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Supersymmetric Lorentz-covariant hyperspaces and self-duality equations in dimensions greater than (4|4)

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Abstract

We generalise the notions of supersymmetry and superspace by allowing generators and coordinates transforming according to more general Lorentz representations than the spinorial and vectorial ones of standard lore. This yields novel $SO(3,1)$-covariant superspaces, which we call hyperspaces, having dimensionality greater than $(4|4)$ of traditional super-Minkowski space. As an application, we consider gauge fields on complexifications of these superspaces; and extending the concept of self-duality, we obtain classes of completely solvable equations analogous to the four-dimensional self-duality equations. © 1997 Elsevier Science B.V.

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1. Introduction

1.1. Hyperspaces

Supersymmetry and *self-duality* have both yielded very fruitful geometric concepts for recent developments in field theory, string physics and differential geometry. It seems that generalisations of both these ideas provide a broader framework for possible further

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applications, both mathematical and physical. The purpose of this paper is to describe certain generalisations of these notions. Specifically, we consider generalised superspaces, which we in general call *hyperspaces*, coordinated by some finite subset from the set of general Lorentz tensors $\{Y^{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\}$ with $a = 0, \frac{1}{2}, \ldots$ and $\dot{a} = 0, \frac{1}{2}, \ldots$, in standard two-spinor notation. These tensors are separately symmetrical in their $2a$ undotted and $2\dot{a}$ dotted indices and transform according to the $(a, \dot{a})$ representation of the Lorentz group. This is a method of parametrising spaces of arbitrary dimensionality in a manifestly four-dimensional Lorentz-covariant fashion. We introduce gauge fields on such hyperspaces and, on complexifications of these spaces, extend the notion of *self-duality* by requiring certain irreducible components of the curvature tensor to vanish, just as the familiar self-duality condition is tantamount to the vanishing of the $(0, 1)$ component of the field strength tensor.

1.2. Construction of hyperspaces

Our construction of hyperspaces is modeled on standard superspace. The latter is constructed as a coset space: the super-Poincaré group over the Lorentz group. This follows the description of Minkowski space as the coset of the Poincaré group by the Lorentz group. Now, factoring out the Lorentz group from the super-Poincaré group yields a space of dimension $(4|4)$ with four odd (fermionic) coordinates $Y^{\alpha}, Y^{\dot{\alpha}}$ transforming according to the spinorial $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ representations of the Lorentz group in addition to the four even (bosonic) vectorial $(\frac{1}{2},\frac{1}{2})$ coordinates $Y^{\alpha\dot{\alpha}}$ of standard Minkowski space. A representation of the super-Poincaré algebra is given by vector fields on superspace. The super translation vector fields $(X_{\alpha}, X_{\dot{\beta}}, X_{\alpha\dot{\beta}})$, built with the same odd $((\frac{1}{2},0)$ and $(0,\frac{1}{2}))$ and even $((\frac{1}{2},\frac{1}{2}))$ representations as the corresponding coordinates, realise the superalgebra

$$\left\{X_{\alpha}, X_{\dot{\beta}}\right\} = 2iX_{\alpha\dot{\beta}}, \tag{1}$$

with all other supercommutators (i.e. commutators between two even tensors and between one even and one odd tensor and anticommutators between two odd tensors) equal to zero. These vector fields together with the elements of the Lorentz algebra realise the super-Poincaré algebra. In the standard coordinate basis, they have the following non-zero supercommutation relations with the superspace coordinates:

$$\begin{gathered}\left[X_{\alpha\dot{\alpha}}, Y^{\beta\dot{\beta}}\right] = \delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}}, \qquad \left\{X_{\alpha}, Y^{\beta}\right\} = \delta_{\alpha}^{\beta}, \qquad \left\{X_{\dot{\alpha}}, Y^{\dot{\beta}}\right\} = \delta_{\dot{\alpha}}^{\dot{\beta}}, \\ \left[X_{\alpha}, Y^{\beta\dot{\beta}}\right] = i\delta_{\alpha}^{\beta}Y^{\dot{\beta}}, \qquad \left[X_{\dot{\alpha}}, Y^{\beta\dot{\beta}}\right] = i\delta_{\dot{\alpha}}^{\dot{\beta}}Y^{\beta},\end{gathered} \tag{2}$$

in virtue of which, the X's can be realised in terms of a holonomic basis of partial derivatives with respect to the Y's thus:

$$X_{\alpha\dot{\beta}} = \frac{\partial}{\partial Y^{\alpha\dot{\beta}}}, \qquad X_{\alpha} = \frac{\partial}{\partial Y^{\alpha}} + iY^{\dot{\beta}}\frac{\partial}{\partial Y^{\alpha\dot{\beta}}}, \qquad X_{\dot{\beta}} = \frac{\partial}{\partial Y^{\dot{\beta}}} + iY^{\alpha}\frac{\partial}{\partial Y^{\alpha\dot{\beta}}}. \tag{3}$$

Now, in the standard case [1] of the super-Poincaré algebra, one insists that the commutator of an odd element $((\frac{1}{2},0)$ or $(0,\frac{1}{2}))$ with the even element $((\frac{1}{2},\frac{1}{2}))$ vanishes. In particular, no $(1,\frac{1}{2})$ element $X_{\alpha\beta\dot{\beta}}$ is allowed to appear in the fashion

$$\left[X_\alpha, X_{\beta\dot{\beta}}\right] = X_{\alpha\beta\dot{\beta}} + \dots \tag{4}$$

This restriction, however, can be lifted and we may indeed think of the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ elements as generating successively higher-spin elements of a generalised superalgebra. So, for instance, the further action of the $(\frac{1}{2},0)$ element on the $(1,\frac{1}{2})$ can yield a $(\frac{3}{2},\frac{1}{2})$ element,

$$\left\{X_\alpha, X_{\beta\gamma\dot{\beta}}\right\} = X_{\alpha\beta\gamma\dot{\beta}} + \dots \tag{5}$$

and so on. The new elements $X_{\alpha\beta\dot{\beta}}, X_{\alpha\beta\gamma\dot{\beta}}, \dots$ may be realised as vector fields on a generalised superspace, with coordinates $Y^{\alpha\beta\dot{\alpha}}, Y^{\alpha\beta\gamma\dot{\alpha}}, \dots$ beyond the traditional $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$ representations. These coordinates are interpreted as coordinates of extra dimensions (in possibly a Kaluza–Klein sense) of a higher-dimensional superspace. The Lorentz invariance, however, remains that of four-dimensional space. This is therefore a way of going to a higher-dimensional space whilst maintaining the four-dimensional Lorentz structure fixed to the coordinate system. For instance, the simple bosonic extension of four-dimensional space with coordinates $(Y^{\alpha\dot{\alpha}}, Y^{\alpha\beta\gamma\dot{\alpha}})$ has dimension $4+8=12$, or one with coordinates $(Y^{\alpha\dot{\alpha}}, Y^{\alpha\beta\dot{\alpha}\dot{\beta}})$ has dimension $4+9=13$.

We thus consider generalised superalgebras $\mathcal{A}$, with elements X taken as a finite subset of the set of general Lorentz tensors of the form $\{X_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\}$ with $a = 0, \frac{1}{2}, \dots$ and $\dot{a} = 0, \frac{1}{2}, \dots$; the same representations possibly appearing more than once. These tensors are, like the corresponding coordinates Y, separately symmetrical in their $2a$ undotted and $2\dot{a}$ dotted indices and transform according to the $(a,\dot{a})$ representation of the Lorentz group. Thus including elements of higher Lorentz spin, represented by vector fields on correspondingly generalised superspaces, we generalise the idea of standard superspace. More precisely, our superalgebras $\mathcal{A}$ can be represented by vector fields on hyperspaces $\mathcal{M}$ with coordinates in one-to-one correspondence with the elements of $\mathcal{A}$ modulo perhaps the Lorentz generators, analogously to the realisation of superspace with coordinates $(Y^{\alpha\dot{\alpha}}, Y^\alpha, Y^{\dot{\alpha}})$ in one-to-one correspondence with the elements of the super-Poincaré algebra modulo the Lorentz algebra.

We shall assume that all Lorentz tensors $\{T(a,\dot{a}) = T_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\}$ take values in a $\mathbb{Z}_2$-graded (super-) vector space. The degree or parity of a tensor with respect to the $\mathbb{Z}_2$-grading, $d(T)$, is defined by $2(a+\dot{a})$ mod 2 and T will be called bosonic (and taken to be grassmann-even) if $d(T)=0$ and fermionic (and taken to be grassmann-odd) if $d(T)=1$. The supercommutator or graded bracket between two tensors is defined by

$$\left[A(a,\dot{a}), B(b,\dot{b})\right] = A(a,\dot{a})B(b,\dot{b}) - (-1)^{d(A)d(B)}B(b,\dot{b})A(a,\dot{a}). \tag{6}$$

It is automatically graded skew-symmetric,

$$\left[A(a,\dot{a}), B(b,\dot{b})\right] = -(-1)^{d(A)d(B)}\left[B(b,\dot{b}), A(a,\dot{a})\right], \tag{7}$$

and satisfies the super-Jacobi identity

$$\begin{aligned}&\big[A(a,\dot a),[B(b,\dot b),C(c,\dot c)]\big]\\&\quad+(-1)^{d(A)(d(B)+d(C))}\big[B(b,\dot b),[C(c,\dot c),A(a,\dot a)]\big]\\&\quad+(-1)^{d(C)(d(A)+d(B))}\big[C(c,\dot c),[A(a,\dot a),B(b,\dot b)]\big]=0.\end{aligned} \tag{8}$$

When both A and B are known to be odd, we shall follow the custom of denoting this bracket by the anticommutator $\{A,B\}$. Although our framework is rather more general than that in which the spin-statistics theorem is valid, we assume that objects with an odd (resp. even) number of (dotted plus undotted) indices are Grassmann-odd or fermionic (resp. Grassmann-even or bosonic). We note that this is not necessarily the case. For instance Lie- (rather than super-) algebra extensions of the d-dimensional Poincaré algebra acting on spaces with Grassmann-even spinorial coordinates have recently been classified [2].

The (Lorentz-invariant) structure constants of the algebra $\mathcal{A}$ encode the non-holonomic nature (torsion) of the vector fields X on $\mathcal{M}$. By solving the super-Jacobi identities for the X's, we shall construct some explicit examples of higher-spin algebras and corresponding superspaces. We shall not a priori insist on Poincaré symmetry, but consider the most general associative superalgebras generated by these Lorentz tensors. The action of the X's on the Y's is given by supercommutation relations generalising (2) and consistent choices of coordinate bases afford determination by solving the super-Jacobi identities involving supercommutations of X's with Y's.

Our consideration is more general than that of [1] in that we allow the appearance of vector fields of spin greater than one and although our construction is Lorentz-covariant, we moreover do not, a priori, demand four-dimensional Poincaré invariance. The classification of [1], which was restricted to extensions of Poincaré symmetry in four dimensions, was generalised in [3] to higher dimensions, where the supersymmetrisations of *higher-dimensional* Poincaré (and de Sitter) symmetry were considered. Our generalisation to higher dimensions, on the other hand, maintains manifest *four-dimensional* Lorentz covariance. In that we allow the existence of generators of spin greater than one, our consideration is close in spirit to that of Fradkin and Vasiliev [4], who were concerned with realising higher-spin superalgebras $\mathcal{A}$ on fields in four-dimensional de Sitter space. They considered the higher-spin generators as giving rise to higher-spin fields, whose consistent dynamics in the curved de Sitter space described by the spin 2 field, however, required $\mathcal{A}$ to be infinite-dimensional, with an action on a chain of fields having spins all the way up to infinity. We however interpret the higher-spin generators of $\mathcal{A}$ as momenta in extra dimensions coordinated by higher-spin coordinates. We do not make any a priori field theoretical or dynamical requirements. In particular, we realise our algebra in flat space. The super-Jacobi identities therefore afford any number of finite-dimensional solutions and the maximal spin can be chosen at will.

1.3. Self-duality

In this paper we do not pursue the interesting possibilities for Lagrangian field theories on our hyperspaces $\mathcal{M}$, nor do we attempt to describe higher-spin dynamics using the algebras $\mathcal{A}$. There are many such exciting possibilities for future work, extending, for instance, the early considerations of Fierz [5] on higher-spin dynamics, or the more recent investigations of Fradkin and Vasiliev [4,6] on the realisation of higher-spin superalgebras $\mathcal{A}$ on interacting fields including gravity. We restrict ourselves here to one simple field-theoretical application: We consider gauge fields on the hyperspaces $\mathcal{M}$. Since the vector fields X act as superderivations on functions of Y, they can be gauge-covariantised by adding a gauge potential A transforming according to the same representation of the Lorentz group as X. Commutators of gauge-covariantised vector fields, i.e. of the $\mathcal{A}$-covariant derivatives, then yield curvatures which decompose into irreducible representations of the Lorentz group. Without pursuing the question of Lagrangian field theories for such generalised gauge fields, we presently investigate the possibility of generalising the very fruitful notion of *self-duality* to hyperspaces $\mathcal{M}$. This yields interesting classes of solvable gauge-invariant systems in superspaces of basically arbitrary dimensionality.

Euclidean space self-duality equations have played a central role in the search for classical solutions to gauge theories in virtue of transforming the second-order field equations into simpler first-order ones. Originally introduced in four-dimensional spaces, the idea of considering algebraic curvature constraints as a means to solving the second-order Yang–Mills equations was extended in a natural way to Euclidean spaces of higher dimensions in [7], where systems of first-order equations for the gauge potential were constructed, which imply the second-order Yang–Mills equations and which are invariant under some subgroup H of the d-dimensional rotation group $SO(d)$. This generalisation of self-duality concerned the construction of a fourth-rank H-invariant tensor $T_{\mu\nu\rho\sigma}$ which could be used instead of the four-dimensional $SO(4)$-invariant tensor $\epsilon_{\mu\nu\rho\sigma}$. Then, the eigenvalue equations for the tensor T, namely

$$T_{\mu\nu\rho\sigma}F^{\rho\sigma} = \lambda F_{\mu\nu}, \tag{9}$$

generalise four-dimensional self-duality in that they are algebraic curvature constraints which imply the Yang–Mills equations in virtue of the Bianchi identities. Projections to distinct eigenspaces of T, with eigenvalues $\{\lambda\}$, correspond to generalisations of self- and anti-self-dual parts of the curvature. This construction therefore generalises the role of the four-dimensional Hodge-duality operator as an endomorphism of the space of two-forms with self- and anti-dual eigenspaces. In the present paper, however, we generalise four-dimensional self-duality in another direction.

In two-spinor notation the commutator of two covariant derivatives manifestly displays the irreducible representation content of the gauge field:

$$\left[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}\right] = \epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta} + \epsilon_{\alpha\beta}F_{\dot{\alpha}\dot{\beta}}. \tag{10}$$

The imposition of self-duality (resp. anti-self-duality) is just a statement of the vanishing of the $(0,1)$ component $F_{\dot{\alpha}\dot{\beta}}$ (resp. the $(1,0)$ component $F_{\alpha\beta}$). Equivalently, one can say that the self-dual curvature contains only the $(1,0)$ Lorentz representation, i.e.

$$\left[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}\right] = \epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} \quad \Leftrightarrow \quad F_{\dot{\alpha}\dot{\beta}} = 0, \tag{11}$$

and analogously for the anti-self-dual case. As these equations show, the use of two-spinor notation is not only a very convenient way of manifestly displaying the irreducible parts of the field-strength tensor, but this decomposition is also revealed to be equivalent to the decomposition in eigenstates of the Hodge-duality operator. This equivalence is central to many of the wonderful mathematical properties of the self-duality equations. So instead of using the above-mentioned T-tensor construction of [7], we could equally generalise the alternative notion that *self-duality* corresponds to the absence of certain irreducible representations in the decomposition of generalised curvature tensors on $\mathcal{M}$. The imposition of such 'coherent' curvature constraints on $\mathcal{M}$ thus maintains the usual four-dimensional rotation group $SO(4)$ as a basic symmetry of the equations. It is this generalisation of self-duality which we pursue in this paper; this generalisation being more immediately applicable to superspaces with both odd and even parts than the T-tensor construction.

Self-duality equations have recently drawn renewed attention as unifying systems for lower-dimensional integrable equations [8] and it has been suggested that the twistor transform could be the 'mother' of lower-dimensional transforms which render the latter completely integrable (like the inverse scattering transform). Many coherent curvature constraints on $\mathcal{M}$ also arise as integrability conditions for linear systems and the advantage of manifest four-dimensional covariance is that a generalised twistor-type transformation is easily constructed. We shall discuss some interesting classes of linear systems, generalising not only those for the self-duality and anti-self-duality equations, but also the linear system for the conventional supercurvature constraints of superspace Yang–Mills theories. The latter, for the $N = 3$ extension, we recall, are equivalent to the full super-Yang–Mills equations [9]. Our discussion of linear systems for coherent curvature constraints generalises Ward's approach to completely solvable curvature constraints in dimensions greater than four [10].

1.4. Plan of paper and notation

The plan of this paper is as follows. In Section 2 we introduce coordinates and $\mathcal{A}$-covariant translation vector fields on $\mathcal{M}$, which satisfy, for consistency, the super-Jacobi identities discussed in Section 3. An explicit novel example of a hyperspace $\mathcal{M}$, containing coordinates and covariant derivatives with spins up to $\frac{3}{2}$ is presented in Section 4. The introduction of gauge fields on such superspaces is discussed in Section 5. Curvature constraints generalising self-duality and integrability conditions for them are discussed in Section 6 and explicit examples are given. We deal mainly with '$N = 1$' superspaces $\mathcal{M}$, with coordinates of any given Lorentz type appearing only once. Some details of the extension to examples with certain Lorentz representations appearing multiply is given in Appendix A. The analogues of self-duality discussed in Section 6 are soluble in the sense that the familiar self-duality equations are: in virtue of a twistor transform to freely specifiable holomorphic data.

We use two-spinor language with dotted and undotted indices raised and lowered by the skew-symmetric symplectic invariant tensors $\epsilon_{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta}, \epsilon^{\alpha\beta}, \epsilon^{\dot\alpha\dot\beta}$, with $\epsilon_{12} = 1 = \epsilon^{21}$. The generators of Lorentz transformations satisfy

$$\begin{aligned}
\left[M_{\alpha\beta}, M_{\gamma\delta}\right] &= \epsilon_{\alpha\gamma}M_{\beta\delta} + \epsilon_{\alpha\delta}M_{\beta\gamma} + \epsilon_{\beta\gamma}M_{\alpha\delta} + \epsilon_{\beta\delta}M_{\alpha\gamma},\\
\left[M^{\dot\alpha\dot\beta}, M^{\dot\gamma\dot\delta}\right] &= \epsilon^{\dot\alpha\dot\gamma}M^{\dot\beta\dot\delta} + \epsilon^{\dot\alpha\dot\delta}M^{\dot\beta\dot\gamma} + \epsilon^{\dot\beta\dot\gamma}M^{\dot\alpha\dot\delta} + \epsilon^{\dot\beta\dot\delta}M^{\dot\alpha\dot\gamma},\\
\left[M_{\alpha\beta}, M^{\dot\gamma\dot\delta}\right] &= 0,
\end{aligned} \tag{12}$$

with $M_{\alpha\beta}$ and $M^{\dot\alpha\dot\beta}$ acting respectively on *undotted* and *dotted* indices. The 2-spinor notation is particularly suited to the description of half-integer spin representations. In fact, it brings to light the fact that when the algebra of the Lorentz group is extended by allowing combinations of generators with complex coefficients, rather than real ones, the algebra can be split formally into two commuting $SU(2)$'s, i.e. that the complex extension of the Lorentz group, $SO(4,\mathbb{C})$, is locally isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. For physical applications, care needs to be taken in imposing appropriate hermiticity conditions at the end so that the theory transforms according to the appropriate real form of $SO(4,\mathbb{C})$. In particular, in the Lorentzian case, when the four-dimensional $Y^{\alpha\dot\alpha}$-subspace, $\mathcal{M}_4$, has $(3,1)$ signature, the real form is the simple group $SL(2,\mathbb{C})$ and dotted and undotted indices are related by complex conjugation. This relation no longer holds if the Lorentz group is taken to be either of the other possible cases $SO(4) = SU(2) \times SU(2)$ (for $\mathcal{M}_4$ having Euclidean $(4,0)$ signature) or $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (corresponding to an $\mathcal{M}_4$ with a kleinian $(2,2)$ signature). We shall, however, deal mainly with the complex extension. In particular, complexification lifts the Minkowski space conjugation between dotted and undotted spinor representations and allows the imposition of constraints like the above $F_{\dot\alpha\dot\beta} = 0$, leaving the choice of hermiticity conditions to be decided later according to the physical application being considered.

We shall use the multi-index notation $[A], [B], [\dot A]$ and $[\dot B]$ to denote sets of, respectively, $2a, 2b, 2\dot a$ and $2\dot b$ symmetrized indices ($a, b, \dot a$ and $\dot b$ being integers or half-integers),

$$\begin{aligned}
[A] &= \alpha_1\alpha_2\cdots\alpha_{2a}, \qquad & [B] &= \beta_1\beta_2\cdots\beta_{2b},\\
[\dot A] &= \dot\alpha_1\dot\alpha_2\cdots\dot\alpha_{2\dot a}, \qquad & [\dot B] &= \dot\beta_1\dot\beta_2\cdots\dot\beta_{2\dot b}.
\end{aligned} \tag{13}$$

Similarly $[A_p]$ (resp. $[\dot A_p]$) will denote the set $[A]$ (resp. $[\dot A]$) with the index α_p (resp. $\dot\alpha_p$) missing. Using the ϵ's we also define multi-index epsilon tensors

$$\epsilon_{[\alpha_s\beta_s]} = \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\cdots\epsilon_{\alpha_s\beta_s}, \qquad \epsilon^{[\dot\alpha_s\dot\beta_s]} = \epsilon^{\dot\alpha_1\dot\beta_1}\epsilon^{\dot\alpha_2\dot\beta_2}\cdots\epsilon^{\dot\alpha_s\dot\beta_s}, \tag{14}$$

where $\epsilon^{[\alpha_0\beta_0]} = \epsilon_{[\alpha_0\beta_0]} = \epsilon^{[\dot\alpha_0\dot\beta_0]} = \epsilon_{[\dot\alpha_0\dot\beta_0]} = 1$, and their inverses

$$\epsilon^{[\beta_s\alpha_s]} = \epsilon^{\beta_1\alpha_1}\epsilon^{\beta_2\alpha_2}\cdots\epsilon^{\beta_s\alpha_s}, \qquad \epsilon_{[\dot\beta_s\dot\alpha_s]} = \epsilon_{\dot\beta_1\dot\alpha_1}\epsilon_{\dot\beta_2\dot\alpha_2}\cdots\epsilon_{\dot\beta_s\dot\alpha_s}, \tag{15}$$

which satisfy $\epsilon^{[\beta_s\alpha_s]}\epsilon_{[\alpha_s\beta_s]} = 2^s$. Moreover, we shall denote by $S[A]$, the symmetrisation operator which symmetrically sums over all the $(2a)!$ permutations of the indices in $[A]$.

Using this notation, an irreducible tensor $T(a,\dot a) \equiv T^{[\dot A]}_{[A]}$, symmetric in its $2a$ undotted and $2\dot a$ dotted indices transforms under (12) as

$$\left[M_{\beta\gamma}, T^{[\dot A]}_{[A]}\right] = \sum_{p=1}^{2a} \epsilon_{\beta\alpha_p} T^{[\dot A]}_{[A_p]\gamma} + \sum_{p=1}^{2a} \epsilon_{\gamma\alpha_p} T^{[\dot A]}_{[A_p]\beta},$$

$$\left[M^{\dot\beta\dot\gamma}, T^{[\dot A]}_{[A]}\right] = \sum_{p=1}^{2\dot a} \epsilon^{\dot\beta\dot\alpha_p} T^{[\dot A_p]\dot\gamma}_{[A]} + \sum_{p=1}^{2\dot a} \epsilon^{\dot\gamma\dot\alpha_p} T^{[\dot A_p]\dot\beta}_{[A]}, \tag{16}$$

and corresponds to an irreducible $(a,\dot a)$ representation of the Lorentz group having dimension $(2a+1)(2\dot a+1)$. When using the multi-indices, we shall write, for visual clarity, the undotted ones lowered and the dotted ones raised. Spinor indices can of course always be raised or lowered Lorentz-covariantly using the epsilon tensors. The 'spin' content of a tensor $T(a,\dot a)$ (i.e. its behaviour under 'space rotation', the diagonal $su(2)$ algebra of (12)), is given by the decomposition

$$a \otimes \dot a = (a+\dot a) \oplus (a+\dot a-1) \oplus \ldots \oplus |a-\dot a|. \tag{17}$$

2. The higher-spin superalgebra $\mathcal{A}$ and the hyperspace $\mathcal{M}$

2.1. Hypersymmetries

We consider a set of Lorentz tensors $\{X(a,\dot a)\}$, transforming according to (16). Incorporating the non-holonomy (1) of the super-Poincaré algebra, we postulate, as defining relations for hypersymmetry algebras $\mathcal{A}$, the most general Lorentz-covariant super-commutation relations having a right-hand side linear in the X's:

$$\left[X^{[\dot A]}_{[A]}, X^{[\dot B]}_{[B]}\right] = \sum_{s=0}^{\min(2a,2b)} \sum_{\dot s=0}^{\min(2\dot a,2\dot b)} t(a,\dot a;\, b,\dot b;\, a+b-s, \dot a+\dot b-\dot s)$$
$$\times S[A]S[\dot A]S[B]S[\dot B]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot\alpha_s\dot\beta_s]} X^{[\dot C(\dot s)]}_{[C(s)]}, \tag{18}$$

where $t(a,\dot a;\, b,\dot b;\, c,\dot c)$ are structure constants (the torsion, or more precisely anholonomy, coefficients) depending on six half-integers and we choose the convention that $t(a,\dot a;\, b,\dot b;\, c,\dot c) = 0$ when $c,\dot c$ are outside the range of the summation. Here we have introduced the multi-indices (for $0 \leqslant s \leqslant \min(2a,2b)$ and $0 \leqslant \dot s \leqslant \min(2\dot a, 2\dot b)$)

$$[C(s)] = \alpha_{s+1}\alpha_{s+2}\cdots\alpha_{2a}\beta_{s+1}\beta_{s+2}\cdots\beta_{2b},$$
$$\left[\dot C(\dot s)\right] = \dot\alpha_{\dot s+1}\dot\alpha_{\dot s+2}\cdots\dot\alpha_{2\dot a}\dot\beta_{\dot s+1}\dot\beta_{\dot s+2}\cdots\dot\beta_{2\dot b}, \tag{19}$$

and we use the fact that the tensor product of irreducible Lorentz representations, $(a, \dot{a}) \otimes (b, \dot{b}) = (a \otimes b, \dot{a} \otimes \dot{b})$, decomposes according to the Clebsch-Gordan rules,

$$\begin{aligned} a \otimes b &= (a+b) \oplus (a+b-1) \oplus \ldots \oplus |a-b|, \\ \dot{a} \otimes \dot{b} &= (\dot{a}+\dot{b}) \oplus (\dot{a}+\dot{b}-1) \oplus \ldots \oplus |\dot{a}-\dot{b}|, \end{aligned} \tag{20}$$

with the Clebsch-Gordan coefficients for these spinor representations being representable by the multi-index ϵ's. The right-hand side of (18) clearly needs to have the symmetry properties of the left under the interchange of the indices $[A], [\dot{A}]$ with $[B], [\dot{B}]$. Taking into account the antisymmetry of the ϵ factors, this leads to the following restrictions on the t parameters

$$\begin{aligned} & t(a, \dot{a};\ b, \dot{b};\ a+b-s, \dot{a}+\dot{b}-\dot{s}) \\ & \quad = (-1)^{4(a+\dot{a})(b+\dot{b})+s+\dot{s}+1} t(b, \dot{b};\ a, \dot{a};\ a+b-s, \dot{a}+\dot{b}-\dot{s}). \end{aligned} \tag{21}$$

We obtain, in particular, that

$$t(a, \dot{a};\ a, \dot{a};\ a-s, \dot{a}-\dot{s}) = 0, \quad \begin{cases} \text{if } 2(a+\dot{a}) \text{ is even and } s+\dot{s} \text{ is even,} \\ \text{if } 2(a+\dot{a}) \text{ is odd and } s+\dot{s} \text{ is odd.} \end{cases} \tag{22}$$

Apart from the appropriate symmetry properties, associativity requires the satisfaction of the relevant super-Jacobi identities, which will be given in the next section. The operators X then form a Z_2-graded superalgebra $\mathcal{A}$, with even (resp. odd) elements having even (resp. odd) $2(a+\dot{a})$.

We note that the natural identifications $X_{\alpha\beta} = M_{\alpha\beta}$, $X^{\dot{\alpha}\dot{\beta}} = M^{\dot{\alpha}\dot{\beta}}$, with the Lorentz generators may be made, though these are by no means necessary requirements. Further, elements $X(a, \dot{a})$ transforming according to any specific representation $(a, \dot{a})$ could, in principle, occur multiply.

2.2. Hyperspaces

The simplest realisations of algebras $\mathcal{A}$ are as infinitesimal translation vector fields on generalisations of standard superspace. To this end, we enlarge Minkowski space to *hyperspace* $\mathcal{M}$, with coordinates $Y(a, \dot{a})$ corresponding to the algebra elements $X(a, \dot{a})$ and also transforming according to (16). For any given finite set $\{(a, \dot{a})\}$, we interpret the (correspondingly even or odd) coordinates $\{Y(a, \dot{a})\}$ as coordinates of $(2a+1)(2\dot{a}+1)$-dimensional (even or odd) subspaces of $\mathcal{M}$. Standard superspace, therefore, with coordinates $Y(\frac{1}{2}, \frac{1}{2})$, $Y(\frac{1}{2}, 0)$ and $Y(0, \frac{1}{2})$, of subspaces of respectively 4 bosonic and $2+2$ fermionic dimension, has total dimension $(4|4)$; and the super-Poincaré algebra has a manifestly covariant action on it. On the other hand, $N = 4$ extended superspace, with four copies of the odd subspaces, has dimension $(4|16)$ and a covariant action of the $N = 4$ extended super-Poincaré algebra. For the simplicity of our exposition, we relegate discussion of analogous spaces $\mathcal{M}$, with certain representations appearing multiply, to Appendix A.

As is usual for coordinates, we assume that they supercommute amongst themselves, i.e.

$$\left[Y^{[\dot{A}]}_{[A]}, Y^{[\dot{B}]}_{[B]}\right] = 0. \tag{23}$$

We now proceed to specify the action of the superalgebras $\mathcal{A}$ on superspaces $\mathcal{M}$ with coordinates Y. The vector fields $X \in \mathcal{A}$ clearly need to act as superderivations on functions of Y. To fulfill, in particular, that the X's map functions of Y to functions of Y, we postulate, as the simplest possibility, that the action of an X on a Y is a linear combination of the Y's, i.e. the hypersymmetry transforms the coordinates at most linearly among themselves. The X's together with the Y's therefore combine to form an enlarged superalgebra, with additional supercommutation relations

$$\begin{aligned}\left[X^{[\dot{A}]}_{[A]}, Y^{[\dot{B}]}_{[B]}\right] = & \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=0}^{\min(2\dot{a},2\dot{b})} u(a,\dot{a};\, b,\dot{b};\, a+b-s, \dot{a}+\dot{b}-\dot{s}) \\ & \times S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot{\alpha}_{\dot{s}}\dot{\beta}_{\dot{s}}]}Y^{[\dot{C}(\dot{s})]}_{[C(s)]} \\ & + c(a,\dot{a})\, \delta_{ab}\delta^{\dot{a}\dot{b}}S[A]S[\dot{A}]\epsilon_{[\alpha_{2a}\beta_{2a}]}\epsilon^{[\dot{\alpha}_{2\dot{a}}\dot{\beta}_{2\dot{a}}]}.\end{aligned} \tag{24}$$

Additional structure constants have been labeled $u(a,\dot{a};\, b,\dot{b};\, c,\dot{c})$ and essential central parameters $c(a,\dot{a})$ have been introduced. If any of the latter are non-zero, they can always be renormalised to 1 by multiplying the X's and/or the Y's by an appropriate factor. Super-Jacobi identities yield quadratic consistency relations among the structure constants $t(a,\dot{a};\, b,\dot{b};\, c,\dot{c}), u(a,\dot{a};\, b,\dot{b};\, c,\dot{c})$ and $c(a,\dot{a})$. These are discussed in the next section.

Given any particular set of X's generating an algebra $\mathcal{A}$ with relations (18), the span of Y's satisfying (23), (24) and the relevant Jacobi identities, can be thought of as the set of coordinates of a Z_2-graded superspace $\mathcal{M}$. For usual Minkowski space, the algebra $\mathcal{A}$ is generated by the Lorentz generators together with the operators $X(\frac{1}{2},\frac{1}{2})$, which are simply realised as partial derivatives $\partial/\partial Y^{\alpha\dot{\alpha}}$. They commute among themselves (the t parameters are zero) and their commutators with the coordinates involve only the central term $c(\frac{1}{2},\frac{1}{2})$, with the u parameters being zero. Now just as Minkowski space can be thought of as the coset space (Poincaré group)/(Lorentz group), we can consider the span of Y's to be basically the coordinates of the supergroup corresponding to the algebra $\mathcal{A}$.

Given a coordinate basis $\{Y\}$, we clearly have a holonomic (supercommuting) basis of infinitesimal translation vector fields constructed from the partial derivatives

$$\partial^{[\dot{A}]}_{[A]} = S[A]S[\dot{A}]\frac{\partial}{\partial Y^{[A]}_{[\dot{A}]}}, \tag{25}$$

satisfying (24) with all t's and u's set to zero and all c's put to unity

$$\left[\partial^{[\dot{A}]}_{[A]}, Y^{[\dot{B}]}_{[B]}\right] = \delta_{ab}\delta^{\dot{a}\dot{b}}S[A]S[\dot{A}]\epsilon_{[\alpha_{2a}\beta_{2a}]}\epsilon^{[\dot{\alpha}_{2\dot{a}}\dot{\beta}_{2\dot{a}}]}. \tag{26}$$

The relations (24) may be realised by $\mathcal{A}$-covariant or *hypercovariant* derivatives, using the notation (19),

$$\begin{aligned} X^{[\dot{A}]}_{[A]} = {} & c(a,\dot{a})\, \partial^{[\dot{A}]}_{[A]} \\ & + \sum_{b=0} \sum_{\dot{b}=0} \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=0}^{\min(2\dot{a},2\dot{b})} u(a,\dot{a};\, b,\dot{b};\, a+b-s, \dot{a}+\dot{b}-\dot{s}) \\ & \times S[A]\, S[\dot{A}]\, \epsilon_{[\alpha_s \beta_s]} \epsilon^{[\dot{\alpha}_{\dot{s}} \dot{\beta}_{\dot{s}}]} Y^{[\dot{C}(\dot{s})]}_{[C(s)]} \partial^{[\dot{B}]}_{[B]} . \end{aligned} \tag{27}$$

This coordinate realisation yields (24) straightaway. However, requiring that these X's satisfy (18) implies quadratic relationships between the t's, the u's and the c's; relationships which, in the abstract setting, arise from the super-Jacobi identities among the X's and Y's. These relations do not necessarily have unique solution and particular solutions correspond to particular choices of coordinate bases: the standard non-chiral and chiral bases for superspace being the simplest example (see Section 4).

The algebra (18), (23), (24) is formally invariant under the following $\mathbb{Z}_2$ 'chiral' transformations

$$\begin{aligned} \text{dotted upper index} &\leftrightarrow \text{undotted lower index}, \\ \text{dotted lower index} &\leftrightarrow \text{undotted upper index}, \\ t(a,\dot{a};\, b,\dot{b};\, c,\dot{c}) &\leftrightarrow t(\dot{a},a;\, \dot{b},b;\, \dot{c},c), \\ u(a,\dot{a};\, b,\dot{b};\, c,\dot{c}) &\leftrightarrow u(\dot{a},a;\, \dot{b},b;\, \dot{c},c), \\ c(a,\dot{a}) &\leftrightarrow c(\dot{a},a). \end{aligned} \tag{28}$$

In particular, the number of independent structure constants and central parameters t, u, and c can be roughly halved by imposing the *maximal non-chirality* condition of $\mathbb{Z}_2$ chiral symmetry, i.e.

$$\begin{aligned} t(a,\dot{a};\, b,\dot{b};\, c,\dot{c}) &= t(\dot{a},a;\, \dot{b},b;\, \dot{c},c), \\ u(a,\dot{a};\, b,\dot{b};\, c,\dot{c}) &= u(\dot{a},a;\, \dot{b},b;\, \dot{c},c), \\ c(a,\dot{a}) &= c(\dot{a},a). \end{aligned} \tag{29}$$

We note that the choice of the set of Y's for a given $\mathcal{A}$ (or more generally of the X's and the Y's) is possibly not unique, since non-linear transformations amongst the generators, preserving their tensorial nature as well as the linearity of the right-hand sides, can be envisaged. We shall not pursue details of such equivalences; rather, we obtain the consistency conditions defining *all* algebras having a given number of X's and Y's. Linear transformations among the generators, on the other hand, are ruled out by Lorentz covariance in the '$N = 1$' cases of at most one element for each Lorentz behaviour.

It is straightforward to extend the algebra to the case where the multiplicity of certain Lorentz representations is greater than one. Then the structure constants t and u depend on three extra multiplicity indices and the central terms c on two extra indices (i.e. they are matrices in the multiplicity space). In these extended cases the elements are defined up to linear transformations amongst the elements transforming similarly under Lorentz transformations. Such transformations can, moreover, be used to diagonalise the central terms. Details are given in Appendix A.

3. The super-Jacobi identities

We now derive the conditions implied by the super-Jacobi identities in order that the vector fields X and coordinates Y form a super-Lie algebra. Since the coordinates commute or anticommute, their Jacobi identities, involving three Y's, are trivially satisfied. Similarly, the super-Jacobi identities involving two Y's and one X are also trivially satisfied.

We first consider the Jacobi identities involving three X's. For the vector fields $X(a,\dot a)$, $X(b,\dot b)$ and $X(c,\dot c)$, all three double-supercommutators yield linear combinations of vector fields $X(f,\dot f)$, with $(f,\dot f)$ belonging to the set obtained in the decomposition of $(a\otimes b\otimes c, \dot a\otimes \dot b\otimes \dot c)$, multiplied by products of ϵ's (realising the Clebsch–Gordan coefficients) and quadratic in the t's. They have terms of the form

$$t(a,\dot a;\ b,\dot b;\ d,\dot d)\ t(d,\dot d;\ c,\dot c;\ f,\dot f)\ S[A]S[\dot A]S[B]S[\dot B]S[C]S[\dot C]\epsilon_{..}\cdots\epsilon_{..}X^{[\dot F]}_{[F]}. \tag{30}$$

The vanishing of the coefficients of the linearly independent tensors $\epsilon_{..}\cdots\epsilon_{..}X^{[\dot F]}_{[F]}$, are quadratic consistency conditions for the t structure constants, which we consider to be the defining relations for algebras $\mathcal{A}$. In constructing the linearly independent tensors, Fierz-type identities based on the spinorial identity $\epsilon_{\alpha\beta}T_\gamma + \epsilon_{\beta\gamma}T_\alpha + \epsilon_{\gamma\alpha}T_\beta = 0$ need to be taken into account. For each $a,\dot a, b,\dot b, c,\dot c, f,\dot f$ and for each linearly independent tensor, we obtain a relation of the form

$$\begin{aligned}\sum_{d,\dot d}\Big(&R_1(d,\dot d)\ t(a,\dot a;\ b,\dot b;\ d,\dot d)\ t(d,\dot d;\ c,\dot c;\ f,\dot f)\\ &+(-1)^{p_2}R_2(d,\dot d)\ t(b,\dot b;\ c,\dot c;\ d,\dot d)\ t(d,\dot d;\ a,\dot a;\ f,\dot f)\\ &+(-1)^{p_3}R_3(d,\dot d)\ t(c,\dot c;\ a,\dot a;\ d,\dot d)\ t(d,\dot d;\ b,\dot b;\ f,\dot f)\Big)=0,\end{aligned} \tag{31}$$

where the numerical constants R_i $(i = 1,2,3)$ depend on the relation involved and where the range of the summation over $d,\dot d$ is given by representations occurring in the tensor products $(a,\dot a)\otimes(b,\dot b)$, $(b,\dot b)\otimes(c,\dot c)$ and $(c,\dot c)\otimes(a,\dot a)$, for the three terms respectively. Here, $p_2 = 4(a+\dot a)(b+\dot b+c+\dot c)$ mod 2 is the parity of the permutation from $(a,\dot a;\ b,\dot b;\ c,\dot c)$ to $(b,\dot b;\ c,\dot c;\ a,\dot a)$ while $p_3 = 4(c+\dot c)(a+\dot a+b+\dot b)$ mod 2 is the parity of the permutation from $(a,\dot a;\ b,\dot b;\ c,\dot c)$ to $(c,\dot c;\ a,\dot a;\ b,\dot b)$. For three bosons, three fermions, or two bosons and one fermion in $(a,\dot a;\ b,\dot b;\ c,\dot c)$ this parity is always 0. For two fermions and one boson, one of them is 0 and the other is 1.

In practice and for simple examples, the linearly independent tensors and the corresponding constant coefficients R_i afford direct determination, using for instance REDUCE. In more generality, a more precise form of (31) may be obtained using $6j$ coefficients. Denoting the individual eigenstates of a Lorentz representation $T(a,\dot{a})$ by

$$T(a,a_3;\,\dot{a},\dot{a}_3),\qquad -a\leqslant a_3\leqslant a,\quad -\dot{a}\leqslant \dot{a}_3\leqslant \dot{a}, \tag{32}$$

where a_3, $\dot{a}_3$ label the eigenstates, the super-commutation relations between say $X(a,a_3;\,\dot{a},\dot{a}_3)$ and $X(b,b_3;\,\dot{b},\dot{b}_3)$ then read

$$\begin{aligned}&\big[X(a,a_3;\,\dot{a},\dot{a}_3),X(b,b_3;\,\dot{b},\dot{b}_3)\big]\\ &\quad=\sum_{c=|a-b|}^{a+b}\ \sum_{\dot{c}=|\dot{a}-\dot{b}|}^{\dot{a}+\dot{b}}\tilde{t}(a,\dot{a};\,b,\dot{b};\,c,\dot{c})\\ &\qquad\times C(a,a_3,b,b_3;\,c,a_3+b_3)\,C(\dot{a},\dot{a}_3,\dot{b},\dot{b}_3;\,\dot{c},\dot{a}_3+\dot{b}_3)\\ &\qquad\times X(c,a_3+b_3;\,\dot{c},\dot{a}_3+\dot{b}_3),\end{aligned}\tag{33}$$

where $C(a,a_3,b,b_3;\,c,a_3+b_3)$ is the $su(2)$ Clebsch–Gordan coefficient coupling the state (a,a_3) with the state (b,b_3) to form the state (c,c_3) (where $c_3=a_3+b_3$ and analogously for the dotted indices) and the $\tilde{t}$'s are renormalised t's. This is an alternative form of the commutation relations (18). Now, the $6j$ recoupling coefficients are defined by the relations

$$\begin{aligned}&C(a,a_3,b,b_3;\,d,a_3+b_3)\,C(d,a_3+b_3,c,c_3;\,f,a_3+b_3+c_3)\\ &\quad=\sum_k R(a,b,c,d,k,f)\,C(a,a_3,c,c_3;\,k,a_3+c_3)\\ &\qquad\times C(k,a_3+c_3,b,b_3;\,f,a_3+b_3+c_3).\end{aligned}\tag{34}$$

If the Clebsch–Gordan coefficient has the symmetry

$$C(a,a_3,b,b_3;\,d,d_3)=C(b,b_3,a,a_3;\,d,d_3),\tag{35}$$

then the $6j$ symbol satisfies

$$R(a,b,c,d,k,f)=R(b,a,c,d,k,f).\tag{36}$$

The super-Jacobi identities are given by

$$\begin{aligned}&\tilde{t}(a,\dot{a};\,b,\dot{b};\,d,\dot{d})\,\tilde{t}(d,\dot{d};\,c,\dot{c};\,f,\dot{f})\\ &\quad+(-1)^{p_2}\sum_{k,\dot{k}}R(b,c,a,k,d,f)\,R(\dot{b},\dot{c},\dot{a},\dot{k},\dot{d},\dot{f})\,\tilde{t}(b,\dot{b};\,c,\dot{c};\,k,\dot{k})\\ &\quad\times\tilde{t}(k,\dot{k};\,a,\dot{a};\,f,\dot{f})\\ &\quad+(-1)^{p_3}\sum_{k,\dot{k}}R(a,c,b,k,d,f)\,R(\dot{a},\dot{c},\dot{b},\dot{k},\dot{d},\dot{f})\,\tilde{t}(c,\dot{c};\,a,\dot{a};\,k,\dot{k})\\ &\quad\times\tilde{t}(k,\dot{k};\,b,\dot{b};\,f,\dot{f})=0.\end{aligned}\tag{37}$$

For given $a, \dot{a}, b, \dot{b}, c, \dot{c}$, i.e. the starting three X operators and the final X operator $f, \dot{f}$ there are as many relations as there are allowed $d, \dot{d}$ sets. This is the precise form of the more schematic relations (31).

The Jacobi identities involving two X's and one Y, say $X(a, \dot{a})$, $X(b, \dot{b})$ and $Y(c, \dot{c})$, imply two classes of conditions. One class involves the structure constants t and u and is linear but inhomogeneous in t and the other involves t, u and c and is strictly linear in c. Amongst t, u we obtain, for each linearly independent combination of tensors for the Y's, a condition of the form

$$\sum_{d,\dot{d}} \Big(S_1(d, \dot{d})\, t(a, \dot{a};\, b, \dot{b};\, d, \dot{d})\, u(d, \dot{d};\, c, \dot{c};\, f, \dot{f}) \\ -S_2(d, \dot{d})\, u(b, \dot{b};\, c, \dot{c};\, d, \dot{d})\, u(a, \dot{a};\, d, \dot{d};\, f, \dot{f}) \\ +(-1)^{q_3} S_3(d, \dot{d})\, u(a, \dot{a};\, c, \dot{c};\, d, \dot{d})\, u(b, \dot{b};\, d, \dot{d};\, f, \dot{f}) \Big) = 0, \tag{38}$$

where the numerical coefficients S_i $(i = 1, 2, 3)$ depend on the relation involved. Here $q_3 = 4(a + \dot{a})(b + \dot{b}) \bmod 2$ is the parity of the permutation from $(a, \dot{a};\, b, \dot{b})$ to $(b, \dot{b};\, a, \dot{a})$ and the allowed values of the summation indices $(d, \dot{d})$ are the same as in (31).

Finally, between t, u and c we obtain the conditions

$$T_1 t(a, \dot{a};\, b, \dot{b};\, c, \dot{c})\, c(c, \dot{c}) - T_2\, u(b, \dot{b};\, c, \dot{c};\, a, \dot{a})\, c(a, \dot{a}) \\ +(-1)^{q_3} T_3 u(a, \dot{a};\, c, \dot{c};\, b, \dot{b})\, c(b, \dot{b}) = 0, \tag{39}$$

with the numerical coefficients T_i $(i = 1, 2, 3)$. A more explicit form of relations (38) and (39), in terms of $6j$ coefficients, may be derived analogously to (37).

Conditions (31), (38), (39) are the only restrictions amongst the structure constants arising from the super-Jacobi identities. If some representations occur multiply, then these conditions of course need to be modified, in order to accommodate the extra labelling indices of t, u and c, as described in Appendix A.

4. Explicit examples of solutions to the Jacobi identities

In this section we present some simple examples of solutions to the conditions (31), (38), (39) and thereby provide explicit examples of hyperspaces $\mathcal{M}$.

Two natural, though by no means necessary, assumptions are:

(a) that the Lorentz generators $M_{\alpha\beta}$ and $M^{\dot{\alpha}\dot{\beta}}$ are identical to the generators $X(1, 0)$ and $X(0, 1)$ respectively; and

(b) that $X(0, 0)$, which is basically a dilatation-type operator, and its corresponding coordinate $Y(0, 0)$ are absent.

The latter may be implemented in the relations for the structure constants (31), (38) and (39) either by ignoring the two operators from the beginning or equivalently by imposing, for all $a, \dot{a}, b, \dot{b}$, the constraints

$$t(0,0;\ a,\dot a;\ b,\dot b) = t(a,\dot a;\ b,\dot b;\ 0,0) = c(0,0) = 0,$$
$$u(0,0;\ a,\dot a;\ b,\dot b) = u(a,\dot a;\ 0,0;\ b,\dot b) = u(a,\dot a;\ b,\dot b;\ 0,0) = 0. \qquad (40)$$

Using (12), assumption (a) fixes unambiguously the following structure constants, for all $a, \dot a, b, \dot b$,

$$\begin{aligned} t(a,\dot a;\ 0,1;\ b,\dot b) &= t(a,\dot a;\ 1,0;\ b,\dot b) = t(0,1;\ a,\dot a;\ b,\dot b) = t(1,0;\ a,\dot a;\ b,\dot b) \\ &= \delta_{ab}\delta_{\dot a\dot b}, \\ u(0,1;\ a,\dot a;\ b,\dot b) &= u(1,0;\ a,\dot a;\ b,\dot b) = \delta_{ab}\delta_{\dot a\dot b}, \\ c(0,1) &= c(1,0) = 0. \end{aligned} \qquad (41)$$

A further natural constraint consistent with (a) is that

$$t(a,\dot a;\ b,\dot b;\ 0,1) = \delta_{a0}\delta_{\dot a1}\delta_{b0}\delta_{\dot b1}, \qquad t(a,\dot a;\ b,\dot b;\ 1,0) = \delta_{a1}\delta_{\dot a0}\delta_{b1}\delta_{\dot b0}. \qquad (42)$$

4.1. General

Of course, any number of abelian examples of $\mathcal{A}$ may be constructed, with arbitrary set of vector fields $\{X\}$, representable in a corresponding coordinate basis $\{Y\}$ by partial derivatives (25). All t's and u's are then zero; and all c's are 1, apart from the vanishing $c(0,1)$ and $c(1,0)$.

4.2. Restrictions

If, apart from the Lorentz generators, the set of operators is restricted to $\{X(\frac12,0), X(0,\frac12), X(\frac12,\frac12), Y(\frac12,0), Y(0,\frac12), Y(\frac12,\frac12)\}$ and the corresponding set of non-zero structure constants is restricted to $\{t(0,\frac12;\ \frac12,0;\ \frac12,\frac12), u(0,\frac12;\ \frac12,\frac12;\ \frac12,0), u(\frac12,0;\ \frac12,\frac12;\ 0,\frac12), c(\frac12,\frac12), c(0,\frac12), c(\frac12,0)\}$ together with those in (42) and (41), the associativity requirements of the previous section correspond to a single relationship amongst the non-zero structure constants, viz.

$$\begin{aligned} t(0,\tfrac12;\ \tfrac12,0;\ \tfrac12,\tfrac12)\, c(\tfrac12,\tfrac12) &= c(0,\tfrac12)\, u(\tfrac12,0;\ \tfrac12,\tfrac12;\ 0,\tfrac12) + c(\tfrac12,0) \\ &\quad \times u(0,\tfrac12;\ \tfrac12,\tfrac12;\ \tfrac12,0). \end{aligned} \qquad (43)$$

If we choose $c(\frac12,\frac12) = c(0,\frac12) = c(\frac12,0) = 1$, we obtain a simple relation for the determination of a coordinate basis consistent with the super-Poincaré relation (1).

4.2.1. Explicit solution

One explicit solution

$$t(0,\tfrac12;\ \tfrac12,0;\ \tfrac12,\tfrac12) = 2i, \qquad u(\tfrac12,0;\ \tfrac12,\tfrac12;\ 0,\tfrac12) = u(0,\tfrac12;\ \tfrac12,\tfrac12;\ \tfrac12,0) = i, \qquad (44)$$

corresponds to the standard super-Poincaré basis (3) with relations (2) and is *maximally non-chiral* (29).

4.2.2. Other solution

Another solution

$$t(0,\tfrac{1}{2};\ \tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2}) = 2i, \qquad u(\tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{1}{2}) = 2i,$$
$$u(0,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},0) = 0, \tag{45}$$

corresponds to the *chiral basis* for super-Poincaré space

$$X_{\alpha\dot\beta} = \frac{\partial}{\partial Y^{\alpha\dot\beta}}, \qquad X_\alpha = \frac{\partial}{\partial Y^\alpha} + 2iY^{\dot\beta}\frac{\partial}{\partial Y^{\alpha\dot\beta}}, \qquad X_{\dot\beta} = \frac{\partial}{\partial Y^{\dot\beta}}. \tag{46}$$

4.2.3. General solution

The general solution to (43) clearly interpolates between these two bases for the super-Poincaré algebra:

$$t(0,\tfrac{1}{2};\ \tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2}) = 2i, \qquad u(\tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{1}{2}) = i(1+r),$$
$$u(0,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},0) = i(1-r), \tag{47}$$

and can be realised as

$$X_{\alpha\dot\beta} = \frac{\partial}{\partial Y^{\alpha\dot\beta}}, \qquad X_\alpha = \frac{\partial}{\partial Y^\alpha} + i(1+r)\,Y^{\dot\beta}\frac{\partial}{\partial Y^{\alpha\dot\beta}},$$
$$X_{\dot\beta} = \frac{\partial}{\partial Y^{\dot\beta}} + i(1-r)\,Y^{\alpha}\frac{\partial}{\partial Y^{\alpha\dot\beta}}. \tag{48}$$

4.3. Simple example

To find a simple example of a non-trivial extension of the super-Poincaré algebra, we have written, using REDUCE, the set of all the conditions when we start with the set of all X's and Y's transforming according to $(a,\dot a)$ representations, with $a+\dot a \leqslant 3$. Even for this simplest extension of the super-Poincaré algebra, the number of algebraic relations among the structure constants is rather large. Using REDUCE, we find, for all representations with $a+\dot a \leqslant 3$, a total of 397 relations of the form (31), 1224 relations of the form (38) and 61 relations of the form (39), which we call, respectively, TT, TU and TC relations. The complete discussion of all allowed possibilities is a formidable task, however, the imposition of certain natural requirements yields a simple specific solution depending on a small number of arbitrary parameters. Imposing

(i) the absence of $X(0,0)$ and $Y(0,0)$, i.e. (40);

(ii) the Lorentz identifications $X_{\alpha\beta} = M_{\alpha\beta}$, $X^{\dot\alpha\dot\beta} = M^{\dot\alpha\dot\beta}$ tantamount to (41), together with (42);

(iii) the condition that the essentially dummy variables $Y(1,0)$ and $Y(0,1)$ are zero;

(iv) the normalisations:

$$c(0,\tfrac{1}{2}) = c(\tfrac{1}{2},0) = c(\tfrac{1}{2},\tfrac{1}{2}) = c(1,\tfrac{1}{2}) = c(\tfrac{1}{2},1) = c(0,\tfrac{3}{2}) = c(\tfrac{3}{2},0) = 1;$$

considerably reduces the number of relations to be satisfied by the non-zero structure constants, though this is still quite large. However, if we in addition insist on the following super-Poincaré properties:

(v) $t(0,\frac{1}{2};\ \frac{1}{2},0;\ \frac{1}{2},\frac{1}{2}) \neq 0$;

(vi) $u(\frac{1}{2},0;\ 0,\frac{1}{2};\ \frac{1}{2},\frac{1}{2}) = u(0,\frac{1}{2};\ \frac{1}{2},0;\ \frac{1}{2},\frac{1}{2}) = 0$;

(vii) $u(\frac{1}{2},\frac{1}{2};\ a,\dot{a};\ b,\dot{b}) = 0$, for all $(b,\dot{b})$ when $2(a+\dot{a})$ is odd;

then the TT-relations imply that $X(\frac{1}{2},\frac{1}{2})$ necessarily commutes with all fermionic X's,

(viii) $t(\frac{1}{2},\frac{1}{2};\ a,\dot{a};\ b,\dot{b}) = 0$, for all $(b,\dot{b})$ when $2(a+\dot{a})$ is odd;

as a consequence of which, all further TT-relations are automatically satisfied. Similarly, the TC-relations imply that

(ix) $u(a,\dot{a};\ b,\dot{b};\ \frac{1}{2},\frac{1}{2}) = 0$, for all $(b,\dot{b})$ when $2(a+\dot{a})$ is odd;

as a consequence of which all TU-relations are resolved. The only conditions then remaining are the following TC-relations amongst the non-zero structure constants (all other structure constants being zero, except of course for the non-zero ones in (41), (42)):

$$
\begin{aligned}
t(0,\tfrac{1}{2};\ \tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2}) &= u(\tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{1}{2}) + u(0,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},0),\\
t(\tfrac{1}{2},0;\ 1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2}) &= u(1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},0) - u(\tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 1,\tfrac{1}{2}),\\
t(0,\tfrac{1}{2};\ \tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2}) &= u(\tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{1}{2}) - u(0,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},1),\\
t(\tfrac{1}{2},1;\ 1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2}) &= u(1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},1) + u(\tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2};\ 1,\tfrac{1}{2}),\\
t(1,\tfrac{1}{2};\ \tfrac{3}{2},0;\ \tfrac{1}{2},\tfrac{1}{2}) &= u(1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{3}{2},0) + u(\tfrac{3}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 1,\tfrac{1}{2}),\\
t(0,\tfrac{3}{2};\ \tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2}) &= u(\tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{3}{2}) + u(0,\tfrac{3}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},1).
\end{aligned}
\tag{49}
$$

Denoting the 12 arbitrary parameters

$$
\begin{aligned}
&u(\tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{1}{2}) = u_1, && u(1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},0) = u_2, && u(\tfrac{1}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 1,\tfrac{1}{2}) = u_3,\\
&u(0,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},0) = \tilde{u}_1, && u(\tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{1}{2}) = \tilde{u}_2, && u(0,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},1) = \tilde{u}_3,\\
&u(1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{3}{2},0) = u_4, && u(\tfrac{3}{2},0;\ \tfrac{1}{2},\tfrac{1}{2};\ 1,\tfrac{1}{2}) = u_5, && u(1,\tfrac{1}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},1) = u_6,\\
&u(\tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2};\ 0,\tfrac{3}{2}) = \tilde{u}_4, && u(0,\tfrac{3}{2};\ \tfrac{1}{2},\tfrac{1}{2};\ \tfrac{1}{2},1) = \tilde{u}_5, && u(\tfrac{1}{2},1;\ \tfrac{1}{2},\tfrac{1}{2};\ 1,\tfrac{1}{2}) = \tilde{u}_6,
\end{aligned}
\tag{50}
$$

we obtain a super-algebra with non-zero commutation relations among the X's (apart from the obvious relations involving the Lorentz generators $X(1,0)$ and $X(0,1)$),

$$
\begin{aligned}
\left\{X_\alpha, X^{\dot\beta}\right\} &= (u_1+\tilde{u}_1)X^{\dot\beta}_{\alpha},\\
\left\{X_\alpha, X^{\dot\beta}_{\beta_1\beta_2}\right\} &= (u_2-u_3)S(\beta_1\beta_2)\,\epsilon_{\alpha\beta_1}X^{\dot\beta}_{\beta_2},\\
\left\{X^{\dot\alpha}, X^{\dot\beta_1\dot\beta_2}_{\beta}\right\} &= (\tilde{u}_2-\tilde{u}_3)S(\dot\beta_1\dot\beta_2)\,\epsilon^{\dot\alpha\dot\beta_1}X^{\dot\beta_2}_{\beta},\\
\left\{X^{\dot\alpha}_{\alpha_1\alpha_2}, X^{\dot\beta_1\dot\beta_2}_{\beta}\right\} &= (u_6+\tilde{u}_6)S(\alpha_1\alpha_2)\ S(\dot\beta_1\dot\beta_2)\,\epsilon_{\alpha_1\beta}\epsilon^{\dot\alpha\dot\beta_1}X^{\dot\beta_2}_{\alpha_2},\\
\left\{X_{\alpha_1\alpha_2\alpha_3}, X^{\dot\beta}_{\beta_1\beta_2}\right\} &= (u_4+2u_5)S(\alpha_1\alpha_2\alpha_3)\,\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}X^{\dot\beta}_{\alpha_3},
\end{aligned}
$$

$$\left\{X_{\alpha}^{\dot{\alpha}_1\dot{\alpha}_2}, X^{\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3}\right\} = (\tilde{u}_4 + 2\tilde{u}_5) S(\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3)\epsilon^{\dot{\alpha}_1\dot{\beta}_1}\epsilon^{\dot{\alpha}_2\dot{\beta}_2}X_{\alpha}^{\dot{\beta}_3}. \tag{51}$$

This algebra clearly has the symmetry corresponding to the $\mathbb{Z}_2$ 'chiral' transformations (28) which in this case reduce to

$$\begin{aligned} &\text{dotted upper index} \leftrightarrow \text{undotted lower index}, \\ &\text{dotted lower index} \leftrightarrow \text{undotted upper index}, \\ &\qquad\qquad u_i \leftrightarrow \tilde{u}_i. \end{aligned} \tag{52}$$

The non-zero commutators between the X's and the Y's are

$$\begin{aligned} \{X_\alpha, Y_\beta\} &= \epsilon_{\alpha\beta}, \\ \left[X_\alpha^{\dot{\alpha}}, Y_\beta^{\dot{\beta}}\right] &= \epsilon_{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}, \\ \left\{X_{\alpha_1\alpha_2}^{\dot{\alpha}}, Y_{\beta_1\beta_2}^{\dot{\beta}}\right\} &= S(\alpha_1\alpha_2)\,\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon^{\dot{\alpha}\dot{\beta}}, \\ \{X_{\alpha_1\alpha_2\alpha_3}, Y_{\beta_1\beta_2\beta_3}\} &= S(\alpha_1\alpha_2\alpha_3)\,\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha_3\beta_3}, \\ \left[X_\alpha, Y_\beta^{\dot{\beta}}\right] &= u_1\epsilon_{\alpha\beta}Y^{\dot{\beta}} + u_3 Y_{\alpha\beta}^{\dot{\beta}}, \\ \left[X_{\alpha_1\alpha_2}^{\dot{\alpha}}, Y_\beta^{\dot{\beta}}\right] &= u_2 S(\alpha_1\alpha_2)\,\epsilon_{\alpha_1\beta}\epsilon^{\dot{\alpha}\dot{\beta}}Y_{\alpha_2} + u_4\epsilon^{\dot{\alpha}\dot{\beta}}Y_{\alpha_1\alpha_2\beta} + u_6 S(\alpha_1\alpha_2)\,\epsilon_{\alpha_1\beta}Y_{\alpha_2}^{\dot{\alpha}\dot{\beta}}, \\ \left[X_{\alpha_1\alpha_2\alpha_3}, Y_\beta^{\dot{\beta}}\right] &= u_5 S(\alpha_1\alpha_2\alpha_3)\,\epsilon_{\alpha_1\beta}Y_{\alpha_2\alpha_3}^{\dot{\beta}}, \end{aligned} \tag{53}$$

together with the six further independent relations obtained by performing the transformations (52).

A coordinate representation of this algebra may be found in terms of a holonomic basis of vector fields $\partial_{[A]}^{[\dot{A}]} = S[A]S[\dot{A}]\partial/\partial Y_{[\dot{A}]}^{[A]}$, having commutation relations corresponding to (51) and (53) with all the u's put to zero. This representation is given by

$$\begin{aligned} X_\alpha^{\dot{\alpha}} &= \partial_\alpha^{\dot{\alpha}}, \\ X_\alpha &= \partial_\alpha + u_1\epsilon_{\dot{\alpha}\dot{\beta}}Y^{\dot{\alpha}}\partial_\alpha^{\dot{\beta}} + u_3\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\beta\gamma}Y_{\alpha\beta}^{\dot{\alpha}}\partial_\gamma^{\dot{\beta}}, \\ X_{\alpha\beta}^{\dot{\alpha}} &= \partial_{\alpha\beta}^{\dot{\alpha}} + u_2 S(\alpha\beta)\,Y_\alpha\partial_\beta^{\dot{\alpha}} + u_4\epsilon^{\gamma\delta}Y_{\alpha\beta\gamma}\partial_\delta^{\dot{\alpha}} + u_6 S(\alpha\beta)\,\epsilon_{\dot{\beta}\dot{\gamma}}Y_\alpha^{\dot{\alpha}\dot{\beta}}\partial_\beta^{\dot{\gamma}}, \\ X_{\alpha\beta\gamma} &= \partial_{\alpha\beta\gamma} + u_5 S(\alpha\beta\gamma)\,\epsilon_{\dot{\alpha}\dot{\beta}}Y_{\alpha\beta}^{\dot{\alpha}}\partial_\gamma^{\dot{\beta}}, \end{aligned} \tag{54}$$

with realisations for the remaining generators, $X^{\dot{\alpha}}, X_\alpha^{\dot{\alpha}\dot{\beta}}, X^{\dot{\alpha}\dot{\beta}\dot{\gamma}}$, being given by the latter three expressions on performing transformations (52).

This example clearly has the super-Poincaré algebra as a subalgebra.

5. Hyperspace gauge fields

We would like to consider gauge fields on the spaces $\mathcal{M}$. To this end we postulate gauge potentials A. These depend on the set of coordinates $\{Y\}$ and are in one-to-one correspondence with the hypercovariant derivatives X, whose Lorentz indices and corresponding transformation properties they carry, allowing the definition of gauge-covariant derivatives,

$$\mathcal{D}^{[\dot{A}]}_{[A]} = X^{[\dot{A}]}_{[A]} + A^{[\dot{A}]}_{[A]}. \tag{55}$$

The gauge potentials take values in some, for instance semi-simple, Lie algebra, with generators $\{\lambda_k;\ k = 1, \dots, N\}$, satisfying

$$[\lambda_k, \lambda_l] = f_{kl}{}^{m}\lambda_m, \qquad \mathrm{tr}(\lambda_k\lambda_l) = \delta_{kl}, \tag{56}$$

where $f_{kl}{}^{m}$ are the structure constants. We can expand the gauge potentials in this Lie algebra basis

$$A^{[\dot{A}]}_{[A]} = \sum_{k=1}^{N} A^{[\dot{A}],k}_{[A]}\lambda_k, \tag{57}$$

with coefficients being extractable thus:

$$A^{[\dot{A}],k}_{[A]} = \mathrm{tr}\left(A^{[\dot{A}]}_{[A]}\lambda_k\right). \tag{58}$$

The covariant transformation law

$$\mathcal{D}^{[\dot{A}]}_{[A]} \to \mathcal{D}'^{[\dot{A}]}_{[A]} = U\mathcal{D}^{[\dot{A}]}_{[A]}U^{-1}, \tag{59}$$

yields the standard inhomogeneous gauge transformations

$$A^{[\dot{A}]}_{[A]} \to A'^{[\dot{A}]}_{[A]} = UA^{[\dot{A}]}_{[A]}U^{-1} - \left[X^{[\dot{A}]}_{[A]}, U\right]U^{-1}, \tag{60}$$

where the (grassmann-even) gauge-group-valued function U depends on the coordinates Y. The infinitesimal version ($U = 1 + \tau + O(\tau^2)$) of this transformation,

$$A^{[\dot{A}]}_{[A]} \to A'^{[\dot{A}]}_{[A]} = A^{[\dot{A}]}_{[A]} - \left[\mathcal{D}^{[\dot{A}]}_{[A]}, \tau\right], \tag{61}$$

where the components of the gauge potential are linear in a dimensionless coupling constant (absorbed into their definition), reveals them to have the same scaling dimension and the same bosonic or fermionic nature as the corresponding X's. Although these potentials have general spin, the gauge transformations remain completely analogous to the usual gauge transformations corresponding to a spin-one gauge degree of freedom, with Lorentz-scalar transformation parameter τ taking values in the gauge algebra. So although the potentials $A^{[\dot{A}]}_{[A]}$ have, in general, higher than spin-one content, no higher-spin gauge invariances and no coupling constants apart from the Yang–Mills ones are introduced.

It is now natural to define generalised gauge fields (curvatures) $\hat{F}$ (corresponding to the sets $[A], [\dot{A}]$ and $[B], [\dot{B}]$ by the equation

$$\hat{F}^{[\dot{A}][\dot{B}]}_{[A][B]} = \left[\mathcal{D}^{[\dot{A}]}_{[A]}, \mathcal{D}^{[\dot{B}]}_{[B]}\right] - \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=0}^{\min(2\dot{a},2\dot{b})} t(a, \dot{a};\ b, \dot{b};\ a+b-s, \dot{a}+\dot{b}-\dot{s})$$
$$\times S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot{\alpha}_s\dot{\beta}_s]}\mathcal{D}^{[\dot{C}(\dot{s})]}_{[C(s)]}. \tag{62}$$

On the right-hand side the second term ensures that the fields $\hat{F}$ are free of differential operators in a gauge-covariant manner. The thus defined curvatures are manifestly gauge-covariant,

$$\hat{F}^{[\dot{A}][\dot{B}]}_{[A][B]} \to \hat{F}'^{[\dot{A}][\dot{B}]}_{[A][B]} = U\hat{F}^{[\dot{A}][\dot{B}]}_{[A][B]}U^{-1}, \tag{63}$$

and decompose under the action of the Lorentz group into irreducible Lorentz representations thus:

$$\hat{F}^{[\dot{A}][\dot{B}]}_{[A][B]} = \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=0}^{\min(2\dot{a},2\dot{b})} S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot{\alpha}_s\dot{\beta}_s]}F^{[\dot{C}(\dot{s})]}_{[C(s)]}. \tag{64}$$

The irreducible components $F^{[\dot{C}(\dot{s})]}_{[C(s)]}$ (which obviously depend on $[A],[B],[\dot{A}],[\dot{B}]$) transforming according to the $(a+b-s, \dot{a}+\dot{b}-\dot{s})$ Lorentz representations may be projected out by contracting the curvatures $\hat{F}$ with the inverse epsilon tensors $\epsilon^{[\beta_s\alpha_s]}$ and $\epsilon_{[\dot{\beta}_s\dot{\alpha}_s]}$ in (15) and symmetrising over the remaining multi-indices $[C(s)]$ and $[\dot{C}(\dot{s})]$ in (19):

$$F^{[\dot{C}(\dot{s})]}_{[C(s)]} = \kappa(s,\dot{s})\, S[C(s)]S[\dot{C}(\dot{s})]\epsilon^{[\beta_s\alpha_s]}\epsilon_{[\dot{\beta}_s\dot{\alpha}_s]}\hat{F}^{[\dot{A}][\dot{B}]}_{[A][B]}, \tag{65}$$

where $\kappa(s,\dot{s})$ are combinatorial factors. The gauge algebra components of the irreducible fields F may be extracted by taking traces as in (58).

For instance, corresponding to the first two commutation relations in (51), we have

$$\hat{F}_{\alpha\dot{\beta}} = \left\{\mathcal{D}_\alpha, \mathcal{D}_{\dot{\beta}}\right\} - (u_1+\tilde{u}_1)\mathcal{D}_{\alpha\dot{\beta}} = F_{\alpha\dot{\beta}}, \tag{66}$$

which is irreducible and

$$\begin{aligned}\hat{F}_{\alpha\beta_1\beta_2\dot{\beta}} &= \left\{\mathcal{D}_\alpha, \mathcal{D}_{\beta_1\beta_2\dot{\beta}}\right\} - (u_2-u_3)S(\beta_1\beta_2)\,\epsilon_{\alpha\beta_1}\mathcal{D}_{\beta_2\dot{\beta}}\\ &= S(\alpha\beta_1\beta_2)\,F_{\alpha\beta_1\beta_2\dot{\beta}} + S(\beta_1\beta_2)\,\epsilon_{\alpha\beta_1}F_{\beta_2\dot{\beta}}.\end{aligned} \tag{67}$$

6. Curvature constraints and integrability conditions

Having the basic ingredients of the previous section, we could now proceed to consider the possibility of constructing consistent Lagrangian gauge field theories on the hyperspaces $\mathcal{M}$. We leave this for future work, concentrating here on systems of equations for gauge fields on $\mathcal{M}$, which generalise the idea of the standard four-dimensional self-duality equations. We shall consider imposing 'coherent' curvature constraints, setting some sets of irreducible gauge fields to vanish. One important class of such constraints arises from demanding that some subset of the commutation relations of the superalgebra $\mathcal{A}$ are preserved under the covariantisation $X \to \mathcal{D}$. This yields relations for higher-spin potentials in terms of lower-spin ones. The simplest example is the 'conventional constraint' of standard super-Yang–Mills theory,

$$F_{\alpha\dot{\beta}} = \mathcal{D}_\alpha A_{\dot{\beta}} + \mathcal{D}_{\dot{\beta}}A_\alpha - (u_1+\tilde{u}_1)A_{\alpha\dot{\beta}} = 0, \tag{68}$$

which determines the vector potential $A_{\alpha\dot{\alpha}}$ in terms of the spinor ones $(A_\alpha, A_{\dot{\alpha}})$ and yields an irreducible representation of the supersymmetry algebra. Another class of constraints are generalisations of the standard four-dimensional self-duality equations. In particular, the self-duality conditions are integrability conditions for a linear system, the starting point for the twistor transform, which establishes formal solubility. The crucial feature is the possibility of writing the constraints, with the non-zero curvature components representing obstructions to Frobenius integrability, in the form of some set of commuting operators.

In order to define linear systems we introduce two commuting spinors, $v^{\oplus\alpha}$ and $u^{+\dot{\alpha}}$. Our notation for these spinors is explained in Appendix B. Let us take a set of representations $\{(a, \dot{a})\}$ of the Lorentz group and associate to each representation two freely specifiable integers: $r(a)$, $0 \leqslant r(a) \leqslant 2a$ and $\dot{r}(\dot{a})$, $0 \leqslant \dot{r}(\dot{a}) \leqslant 2\dot{a}$. We consider the linear systems

$$v^{\oplus\alpha_1} \cdots v^{\oplus\alpha_{r(a)}} u^{+\dot{\alpha}_1} \cdots u^{+\dot{\alpha}_{\dot{r}(\dot{a})}} \mathcal{D}_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}} \varphi = 0. \tag{69}$$

When $r(a) = 0$ (resp. $\dot{r}(\dot{a}) = 0$), we mean that v's (resp. u's) do not appear.

The integrability of these linear systems for *arbitrary choices* of u's and v's is equivalent to Lorentz-covariant curvature constraints. Now, for field theories in dimensions greater than two, there is no clear-cut definition of 'complete integrability', but for the cases where this notion has any meaning, four-dimensional self-duality being the paradigm, the existence of a linear system is crucial. In particular, it is central to solution generating transforms.

Of particular significance amongst the systems (69), are those with $r(a)$ taking values 0 or $2a$ only and $\dot{r}(\dot{a})$ taking values 0 or $2\dot{a}$ only. In other words, if, for any $(a, \dot{a})$, u- or v-type spinors exist, they saturate *all* the corresponding indices on $\mathcal{D}_{[A][\dot{A}]}$. These linear systems therefore correspond to representations with indices $(a, \dot{a})$ falling into four disjoint sets $\Lambda_{ij}, i, j = 0, 1$ (where $i = r(a) \bmod 2a$ and $j = \dot{r}(\dot{a}) \bmod 2\dot{a}$)

$$\Lambda_{00} = \{(a, \dot{a}) \mid \mathcal{D}_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\varphi = 0\}, \tag{70}$$

$$\Lambda_{01} = \left\{(a, \dot{a}) \mid 2\dot{a} \geqslant 1,\ u^{+\dot{\alpha}_1} \cdots u^{+\dot{\alpha}_{2\dot{a}}} \mathcal{D}_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\varphi = 0\right\}, \tag{71}$$

$$\Lambda_{10} = \left\{(a, \dot{a}) \mid 2a \geqslant 1,\ v^{\oplus\alpha_1} \cdots v^{\oplus\alpha_{2a}} \mathcal{D}_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\varphi = 0\right\}, \tag{72}$$

$$\Lambda_{11} = \left\{(a, \dot{a}) \mid 2a \geqslant 1,\ 2\dot{a} \geqslant 1,\ v^{\oplus\alpha_1} \cdots v^{\oplus\alpha_{2a}} u^{+\dot{\alpha}_1} \cdots u^{+\dot{\alpha}_{2\dot{a}}} \mathcal{D}_{\alpha_1\cdots\alpha_{2a}\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{a}}}\varphi = 0\right\}. \tag{73}$$

The constraints which arise as integrability conditions for these systems may be formally solved following the generalised twistor construction given in the appendix of [10]. The harmonic space description of solutions may also be given on the lines of the construction reviewed for instance in [11-13]. The integrability conditions for (70)-(73) therefore provide classes of systems which are completely solvable in the sense of the standard self-duality equations and generalise the systems of higher-dimensional solvable gauge field constraints classified in [10]. They fall into two broad classes:

(I) with either only u-type or only v-type spinors present, which are exact analogues of self-dual or respectively anti-self-dual systems on $\mathcal{M}$.
(II) with both u-type and v-type spinors present, which are analogues of light-like integrable systems.

Class I. Generalised self-duality

For gauge fields on an arbitrary hyperspace $\mathcal{M}$, we define self-duality as the condition that the only non-zero irreducible curvatures are those appearing in (64) with a coefficient including at least one ϵ-tensor with dotted indices, i.e. those in the decomposition

$$\hat{F}^{[\dot{A}][\dot{B}]}_{[A][B]} = \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=1}^{\min(2\dot{a},2\dot{b})} S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot{\alpha}_{\dot{s}}\dot{\beta}_{\dot{s}}]} F^{[\dot{C}(\dot{s})]}_{[C(s)]}. \qquad (74)$$

The vanishing of the curvatures appearing in (64) but not in (74) is a natural generalisation of the self-duality equations.

For a given superspace $\mathcal{M}$, with coordinates enumerated by some set of Lorentz indices $\Lambda = \{(a,\dot{a})\}$, the vanishing of these curvatures are integrability conditions for linear systems containing equations of the form (70) with $2\dot{a} = 0$ and (71) only. The prototypical examples are standard self-duality and super-self-duality. Obviously analogous *anti*-self-dual systems containing equations of the form (70) with $2a = 0$ and (72) only can also be considered. In the following examples we restrict ourselves to the former self-dual systems.

Example I-1. $\Lambda_{01} = \{(\frac{1}{2},\frac{1}{2})\}$, with commutative algebra of the four components of $X_{\alpha\dot{\alpha}}$ (i.e. $\mathcal{A}$ is the Poincaré algebra). The linear system ($r = 0, \dot{r} = 1$)

$$u^{+\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}\varphi = 0 \qquad (75)$$

is precisely the Belavin–Zakharov–Ward linear system for the self-duality equations. The consistency conditions are

$$\left[u^{+\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}, u^{+\dot{\beta}}\mathcal{D}_{\beta\dot{\beta}}\right] = u^{+\dot{\alpha}}u^{+\dot{\beta}}\left[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}\right] = \epsilon_{\alpha\beta}u^{+\dot{\alpha}}u^{+\dot{\beta}}F_{\dot{\alpha}\dot{\beta}} = 0. \qquad (76)$$

If we insist that these hold for any choice of u, which is tantamount to requiring their Lorentz covariance, we obtain

$$\left[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}\right] = \epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta} \quad \Leftrightarrow \quad F_{\dot{\alpha}\dot{\beta}} = 0, \qquad (77)$$

i.e. the self-duality constraints.

Example I-2. $\Lambda_{00} = \{(\frac{1}{2},0)\}$, $\Lambda_{01} = \{(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\}$, with algebra of the vector fields $(X_\alpha, X_{\dot{\alpha}}, X_{\alpha\dot{\alpha}})$ having non-zero supercommutator, $\{X_\alpha, X_{\dot{\alpha}}\} = 2X_{\alpha\dot{\alpha}}$ (i.e. $\mathcal{A}$ is the super-Poincaré algebra). The system

$$\mathcal{D}_\alpha\varphi = 0, \qquad u^{+\dot{\beta}}\mathcal{D}_{\dot{\beta}}\varphi = 0, \qquad u^{+\dot{\beta}}\mathcal{D}_{\alpha\dot{\beta}}\varphi = 0 \qquad (78)$$

implies the super-self-duality constraints.

N-extended supersymmetrisations, on superspaces with odd coordinates having multiplicity N, $\{Y^{\alpha\dot\alpha}, Y^\alpha_m, Y^{m\dot\alpha}; m = 1,\dots,N\}$, and vector fields $\{X_{\alpha\dot\alpha} X^m_\alpha, X_{m\dot\alpha}; m = 1,\dots,N\}$ satisfying the algebra

$$\left\{X^m_\alpha, X_{n\dot\beta}\right\} = 2\delta^m_n X_{\alpha\dot\beta}, \tag{79}$$

with all other supercommutators amongst the X's vanishing and Jacobi-allowed X, Y supercommutators exist for arbitrary N [15] and the linear system

$$\mathcal{D}^m_\alpha \varphi = 0, \qquad u^{+\dot\beta}\mathcal{D}_{m\dot\beta}\varphi = 0, \qquad u^{+\dot\beta}\mathcal{D}_{\alpha\dot\beta}\varphi = 0 \tag{80}$$

implies the following curvature constraints equivalent to the N-extended super-self-duality equations:

$$\begin{aligned}
\{\mathcal{D}^m_\alpha, \mathcal{D}^n_\beta\} &= 0 & &\Leftrightarrow & &F^{mn}_{\alpha\beta} = 0, \quad F^{mn} = 0,\\
\{\mathcal{D}^m_\alpha, \mathcal{D}_{n\dot\alpha}\} &= 2\delta^m_n \mathcal{D}_{\alpha\dot\alpha} & &\Leftrightarrow & &F^m_{n\alpha\dot\alpha} = 0,\\
\left[\mathcal{D}^m_\alpha, \mathcal{D}_{\beta\dot\beta}\right] &= 0 & &\Leftrightarrow & &F^m_{\alpha\beta\dot\beta} = 0, \quad F^m_{\dot\beta} = 0,\\
\left\{\mathcal{D}_{m\dot\alpha}, \mathcal{D}_{n\dot\beta}\right\} &= \epsilon_{\dot\alpha\dot\beta} F_{mn} & &\Leftrightarrow & &F_{mn\dot\alpha\dot\beta} = 0,\\
\left[\mathcal{D}_{m\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] &= \epsilon_{\dot\alpha\dot\beta} F_{m\beta} & &\Leftrightarrow & &F_{m\beta\dot\alpha\dot\beta} = 0,\\
\left[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] &= \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta} & &\Leftrightarrow & &F_{\dot\alpha\dot\beta} = 0,
\end{aligned} \tag{81}$$

where, by construction, F_{mn} and F^{mn} are distinct curvatures.

Example I-3. $\Lambda_{01} = \{(\frac{1}{2}, \frac{n}{2})\}$, for fixed odd integer $n > 1$, with commutative algebra of the elements $X_{\alpha\dot\alpha_1\cdots\dot\alpha_n}$. In the bosonic space of dimension $2(n+1)$ with coordinates $Y^{\alpha\dot\alpha_1\cdots\dot\alpha_n}$, the irreducible parts of the gauge curvature are given by

$$\begin{aligned}
\left[\mathcal{D}_{\alpha\dot\alpha_1\cdots\dot\alpha_n}, \mathcal{D}_{\beta\dot\beta_1\cdots\dot\beta_n}\right] &= S(\dot\alpha_1\cdots\dot\alpha_n)\, S(\dot\beta_1\cdots\dot\beta_n)\\
&\quad\times\Bigg(\epsilon_{\alpha\beta}\sum_{m=0}^{\frac{1}{2}(n-1)} \epsilon_{[\dot\alpha_{2m}\dot\beta_{2m}]} F_{\dot\alpha_{2m+1}\cdots\dot\alpha_n\dot\beta_{2m+1}\cdots\dot\beta_n}\\
&\qquad + \sum_{m=0}^{\frac{1}{2}(n-1)} \epsilon_{[\dot\alpha_{2m+1}\dot\beta_{2m+1}]} F_{\alpha\beta\dot\alpha_{2m+2}\cdots\dot\alpha_n\dot\beta_{2m+2}\cdots\dot\beta_n}\Bigg),
\end{aligned} \tag{82}$$

where at the upper limit in the second sum $F_{\alpha\beta\dot\alpha_{n+1}\cdots\dot\alpha_n\dot\beta_{n+1}\cdots\dot\beta_n}$ simply stands for $F_{\alpha\beta}$.

The linear system [10]

$$u^{+\dot\alpha_1}\cdots u^{+\dot\alpha_n}\mathcal{D}_{\alpha\dot\alpha_1\cdots\dot\alpha_n}\varphi = 0 \tag{83}$$

yields the Lorentz-covariant constraint

$$F_{\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n} = 0, \tag{84}$$

which Ward considered as an example of a soluble system generalising self-duality [10].

We note that the linear systems

$$u^{+\dot\alpha_1}\cdots u^{+\dot\alpha_{\dot r}}\mathcal{D}_{\alpha\dot\alpha_1\cdots\dot\alpha_n}\varphi = 0, \qquad \text{for some fixed } \dot r, \quad 0 < \dot r < n, \tag{85}$$

yield integrability conditions with more of the irreducible pieces of the curvature in the decomposition (82) vanishing. These gauge field constraints are however not amenable to the twistor transform, the mapping to twistor space variables not being invertible for $0 < s < n$. It is therefore unclear to what extent such equations, arising as integrability conditions for linear systems, are actually 'integrable'. They do not appear to be exactly solvable in the sense of the standard self-duality equations.

Example I-4. $\Lambda_{01} = \{(\frac{n}{2},\frac{1}{2})\}$, for fixed odd integer $n > 1$. This example is clearly a mirror image of the previous one. The irreducible parts of the gauge curvature are

$$\begin{aligned}\left[\mathcal{D}_{\alpha_1\cdots\alpha_n\dot\alpha},\mathcal{D}_{\beta_1\cdots\beta_n\dot\beta}\right] &= S(\alpha_1\cdots\alpha_n)\,S(\beta_1\cdots\beta_n)\\ &\times\left(\epsilon_{\dot\alpha\dot\beta}\sum_{m=0}^{\frac{1}{2}(n-1)}\epsilon_{[\alpha_{2m}\beta_{2m}]}F_{\alpha_{2m+1}\cdots\alpha_n\beta_{2m+1}\cdots\beta_n}\right.\\ &\left.+\sum_{m=0}^{\frac{1}{2}(n-1)}\epsilon_{[\alpha_{2m+1}\beta_{2m+1}]}F_{\alpha_{2m+2}\cdots\alpha_n\beta_{2m+2}\cdots\beta_n\dot\alpha\dot\beta}\right).\end{aligned} \tag{86}$$

The linear system [10]

$$u^{+\dot\alpha}\mathcal{D}_{\alpha_1\cdots\alpha_n\dot\alpha}\varphi = 0 \tag{87}$$

yields the constraints

$$\begin{aligned}\left[\mathcal{D}_{\alpha_1\cdots\alpha_n\dot\alpha},\mathcal{D}_{\beta_1\cdots\beta_n\dot\beta}\right] &= S(\alpha_1\cdots\alpha_n)\,S(\beta_1\cdots\beta_n)\,\epsilon_{\dot\alpha\dot\beta}\\ &\times\sum_{m=0}^{\frac{1}{2}(n-1)}\epsilon_{[\alpha_{2m}\beta_{2m}]}F_{\alpha_{2m+1}\cdots\alpha_n\beta_{2m+1}\cdots\beta_n},\end{aligned} \tag{88}$$

which are equivalent to

$$F_{\dot\alpha\dot\beta} = 0, \tag{89}$$

together with

$$F_{\alpha_{2m}\cdots\alpha_n\beta_{2m}\cdots\beta_n\dot\alpha\dot\beta} = 0, \qquad \text{for all } m = 1,\ldots,\tfrac{1}{2}(n-1). \tag{90}$$

Example I-5. This example provides self-duality conditions for the hyperspace $\mathcal{M}$ of Section 4.3. We take $\Lambda_{00} = \{(\frac{1}{2},0),(\frac{3}{2},0)\}$ and $\Lambda_{01} = \{(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2}),(\frac{1}{2},1),(0,\frac{3}{2})\}$.

The seven equations of the corresponding linear system have 28 consistency conditions which imply that all the curvatures vanish except for 19 curvatures appearing with a coefficient consisting of at least one ϵ with dotted indices.

Class II. Light-like integrable systems

Including u and v spinors simultaneously, we may consider linear systems of the form (70)–(73), with all spinorial indices saturated,

$$\begin{aligned} \mathcal{D}\varphi &= 0, && \\ u^{+\dot\alpha_1}\cdots u^{+\dot\alpha_{2\dot a}}\mathcal{D}_{\dot\alpha_1\cdots\dot\alpha_{2\dot a}}\varphi &= 0, && \text{for } 2\dot a \geqslant 1, \\ v^{\oplus\alpha_1}\cdots v^{\oplus\alpha_{2a}}\mathcal{D}_{\alpha_1\cdots\alpha_{2a}}\varphi &= 0, && \text{for } 2a \geqslant 1, \\ v^{\oplus\alpha_1}\cdots v^{\oplus\alpha_{2a}}u^{+\dot\alpha_1}\cdots u^{+\dot\alpha_{2\dot a}}\mathcal{D}_{\alpha_1\cdots\alpha_{2a}\dot\alpha_1\cdots\dot\alpha_{2\dot a}}\varphi &= 0, && \text{for } 2a \geqslant 1, \quad 2\dot a \geqslant 1. \end{aligned} \tag{91}$$

Their consistency conditions express integrability along certain lines in $\mathcal{M}$, analogues of 'super null lines' in Minkowski space. These correspond to the vanishing of all curvatures which appear in the decomposition (64) with a coefficient including at least one ϵ-tensor (with either dotted or undotted indices). In other words, the only non-zero irreducible curvatures are those appearing in

$$\hat F^{[\dot A][\dot B]}_{[A][B]} = \sum_{s=0}^{\min(2a,2b)} \sum_{\dot s=1}^{\min(2\dot a,2\dot b)} S[A]S[\dot A]S[B]S[\dot B]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot\alpha_{\dot s}\dot\beta_{\dot s}]}F^{[\dot C(\dot s)]}_{[C(s)]}, \tag{92}$$

or in

$$\hat F^{[\dot A][\dot B]}_{[A][B]} = \sum_{s=1}^{\min(2a,2b)} \sum_{\dot s=0}^{\min(2\dot a,2\dot b)} S[A]S[\dot A]S[B]S[\dot B]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot\alpha_{\dot s}\dot\beta_{\dot s}]}F^{[\dot C(\dot s)]}_{[C(s)]} \tag{93}$$

and the zero curvatures are those appearing in (64) but not in (92) or in (93).

Example II-1. The linear equations in standard Minkowski space

$$v^{\oplus\alpha}u^{+\dot\beta}\mathcal{D}_{\alpha\dot\beta}\varphi = 0 \tag{94}$$

clearly imply

$$\left[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] = \epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} \tag{95}$$

and hence impose no constraints on the curvature. More generally, this conclusion follows from (91) whenever Λ contains only one bosonic representation.

Example II-2. The linear equations on N-extended super-Minkowski space corresponding to algebra (79),

$$v^{\oplus\alpha}\mathcal{D}^m_\alpha\varphi = 0, \qquad u^{+\dot\beta}\mathcal{D}_{n\dot\beta}\varphi = 0, \qquad v^{\oplus\alpha}u^{+\dot\beta}\mathcal{D}_{\alpha\dot\beta}\varphi = 0, \tag{96}$$

have as integrability conditions the conventional superspace constraints for $N = 1,2,3$ super-Yang-Mills theories [16],

$$
\begin{aligned}
\left\{\mathcal{D}^m_\alpha, \mathcal{D}^n_\beta\right\} &= \epsilon_{\alpha\beta} F^{mn} &\Leftrightarrow\quad & F^{mn}_{\alpha\beta} = 0,\\
\left\{\mathcal{D}^m_\alpha, \mathcal{D}_{n\dot\alpha}\right\} &= 2\delta^m_n \mathcal{D}_{\alpha\dot\alpha} &\Leftrightarrow\quad & F^m_{n\alpha\dot\alpha} = 0,\\
\left[\mathcal{D}^m_\alpha, \mathcal{D}_{\beta\dot\beta}\right] &= \epsilon_{\alpha\beta} F^m_{\dot\beta} &\Leftrightarrow\quad & F^m_{\alpha\beta\dot\beta} = 0,\\
\left\{\mathcal{D}_{m\dot\alpha}, \mathcal{D}_{n\dot\beta}\right\} &= \epsilon_{\dot\alpha\dot\beta} F_{mn} &\Leftrightarrow\quad & F_{mn\dot\alpha\dot\beta} = 0,\\
\left[\mathcal{D}_{m\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] &= \epsilon_{\dot\alpha\dot\beta} F_{m\beta} &\Leftrightarrow\quad & F_{m\beta\dot\alpha\dot\beta} = 0,\\
\left[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] &= \epsilon_{\alpha\beta} F_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta}. & &
\end{aligned}
\tag{97}
$$

These are needed in order that the superfield carries an irreducible representation of the supersymmetry algebra. For $N = 1,2$ these constraints do not have any dynamical consequences, but for $N = 3$ they turn out to be *equivalent* to the full (second-order) $N = 3$ Yang-Mills equations for the component fields in ordinary Minkowski space [16,9].

For $N = 4$, a further constraint is necessary to have an irreducible supermultiplet, viz.

$$F^{ij} = \tfrac{1}{2}\epsilon^{ijkl} F_{kl} \tag{98}$$

and these together with (97) are similarly equivalent to the full super-Yang-Mills equations for the $N = 4$ Yang-Mills multiplet. This further constraint does not arise as an integrability condition of the type considered here, so this is one example of an interesting set of constraints which is *not* a consequence of (69).

Example II-3. $\Lambda_{11} = \{(\frac{1}{2}, \frac{n}{2})\}$, for fixed even integer $n > 1$. The curvatures in this purely fermionic space are given by

$$
\begin{aligned}
\left\{\mathcal{D}_{\alpha\dot\alpha_1\cdots\dot\alpha_n}, \mathcal{D}_{\beta\dot\beta_1\cdots\dot\beta_n}\right\} = {} & S(\dot\alpha_1\cdots\dot\alpha_n)\, S(\dot\beta_1\cdots\dot\beta_n)\\
& \times\Bigg(\epsilon_{\alpha\beta}\sum_{m=0}^{\frac{1}{2}(n-2)} \epsilon_{[\dot\alpha_{2m+1}\dot\beta_{2m+1}]} F_{\dot\alpha_{2m+2}\cdots\dot\alpha_n\dot\beta_{2m+2}\cdots\dot\beta_n}\\
& + \sum_{m=0}^{\frac{1}{2}n} \epsilon_{[\dot\alpha_{2m}\dot\beta_{2m}]} F_{\alpha\beta\dot\alpha_{2m+1}\cdots\dot\alpha_n\dot\beta_{2m+1}\cdots\dot\beta_n}\Bigg),
\end{aligned}
\tag{99}
$$

where at the upper limit in the second sum $F_{\alpha\beta\dot\alpha_{n+1}\cdots\dot\alpha_n\dot\beta_{n+1}\cdots\dot\beta_n}$ stands for $F_{\alpha\beta}$.

The linear system

$$v^{\oplus\alpha} u^{+\dot\alpha_1}\cdots u^{+\dot\alpha_n}\mathcal{D}_{\alpha\dot\alpha_1\cdots\dot\alpha_n}\varphi = 0 \tag{100}$$

yields the Lorentz-covariant constraint

$$F_{\alpha\beta\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n} = 0, \tag{101}$$

similarly to (84).

7. Concluding remarks and outlook

We have considered generalised supersymmetry algebras, including elements transforming according to general Lorentz representations $(a, \dot{a})$ possibly having spin greater than one.

As an application of our analysis, we have generalised the notion of self-duality, the paradigm of solvability for systems with four independent variables, to the hyperspaces $\mathcal{M}$. We have thus obtained a hierarchy of solvable systems in an arbitrarily large number of variables.

There are several directions in which our considerations afford generalisation.

(a) In the complex setting, having in mind physical considerations, we have worked with representations $(a, \dot{a})$ of the algebra $su(2) \oplus su(2) \simeq sp(1) \oplus sp(1)$ of the group $SO(4)$. As a possible generalisation, this basic algebra could be extended to the direct sum of two algebras $g \oplus h$, for example, $sp(1) \oplus sp(n)$. Generalisations involving algebras which are not direct sums may also be considered.

(b) We have worked with supercommuting coordinates satisfying (23). Our formalism suggests generalisations to either non-commutative geometry and/or to quantum type (deformed) commutation relations, i.e. appropriate deformations of the superalgebra of the X's and Y's and of the gauge group.

(c) We could extend the superalgebra of X's and Y's to associative algebras containing suitable non-linear terms in the right-hand sides of the supercommutators. However, non-linear terms in the algebra $\mathcal{A}$ yield hyperspaces on which the minimal coupling of gauge fields is not well defined.

(d) We have restricted ourselves to flat spaces. A generalisation to curved spaces using *diffeomorphism-covariant* derivatives may also be considered, with the X's in $\mathcal{A}$ interpreted as vector fields spanning the tangent space.

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Appendix A. Multiplicities

A.1. *First case*

In this appendix we give the main formulas of the text modified in such a way as to allow for multiplicities of the different operators corresponding to the coordinates and the vector fields. Let $N^Z_{a,\dot{a}}$ be the multiplicity of the operators $Z = Y$ or $Z = X$ of given behaviour $a, \dot{a}$ under the Lorentz algebra. The set of coordinates $Y_{[A]}^{[\dot{A}]}(m)$ (resp. the set of differential vector operators $X_{[A]}^{[\dot{A}]}(m)$) is indexed by an integer m with $1 \leqslant m \leqslant N_{a,\dot{a}}^{Y}$ (resp. $1 \leqslant m \leqslant N_{a,\dot{a}}^{X}$). We expect that in the interesting cases $N_{a,\dot{a}}^{X} = N_{a,\dot{a}}^{Y}$ i.e. that the number of coordinates is exactly equal to the number of generalised derivatives (except perhaps for the Lorentz generators themselves which don't necessarily need their coordinate counterparts).

Using this notation, (23) becomes

$$\left[Y_{[A]}^{[\dot{A}]}(m), Y_{[B]}^{[\dot{B}]}(n)\right] = 0, \tag{A.1}$$

and (18) becomes

$$\begin{aligned} &\left[X_{[A]}^{[\dot{A}]}(m), X_{[B]}^{[\dot{B}]}(n)\right] \\ &= \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=0}^{\min(2\dot{a},2\dot{b})} \sum_{p=1}^{(N^{X}_{a+b-s,\dot{a}+\dot{b}-\dot{s}})} t(a,\dot{a},m;\, b,\dot{b},n;\, a+b-s, \dot{a}+\dot{b}-\dot{s}, p) \\ &\quad \times S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot{\alpha}_s\dot{\beta}_s]} X_{[C(s)]}^{[\dot{C}(\dot{s})]}(p), \end{aligned} \tag{A.2}$$

where the structure constants t not only have the Lorentz labels of the three tensors involved but also on their multiplicity indices. Finally, (24) becomes

$$\begin{aligned} &\left[X_{[A]}^{[\dot{A}]}(m), Y_{[B]}^{[\dot{B}]}(n)\right] \\ &= \sum_{s=0}^{\min(2a,2b)} \sum_{\dot{s}=0}^{\min(2\dot{a},2\dot{b})} \sum_{p=1}^{(N^{Y}_{a+b-s,\dot{a}+\dot{b}-\dot{s}})} u(a,\dot{a},m;\, b,\dot{b},n;\, a+b-s, \dot{a}+\dot{b}-\dot{s}, p) \\ &\quad \times S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[\alpha_s\beta_s]}\epsilon^{[\dot{\alpha}_s\dot{\beta}_s]} Y_{[C(s)]}^{[\dot{C}(\dot{s})]}(p) \\ &\quad + c(a,\dot{a};\, m,n)\, \delta^{ab}\delta_{\dot{a}\dot{b}} S[A]S[\dot{A}]\epsilon_{[\alpha_{2a}\beta_{2a}]}\epsilon^{[\dot{\alpha}_{2\dot{a}}\dot{\beta}_{2\dot{a}}]}, \end{aligned} \tag{A.3}$$

where the structure constants u depend also on the multiplicity indices involved. For a given a and a given $\dot{a}$, the central term parameters $c(a,\dot{a};\, m,n)$ form an $N_{a,\dot{a}}^{X} \times N_{a,\dot{a}}^{Y}$ matrix. We expect that in the interesting cases this matrix is square and non-singular and that it can be brought to the unit matrix δ_{mn} by redefining suitable linear combinations of the X's and of the Y's as the basic operators.

As far as the Jacobi identities are concerned, the multiplicity indices have to be taken into account. In particular, a summation on the 'internal' multiplicity index has to be included in the generalisation of, for instance, (31).

A.2. Second case

When a certain Lorentz representation $(a, \dot{a})$ occurs multiply, the corresponding tensors can be linearly combined. Using the matrices $A \in GL(N_{a\dot{a}}^{X}, \mathbb{C})$ for the X's and $B \in GL(N_{a\dot{a}}^{Y}, \mathbb{C})$ for the Y's, the allowed transformations are

$$X'(m') = A(m',m)\,X(m), \qquad Y'(n') = B(n',n)\,Y(n), \tag{A.4}$$

where we have indicated only the multiplicity index. (Note that, in the real setting, the field $\mathbb{R}$ should be used rather than $\mathbb{C}$).

We may use this freedom, for instance, to put the, a priori complex, matrix $c(a,\dot{a}; m,n)$, which transforms as

$$c'(m',n') = A(m',m)\,B(n',n)\,c(m,n) \quad \Leftrightarrow \quad c' = AcB^t, \tag{A.5}$$

in the canonical form

$$\tilde{c}(m,n) = \begin{pmatrix} 1_{(r,r)} & 0_{(r,s)} \\ 0_{(t,r)} & 0_{(t,s)} \end{pmatrix}, \tag{A.6}$$

where $1_{(r,r)}$ is the unit $r \times r$ matrix and $0_{(p,q)}$ is the zero matrix with p rows and q columns. The number of X's, i.e. $(r+t)$, can be different from the number of Y's, i.e. $(r+s)$. The stability group of this canonical $\tilde{c}(m,n)$ has matrices of the form

$$A = \begin{pmatrix} A^1_{(r,r)} & A^2_{(r,t)} \\ 0_{(t,r)} & A^4_{(t,t)} \end{pmatrix}, \qquad B = \begin{pmatrix} B^1_{(r,r)} & B^2_{(r,s)} \\ 0_{(s,r)} & B^4_{(s,s)} \end{pmatrix}, \qquad B^{1t} = (A^1)^{-1}, \tag{A.7}$$

where A^1, A^4 and B^4 are arbitrary invertible matrices.

This stability group can be used to put the t's and/or the u's which transform as (only indicating the multiplicity index)

$$\begin{aligned} t'(m',n',p') &= A(m',m)\,A(n'n)\,A^{-1}(p',p)\,t(m,n,p), \\ u'(m',n',p') &= A(m',m)\,B(n'n)\,B^{-1}(p',p)\,u(m,n,p), \end{aligned} \tag{A.8}$$

into some canonical form.

We could also consider non-linear transformations among the X's or the Y's which preserve the Lorentz transformation properties. However we feel that such transformations have little physical significance.

Appendix B. The $SU(2) \otimes SU(2)$ harmonics

In this appendix we outline the harmonic space notation [14]. We use two sets of harmonics parametrising auxiliary spaces. These are commuting spinors $u^+_{\dot\alpha}, u^-_{\dot\alpha}$ and $v^\oplus_\alpha, v^\ominus_\alpha$, which satisfy the constraints

$$u^{+\dot\alpha} u^-_{\dot\alpha} = 1, \qquad v^{\oplus\alpha} v^\ominus_\alpha = 1. \tag{B.1}$$

In the Euclidian case, an $SU(2)$ matrix can be written

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}, \qquad \text{with } |\alpha|^2 + |\beta|^2 = 1. \tag{B.2}$$

Let

$$V = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{B.3}$$

be a $U(1)$ subgroup of $SU(2)$. The quotient $SU(2)/U(1)$ given by the equivalence classes is a 2-sphere and can be described by the spinors

$$u^{+\dot\alpha} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad u^-_{\dot\alpha} = u^{+\dot\alpha *} \Leftrightarrow u^{-\dot\alpha} = \begin{pmatrix} -\beta^* \\ \alpha^* \end{pmatrix}, \tag{B.4}$$

up to their respective phases $u^{+\dot\alpha} \equiv \exp(i\phi)u^{+\dot\alpha}$ and $u^{-\dot\alpha} \equiv \exp(-i\phi)u^{-\dot\alpha}$.

Vector fields on these auxiliary spaces are given by

$$\begin{aligned} D^{++} &= u^{+\dot\alpha}\frac{\partial}{\partial u^{-\dot\alpha}}, & D^{\oplus\oplus} &= v^{\oplus\alpha}\frac{\partial}{\partial v^{\ominus\alpha}}, \\ D^{--} &= u^{-\dot\alpha}\frac{\partial}{\partial u^{+\dot\alpha}}, & D^{\ominus\ominus} &= v^{\ominus\alpha}\frac{\partial}{\partial v^{\oplus\alpha}}, \\ D^{+-} &= u^{+\dot\alpha}\frac{\partial}{\partial u^{+\dot\alpha}} - u^{-\dot\alpha}\frac{\partial}{\partial u^{-\dot\alpha}}, & D^{\oplus\ominus} &= v^{\oplus\alpha}\frac{\partial}{\partial v^{\oplus\alpha}} - v^{\ominus\alpha}\frac{\partial}{\partial v^{\ominus\alpha}}, \end{aligned} \tag{B.5}$$

and they satisfy two commuting $SU(2)$ algebras.

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