(y)\} &= -\frac{1}{2\rho}\left(P^A_0 + \mathrm{i}\Gamma^I_{A\dot B}\left(\psi^I_{2+}\chi^{\dot B}_+ - \psi^I_{2-}\chi^{\dot B}_-\right)\right)\delta(x-y)\,.
\end{aligned}
\tag{3.9}
$$

In the fermionic sector we find the following Dirac brackets

$$
\begin{aligned}
\left\{\chi^{\dot A}_\pm(x)\,,\chi^{\dot B}_\pm(y)\right\} &= -\frac{\mathrm{i}}{2\rho}\,\delta^{\dot A\dot B}\,\delta(x-y)\,,\\
\{\psi^I_\pm(x)\,,\psi^J_{2\pm}(y)\} &= \mp\frac{\mathrm{i}}{2\rho}\,\delta^{IJ}\,\delta(x-y)\,.
\end{aligned}
\tag{3.10}
$$

Owing to the explicit appearance of the bosonic fields in the fermionic second class constraints there are also non-vanishing mixed brackets

$$
\begin{aligned}
\left\{\pi_\rho(x)\,,\chi^{\dot A}_\pm(y)\right\} &= \frac{1}{2\rho}\,\chi^{\dot A}_\pm\,\delta(x-y)\,,\\
\{\pi_\rho(x)\,,\psi^I_{2\pm}(y)\} &= \frac{1}{\rho}\,\psi^I_{2\pm}\,\delta(x-y)\,,
\end{aligned}
\tag{3.11}
$$

while the form of P^A_0 in (3.6) gives rise to

$$
\left\{P^A_0(x)\,,\chi^{\dot B}_\pm(y)\right\} = \pm\frac{1}{\rho}\Gamma^I_{A\dot B}\psi^I_{2\pm}\,\delta(x-y)\,,
$$

$$\{P_0^A(x), \psi_\pm^I(y)\} = -\frac{1}{\rho}\Gamma^I_{A\dot{B}}\chi_\pm^{\dot{B}}\,\delta(x-y)\,. \tag{3.12}$$

Alternatively, the necessity of (3.11) can be inferred from the presence of ρ on the r.h.s. of (3.10).

The above brackets are slightly simplified when written in terms of canonical variables with vanishing mixed brackets. For this purpose, we introduce

$$\widetilde{P}_0^A := \rho P_0^A - 2\mathrm{i}\rho\psi_{2+}^I\chi_+^{\dot{A}}\Gamma^I_{A\dot{A}} + 2\mathrm{i}\rho\psi_{2-}^I\chi_-^{\dot{A}}\Gamma^I_{A\dot{A}}\,, \tag{3.13}$$

together with

$$\widetilde{P}_\pm^A := \tfrac{1}{2}(\widetilde{P}_0^A \pm P_1^A)\,.$$

These are the variables which commute with all the fermions and with $\partial_\pm\sigma$. Moreover, we notice that ψ^I and the rescaled fermions $\rho\psi_2^I$ and $\rho^{1/2}\chi^{\dot{A}}$ commute with π_ρ, and hence $\partial_\pm\sigma$ as well.

4. Constraint superalgebra

In this section we establish the constraint superalgebra underlying the superconformal transformations (2.18) which remain after imposing the superconformal gauge (2.8). They are shown to close into a superconformal algebra which in addition contains the conformal transformations generated by (2.14) and the $SO(16)$ gauge transformations (3.7).

Our most important result in this section is the expression for the supersymmetry constraint generators $S_\pm^I$ (4.2) below, which is complete and includes all cubic fermionic terms. This is the crucial operator because all other constraints can be completely determined from the commutator of two supersymmetry generators (in principle we could thus compute the quartic spinorial contributions to $T_{\pm\pm}$, but the explicit expressions are not very illuminating). In the next section we will make use of these results and demonstrate that the integrals of motion associated with the affine $E_{9(+9)}$ symmetry of the equations of motion weakly commute with the supersymmetry constraint. This calculation provides a stringent consistency check on the correctness of the cubic spinor terms in $S_\pm^I$.

As discussed above, the energy momentum (Virasoro) constraints $T_{\pm\pm}$ descend from (2.6) before going into the superconformal gauge (2.8). For the determination of canonical brackets and the constraint algebra it is, however, necessary to express $T_{\pm\pm}$ and the other constraints entirely in terms of canonical variables. In other words, all time derivatives implicit in the derivatives $\partial_\pm$ must be converted into momenta and spatial derivatives of the canonical variables by means of their equations of motion. More specifically, the fermionic equations of motion (2.12) can be invoked to derive

$$D_\pm(\rho^{1/2}\chi_\pm^{\dot{A}}) = \pm D_1(\rho^{1/2}\chi_\pm^{\dot{A}}) \pm \tfrac{1}{2}\rho^{1/2}\psi_{2\pm}^I\Gamma^I_{A\dot{A}}P_\mp^A\,,$$
$$D_\pm\psi_\pm^I = \pm D_1\psi_\pm^I - \tfrac{1}{2}\chi_\pm^{\dot{A}}\Gamma^I_{A\dot{A}}P_\mp^A\,,$$

$$D_\pm(\rho\psi^I_{2\pm}) = \pm D_1(\rho\psi^I_{2\pm}) ,$$

where the l.h.s. is to be replaced by the expression on the r.h.s. before computing any bracket. The resulting form of the constraint will be referred to as the "canonical form". Nevertheless, writing the constraints in "non-canonical" form (2.14), (2.15) has the advantage that the conformal covariance properties become manifest. Consequently, we will use both forms of the constraints according to convenience.

The canonical form of the energy momentum constraint is given by

$$\begin{aligned} T_{\pm\pm} = {} & \tfrac{1}{2}\rho P^A_\pm P^A_\pm - \partial_\pm\rho\,\partial_\pm\sigma \pm \tfrac{1}{2}\partial_1\partial_\pm\rho \mp \mathrm{i}\rho P^A_1 \Gamma^I_{A\dot B}\psi^I_{2\pm}\chi^{\dot B}_\pm \\ & \mp\mathrm{i}\rho\chi^{\dot A}_\pm D_1\chi^{\dot A}_\pm - \mathrm{i}\psi^I_\pm D_1(\rho\psi^I_{2\pm}) - \mathrm{i}\rho\psi^I_{2\pm}D_1(\psi^I_\pm) , \end{aligned} \tag{4.1}$$

again up to quartic fermionic terms. The supersymmetry constraint reads in canonical form

$$\begin{aligned} S^I_\pm = {} & \pm D_1(\rho\psi^I_{2\pm}) - \rho\partial_\pm\sigma\,\psi^I_{2\pm} \mp \rho\chi^{\dot A}_\pm\Gamma^I_{A\dot A}P^A_\pm \pm \partial_\pm\rho\,\psi^I_\pm \\ & \mp\mathrm{i}\rho\psi^J_\pm\,\chi_\pm\Gamma^{IJ}\chi_\pm - \tfrac{1}{2}\mathrm{i}\,\rho\psi^J_{2\pm}\Gamma^{IJ}_{\dot A\dot B}\left(\chi^{\dot A}_\pm\chi^{\dot B}_\pm - \chi^{\dot A}_\mp\chi^{\dot B}_\mp\right) \\ & +2\mathrm{i}\rho\psi^I_\pm\psi^J_\pm\psi^J_{2\pm} \pm 2\mathrm{i}\rho\psi^I_\mp\psi^J_{2\pm}\psi^J_{2\mp} \mp 2\mathrm{i}\rho\psi^I_{2\mp}\psi^J_\mp\psi^J_{2\pm} - 2\mathrm{i}\rho\psi^I_{2\mp}\psi^J_{2\pm}\psi^J_{2\mp} . \end{aligned} \tag{4.2}$$

In contradistinction to the formula for $T_{\pm\pm}$ we have given the complete expression including all cubic fermionic terms. These extra terms have been derived by requiring closure of the superalgebra, see (4.4), (4.5) below, and it is important here that this method does fix the higher order terms uniquely. Alternatively, they could have been determined from the full equations of motion, but with more effort, since this would have required analyzing the quartic terms resulting from the elimination of the Kaluza–Klein–Maxwell vector as well.

Local conformal $N = 16$ supersymmetry variations are generated according to

$$\delta_\pm\varphi = 2\mathrm{i}\int dx\,\epsilon^I_\pm(x)\,\left\{S^I_\pm(x)\,,\,\varphi\right\} , \tag{4.3}$$

Readers are invited to check that the resulting variations coincide with the ones stated in (2.18) to the relevant order.

The local supersymmetry generators $S^I_\pm$ satisfy the constraint algebra

$$\begin{aligned} \left\{S^I_\pm(x)\,,S^J_\pm(y)\right\} = {} & -\delta^{IJ}\left(\mathrm{i}T_{\pm\pm} \mp 2\psi^K_\pm S^K_\pm - \tfrac{1}{4}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\Gamma^{KL}_{\dot A\dot B}\Phi^{KL}\right)\delta(x-y) \\ & \mp\left(\psi^I_\pm S^J_\pm + \psi^J_\pm S^I_\pm\right)\delta(x-y) \\ & +\tfrac{1}{2}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\left(\Gamma^{IK}_{\dot A\dot B}\Phi^{KJ} + \Gamma^{JK}_{\dot A\dot B}\Phi^{KI}\right)\delta(x-y) , \end{aligned} \tag{4.4}$$

$$\begin{aligned} \left\{S^I_+(x)\,,S^J_-(y)\right\} = {} & -\delta^{IJ}\left(\psi^K_{2+}S^K_- + \psi^K_{2-}S^K_+\right)\delta(x-y) \\ & +\left(\psi^I_{2-}S^J_+ + \psi^J_{2+}S^I_-\right)\delta(x-y) \\ & +\tfrac{1}{4}\chi^{\dot A}_+\chi^{\dot B}_-\,\Gamma^I_{A\dot A}\Gamma^{KL}_{AB}\Gamma^J_{B\dot B}\,\Phi^{KL}\,\delta(x-y) . \end{aligned} \tag{4.5}$$

This is the complete result, valid in all fermionic orders. As mentioned above, the constraint superalgebra closes into the energy momentum constraints $T_{\pm\pm}$ and the $SO(16)$ constraints Φ^{IJ}. The closure of the algebra is, of course, guaranteed by general consistency arguments, but the canonical algebra has so far not been exhibited explicitly for this theory. It is conceivable that it can be further simplified by making field-dependent redefinitions of the constraints.

Conformal coordinate transformations with parameters $\xi^{\pm} = \xi^{\pm}(x^{\pm})$ are generated by

$$\delta_{\xi^{\pm}}\varphi = \int dx\, \xi^{\pm}(x)\, \{T_{\pm\pm}(x)\,,\varphi\} \;= -h_{\varphi}^{\pm}\,\partial_{\pm}\xi^{\pm}\,\varphi + \xi^{\pm}\,D_{\pm}\varphi\,, \tag{4.6}$$

where $h_{\varphi}^{\pm}$ denotes the conformal dimensions of the field φ. This formula illustrates the interplay between the canonical and the covariant framework. Canonically, the gauge parameters $\xi^{\pm}$ are defined as functions of and integrated over the spatial coordinate x. Upon using the equations of motion for φ and restoring the time dependence of $\xi^{\pm}$ according to $\partial_{\pm}\xi^{\mp} = 0$, the r.h.s. of (4.6) takes a conformally covariant form. Thus, $T_{\pm\pm}$ indeed generates translations along the $x^{\pm}$ coordinates modulo local $SO(16)$ transformation with field-dependent parameter Q_1^{IJ} (here we have tacitly adopted the Coulomb gauge $Q_0^{IJ} = 0$). The algebra (4.4) permits us to calculate (4.6) and thus the equations of motion in all fermionic orders by means of the super-Jacobi identities.

The constraints (3.7) generate the $SO(16)$ transformations (2.7) via

$$\delta_{\omega}\varphi \equiv \int dx\, \omega^{IJ}(x)\, \left\{\Phi^{IJ}(x)\,,\varphi\right\} \tag{4.7}$$

and satisfy the $SO(16)$ algebra:

$$\left\{\Phi^{IJ}(x)\,,\Phi^{KL}(y)\right\} = \left(\delta^{JK}\Phi^{IL} - \delta^{IK}\Phi^{JL} + \delta^{IL}\Phi^{JK} - \delta^{JL}\Phi^{IK}\right)\delta(x-y)\,. \tag{4.8}$$

The remaining commutation relations of the superconformal algebra are listed below

$$\begin{aligned}
\{T_{\pm\pm}(x)\,,T_{\pm\pm}(y)\} &= \mp\left(T_{\pm\pm}(x) + T_{\pm\pm}(y)\right)\delta'(x-y)\,,\\
\{T_{\pm\pm}(x)\,,T_{\mp\mp}(y)\} &= \tfrac{1}{4}\,\Gamma^{IJ}_{AB}P_{+}^{A}P_{-}^{B}\Phi^{IJ}\,\delta(x-y)\,,\\
\left\{T_{\pm\pm}(x)\,,S_{\pm}^{I}(y)\right\} &= \mp\tfrac{3}{2}S_{\pm}^{I}(y)\,\delta'(x-y) + D_{\pm}S_{\pm}^{I}\,\delta(x-y)\\
&\quad \mp\tfrac{1}{4}\left(\Gamma^{KL}\Gamma^{I}\right)_{A\dot{A}}P_{\pm}^{A}\chi_{\pm}^{\dot{A}}\Phi^{KL}\,\delta(x-y)\,,\\
\left\{T_{\pm\pm}(x)\,,S_{\mp}^{I}(y)\right\} &= \pm\tfrac{1}{4}\left(\Gamma^{KL}\Gamma^{I}\right)_{A\dot{A}}P_{\pm}^{A}\chi_{\mp}^{\dot{A}}\Phi^{KL}\,\delta(x-y)\,,\\
\left\{\Phi^{IJ}(x)\,,S_{\pm}^{K}(y)\right\} &= \tfrac{1}{2}\left(\delta^{IK}S_{\pm}^{J} - \delta^{JK}S_{\pm}^{I}\right)\delta(x-y)\,,\\
\left\{\Phi^{IJ}(x)\,,T_{\pm\pm}(y)\right\} &= 0\,.
\end{aligned} \tag{4.9}$$

We have not worked out the higher order fermionic contributions even though in principle all of them can be determined straightforwardly (though tediously) from (4.2) and the super-Jacobi identities.

The constraint superalgebra (4.4), (4.5), (4.9) is a superconformal extension of the Virasoro algebra (4.9) with $N = 16$ supersymmetry. Its existence does not contradict the well-known absence of the standard superconformal algebras with $N > 4$ [21], because in comparison with the algebras which have been studied in superconformal field theory, it exhibits some unusual features. Let us briefly comment on the differences.

First of all, unlike the usual superconformal algebras, the present model does not completely factorize into two chiral halves: this is already evident from the equations of motion, but also reflected by the fact that the chiral supercharges S_+ and S_- in (4.5) do not commute with one another.

Secondly, as already pointed out before, the brackets (4.4) and (4.5) do not close into a linear algebra. Rather, on the r.h.s the constraints $S^I_\pm$ appear with coefficients that explicitly depend on the fermionic fields ψ^I and ψ^I_2. This was of course to be expected in view of the general result that the algebras arising in supergravity are usually "soft" gauge algebras [22,23]. The important new feature here is that we can nevertheless give a complete canonical characterization.

Finally, we stress the conspicuous absence of internal chiral current algebras which appear in the standard extended superconformal algebras [21]. Namely, a linear superconformal chiral algebra with $N \leqslant 4$ supercharges requires chiral internal bosonic currents multiplying $\delta'(x-y)$ on the r.h.s. of (4.4). This fact can be immediately deduced from the super-Jacobi identities involving $\{S^I, \{S^J, S^K\}\}$, whose δ' terms only cancel with the contribution from the additional current. By contrast, we here have only one "vectorlike" $SO(16)$ current Φ^{IJ}; moreover, in the algebra this current appears only in second order in the fermions and multiplies the δ-function rather than its derivative. The terms required for consistency of the super-Jacobi identity now originate from the additional contributions due to the field-dependent structure constants on the r.h.s. of (4.4). Another distinctive feature is the invariance of the generators $T_{\pm\pm}$ under Φ^{IJ}; the $SO(16)$ constraint generator thus carries zero conformal weight, unlike the chiral currents of standard superconformal field theory.

A further extension of our results which we postpone to later investigations would involve relaxing the superconformal gauge and thus entail a canonical treatment (and quantization) of the topological degrees of freedom as well.

5. Conserved charges

Our main purpose in this section is to investigate the infinitely many conserved charges associated with the classical $E_{9(+9)}$ symmetry of the equations of motion, their algebra and the infinite dimensional symmetries they generate. Furthermore, we demonstrate the supersymmetry invariance of these charges by showing that they weakly commute with the full supersymmetry constraints, and hence with all other constraints.

5.1. Linear system

As shown in [1–3] the supergravity equations of motion can be obtained as the compatibility condition of a linear system (or Lax pair) for an E_8-valued matrix $\widehat{\mathcal{V}}(x;\gamma)$.

Here γ is a spectral parameter which depends explicitly on the coordinates via the dilaton and the axion fields, as we will shortly explain. The field dependence of γ is in marked contrast to the constancy of the spectral parameter appearing in flat space integrable systems such as non-linear σ-models and their rigidly supersymmetric extensions.

In the version of [2], the linear system takes the form

$$\widehat{\mathcal{V}}^{-1}\partial_\pm\widehat{\mathcal{V}}(\gamma) = L_\pm(\gamma) \equiv \tfrac{1}{2}\widehat{Q}^{IJ}_\pm(\gamma)X^{IJ} + \widehat{P}^A_\pm(\gamma)Y^A\,, \tag{5.1}$$

with the connection coefficients

$$\begin{aligned}\widehat{Q}^{IJ}_\pm(\gamma) &= Q^{IJ}_\pm - \frac{2\mathrm{i}\gamma}{(1\pm\gamma)^2}\left(8\psi^{[I}_{2\pm}\psi^{J]}_\pm \pm \Gamma^{IJ}_{\dot A\dot B}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\right) - \frac{32\mathrm{i}\gamma^2}{(1\pm\gamma)^4}\,\psi^I_{2\pm}\psi^J_{2\pm}\,,\\ \widehat{P}^A_\pm(\gamma) &= \frac{1\mp\gamma}{1\pm\gamma}\,P^A_\pm + \frac{4\mathrm{i}\gamma(1\mp\gamma)}{(1\pm\gamma)^3}\,\Gamma^I_{A\dot B}\psi^I_{2\pm}\chi^{\dot B}_\pm\,.\end{aligned}$$

Despite the occurrence of cubic and quartic spinor terms in the fermionic and bosonic equations of motion, the linear system (5.1) does not receive any higher order corrections but is at most quadratic in the fermionic fields. In other words, *the linear system generates all required higher order fermionic terms by itself.* So far, this had only been demonstrated in the "super-Weyl gauge" where only the $(\bar\chi\chi)^2$ terms had to be checked [1]. The more general result, which also includes the higher order contributions involving the gravitinos and dilatinos, is a consequence of the result (5.16) in the next section.

The spectral parameter γ is subject to the differential equations [24–26]

$$\gamma^{-1}\partial_\pm\gamma = \frac{1\mp\gamma}{1\pm\gamma}\,\rho^{-1}\partial_\pm\rho\,. \tag{5.2}$$

At a practical level, this field dependence is due to the appearance of the non-constant dilaton field ρ in the equations of motion listed in Section 2. At a more fundamental level it is linked to the presence of a hidden Witt–Virasoro symmetry of the theory and the fact that (a reformulation of) equation (5.2) may be interpreted as a linear system for the dilaton itself [20]. An important consequence of the dependence of γ on ρ and $\tilde\rho$ is that in this way the spectral parameter becomes a canonical variable of its own, having non-vanishing canonical brackets with the conformal factor.

The solution of (5.2) depends not only on the fields ρ and $\tilde\rho$, but also on an integration constant w, that is sometimes called "constant spectral parameter" (because it is the parameter relevant for the $\mathfrak{e}_9$ current algebra). It is given by

$$\gamma(\rho,\tilde\rho;w) = \frac{1}{\rho}\left(w+\tilde\rho-\sqrt{(w+\tilde\rho)^2-\rho^2}\right) \Longleftrightarrow w = \tfrac{1}{2}\rho\left(\gamma+\frac{1}{\gamma}\right)-\tilde\rho\,. \tag{5.3}$$

The function $\gamma(\rho,\tilde\rho;w)$ lives on the two-sheeted covering of the complex w plane with a field-dependent branch cut connecting the points $w_\pm = -\tilde\rho\pm\rho$ on the real w-axis. This cut disappears in the limits $\rho\to 0$ or $\tilde\rho\to\infty$, whereas it extends over the whole real axis for $\rho\to\infty$, in which case the two sheets disconnect from one another. For the later treatment, we will be particularly interested in the critical points where the function $\gamma(\rho, \tilde{\rho}; w)$ becomes independent of w. This happens at

$$\gamma(\rho \to 0) \to \begin{cases} 0 \\ \infty \end{cases}, \quad \gamma(\rho \to \infty) \to \begin{cases} \mathrm{i} \\ -\mathrm{i} \end{cases}, \quad \gamma(\tilde{\rho} \to \pm\infty) \to \begin{cases} 0 \\ \infty \end{cases}, \tag{5.4}$$

where the two values correspond to the two sheets of the covering. The transition between the two sheets is performed by $\gamma \mapsto \frac{1}{\gamma}$.

The algebra involution τ from (A.8) can be extended to an involution τ^∞ which acts on E_8-valued functions of the spectral parameter γ by combining the action on E_8 with a transition between the two sheets of γ [27,26]

$$\tau^\infty\Big(U(\gamma)\Big) \equiv \tau\left(U\left(\frac{1}{\gamma}\right)\right). \tag{5.5}$$

This involution leaves the linear system (5.1) invariant.

5.2. Non-local conserved charges

The non-local charges are obtained from the transition matrices [5]

$$U(t, x, y; w) \equiv \mathcal{P}\exp\int_x^y dz\, L_1\Big(t, z; \gamma(t, z; w)\Big) \tag{5.6}$$

$$\equiv \widehat{\mathcal{V}}^{-1}\big(t, x; \gamma(t, x; w)\big)\widehat{\mathcal{V}}\big(t, y; \gamma(t, y; w)\big) \quad \in E_8, \tag{5.7}$$

associated with the linear system (5.1). For $w \notin \mathbb{R}$ (so as to avoid any ambiguities caused by the branch cut on the real line) these transition matrices are defined uniquely and like γ live on a two-sheeted covering of the complex w-plane. Like the connection of the linear system they are invariant under the generalized involution τ^∞ (5.5). Unless specified otherwise, we will consider the sheet underlying the unit disc of the complex γ-plane.

We further define the following objects:

$$\widetilde{U}(t, x, y; w) \equiv \mathcal{V}(t, x)U(t, x, y; w)\mathcal{V}^{-1}(t, y), \quad \text{for } w \notin \mathbb{R}, \tag{5.8}$$

and the monodromy matrix [26]

$$\mathcal{M}(t, z; w) \equiv \lim_{\epsilon \to +0}\left(\mathcal{V}(z)U(t, z, y; w + \mathrm{i}\epsilon)\,\tau\big(U(t, y, z; w - \mathrm{i}\epsilon)\mathcal{V}(z)^{-1}\big)\right), \tag{5.9}$$

$$\text{for } w \in \mathbb{R}, \quad |w + \tilde{\rho}(t, z)| > |\rho(t, z)|, \quad \text{and} \quad |w + \tilde{\rho}(t, y)| < |\rho(t, y)|.$$

Unless ρ and $\tilde{\rho}$ are constant fields (in which case the solution becomes trivial anyway) one may always find points z, y which satisfy the last two conditions. Mathematically, these conditions mean, that with coordinates (t, z) the spectral parameter γ is single

[5] In this equation, we write $x \equiv x^1$, $y \equiv y^1$, etc., for the space coordinates as before.

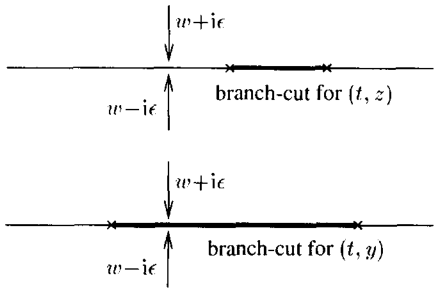

Fig. 1. Illustrating the definition of the monodromy matrix $\mathcal{M}$ in the w-plane.

valued in a neighborhood of w, whereas with coordinates (t,y) the parameter w lies on the branch cut of (5.3), implying that $w+\mathrm{i}\epsilon$ and $w-\mathrm{i}\epsilon$ tend to $\gamma(y,w)$ and $\gamma^{-1}(y,w)$, respectively, see Fig. 1. The condition on y thus guarantees that the definition of $\mathcal{M}$ indeed does not depend on y; the condition on z then basically ensures the non-triviality of $\mathcal{M}$.

According to their definition, $\widetilde{U}(t,x,y;w)$ and $\mathcal{M}(t,z;w)$ have the following time dependence:

$$\partial_t \widetilde{U}(t,x,y;w) = -\widetilde{L}_0(t,x,\gamma(t,x,w))\,\widetilde{U} + \widetilde{U}\,\widetilde{L}_0(t,y,\gamma(t,y,w)), \tag{5.10}$$

$$\partial_t \mathcal{M}(t,z;w) = -\widetilde{L}_0(t,z,\gamma(t,z,w))\,\mathcal{M} + \mathcal{M}\,\tau\left(\widetilde{L}_0(t,z,\gamma(t,z,w))\right), \tag{5.11}$$

with

$$\widetilde{L}_0 = \mathcal{V}L_0\mathcal{V}^{-1} - \partial_0\mathcal{V}\mathcal{V}^{-1}.$$

Thus, the modified transition matrix $\widetilde{U}(t,x_0,y_0;w)$ becomes time independent, and hence an integral of motion, if it connects two points x_0 and y_0 at both of which $\widetilde{L}_0$ vanishes. Similarly, the monodromy matrix $\mathcal{M}(z_0;w)$ turns into an integral of motion if $\widetilde{L}_0$ vanishes at z_0. There are two ways in which this can happen. Either the physical fields vanish at these points (as they would for instance at spatial infinity), or otherwise the spectral parameter γ vanishes (cf. (5.4)) with the physical fields remaining regular. For instance, both situations are realized for cylindrical gravitational waves, where $\rho=0$ corresponds to the origin, and $\rho=\infty$ to spatial infinity. What we would like to emphasize, however, is that the present framework allows for more general possibilities: depending on the behavior of the dilaton field (i.e. the zeroes and poles of the functions ρ and $\tilde{\rho}$) there might even exist several conserved charges. [6]

The matrix $\mathcal{M}(z_0;w)$ is of special interest. As a function of real w it is single-valued, real and satisfies

$$\mathcal{M}(w) = \tau\left(\mathcal{M}^{-1}(w)\right). \tag{5.12}$$

[6] From the Kaluza-Klein point of view, the values $\rho=0$ and ∞, respectively, correspond to internal manifolds of zero or infinite size.

We may further introduce its Riemann–Hilbert decomposition

$$\mathcal{M}(w) \equiv U_+(w)\, \tau\left(U_-^{-1}(w)\right) , \tag{5.13}$$

into E_8-valued functions $U_\pm(w)$ which are holomorphic in the upper and the lower half of the complex w-plane, respectively. For instance, with the gravitational wave boundary conditions mentioned above, the functions $U_\pm(w)$ are related to the modified transition matrices (5.8) [8] by

$$U_\pm(w) = \widetilde{U}(0,\infty;w) \quad \text{for} \quad \operatorname{Im} w \gtrless 0 . \tag{5.14}$$

It is important here that the analytic continuation of U_+ into the lower half plane does not coincide with U_-, and vice versa. Rather they are related by

$$U_+(w) = \overline{U_-(\bar{w})} .$$

The monodromy matrix $\mathcal{M}(w)$ in this case yields the values of the original physical fields on the symmetry axis $\rho = 0$ for $\tilde{\rho} = w \in \mathbb{R}$.

We note two basic differences with the flat space integrable models. First, the coordinate dependence of the spectral parameter γ has given rise to the definition of the monodromy matrix $\mathcal{M}$ (5.9) which has no analog in flat space. Secondly, (5.10) shows that for spatially periodic boundary conditions, the eigenvalues of the transition matrices are not necessarily integrals of motion. The reason is that periodicity of the physical fields is not enough to ensure periodicity of the dual potentials because even if we choose the dilaton ρ to be a periodic function, the spectral parameter γ will not be periodic due to the non-periodicity of the dual axion field $\tilde{\rho}$. It remains an open problem to reconcile periodicity with the existence of infinite dimensional duality symmetries.

5.3. *Supersymmetry of non-local charges*

As we have already pointed out, in a theory with *local* supersymmetry, time independence is not quite enough to distinguish reasonable observables (in the sense of Dirac). In addition, these must weakly commute with the full gauge algebra (4.4)–(4.9). The particular constraint associated with time translations (alias the Wheeler–DeWitt operator in the quantized theory) is just one part of this gauge algebra, and follows from the commutation of two supersymmetry constraints.

In this section we will show that the integrals of motion identified above are indeed invariant under the full gauge algebra. We first note that the modified transition matrices (5.8) and the monodromy matrix (5.9) for arbitrary values of x, y and z are invariant under the $SO(16)$ gauge transformations (2.7) generated by Φ^{IJ} from (3.7):

$$\left\{\Phi^{IJ}(u)\,, \widetilde{U}(x,y;w)\right\} = \left\{\Phi^{IJ}(u)\,, \mathcal{M}(z;w)\right\} = 0 . \tag{5.15}$$

The main result in this section is the behavior of the transition matrices (5.6) under supersymmetry transformations:

$$
\begin{aligned}
&\Big\{U(x,y;w)\,,S^I_\pm(z)\Big\}\\
&=\frac{4\gamma\,\theta(x,z,y)}{\rho(1\pm\gamma)^2}\,U(x,z;w)X^{IJ}S^J_\pm(z)U(z,y;w)\\
&\pm\frac{\gamma\,\theta(x,z,y)}{\rho(1-\gamma^2)}\,\chi^{\dot B}_\pm\left(\Gamma^I\Gamma^{JK}\right)_{\dot BA}\Phi^{JK}U(x,z;w)Y^AU(z,y;w)\\
&\pm\frac{1\mp\gamma}{1\pm\gamma}\,\chi^{\dot B}_\pm\,\Gamma^I_{A\dot B}\left(U(x,y;w)Y^A\delta(z-y)-Y^AU(x,y;w)\delta(z-x)\right)\\
&\mp\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\pm}\left(U(x,y;w)X^{IJ}\delta(z-y)-X^{IJ}U(x,y;w)\delta(z-x)\right)\,,
\end{aligned}
\tag{5.16}
$$

with

$$
\theta(x,z,y):=\begin{cases}1 & \text{for } x<z<y\\ 0 & \text{else }\ (x\neq y\neq z)\end{cases}
\tag{5.17}
$$

This result is valid in all orders of fermions, i.e. including all the cubic fermionic terms from (4.2). For the modified transition matrices $\widetilde{U}$ and the monodromy matrix $\mathcal{M}$ it implies:

$$
\begin{aligned}
&\Big\{\widetilde{U}(x,y;w)\,,S^I_\pm(z)\Big\}\\
&\approx\frac{2\gamma}{1\pm\gamma}\,\chi^{\dot B}_\pm\,\Gamma^I_{A\dot B}\,\left(\mathcal{V}Y^A\mathcal{V}^{-1}\right)\widetilde{U}(x,y;w)\,\delta(z-x)\\
&-\frac{2\gamma}{1\pm\gamma}\,\chi^{\dot B}_\pm\,\Gamma^I_{A\dot B}\,\widetilde{U}(x,y;w)\,\left(\mathcal{V}Y^A\mathcal{V}^{-1}\right)\,\delta(z-y)\\
&\pm\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\pm}\,\left(\mathcal{V}X^{IJ}\mathcal{V}^{-1}\right)\widetilde{U}(x,y;w)\,\delta(z-x)\\
&\mp\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\pm}\,\widetilde{U}(x,y;w)\,\left(\mathcal{V}X^{IJ}\mathcal{V}^{-1}\right)\,\delta(z-y)\,,
\end{aligned}
\tag{5.18}
$$

and

$$
\begin{aligned}
&\Big\{\mathcal{M}(z;w)\,,S^I_\pm(x)\Big\}\\
&\approx\frac{2\gamma}{1\pm\gamma}\,\chi^{\dot B}_\pm\,\Gamma^I_{A\dot B}\Big(\left(\mathcal{V}Y^A\mathcal{V}^{-1}\right)\mathcal{M}-\mathcal{M}\,\tau\left(\mathcal{V}Y^A\mathcal{V}^{-1}\right)\Big)\,\delta(x-z)\\
&\pm\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\pm}\Big(\left(\mathcal{V}X^{IJ}\mathcal{V}^{-1}\right)\mathcal{M}+\mathcal{M}\,\tau\left(\mathcal{V}X^{IJ}\mathcal{V}^{-1}\right)\Big)\,\delta(x-z)\,.
\end{aligned}
$$

The r.h.s. of these equations vanishes under the very same conditions that have been discussed in (5.10), (5.11) for the vanishing of $\widetilde{L}_0$. This shows that the integrals of motion obtained in the previous section are indeed superconformally invariant. Due to the form of the constraint superalgebra (4.4), the brackets of these charges with $T_{\pm\pm}$ then also vanish weakly. They are thus invariant under the full gauge algebra.

In particular, this implies our previous claim that the linear system (5.1) does not receive any quartic corrections but captures the full content of the theory. Since by supersymmetry transformations (2.18) any solution can be fixed to obey the super-Weyl gauge, the invariance of the linear system under supersymmetry shows that indeed no quartic corrections arise in the general case.

The rest of this section is devoted to an outline of the proof of (5.16). It is obtained from the general formula

$$\left\{U(x,y;v)\,,S^I_\pm(z')\right\} = \int_x^y dz\, U(x,z;v)\left\{L_1\big(z,\gamma(z;v)\big)\,,S^I_\pm(z')\right\}U(z,y;v)\,,$$

or equivalently

$$U(z',x,v)\left\{U(x,y;v)\,,S^I_\pm(z')\right\}U(y,z';v)$$
$$= \int_x^y dz\, U(z',z,v)\left\{L_1\big(z,\gamma(z;v)\big)\,,S^I_\pm(z')\right\}U(z,z',v)\,. \tag{5.19}$$

It is straightforward although lengthy to evaluate (5.19) using the form of the supersymmetry generator (4.2) and the fundamental Poisson brackets (3.2)–(3.12). Up to the higher order terms in the fermions this result had already been given in [3]. Thus it remains to check the cubic fermionic terms. As we can show, all the extra cubic terms cancel with the exception of those required to complete the supersymmetry generators in (5.16) to the full expressions given in (4.2).

There are altogether four different sources yielding cubic fermionic terms. First such terms come from the brackets involving cubic terms in the supersymmetry generators $S^I_\pm$, second from bilinear fermionic terms in the Poisson brackets (3.8) between P_0 and P_0. Third, they arise from the Poisson brackets involving $\partial_\pm\sigma$ in $S^I_\pm$ and at last, cubic terms enter when partial integration of the δ' terms in (5.19) leads to the appearance of the connection L_1 again.

To give an idea of the calculation we display the cancellation of the cubic terms proportional to $\psi_{2\pm}\psi_{2\pm}\chi_\pm$ in (5.19). According to (3.10) and (3.12) we have

$$\left\{L_1(\gamma)\,,\chi^{\dot A}_\pm\right\} = \frac{\gamma}{\rho(1\pm\gamma)^2}\,\Gamma^{IJ}_{\dot A\dot B}\chi^{\dot B}_\pm X^{IJ}\delta(z-z')$$
$$-\frac{8\mathrm{i}\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^K_{A\dot A}\psi^K_{2\pm}Y^A\delta(z-z')\,,$$

such that the cubic term $\psi_{2\pm}\chi_\pm\chi^{\dot B}_\pm$ from (4.2) gives the contribution

$$\left\{L_1(\gamma)\,,-\tfrac{1}{2}\mathrm{i}\,\rho\psi^K_{2\pm}\chi_\pm\Gamma^{IK}\chi_\pm\right\} \to \frac{-8\mathrm{i}\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\left(\Gamma^{IK}\Gamma^L\right)_{\dot A A}\psi^K_{2\pm}\psi^L_{2\pm}\chi^{\dot A}Y^A\,, \tag{5.20}$$

to the r.h.s of (5.19). Next, there comes a contribution from the bracket between P_0 in $L_1(\gamma)$ and the $\rho\chi_\pm P_\pm$ part of the supersymmetry constraint (4.2), which is due to the quadratic fermionic terms in (3.8) and reads

$$\left\{L_1(\gamma)\,,\mp\rho\chi_\pm^{\dot A}\Gamma^I_{A\dot A}P^A_\pm\right\} \rightarrow \frac{-8\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\left(\Gamma^{KL}\Gamma^I\right)_{A\dot A}\psi^K_{2\pm}\psi^L_{2\pm}\chi^{\dot A}Y^A\,. \tag{5.21}$$

Making use of (A.6) the two terms (5.20) and (5.21) sum up to

$$\frac{8i\gamma^2}{(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^K_{A\dot A}\,\psi^I_{2\pm}\psi^K_{2\pm}\chi^{\dot A}Y^A\,. \tag{5.22}$$

Several further relevant terms arise from the Poisson brackets involving the $\rho\psi_{2\pm}\partial_\pm\sigma$ term in (4.2). Namely, $\{L_1(\gamma),\pi_\rho\}$ gives rise to several bilinear fermionic terms due to the brackets (3.9), (3.11) and eventually also due to

$$\{\gamma(z)\,,\partial_\pm\sigma(z')\} = \frac{\gamma(1\mp\gamma)}{\rho(1\pm\gamma)}\,\delta(z-z')\,.$$

Altogether they sum up to

$$\left\{L_1(\gamma)\,,-\rho\partial_\pm\sigma\,\psi^I_{2\pm}\right\} \rightarrow \frac{16i\gamma^2(1\mp4\gamma+\gamma^2)}{1\pm\gamma)^4(1-\gamma^2)}\,\Gamma^K_{A\dot A}\psi^I_{2\pm}\psi^K_{2\pm}\chi^{\dot A}Y^A\,. \tag{5.23}$$

Finally, the integrand of (5.19) has terms proportional to $\delta'(z-z')$ due to

$$\left\{L_1(t)\,,\mp\rho\chi_\pm^{\dot A}\Gamma^I_{A\dot A}P^A_\pm\right\} \rightarrow \pm\frac{1\mp\gamma}{1\pm\gamma}\,\Gamma^I_{A\dot A}\chi^{\dot A}_\pm Y^A\,\delta'(z-z')\,,$$

and

$$\left\{L_1(t)\,,\pm\rho\partial_1\psi^I_{2\pm}\right\} \rightarrow \pm\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^K_{2\pm}X^{KI}\,\delta'(z-z')\,.$$

Upon partial integration in (5.19) and using (5.1) they give rise to

$$\mp\frac{1\mp\gamma}{2(1\pm\gamma)}\,\widehat{Q}^{KL}_\pm(\gamma)\Gamma^I_{A\dot A}\chi^{\dot A}_\pm\,\left[X^{KL},Y^A\right] \rightarrow \frac{8i\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\left(\Gamma^{KL}\Gamma^I\right)_{A\dot A}\psi^K_{2\pm}\psi^L_{2\pm}\chi^{\dot A}Y^A \tag{5.24}$$

and

$$\mp\frac{2\gamma}{(1\pm\gamma)^2}\,\widehat{P}^A_\pm(\gamma)\psi^K_{2\pm}\,\left[Y^A,X^{KI}\right] \rightarrow \frac{8i\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\left(\Gamma^{KI}\Gamma^L\right)_{A\dot A}\psi^K_{2\pm}\psi^L_{2\pm}\chi^{\dot A}Y^A\,. \tag{5.25}$$

The sum of (5.24) and (5.25) then yields (again with some Γ-matrix algebra (A.6))

$$\frac{-24i\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\,\Gamma^K_{A\dot A}\psi^I_{2\pm}\psi^K_{2\pm}\chi^{\dot A}Y^A\,. \tag{5.26}$$

Adding the different terms (5.22), (5.23) and (5.26) finally leads to

$$\frac{8i\gamma^2}{(1\pm\gamma)^2(1-\gamma^2)}+\frac{16i\gamma^2(1\mp4\gamma+\gamma^2)}{(1\pm\gamma)^4(1-\gamma^2)}-\frac{24i\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}=0. \tag{5.27}$$

We see, how the terms of the type $\psi_{2\pm}\psi_{2\pm}\chi_\pm$ from all the different sources eventually cancel. In a similar way all the unwanted cubic fermionic terms in (5.19) can be shown to drop out.

We have thus found an infinite number of observables in the sense of Dirac. We note that a similar transformation behavior has been observed in the supersymmetric extension of the non-linear σ-model [28–31]. There, with suitable boundary conditions the bosonic non-local charges are invariant under global supersymmetry. In our model, invariance under the local supersymmetry is an indispensable condition for meaningful observables, since supersymmetry appears as a constraint.

6. Algebra of charges and symmetries

6.1. Algebra of conserved charges

We now calculate the Poisson algebra of the conserved charges that we have derived above. As it turns out, it is entirely sufficient to compute the brackets for the connection coefficients entering the linear system (5.1). Our key result is that the fermionic contributions conspire in precisely such a way that the canonical brackets (6.3) below *are identical with the ones obtained for the purely bosonic theory*! This implies that, as far as the analysis of conserved non-local charges and their algebra is concerned, we can take over the analysis of the bosonic case in [8] practically without modification. However, the realization of the algebra and hence the quantum spaces on which the algebra eventually acts will be very different in the two cases.

The starting point of the computation is the well-known formula

$$\begin{aligned}\left\{\overset{1}{U}(x,y;v)\,,\overset{2}{U}(x',y';w)\right\}=\int_x^y dz\int_{x'}^{y'}dz'\Big(&\overset{1}{U}(x,z;v)\,\overset{2}{U}(x',z';w)\Big)\\ &\times\left\{\overset{1}{L_1}(z,\gamma(z,v))\,,\overset{2}{L_1}(z',\gamma(z',w))\right\}\\ &\times\Big(\overset{1}{U}(z,y;v)\,\overset{2}{U}(z',y';w)\Big),\end{aligned} \tag{6.1}$$

which we have given in the tensor notation explained in Appendix A.3.

Further evaluation requires the following canonical brackets between the connection coefficients in the linear system (5.1):

$$\left\{\widehat{Q}_1^{IJ}(\gamma_1),\widehat{Q}_1^{KL}(\gamma_2)\right\} = -\frac{4\gamma_1\gamma_2}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\,\delta(x-y)$$
$$\times\Big(\delta^{JK}\left(\widehat{Q}_1^{IL}(\gamma_1)-\widehat{Q}_1^{IL}(\gamma_2)\right)-\delta^{IK}\left(\widehat{Q}_1^{JL}(\gamma_1)-\widehat{Q}_1^{JL}(\gamma_2)\right)$$
$$+\delta^{IL}\left(\widehat{Q}_1^{JK}(\gamma_1)-\widehat{Q}_1^{JK}(\gamma_2)\right)-\delta^{JL}\left(\widehat{Q}_1^{IK}(\gamma_1)-\widehat{Q}_1^{IK}(\gamma_2)\right)\Big),$$
$$\left\{\widehat{Q}_1^{IJ}(\gamma_1),\widehat{P}_1^{A}(\gamma_2)\right\} = \frac{2\gamma_2^2(1-\gamma_1^2)}{\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\,\Gamma^{IJ}_{AB}\widehat{P}_1^{B}(\gamma_1)\,\delta(x-y)$$
$$-\frac{2\gamma_1\gamma_2}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\,\Gamma^{IJ}_{AB}\widehat{P}_1^{B}(\gamma_2)\,\delta(x-y),$$
$$\left\{\widehat{P}_1^{A}(\gamma_1),\widehat{P}_1^{B}(\gamma_2)\right\} = -\frac{(1-\gamma_1^2)\gamma_2^2}{\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\,\Gamma^{IJ}_{AB}\widehat{Q}_1^{IJ}(\gamma_1)\,\delta(x-y)$$
$$+\frac{(1-\gamma_2^2)\gamma_1^2}{\rho(1-\gamma_1^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\,\Gamma^{IJ}_{AB}\widehat{Q}_1^{IJ}(\gamma_2)\,\delta(x-y)$$
$$+\frac{4\delta^{AB}}{(1-\gamma_1^2)(1-\gamma_2^2)}\left(\frac{\gamma_1(1+\gamma_2^2)}{\rho(x)}+\frac{\gamma_2(1+\gamma_1^2)}{\rho(y)}\right)\delta'(x-y)$$
$$+\frac{4\gamma_1\gamma_2}{\rho^2(1-\gamma_1^2)(1-\gamma_2^2)}\,\Gamma^{IJ}_{AB}\,\Phi_{IJ}\,\delta(x-y).\qquad(6.2)$$

Here we have introduced the shorthand notation $\gamma_1 \equiv \gamma(x,v)$, $\gamma_2 \equiv \gamma(y,w)$ (as usual we suppress the dependence on the time coordinate t here). It is convenient to combine these equations into a single one by means of the index-free tensor notation introduced in Appendix A.3:

$$\left\{\overset{1}{L_1}(\gamma_1),\overset{2}{L_1}(\gamma_2)\right\}$$
$$\approx -\frac{4\,\Omega_{\mathfrak{k}}}{(1-\gamma_1^2)(1-\gamma_2^2)}\left(\frac{\gamma_1(1+\gamma_2^2)}{\rho(x)}+\frac{\gamma_2(1+\gamma_1^2)}{\rho(y)}\right)\delta'(x-y)$$
$$-\frac{4\gamma_1\gamma_2}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\left[\Omega_{so(16)},\overset{1}{L_1}(\gamma_1)+\overset{2}{L_1}(\gamma_2)\right]\delta(x-y)$$
$$-\frac{4\gamma_2^2(1-\gamma_1^2)}{\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\left[\Omega_{\mathfrak{k}},\overset{1}{L_1}(\gamma_1)\right]\delta(x-y)$$
$$-\frac{4\gamma_1^2(1-\gamma_2^2)}{\rho(1-\gamma_1^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\left[\Omega_{\mathfrak{k}},\overset{2}{L_1}(\gamma_2)\right]\delta(x-y),\qquad(6.3)$$

where we have dropped the contribution containing Φ^{IJ} so that the equality holds only on the constraint hypersurface. These brackets coincide with the ones of the purely bosonic theory [8]. This fortuitous circumstance enables us to take over the result for the bracket of two transition matrices (5.8) from [8]. Namely, inserting the above relations into (6.1) and taking into account the additional contributions of the type

$$\left\{\overset{1}{U}(x,y;w)\,,\,\overset{2}{\mathcal{V}}(z)\right\} = \frac{2\gamma(z;w)}{\rho(z)(1-\gamma^2(z;w))} \times\theta(x,z,y)\,\overset{1}{U}(x,z;w)\,\overset{2}{\mathcal{V}}(z)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(z,y;w)\,,$$

finally leads to the following algebra:

$$\begin{aligned}
&\overset{1}{\mathcal{V}^{-1}}(x)\,\overset{2}{\mathcal{V}^{-1}}(x')\left\{\overset{1}{\widetilde{U}}(x,y;v)\,,\,\overset{2}{\widetilde{U}}(x',y';w)\right\}\overset{1}{\mathcal{V}}(y)\,\overset{2}{\mathcal{V}}(y')\\
&= \frac{2}{v-w}\times\Big\{\;\theta(x,x',y)\left(\overset{1}{U}(x,x';v)\;\Omega_{\mathfrak{so}(16)}\;\overset{1}{U}(x',y;v)\,\overset{2}{U}(x',y';w)\right)\\
&\quad+\theta(x',x,y')\left(\overset{2}{U}(x',x;w)\;\Omega_{\mathfrak{so}(16)}\;\overset{1}{U}(x,y;v)\,\overset{2}{U}(x,y';w)\right)\\
&\quad-\theta(x,y',y)\left(\overset{1}{U}(x,y';v)\,\overset{2}{U}(x',y';w)\;\Omega_{\mathfrak{so}(16)}\;\overset{1}{U}(y',y;v)\right)\\
&\quad-\theta(x',y,y')\left(\overset{1}{U}(x,y;v)\,\overset{2}{U}(x',y;w)\;\Omega_{\mathfrak{so}(16)}\;\overset{2}{U}(y,y';w)\right)\Big\}\\
&\quad+\frac{2\theta(x,x',y)}{v-w}\,f(x';w,v)\left(\overset{1}{U}(x,x';v)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(x',y;v)\,\overset{2}{U}(x',y';w)\right)\\
&\quad+\frac{2\theta(x',x,y')}{v-w}\,f(x;v,w)\left(\overset{2}{U}(x',x;w)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(x,y;v)\,\overset{2}{U}(x,y';w)\right)\\
&\quad-\frac{2\theta(x,y',y)}{v-w}\,f(y';w,v)\left(\overset{1}{U}(x,y';v)\,\overset{2}{U}(x',y';w)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(y',y;v)\right)\\
&\quad-\frac{2\theta(x',y,y')}{v-w}\,f(y;v,w)\left(\overset{1}{U}(x,y;v)\,\overset{2}{U}(x',y;w)\,\Omega_{\mathfrak{k}}\,\overset{2}{U}(y,y';w)\right).
\end{aligned}\tag{6.4}$$

with θ from (5.17) and

$$f(x;v,w) \equiv \frac{1-2\gamma(x;w)\gamma(x;v)+\gamma^2(x;w)}{1-\gamma^2(x;w)}\,.$$

Let us recall that the limits of these expressions for the corresponding flat space models do not exist as the result depends on the order in which the limits are taken due to the different coefficients of the θ functions [7]. For the gravitationally coupled models, however, these ambiguities disappear altogether by virtue of the coordinate dependence of the spectral parameter if we have $\partial_w\gamma(\rho,\tilde{\rho};w) = 0$ at the limit points [8]! This is indeed the case with appropriate boundary conditions on the dilaton and its axionic partner as we have already seen in (5.4).

Thus, (6.4) yields a well-defined algebra for the modified transition matrices $\widetilde{U}$ connecting two of these critical points x_0, y_0. With

$$U(w) \equiv \widetilde{U}(x_0,y_0;w)\,, \tag{6.5}$$

the final Poisson algebra takes the form

$$\left\{ \overset{1}{U}(v), \overset{2}{U}(w) \right\} = \left[\frac{2\Omega_{\mathfrak{e}_8}}{v-w}, \overset{1}{U}(v) \overset{2}{U}(w) \right], \tag{6.6}$$

for coinciding signs of the imaginary parts of v and w. The Poisson brackets between transition matrices for v and w from different halves of the complex planes depend on the concrete behavior of ρ and $\tilde{\rho}$ in the end-points. They may e.g. coincide with (6.6) or with (6.9) below.

In contrast, the Poisson structure of the monodromy matrix $\mathcal{M}$ is universal and may be computed from (6.4) and the definition (5.9) to

$$\begin{aligned} \left\{ \overset{1}{\mathcal{M}}(v), \overset{2}{\mathcal{M}}(w) \right\} = {} & \frac{2\Omega_{\mathfrak{e}_8}}{v-w} \overset{1}{\mathcal{M}}(v) \overset{2}{\mathcal{M}}(w) + \overset{1}{\mathcal{M}}(v) \overset{2}{\mathcal{M}}(w) \frac{2\Omega_{\mathfrak{e}_8}}{v-w} \\ & - \overset{1}{\mathcal{M}}(v) \frac{2\Omega^T_{\mathfrak{e}_8}}{v-w} \overset{2}{\mathcal{M}}(w) - \overset{2}{\mathcal{M}}(w) \frac{2\Omega^T_{\mathfrak{e}_8}}{v-w} \overset{1}{\mathcal{M}}(v), \end{aligned} \tag{6.7}$$

with $\Omega_{\mathfrak{e}_8}$ and $\Omega^T_{\mathfrak{e}_8}$ from (A.10) and (A.11), respectively. One may check that indeed these brackets are compatible with the symmetry (5.12), as required for consistency. For the purpose of quantization and representation of (6.7) it is further convenient to decompose this structure according to (5.13) into the following brackets:

$$\left\{ \overset{1}{U}_\pm(v), \overset{2}{U}_\pm(w) \right\} = \left[\frac{2\Omega_{\mathfrak{e}_8}}{v-w}, \overset{1}{U}_\pm(v) \overset{2}{U}_\pm(w) \right], \tag{6.8}$$

$$\left\{ \overset{1}{U}_\pm(v), \overset{2}{U}_\mp(w) \right\} = \frac{2\Omega_{\mathfrak{e}_8}}{v-w} \overset{1}{U}_\pm(v) \overset{2}{U}_\mp(w) - \overset{1}{U}_\pm(v) \overset{2}{U}_\mp(w) \frac{2\Omega^T_{\mathfrak{e}_8}}{v-w}, \tag{6.9}$$

which may be easier to handle due to the similarity of (6.8) with the well-known Yangian algebra $Y(\mathfrak{e}_8)$ [32–34].

6.2. *Hidden symmetries: the Lie–Poisson action of $E_{9(+9)}$*

Previous studies of the Geroch group and its generalizations have been mostly concerned with the non-linear and non-local realization of these groups on the physical fields and their dual potentials at the level of the equations of motion (recall that duality symmetries in even dimensions are always on-shell). Now, for the flat space σ-models it has been known for a long time that this action is not symplectic [35,36]. Rather, it represents a Lie–Poisson action of the associated symmetry group [37,9]. Here, a similar picture emerges, since the hidden symmetries of dimensionally reduced supergravity can be recovered via a Lie–Poisson action of $\mathfrak{e}_9$, which is canonically generated by the integrals of motion $U(w)$ from (6.5) [38,8].[7]

In this section, we show how this result fits into the canonical framework established in the foregoing sections. In particular, we give the action of $\mathfrak{e}_9$ on all the fermionic fields involved in the model. Our basic objects are the transition matrices $U(w)$ from (6.5), where we assume that x_0 and y_0 are the spatial boundaries and critical in the sense

[7] See e.g. [25,26,39] for discussions of the Geroch group within the general framework of dressing transformations.

discussed above, namely that $\widetilde{L}_0$ vanishes at these points. In particular, the transition matrix $U(t, x_0, x; w)$ then provides a solution of the linear system (5.1) which we denote by $\widehat{\mathcal{V}}_0$.

We define the following matrix-valued symmetry generator

$$G(v) \equiv \mathrm{ad}_{U(v)} U^{-1}(v) , \tag{6.10}$$

where "ad" denotes the adjoint action via the canonical Poisson structure. In matrix components the action of (6.10) on an arbitrary phase space function f reads

$$G^{ab}(v)\, f \equiv \{U^{ac}(v)\, , f\} \left(U^{-1}(v)\right)^{cb} ,$$

with $E_{8(+8)}$ indices $a, b, \ldots$ (cf. Appendix A.3). Making use of (6.4) we can determine the action of (6.10) on the monodromy matrix $\mathcal{M}$:

$$\overset{1}{G}(v)\ \overset{2}{\mathcal{M}}(w) = \frac{1}{v-w}\left(\Omega_{\mathfrak{e}_8}\ \overset{2}{\mathcal{M}}(w) - \overset{2}{\mathcal{M}}(w)\ \Omega^T_{\mathfrak{e}_8}\right) . \tag{6.11}$$

This motivates the definition of the following symmetry operator

$$G[\Lambda] \equiv \oint_\ell \frac{dw}{2\pi \mathrm{i}} \operatorname{tr}\left(\Lambda(w)\, G(w)\right) , \tag{6.12}$$

parametrized by an algebra-valued function $\Lambda(w) \in \mathfrak{e}_8$, regular along the real w-axis and vanishing at $w \to \infty$. The path ℓ is chosen to encircle the real w-axis, such that $\Lambda(w)$ is holomorphic inside the enclosed area. The monodromy matrix transforms under the action of $G[\Lambda]$ as

$$G[\Lambda]\, \mathcal{M}(w) = \Lambda(w)\mathcal{M}(w) - \mathcal{M}(w)\, \tau(\Lambda(w)) . \tag{6.13}$$

The orbit of the monodromy matrix under the symmetry group fills the complete set of E_8-valued functions with symmetry (5.12) and the assumed analyticity properties on the real axis. The symmetry group thus acts transitively if the monodromy matrix parametrizes the full phase space (which e.g. is the case for cylindrical gravitational waves).

With the general formula

$$\{U(x, y; v)\, , f\} = \int_x^y dx' U(x, x'; v)\, \{L_1(x', \gamma(x', v))\, , f\}\, U(x', y; v) , \tag{6.14}$$

and the fundamental Poisson brackets (3.2)–(3.12) we can directly compute the symmetry action on the physical fields. It turns out that the relevant parameter which describes this action is the combination

$$\widehat{\mathcal{V}}_0^{-1} \Lambda \widehat{\mathcal{V}}_0 \equiv \tfrac{1}{2}\widehat{\Lambda}^{IJ} X^{IJ} + \widehat{\Lambda}^A Y^A , \tag{6.15}$$

(note that $\widehat{\Lambda}$ depends on both w and the fields, whereas $\Lambda(w)$ is coordinate independent). The symmetry action on the bosonic matrix $\mathcal{V}(x)$ becomes

$$G[\Lambda]\,\mathcal{V}(x) = \oint_{\ell} \frac{dw}{2\pi i} \left(\frac{2\gamma}{\rho(1-\gamma^2)}\,\mathcal{V}(x)\,\widehat{\Lambda}^A(\gamma(w))Y^A \right)$$

$$= -\oint_{\gamma(\ell)} \frac{d\gamma}{2\pi i\gamma} \left(\mathcal{V}(x)\,\widehat{\Lambda}^A(\gamma)Y^A \right) . \qquad (6.16)$$

The action on the fermionic fields of the model is given by

$$G[\Lambda]\,\psi^I_{2\pm} = -\oint_{\ell} \frac{dv}{2\pi i} \left(\frac{4\gamma}{\rho(1\pm\gamma)^2}\,\widehat{\Lambda}^{IJ}(\gamma)\psi^J_{2\pm} \right) ,$$

$$G[\Lambda]\,\chi^{\dot A}_\pm = -\oint_{\ell} \frac{dv}{2\pi i} \left(\frac{\gamma}{\rho(1\pm\gamma)^2}\,\Gamma^{IJ}_{\dot A\dot B}\,\widehat{\Lambda}^{IJ}(\gamma)\,\chi^{\dot B}_\pm + \frac{8\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^I_{A\dot A}\,\widehat{\Lambda}^A(\gamma)\psi^I_{2\pm} \right),$$

$$G[\Lambda]\,\psi^I_\pm = -\oint_{\ell} \frac{dv}{2\pi i} \left(\frac{4\gamma}{\rho(1\pm\gamma)^2}\,\widehat{\Lambda}^{IJ}(\gamma)\psi^J_\pm + \frac{16\gamma^2}{\rho(1\pm\gamma)^4}\,\widehat{\Lambda}^{IJ}(\gamma)\psi^J_{2\pm} \right)$$

$$\pm \oint_{\ell} \frac{dv}{2\pi i} \left(\frac{8\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^I_{A\dot B}\,\widehat{\Lambda}^A(\gamma)\,\chi^{\dot B}_\pm \right) . \qquad (6.17)$$

These transformations preserve the chirality of the fermionic fields in the following sense. The equations of motion (2.12) admit chiral solutions, i.e. solutions with either all + or all − fermionic components switched off. The action (6.17) then takes place within these sectors. The respective first terms in (6.17) are pure $SO(16)$ gauge transformations (2.7) when acting on a chiral solution. In general, they can not be absorbed by this gauge freedom owing to the different coefficients for the chiral halves. Observe also that the transformations preserve the super-Weyl gauge, where $\psi^I_2 = 0$. [8]

The homogeneous action of the symmetry on the fermionic fields has the important consequence that it cannot act transitively on the space of all solutions. It is not possible to generate fermionic from purely bosonic solutions of the classical equations of motion in this way.

The action on the conformal factor σ is given by

$$G[\Lambda]\,\sigma = \oint_{\ell} \frac{dw}{2\pi i}\,\mathrm{tr}\left(\Lambda\partial_w\widehat{\mathcal{V}}_0\widehat{\mathcal{V}}_0^{-1} \right) , \qquad (6.18)$$

in agreement with the result derived in [40]. Formula (6.18) is easily obtained from

$$G(w)\,\partial_1\sigma(x) = -\widehat{\mathcal{V}}_0(x,\gamma(w))\,\partial_w L_1(x,\gamma(w))\,\widehat{\mathcal{V}}_0^{-1}(x,\gamma(w)) ,$$

which in turn follows from (3.3), (6.14) and the fact that $\partial_w L_1 = \partial_{\tilde\rho} L_1$ (cf. (5.3)).

[8] In this gauge, the formula for the variation of $\chi^{\dot A}_\pm$ coincides with (4.2.29) of [40], which is obtained from the particular values $\Upsilon_\Lambda(x,\gamma=\mp 1)$ of the compensating $\mathfrak{h}^\infty$-rotation from (6.19) below.

Finally, we can also give the transformation behavior of the solution $\widehat{\mathcal{V}}_0$ of the linear system. Evaluating (6.4) we obtain

$$G[\Lambda]\,\widehat{\mathcal{V}}_0(x,\gamma(w)) = \Lambda(w)\widehat{\mathcal{V}}_0(x,\gamma(w)) - \widehat{\mathcal{V}}_0(x,\gamma(w))\,\mathcal{T}_\Lambda(x,\gamma(w))\,, \tag{6.19}$$

where

$$\mathcal{T}_\Lambda(x,\gamma(w)) = \oint_\ell \frac{dv}{2\pi\mathrm{i}(v-w)}\,\tfrac{1}{2}\widehat{\Lambda}^{IJ}X^{IJ} + \frac{1-\gamma^2(w)}{\gamma(w)}\oint_\ell \frac{dv}{2\pi\mathrm{i}(v-w)}\,\frac{\gamma(v)}{1-\gamma^2(v)}\,\widehat{\Lambda}^A Y^A\,,$$

The matrix $\Lambda(w)$ depends on the constant spectral parameter w; whereas $\mathcal{T}_\Lambda(x,t,\gamma(w))$ depends on the variable spectral parameter γ and satisfies

$$\mathcal{T}_\Lambda(x,\gamma(w)) = \tau^\infty\left(\mathcal{T}_\Lambda(x,\gamma(w))\right) = \tau\left(\mathcal{T}_\Lambda(x,\gamma^{-1}(w))\right)\,. \tag{6.20}$$

This result provides the link to the traditional realization of the Geroch group via the linear system. There, the space of classical solutions is formally identified with the infinite dimensional coset space $\mathbf{G}^\infty/\mathbf{H}^\infty$, where the underlying algebra $\mathfrak{g}^\infty$ is generated by $\mathfrak{e}_8$ valued functions in the w-plane while $\mathfrak{h}^\infty$ is defined as the set of $\mathfrak{e}_8$ valued functions in the γ-plane satisfying (6.20). Like in (6.12) the symmetry algebra is parametrized by a function $\Lambda(w)\in\mathfrak{g}^\infty$. Starting from a solution $\widehat{\mathcal{V}}_0(\gamma)$ which is holomorphic in the unit disc $|\gamma|\leqslant 1$, the function $\mathcal{T}_\Lambda\in\mathfrak{h}^\infty$ in (6.19) is then uniquely defined so as to restore this holomorphy which is violated by the pure action of $\Lambda(w)$. Indeed, it follows from the form of $\mathcal{T}_\Lambda$ and (6.15), that in (6.19) the r.h.s. multiplication of $\widehat{\mathcal{V}}_0$ with $\mathcal{T}_\Lambda$ removes all singularities caused by the l.h.s. multiplication with $\Lambda(w)$ from the unit disc (note that the path ℓ surrounds the unit disc in the γ-plane).

We close this section with some remarks on the algebraic structure of the symmetry. The canonical realization given in (6.12) is parametrized by meromorphic $\mathfrak{e}_8$-valued functions and yields closed expressions for the action on the physical fields. The algebra of these operators is most conveniently obtained from (6.13), which immediately shows

$$\Big[G[\Lambda_1],G[\Lambda_2]\Big] = G\Big[[\Lambda_1,\Lambda_2]\Big]\,. \tag{6.21}$$

Half of the affine algebra $\mathfrak{e}_9$ may be recovered by formal Laurent expansion around $w=\infty$:[9]

$$\Lambda(w) = I + \Lambda_1 w + \Lambda_2 w^2 + \dots\,, \tag{6.22}$$

The action of these modes on the physical fields follows from expansion of the respective closed formulas (6.16)–(6.17). In a rather formal sense (6.22) may be related to the expansion

[9] There is a slight subtlety here, since strictly speaking the functions $\Lambda(w)=\Lambda_n w^n$ do not belong to the class of functions for which we have defined (6.12). Since the integrand is singular at infinity, definition (6.12) depends on the precise choice of the contour in this region, which has not been specified above.

$$G(w) = I + \frac{1}{w} G_1 + \frac{1}{w^2} G_2 + \dots , \tag{6.23}$$

obtained from (6.10) by handling the linear system (5.1) as a formal power series in w^{-1}. Observe, however, that $G(w)$ does not allow a Laurent expansion around $w = \infty$ when the branch point moves to infinity.

The conventional realization of $\mathfrak{e}_9$ also includes a representation of the other half of the affine algebra; this is achieved by the introduction of infinitely many additional gauge degrees of freedom associated with the "maximal compact subgroup" $\mathbf{H}^\infty$ of $E_{9(+9)}$ [26,20]. Furthermore, this group possesses a central extension which acts trivially on the physical fields but shifts the conformal factor σ by a constant [27]. Although so far there does not exist a canonical formulation with an enlarged phase space capturing these extra symmetries, the proper action of the central term is implied by formula (6.18), which itself is canonically generated.

The form of the symmetry operators (6.10) shows that this action is not symplectic but satisfies

$$G(w)\, \{f_1, f_2\} = \{G(w) f_1, f_2\} + \{f_1, G(w) f_2\} + \left[G(w) f_1\,, G(w) f_2\right] , \tag{6.24}$$

on any two-phase space functions f_1, f_2, where the commutator on the r.h.s. is understood for the matrix-valued action of $G(w)$. This is an example of a Lie–Poisson action, i.e. it does not preserve the Poisson structure on the phase space but on the direct product of the phase space with the symmetry group [37,9]. This fact becomes crucial upon quantizing the structure, since it is not $\mathfrak{e}_9$ but its quadratic deformation underlying (6.6) according to the representations of which the spectrum of physical states will have to be classified.

7. Outlook

Our ultimate interest in developing the canonical framework is in quantizing $d = 2$, $N = 16$ supergravity, and this not only in view of constructing exactly solvable and "sufficiently complicated" models of matter coupled quantum gravity in two dimensions. We are equally motivated by recent developments in non-perturbative string theories, where (finite dimensional and discrete) duality symmetries play a central role.

In standard canonical quantization one converts the Poisson (or Dirac) brackets of the basic fields into (anti)commutators and the constraints into operators acting in a suitable Hilbert space. Irrespective of the possible ambiguities in this procedure, one must ensure that these operator constraints are indeed well defined in the sense that their matrix elements exist between any two states; this may require some "renormalization", such as normal ordering in string theory. The next step would be to search for physical states which by definition are annihilated by the quantum supersymmetry generators. This is a difficult task because $N = 16$ supergravity is a fully interacting theory on the world-sheet, unlike the conformal supergravities giving rise to superstring theories. If the classical constraint algebra can be transferred to the quantum theory (possibly modulo certain anomalies), the supersymmetry algebra would ensure that the physical states are also annihilated by the Hamiltonian (Wheeler–DeWitt operator), as well as the diffeomorphism and $SO(16)$ constraints.

Now, it would seem rather foolhardy to hope to be able to carry through such an ambitious program for the Lagrangian (2.6) directly. However, the integrability of the model, which is encapsulated in the existence of infinitely many conserved non-local charges, may open an alternative representation theoretic route to its quantization. Clearly, the physical states should transform as a representation of the (quantum) algebra generated by the conserved charges. In this way, these charges would give rise to a kind of spectrum generating algebra akin to the one of string theory, which is generated by the DDF operators. To properly work out these ideas may involve an extension of the techniques that have been successfully applied to the computation of form factors of the flat space non-linear $SO(3)$ σ-model [41]. [10] The crucial difference with the flat space theories is the requirement of consistency with local supersymmetry.

According to the discussion in Subsection 6.2, and contrary to one's naive expectations the physical states cannot be expected to fall into standard $\mathfrak{e}_9$-multiplets because the action of $\mathfrak{e}_9$ is not symplectic. Even if this were the case, we note that almost nothing is known about the unitary irreducible representations of $E_{9(+9)}$; even for its finite dimensional subgroup $E_{8(+8)}$, the representation theory is still only rudimentary. At any rate, a minimum requirement for analyzing the physical spectrum will thus be to find the quantum algebra underlying the classical Poisson algebras (6.7) and (6.8), (6.9). These structures are closely related to the Yangian algebra $Y(\mathfrak{e}_8)$ [32–34], however with different analyticity properties, the novel "twist" structure of (6.9) and the additional symmetry (5.12). Quantization must respect these properties. For the models with coset space $SL(N,\mathbb{R})/SO(N)$ the problem of directly quantizing (6.7) has been solved in [8] by use of several well-known results on the corresponding Yangians. Although $E_{8(+8)}$ is a much more complicated group, it turns out that the corresponding R-matrix has already been worked out in the mathematical literature [42]. Curiously, this R-matrix exists only if an extra singlet is added to the **248** of $E_{8(+8)}$; in other words, the quantum U-matrix must be extended by one row and one column to a 249 by 249 matrix if it is to obey all consistency conditions. This seems to indicate that the full quantum symmetry will involve additional degrees of freedom from the gravitational sector.

The quantization of $N = 16$ supergravity may resolve various puzzles related to the presence of fermions. While the Geroch group and its analogs are known to act transitively on the space of classical bosonic solutions (modulo some technical assumptions), we do not know how to generate fermionic solutions from purely bosonic ones (the vacuum, in particular), see the remarks accompanying (6.17). This would require a superextension of $E_{9(+9)}$. In principle, local supersymmetry can give rise to global supercharges, but only in very special backgrounds admitting Killing spinors. Performing a

[10] Another object of interest in flat space integrable quantum field theories is the exact S-matrix. However, in a theory of quantum gravity, the very notion of an S-matrix is a priori meaningless, except in those special circumstances corresponding to asymptotically flat space-times which allow for asymptotic states to exist.

Geroch transformation on such a background will destroy this property in general. This makes the existence of a superalgebra containing $\mathfrak{e}_9$ as its maximal bosonic subalgebra somewhat unlikely. A further indication that something is amiss here is that a supersymmetric generalization of the Breitenlohner–Maison cocycle formula [26] for the conformal factor has so far not been found. Lastly, it is not clear what physical significance should be attached to classical solutions depending on anticommuting c-numbers. In the quantum theory, all these problems may dissolve by themselves. The first because the quantum theory may not admit purely bosonic physical states, like simple exactly solvable models of quantum supergravity in three dimensions [43] or perturbatively treated canonical $N = 1$ quantum supergravity in four dimensions [44]. The others because the fermions become operators, and only expectation values of observables (such as the conserved charges) are physically meaningful quantities in quantum gravity.

Finally, the Geroch group and its generalizations represent infinite dimensional extensions of the duality symmetries that have played such an important role in recent developments of string theory. We have not yet studied the case of bounded "open" world-sheets and the effect of these symmetries on the boundary conditions (which in the case of open strings have lead to the discovery of D branes [45]). A first step in this direction would be to find out whether the Geroch group can be implemented with periodic boundary conditions on the space coordinates. Unlike T-duality, which simply involves an interchange of x^0 and x^1, and thus of Neumann and Dirichlet boundary conditions, we here face the challenge of making an infinite number of duality transformations compatible with the boundary conditions.

Acknowledgements

H.S. is grateful to Studienstiftung des deutschen Volkes for financial support. H.N. has benefitted from discussions with B. Julia on Lie–Poisson structures, and would like to thank B. Zumino for pointing out the possible relevance of Yangians for dimensionally reduced gravity already some time ago. We wish to thank K. Koepsell for discussions on $Y(\mathfrak{e}_8)$.

Appendix A. Conventions

A.1. Metric and spinor conventions

Throughout this paper we work with the flat metric $\eta_{\alpha\beta} = \mathrm{diag}(+-)$. For any vector V_α and covector V^α, we define the light-cone components

$$V_\pm := \tfrac{1}{2}(V_0 \pm V_1)\,, \qquad V^\pm := V^0 \pm V^1 = V_0 \mp V_1\,, \tag{A.1}$$

respectively, so that $V^\alpha W_\alpha = V_+ W_- + V_- W_+$.

The γ-matrices obey

$$\gamma_\alpha\gamma_\beta = \eta_{\alpha\beta} + \epsilon_{\alpha\beta}\gamma^3\,, \qquad \gamma^3\gamma_\alpha = \epsilon_{\alpha\beta}\gamma^\beta\,, \qquad \gamma^3\gamma_\pm = \mp\gamma_\pm \tag{A.2}$$

with $\epsilon_{01} = -\epsilon^{01} = 1$, and are explicitly given by

$$\gamma_0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}$$

We thus make use of the Majorana representation where the charge conjugation matrix is $\mathcal{C} = \gamma_0$, such that a Majorana spinor obeying $\overline{\psi} = \psi^T\mathcal{C}$ has two real components. Any Majorana–Weyl spinor decomposes as

$$\tfrac{1}{2}(1 \pm \gamma^3)\psi \equiv \begin{pmatrix} \psi_\pm \\ \pm\psi_\pm \end{pmatrix} \Longrightarrow \gamma_\pm(1 \mp \gamma^3)\psi = 0\,. \tag{A.4}$$

The one-component spinors $\psi_\pm$, etc., are thus to be treated as real anticommuting variables. Let us also give some useful rules for the transcription between two-component and one-component notation:

$$\overline{\psi}\chi = 2\mathrm{i}(\psi_+\chi_- - \psi_-\chi_+)\,, \qquad \overline{\psi}\gamma^3\chi = -2\mathrm{i}(\psi_+\chi_- + \psi_-\chi_+)\,,$$
$$\overline{\psi}\gamma_+\chi = 2\psi_+\chi_+, \qquad \overline{\psi}\gamma_-\chi = 2\psi_-\chi_-.$$

A.2. E_8 conventions

Under its $SO(16)$ subgroup, the fundamental (=adjoint) representation of $E_{8(+8)}$ decomposes as $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$. We denote the 120 generators of the maximal compact subgroup $SO(16)$ by $X^{IJ} = -X^{JI}$ with $SO(16)$ vector indices $I, J, \ldots = 1, \ldots, 16$, and the non-compact generators by Y^A, where $A, B, \ldots$ (and $\dot{A}, \dot{B}, \ldots$) $= 1, \ldots, 128$ label the left (right) handed spinor representation of $SO(16)$. The defining relations for the Lie algebra $\mathfrak{e}_8$ are

$$\begin{aligned} \left[X^{IJ}, X^{KL}\right] &= \delta^{JK}X^{IL} - \delta^{IK}X^{JL} + \delta^{IL}X^{JK} - \delta^{JL}X^{IK}\,, \\ \left[X^{IJ}, Y^A\right] &= -\tfrac{1}{2}\Gamma^{IJ}_{AB}Y^B\,, \qquad \left[Y^A, Y^B\right] = \tfrac{1}{4}\Gamma^{IJ}_{AB}X^{IJ}\,, \end{aligned} \tag{A.5}$$

where the Γ^{IJ}_{AB} denote the $SO(16)$-Γ-matrices which fulfill

$$\Gamma^I_{A\dot{A}}\Gamma^J_{\dot{A}B} = \delta^{IJ}_{AB} + \Gamma^{IJ}_{AB}\,. \tag{A.6}$$

In the adjoint representation the generators are normalized such that

$$\mathrm{tr}\,\left(X^{IJ}X^{KL}\right) = -120\,\delta^{IJ}_{KL}\,, \qquad \mathrm{tr}\,\left(Y^AY^B\right) = 60\,\delta^{AB}\,.$$

For the current

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \tfrac{1}{2}Q^{IJ}_\mu X^{IJ} + P^A_\mu Y^A\,,$$

this yields

$$Q^{IJ}_\mu = -\tfrac{1}{60}\mathrm{tr}\,\left(X^{IJ}\mathcal{V}^{-1}\partial_\mu\mathcal{V}\right)\,, \qquad P^A_\mu = \tfrac{1}{60}\mathrm{tr}\,\left(Y^A\mathcal{V}^{-1}\partial_\mu\mathcal{V}\right)\,.$$

We denote the splitting of $\mathfrak{e}_8$ into its compact and non-compact part as

$$\mathfrak{e}_8 = \mathfrak{so}(16) \oplus \mathfrak{k}, \tag{A.7}$$

where $\mathfrak{k}$ is (as a vector space) generated by the Y^A. This splitting defines the involutive algebra-automorphism τ

$$\tau\left(X^{IJ}\right) = X^{IJ}, \qquad \tau\left(Y^A\right) = -Y^A. \tag{A.8}$$

By exponentiation this involution is lifted to the group E_8.

A.3. Tensor conventions

Finally, we explain the index-free tensor notation for $E_{8(+8)}$, in terms of which some formulas in the main text can be cast into a more compact form. We consider the 248-dimensional adjoint matrix representation of $\mathfrak{e}_8$, on which by integration also $E_{8(+8)}$ is represented. We label the matrix indices by $a, b, c, \ldots = 1, \ldots, 248$.

For any matrix A^{ab} we define the corresponding matrices acting in the tensor product of two representation spaces:

$$\overset{1}{A} \equiv A \otimes I \qquad \text{and} \qquad \overset{2}{A} \equiv I \otimes A.$$

In components this takes the form $(A \otimes I)^{ab,cd} \equiv A^{ab}\delta^{cd}$ and $(I \otimes A)^{ab,cd} \equiv \delta^{ab}A^{cd}$. Following [46] we then introduce the matrix notation for Poisson brackets:

$$\left\{ \overset{1}{A}, \overset{2}{B} \right\}^{ab,cd} \equiv \left\{ A^{ab}, B^{cd} \right\}, \tag{A.9}$$

for matrices A^{ab}, B^{cd}. In this notation the canonical brackets (3.5) become

$$\left\{ \overset{1}{\mathcal{V}^{-1}\partial_1 \mathcal{V}} (x), \overset{2}{\Pi} (y) \right\} = \Omega_{\mathfrak{e}_8} \delta(x-y),$$

where the Casimir element $\Omega_{\mathfrak{e}_8}$ of $\mathfrak{e}_8$ is defined as

$$\Omega_{\mathfrak{e}_8} \equiv \Omega_{\mathfrak{so}(16)} + \Omega_{\mathfrak{k}} \equiv \tfrac{1}{2} X^{IJ} \otimes X^{IJ} - Y^A \otimes Y^A \in \mathfrak{e}_8 \otimes \mathfrak{e}_8. \tag{A.10}$$

In addition, we need the following "twisted" Casimir element which appears in the Poisson brackets (6.7), (6.9):

$$\Omega^{\tau}_{\mathfrak{e}_8} \equiv \Omega_{\mathfrak{so}(16)} - \Omega_{\mathfrak{k}} \equiv \tfrac{1}{2} X^{IJ} \otimes X^{IJ} + Y^A \otimes Y^A \in \mathfrak{e}_8 \otimes \mathfrak{e}_8. \tag{A.11}$$

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